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# Algorithm 28

# Algorithm for the computation of Bessel function integrals *

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*Abstract:* A FORTRAN subroutine is given for the computation of integrals of the form $\int_0^c f(x) J_\nu(\alpha x)\,\mathrm{d}x$, where $\nu = 0, 1, \ldots, 10$.

*Keywords:* Numerical integration, Bessel function integrals, Fourier–Bessel transform, Chebyshev polynomials.

## 1. Introduction

We consider the numerical computation of

$$I(\alpha) = \int_0^c f(x) J_\nu(\alpha x)\,\mathrm{d}x \tag{1}$$

where $J_\nu(x)$ is the Bessel function of the first kind and of integer order $\nu (0 \leqslant \nu \leqslant 10)$, and where $\alpha$ and $c$ are positive real numbers. If the product $\alpha c$ is large, the integrand of (1) is rapidly oscillatory so that classical methods of numerical integration are uneconomical and unreliable. We present here a FORTRAN program for evaluating (1), using the modified Clenshaw–Curtis method described in [3]. This method is useful only if $f(x)$ is a smooth function on $[0, c]$. It is based on the approximation of $f(x)$ by a truncated series of shifted Chebyshev polynomials

$$f(x) \simeq {\sum_{k=0}^{N}}' a_k T_k^*(x/c) \tag{2}$$

Substituting (2) into (1) yields

$$I(\alpha) \simeq c {\sum_{k=0}^{N}}' a_k M_k \tag{3}$$

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where

$$M_k = \int_0^1 J_\nu(\alpha c x) T_k^*(x)\, \mathrm{d}x \tag{4}$$

The sequence $M_k$, $k = 0, 1, 2, \ldots$ satisfies a linear recurrence relation of order 8. The computation of $M_k$ is discussed in [3].

## 2. Implementation of the algorithm

The main subroutine for the computation of (1) is subroutine BESCHE. Since in practical computations, $I(\alpha)$ is usually desired for several values of $\alpha$, the user has to provide a sequence $\alpha_i$, $i = 1, 2, \ldots,$ NUM and the subroutine returns approximations to $I(\alpha_i)$. BESCHE calls two subroutines: CHEBY and BBMOM. Subroutine CHEBY computes the Chebyshev series coefficients $a_k$. The method of computation is based on the fast Fourier transform algorithm, as described by Stoer [4, p. 66]. CHEBY needs a subroutine for performing the fast Fourier transform. We have used subroutine FFT, written by de Boor [1]. CHEBY may be replaced by any other subroutine for constructing Chebyshev series expansions. Subroutine BBMOM evaluates the modified moments $M_k$, using the recurrence relation given in [3]. BBMOM calls the function subprograms BES9, BES10 and BESINT, which are needed for the computation of the starting values of the recurrence relation. BES9 and BES10 call RATEV, a function subprogram for evaluating rational functions. In BBMOM, a band system of linear equations is solved by subroutine DGBFA and subroutine DGBSL of LINPACK [1].

## 3. Sample program illustrating the use of BESCHE

We have computed

$$\int_0^\infty e^{-2x} J_\nu(\alpha x)\, \mathrm{d}x = \alpha^{-\nu}\left[(4+\alpha^2)^{1/2} - 2\right]^\nu (4+\alpha^2)^{-1/2}$$

for $\nu = 0, 1, \ldots, 10$ and $\alpha = 1, 10, 10^2, 10^3, 10^4$ and $10^5$, using BESCHE and after truncation of the infinite interval to the finite interval [0,30]. The driver program and the output are shown together with the integration routines. The computations are carried out on a PDP 11/60.

## References

[1] C. de Boor, FFT as nested multiplication, with a twist, *SIAM J. Sci. Statist. Comput.* **1** (1980).

[2] J.J. Dongarra, J.R. Bunch, C.B. Moler and G.W. Stewart, *LINPACK Users' Guide* (SIAM, Philadelphia 1979).

