$$p_k(x) = \frac{\overbrace{f_k(x)}^{\to\ \mathcal{N}(\mu_k, \sigma^2)} \cdot \pi_k}{\text{const}}$$

from assuming $f_k(x) \sim \mathcal{N}(\mu_k, \sigma^2)$

$$\log(p_k(x)) = \underbrace{\log(\pi_k) + \overbrace{\frac{1}{2}\frac{2\mu_k x}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2}}}_{=: \delta_k(x) \text{ discriminant score}} + \underline{\text{const.(wrt } k)}$$

Boundary between cat. 1 and 2: $\delta_1(x) = \delta_2(x)$

$K=2$, $\underbrace{\pi_1 = \pi_2}_{\text{assumption}}$

In this case, decision boundary $(p=1)$

$$\delta_1(x) = \delta_2(x)$$

$$\cancel{\log(\pi_1)} + \frac{1}{\cancel{2}}\frac{\cancel{2}\mu_1 x}{\cancel{\sigma^2}} - \frac{\mu_1^2}{2\cancel{\sigma^2}} = \cancel{\log(\pi_2)} + \frac{\mu_2 x}{\cancel{\sigma^2}} - \frac{\mu_2^2}{2\cancel{\sigma^2}}$$

$$\Rightarrow x\,(\cancel{\mu_1 - \mu_2}) = \frac{\mu_1^2}{2} - \frac{\mu_2^2}{2} = \frac{1}{2}(\mu_1 + \mu_2)(\cancel{\mu_1 - \mu_2})$$

$$x = \frac{1}{2}(\mu_1 + \mu_2)$$

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More general: $\pi_1 \neq \pi_2$

$$\delta_1(x) = \delta_2(x)$$

$$x\frac{\mu_1}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} + \log(\pi_1) = x\frac{\mu_2}{\sigma^2} - \frac{\mu_2^2}{2\sigma^2} + \log(\pi_2)$$

$$\underset{\text{reformulate}}{\Rightarrow} \quad x = \underbrace{\frac{\mu_1+\mu_2}{2} + \sigma^2 \frac{\log(\pi_2)-\log(\pi_1)}{(\mu_1-\mu_2)}}$$

---

$$\underbrace{(x_1\ x_2)\cdot\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}}\underbrace{\begin{pmatrix}-2\\-2\end{pmatrix}}_{\mu_A-\mu_B} - \frac{1}{2}\underbrace{(1\ \ 1)\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$+\frac{1}{2}\underbrace{(3\ 3)}_{\mu_B^T}\underbrace{\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}}_{\Sigma^{-1}}\underbrace{\begin{pmatrix}3\\3\end{pmatrix}}_{\mu_B} + \underbrace{\log(\pi_1) - \log(\pi_2)}_{=0} = 0$$

$$\xrightarrow[\text{Matrix algebra}]{} \quad \begin{pmatrix}\frac{x_1}{2} & \frac{x_2}{2}\end{pmatrix}\begin{pmatrix}-2\\-2\end{pmatrix} - \frac{1}{2}1 + \frac{1}{2}\underbrace{\begin{pmatrix}\frac{3}{2} & \frac{3}{2}\end{pmatrix}\begin{pmatrix}3\\3\end{pmatrix}}_{9} = 0$$

$$\Rightarrow -x_1 - x_2 - \frac{1}{2} + \frac{9}{2} = 0$$

$$\Rightarrow \quad x_2 = 4 - x_1$$

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$$\log\left(\frac{p_1(x)}{1-p_1(x)}\right) = \log\left(\frac{p_1(x)}{p_2(x)}\right)$$

$p_1(x)$: $\pi_1 \cdot f_1(x)$

$p_2(x)$: $\pi_2 f_2(x)$

$$\underset{\text{ignoring constants}}{=} \log(\pi_1) - \log(\pi_2) + \frac{1}{2\sigma^2}(x-\mu_1)^2 + \frac{1}{2\sigma^2}(x-\mu_2)^2 + \text{const}$$

$$= \underbrace{c}_{\beta_0} + \underbrace{\frac{1}{2\sigma^2}\left(2\mu_1 - 2\mu_2\right)}_{\beta_1} \cdot x$$

$x^2$ terms cancel out