[3] R. Piessens and M. Branders, Modified Clenshaw–Curtis method for the computation of Bessel function integrals, *BIT* **23** (1983) 370–381.

[4] J. Stoer, *Einführung in die Numerische Mathematik* **I** (Springer, Heidelberg, 1972).

## Subroutine BESCHE

```
      SUBROUTINE BESCHE(F,C,ALFA,NUM,NU,N,RESULT,INFO)
C
C   COMPUTATION OF THE INTEGRALS OVER (0,C) OF THE FUNCTIONS
C   F(X)*J(ALFA(I)*X,NU), WHERE J(X,NU) IS THE BESSEL FUNCTION
C   OF THE FIRST KIND AND OF ORDER NU, AND I=1,2,...,NUM.
C
C   INPUT PARAMETERS
C      F      - DOUBLE PRECISION FUNCTION, DEFINING THE INTEGRAND
C               FUNCTION F(X). THE ACTUAL NAME FOR F NEEDS TO
C               BE DECLARED EXTERNAL IN THE CALLING PROGRAM
C      C      - DOUBLE PRECISION
C               UPPER LIMIT OF THE INTEGRATION INTERVAL
C      ALFA   - ONE-DIMENSIONAL ARRAY OF DOUBLE PRECISION
C               VARIABLES WHICH CONTAINS THE VALUES OF THE
C               PARAMETER IN THE ARGUMENT OF THE BESSEL FUNCTION
C               IF DABS(C*ALFA(I)).LT.1.0D-2, THE  INTEGRAL IS NOT
C               COMPUTED AND  INFO(I)=-2
C               THE DIMENSION OF ALFA IS NUM
C      NUM    - INTEGER
C               NUMBER OF INTEGRALS TO BE COMPUTED
C      NU     - INTEGER
C               ORDER OF THE BESSEL FUNCTION
C               IF NU.LT.0 .OR. NU.GT.10, THE ROUTINE WILL END
C               WITH INFO(I)=-1, I=1,2,...,NUM
C      N      - INTEGER
C               DEGREE OF THE CHEBYSHEV SERIES APPROXIMATION OF F(X)
C               IF N.LT.3 .OR. N.GT.50, THE ROUTINE WILL END WITH
C               INFO(I)=-3, I=1,2,...,NUM
C               THE VALUE OF N DETERMINES THE DIMENSION OF THE
C               VECTORS B AND Q,  ( B(2*N),Q(N+1))
C
C   OUTPUT PARAMETERS
C
C      RESULT- DOUBLE PRECISION ONE-DIMENSIONAL VECTOR
C               CONTAINING THE APPROXIMATIONS TO THE INTEGRALS
C               THE DIMENSION IS NUM
C       INFO  - INTEGER ONE-DIMENSIONAL VECTOR OF DIMENSION
C               NUM
C               INFO(I) CONTAINS INFORMATION ON THE
C               COMPUTATION OF RESULT(I)
C                INFO(I) = -3  INVALID VALUE OF N
C                INFO(I) = -2  INVALID VALUE OF C AND/OR ALFA
C                INFO(I) = -1  INVALID VALUE OF NU
C                INFO(I) = 0   NORMAL TERMINATION OF THE ROUTINE
C                INFO(I).GT.0  DIFFICULTIES IN BBMOM
C
C   SUBROUTINES OR FUNCTIONS NEEDED
C     CHEBY   - A SUBROUTINE FOR THE COMPUTATION OF THE
C               APPROXIMATE CHEBYSHEV SERIES COEFFICIENTS OF A
C               FUNCTION OVER A FINITE INTERVAL
C               CHEBY CALLS FFT AND FFTSTP
```

```
C     BBMOM
C     DABS (FORTRAN)
C
      DOUBLE PRECISION ALFA,B,C,F,Q,RESULT
      DOUBLE PRECISION DABS
      DIMENSION ALFA(NUM),B(100),Q(51),RESULT(NUM)
      INTEGER I,IER,INFO,K,N,NU,NUM,N1
      DIMENSION INFO(NUM)
      EXTERNAL F
      DO 10 I=1,NUM
        INFO(I) = 0
        RESULT(I) = 0.0D+00
   10 CONTINUE
      IF (C.EQ.0.0D+00) GO TO 999
      DO 20 I=1,NUM
        INFO(I) = -1
   20 CONTINUE
      IF (NU.LT.0.OR.NU.GT.10) GO TO 999
      DO 30 I=1,NUM
        INFO(I) = -3
   30 CONTINUE
C  COMPUTATION OF THE CHEBYSHEV SERIES COEFFICIENTS
      CALL CHEBY(F,0.0D+00,C,N,Q,IER)
      IF (IER.NE.0) GO TO 999
C  MAIN DO-LOOP
      DO 50 I=1,NUM
        INFO(I) = -2
        IF (DABS(ALFA(I)*C).LT.1.0D-02) GO TO 50
C  COMPUTATION OF THE MODIFIED MOMENTS
        CALL BBMOM(ALFA(I)*C,NU,N,B,IER)
        INFO(I) = IER
        IF (IER.NE.0) GO TO 50
C  COMPUTATION OF THE INTEGRALS
        RESULT(I)=0.5D+00*B(1)*Q(1)
        N1= N+1
        DO 40 K=2,N1
          RESULT(I)=RESULT(I)+B(K)*Q(K)
   40   CONTINUE
        RESULT(I)=0.5D+00*C*RESULT(I)
   50 CONTINUE
  999 RETURN
      END


      SUBROUTINE BBMOM(A,NU,N,B,INFO)
C
C  COMPUTATION OF INTEGRALS OVER (0,1) OF THE FUNCTION
C  J(A*X,NU)*TS(X,K), WHERE J(X,NU) IS THE BESSEL FUNCTION
C  OF THE FIRST KIND AND ORDER NU, AND TS(X,K) IS THE SHIFTED
C  CHEBYSHEV POLYNOMIAL OF DEGREE K, AND WHERE K=0,1,2,...,N.
C  THESE INTEGRALS ARE STORED IN  B(1),B(2),...B(N+1).
C
C
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```
C   INPUT PARAMETERS
C      A       - DOUBLE PRECISION
C                PARAMETER IN THE ARGUMENT OF THE BESSEL FUNCTION
C                THE ABSOLUTE VALUE OF A MUST BE GREATER THAN 0.01
C      NU      - INTEGER,  ORDER OF THE BESSEL FUNCTION
C                NU MUST BE GREATER THAN -1 AND LESS THAN 11
C      N       - INTEGER, MAXIMAL DEGREE OF THE CHEBYSHEV POLYNOMIAL
C                N MUST BE LESS THAN OR EQUAL TO 100
C
C   OUTPUT PARAMETERS
C      B       - ONE-DIMENSIONAL VECTOR OF DOUBLE PRECISION NUMBERS
C                WHICH CONTAINS THE INTEGRALS
C      INFO    - INTEGER
C                = -1   INVALID INPUT PARAMETERS
C                =  0   NORMAL COMPUTATION
C                .GT.0 SINGULARITY OF THE SYSTEM OF LINEAR EQUATIONS
C

C   SUBROUTINES OR FUNCTIONS NEEDED
C      BESINT
C      BES9
C      BES10
C      RATEV
C      DGBFA, DGBSL (LINPACK)
C      DAXPY,DDOT,DSCAL,IDAMAX (LINPACK)
C      MAX0,DABS (FORTRAN)
C
C
      DOUBLE PRECISION A,ABD,AC,AFA0,AFA1,AFA2,AFA3,AFN0,AFN1,
     *  AFN2,AFN3,AK,AL,AM,AN,ANU,ANU2,A2,A3,B,BB,BES,BESINT,
     *  BES9,BES10,BIN,DK,E,F,FACT,FACTOR,G,H,PSIN,PSIP,SIGN,TERM
      INTEGER I,IB,INFO,J,K,L,M,N,NA,NN,NU,NV
      DIMENSION ABD(15,200),AM(4),B(200),BB(4),BES(11),IPVT(200),
     *  PSIN(4),PSIP(4),TERM(4)
C
C
      INFO = -1
C   TEST ON THE INPUT PARAMETERS A AND NU
      IF (DABS(A).LT.1.0D-2 .OR. NU.LT.0 .OR. NU.GT.10) GO TO 999
C   TEST ON THE VALUE OF N. IF A VALUE OF N.GT.100 IS NEEDED,
C   THIS TEST MAY BE CHANGED, BUT THE DIMENSIONS OF THE ARRAYS
C   ABD,B AND IPVT MUST BE CHANGED ACCORDINGLY (=2*N)
      IF (N.GT.100) GO TO 999
      INFO = 0
      ANU = NU
      A2 = 2.0D+00/A
      NA = 0.5D+00*DABS(A)
      NV = N
      IF(N.GE.NA) NV = 3
C   COMPUTATION OF THE STARTING VALUES
      BES(11) = BES10(A)
      BES(10) = BES9(A)
      AK = 9.0D+00
      DO 10 I=1,9
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        L = 10-I
        BES(L) = A2*AK*BES(L+1)-BES(L+2)
        AK = AK-1.0D+00
 10   CONTINUE
      BIN = BESINT(A)
      IF (NU.EQ.0) GO TO 70
      F = BES(NU)
      IF (NU.EQ.1) GO TO 60
      IF (2*(NU/2).EQ.NU) GO TO 30
      AM(1) = 1.0D+00-BES(1)
      DO 20 L=2,NU,2
        BIN = BIN-2.0D+00*BES(L)
        AM(1) = AM(1)-2.0D+00*BES(L+1)
 20   CONTINUE
      GO TO 50
 30   AM(1) = BIN
      BIN = 1.0D+00+BES(1)
      DO 40 L=2,NU,2
        AM(1) = AM(1)-2.0D+00*BES(L)
        BIN = BIN-2.0D+00*BES(L-1)
 40   CONTINUE
 50   AM(2) = ANU*BIN-A*BES(NU)
      GO TO 80
 60   AM(1) = 1.0D+00-BES(1)
      AM(2) = ANU*BIN-A*BES(1)
      GO TO 80
 70   AM(1) = BIN
      F = -BES(2)
      AM(2) = -A*F
 80   F = A*F
      G = A
      DO 90 I=3,4
        E = I-2
        AM(I) = -(E*E-ANU*ANU)*AM(I-2)+((E+ANU)*BES(NU+1)-F)*G
        G = A*G
 90   CONTINUE
      B(1) = A2*AM(1)
      B(2) = (A2*AM(2)-AM(1))*A2
      B(3) = ((2.D+00*A2*AM(3)-4.D+00*AM(2))*A2+AM(1))*A2
      B(4) = (((4.0D+00*A2*AM(4)-12.0D+00*AM(3))*A2+9.0D+00*AM(2))
     *    *A2-AM(1))*A2
      IF (N.LE.3) GO TO 999
      E = -16.0D+00/(A*A)
      F = ANU*ANU+0.25D+00*A*A
      G = 4.0D+00*ANU*ANU
      H = 6.0D+00*(ANU*ANU-1.0D+00)-3.0D+00*A*A/8.0D+00
      DO 100 K=1,4
        L = 8-K
        B(L) = B(L-3)
 100  CONTINUE
      DO 110 K=1,4
        L = 8-K
        B(K) = B(L)
        BB(K) = B(K+3)
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  110 CONTINUE
      B(8) = ((9.0D+00-F)*B(6)+(4.0D+00+G)*B(5)-0.5D+00*H*B(4))*E
      IF (NV.EQ.4) GO TO 130
      NN = NV-4
C  FORWARD RECURSION
      DO 120 K = 1,NN
        DK = K
        B(K+8) = (((DK+3.0D+00)**2-F)*B(K+6)+(G+2.0D+00*(DK+
     *   2.0D+00))*B(K+5)-(2.0D+00*DK*DK+H)*B(K+4)+(G-2.0D+00*
     *   (DK-2.0D+00))*B(K+3)+((DK-3.0D+00)**2-F)*B(K+2))*E-B(K)
  120 CONTINUE
  130 NN = NV+1
      DO 140 K=1,NN
        B(K) = B(K+3)
  140 CONTINUE
      IF (N.LT.NA) GO TO 999
C  CONVERSION OF THE RECURRENCE RELATION INTO A BAND SYSTEM
C  OF LINEAR EQUATIONS. THE ARRAY ABD CONTAINS THE ELEMENTS
C  OF THE BAND MATRIX. THE VECTOR B CONTAINS THE RIGHT HAND SIDE
      IB = MAX0(N+19,2*NA)
      M = IB-4
      A2 = A*A
      A3 = A*A2
      ANU2 = ANU*ANU
      AC = A2/16.0D+00
      AL = ANU2+A2/4.0D+00
      AM = 4.0D+00*(ANU2+1.0D+00)
      AN = 6.0D+00*(ANU2-1.0D+00)-3.0D+00*A2/8.0D+00
      DO 150 I=1,M
        AK = I+1
        ABD(7,I) = AC
        ABD(8,I) = 0.0D+00
        ABD(9,I) = (AK+3.0D+00)**2-AL
        ABD(10,I) = AM+2.0D+00*(AK+1.0D+00)
        ABD(11,I) = -2.0D+00*(AK+2.0D+00)**2-AN
        ABD(12,I) = AM-2.0D+00*(AK+3.0D+00)
        ABD(13,I) = (AK+1.0D+00)**2-AL
        ABD(14,I) = 0.0D+00
        ABD(15,I) = AC
        B(I) = 0.0D+00
  150 CONTINUE
      B(1) = -(4.0D+00+AM)*BB(4)+(8.0D+00+AN-AC)*BB(3)+(4.0D+00-
     *  AM)*BB(2)-(1.0D+00-AL)*BB(1)
      B(2) = (18.0D+00+AN)*BB(4)+(6.0D+00-AM)*BB(3)+(AL-AC)*BB(2)
      B(3) = (8.0D+00-AM)*BB(4)-(1.0D+00-AL)*BB(3)-AC*BB(1)
      B(4) = -(4.0D+00-AL)*BB(4)-AC*BB(2)
      B(5) = -AC*BB(3)
      B(6) = -AC*BB(4)
      AFN0 = 0.0D+00
      AFN1 = 0.0D+00
      AFN2 = 0.0D+00
      AFN3 = 0.0D+00
      AFA0 = BES(NU+1)
      IF (NU.EQ.0) GO TO 180
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      AFA1 = A*BES(NU)-ANU*AFA0
      IF (NU.EQ.1) GO TO 170
      IF (NU.EQ.2) GO TO 160
      IF (NU.GT.3) GO TO 190
      AFN3 = 0.125D+00*A3
      GO TO 190
  160 AFN2 = 0.25D+00*A2
      GO TO 190
  170 AFN1 = 0.5D+00*A
      AFN3 = -0.375D+00*A3
      GO TO 190
  180 AFN0 = 1.0D+00
      AFN2 = -0.5D+00*A2
      AFA1 = -A*BES(2)
  190 AFA2 = (ANU2-A2)*AFA0-AFA1
      AFA3 = -2.0D+00*A2*AFA0+(ANU2-1.0D+00-A2)*AFA1-3.0D+00*AFA2
      PSIP(1) = -AFN0
      PSIN(1) = AFA0
      PSIP(2) = AFN0-1.5D+00*AFN1
      PSIN(2) = -AFA0-1.5D+00*AFA1
      PSIP(3) = -AFN0+7.5D+00*AFN1-3.75D+00*AFN2
      PSIN(3) = AFA0+7.5D+00*AFA1+3.75D+00*AFA2
      PSIP(4) = AFN0-31.5D+00*AFN1+52.5D+00*AFN2-13.125D+00*AFN3
      PSIN(4) = -AFA0-31.5D+00*AFA1-52.5D+00*AFA2-13.125D+00*AFA3
      SIGN = -1.0D+00
      IF (2*(IB/2).EQ.IB) SIGN = 1.0D+00
      J = M-1
      DO 210 K=1,2
        FACT = IB+K-1
        FACT = 1.0D+00/(FACT*FACT)
        FACTOR = FACT
        B(J) = 0.0D+00
        DO 200 I=1,4
          TERM(I) = (SIGN*PSIP(I)-PSIN(I))*FACTOR
          FACTOR = -FACTOR*FACT
          B(J) = B(J)+TERM(I)
  200   CONTINUE
        B(J) = -AC*B(J)
        J = J+1
        SIGN = -SIGN
  210 CONTINUE
C  SOLUTION OF THE BAND SYSTEM
      CALL DGBFA(ABD,15,M,6,2,IPVT,INFO)
      IF (INFO.NE.0) GO TO 999
      CALL DGBSL(ABD,15,M,6,2,IPVT,B,0)
      DO 220 K=4,N
        J = N+5-K
        B(J) = B(J-4)
  220 CONTINUE
      DO 230 K=1,4
        B(K) = BB(K)
  230 CONTINUE
  999 RETURN
      END
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      DOUBLE PRECISION FUNCTION BES9(X)
C
C  THIS FUNCTION EVALUATES THE BESSEL FUNCTION OF THE
C  FIRST KIND AND OF ORDER 9 (J9(X)), USING PIECEWISE
C  RATIONAL APPROXIMATIONS OVER THE INTERVALS (0,9),
C  (9,15),(15,20),(20,24),(24,27) AND (27,INFINITY).
C  IF (X.GT.-0.0005).AND.(X.LT.0.0005), THEN THE VALUE
C  OF BES9 IS SET EQUAL TO ZERO.
C
C  SUBROUTINES OR FUNCTIONS NEEDED:
C    DOUBLE PRECISION FUNCTION RATEV
C    DABS,DSIN,DSQRT (FORTRAN)
C
      DOUBLE PRECISION AX,A0,A1,A2,A3,A4,B1,B2,B3,B4,DABS,DSIN,
     *  DSQRT,PHI,PI,P1,P2,P3,P4,P5,P6,Q1,Q2,Q3,Q4,Q5,Q6,RATEV,X,
     *  XX,X2,Z
      DIMENSION P1(10),P2(11),P3(12),P4(12),P5(12),P6(4),Q1(4),Q2(5),
     *  Q3(6),Q4(6),Q5(6),Q6(5)
      DATA P1(1),P1(2),P1(3),P1(4),P1(5),P1(6),P1(7),P1(8),P1(9),
     *  P1(10)/
     *  0.1854902491017173D-08, -0.1842587910613865D-08,
     *  0.7837387220977662D-09, -0.1917167436889884D-09,
     *  0.3036760694047475D-10, -0.3300478546655368D-11,
     *  0.2519154646726653D-12, -0.1337607053441661D-13,
     *  0.4635731878134301D-15, -0.8356082103682622D-17/
      DATA Q1(1),Q1(2),Q1(3),Q1(4)/
     *  0.1000000000000000D+01, 0.1320257938321785D+00,
     *  0.6668676928897496D-02, 0.1302372209415135D-03/
      DATA P2(1),P2(2),P2(3),P2(4),P2(5),P2(6),P2(7),P2(8),P2(9),
     *  P2(10),P2(11)/
     *  0.2618222749327460D-10, -0.1161824065060082D-09,
     *  0.1469078535022866D-09, -0.9087888423672116D-10,
     *  0.3335054713962859D-10, -0.7972868764383670D-11,
     *  0.1302209139177694D-11, -0.1477434705098757D-12,
     *  0.1145923352692998D-13, -0.5652432743472362D-15,
     *  0.1394074067743551D-16/
      DATA Q2(1),Q2(2),Q2(3),Q2(4),Q2(5)/
     *  0.1000000000000000D+01, 0.2658685386934859D+00,
     *  0.3009635304371910D-01, 0.1734261540798329D-02,
     *  0.4334871460773923D-04/
      DATA P3(1),P3(2),P3(3),P3(4),P3(5),P3(6),P3(7),P3(8),P3(9),
     *  P3(10),P3(11),P3(12)/
     *  0.4436970719901405D-12, 0.1964948036022380D-11,
     * -0.3218776725138120D-11, 0.8676034786908869D-12,
     *  0.7326082305792477D-12,-0.6214954703731785D-12,
     *  0.2180296874003063D-12,-0.4528872682576228D-13,
     *  0.6071512754998844D-14,-0.5295741422349722D-15,
     *  0.2818532216445706D-16,-0.7194188457820855D-18/
      DATA Q3(1),Q3(2),Q3(3),Q3(4),Q3(5),Q3(6)/
     *  0.1000000000000000D+01, 0.3564698886755565D+00,
     *  0.5742302962424486D-01, 0.5270100530250009D-02,
     *  0.2783077213425045D-03, 0.6844202134343295D-05/
      DATA P4(1),P4(2),P4(3),P4(4),P4(5),P4(6),P4(7),P4(8),P4(9),
     *  P4(10),P4(11),P4(12)/
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     *  -0.1302876341371364D-12,-0.4083297892338410D-13,
     *   0.2433208347020279D-12,-0.7270771537877821D-13,
     *  -0.3958068269085081D-13, 0.2269105652033925D-13,
     *  -0.2479343215446627D-14,-0.7892761013771215D-15,
     *   0.2949656625862532D-15,-0.4306637451949708D-16,
     *   0.3244590478591921D-17,-0.1075826173718991D-18/
      DATA Q4(1),Q4(2),Q4(3),Q4(4),Q4(5),Q4(6)/
     *   0.1000000000000000D+01, 0.3989548431108367D+00,
     *   0.7303607692830352D-01, 0.7771284644526691D-02,
     *   0.4889347112413302D-03, 0.1493607295871117D-04/
      DATA P5(1),P5(2),P5(3),P5(4),P5(5),P5(6),P5(7),P5(8),P5(9),
     *   P5(10),P5(11),P5(12)/
     *   0.3311431839875282D-13, 0.1146977254616786D-13,
     *  -0.3440683419506924D-13, 0.1940398454964233D-14,
     *   0.4879634850948548D-14,-0.8291459059696609D-15,
     *  -0.1585593490156917D-15, 0.4948390039514027D-16,
     *  -0.1780410670498951D-17,-0.6684547383640223D-18,
     *   0.9136267508277833D-19,-0.3873432504388453D-20/
      DATA Q5(1),Q5(2),Q5(3),Q5(4),Q5(5),Q5(6)/
     *   0.1000000000000000D+01, 0.3975806314473493D+00,
     *   0.7242889514463364D-01, 0.7920655363995862D-02,
     *   0.5277540591769882D-03, 0.1933995155392605D-04/
      DATA P6(4),P6(3),P6(2),P6(1)/
     *  -0.1275004684511672D+00,-0.1163735571810356D+00,
     *   0.2474650965042455D-02,-0.2659305339394706D-04/
      DATA Q6(5),Q6(4),Q6(3),Q6(2),Q6(1)/
     *   0.1000000000000000D+01, 0.2357819796287977D+00,
     *  -0.1138193550158312D-01, 0.1870900085917186D-03,
     *  -0.1317302954498964D-05/
      DATA A0,A1,A2,A3,A4,B1,B2,B3,B4/
     *   0.2381156206599237D+02,-0.3598191873462367D+01,
     *   0.8232476093270257D-01,-0.8252417594349535D-03,
     *   0.7976035987759812D-09,-0.1004064583312075D+00,
     *   0.2166322581032705D-02,-0.2043954743877925D-04,
     *   0.1975488789534709D-10/
      DATA PI/3.141592653589793D+00/
      BES9 = 0.0D+00
      AX = DABS(X)
      IF(AX.LT.0.5D-03) GO TO 999
      X2 = X*X
      IF(AX.GT.9.0D+00) GO TO 10
      XX = 0.2469135802469136D-01*(X2-40.5D+00)
      BES9 = X*X2*X2*X2*X2*RATEV(10,4,P1,Q1,XX)
      GO TO 999
 10   IF(AX.GT.15.0D+00) GO TO 20
      XX = 0.1388888888888889D-01*(X2-153.0D+00)
      BES9 = X*X2*X2*X2*X2*RATEV(11,5,P2,Q2,XX)
      GO TO 999
 20   IF(AX.GT.20.0D+00) GO TO 30
      XX = 0.1142857142857143D-01*(X2-312.5D+00)
      BES9 = X*X2*X2*X2*X2*RATEV(12,6,P3,Q3,XX)
      GO TO 999
 30   IF(AX.GT.24.0D+00) GO TO 40
      XX = 0.1136363636363636D-01*(X2-488.0D+00)
```

```
      BES9 = X*X2*X2*X2*X2*RATEV(12,6,P4,Q4,XX)
      GO TO 999
 40   IF(AX.GT.27.0D+00) GO TO 50
      XX = 0.1307189542483660D-01*(X2-652.5D+00)
      BES9 = X*X2*X2*X2*X2*RATEV(12,6,P5,Q5,XX)
      GO TO 999
 50   Z = 1.0D+00/X2
      PHI = (((((A0*Z+A1)*Z+A2)*Z+A3)*Z+A4)/
     *  ((((Z+B1)*Z+B2)*Z+B3)*Z+B4))/AX-4.25D+00*PI+AX
      BES9 = DSQRT(2.0D+00/(PI*AX))*DSIN(PHI)*(1.0D+00+
     *  Z*RATEV(4,5,P6,Q6,Z))
      IF (X.LT.0.0D+00) BES9 = -BES9
 999  RETURN
      END


      DOUBLE PRECISION FUNCTION BES10(X)
C
C  THIS FUNCTION EVALUATES THE BESSEL FUNCTION OF THE FIRST
C  KIND AND OF ORDER 10 (J10(X)), USING PIECEWISE RATIONAL
C  APPROXIMATIONS OVER THE INTERVALS (0,6),(6,10),(10,16),
C  (16,21),(21,25),(25,28) AND (28,INFINITY).
C  IF (X.GT.-0.002).AND.(X.LT.0.002) THEN BES10 IS SET EQUAL
C  TO ZERO.
C
C  SUBROUTINES OR FUNCTIONS NEEDED :
C     DOUBLE PRECISION FUNCTION RATEV
C     DABS,DSIN,DSQRT (FORTRAN)
C
C
      DOUBLE PRECISION AX,A0,A1,A2,A3,A4,B1,B2,B3,B4,DABS,DSIN,
     *  DSQRT,PHI,PI,P1,P2,P3,P4,P5,P6,P7,Q1,Q2,Q3,Q4,Q5,Q6,Q7,
     *  RATEV,X,XX,X2,Z
      DIMENSION P1(9),P2(10),P3(12),P4(12),P5(12),P6(12),P7(4),
     *  Q1(3),Q2(4),Q3(6),Q4(6),Q5(6),Q6(6),Q7(5)
      DATA P1(1),P1(2),P1(3),P1(4),P1(5),P1(6),P1(7),P1(8),P1(9)/
     *  0.1774630994301248D-09,-0.6686904408975376D-10,
     *  0.1110361330353677D-10,-0.1084017298554262D-11,
     *  0.6964413144397730D-13,-0.3097133375204923D-14,
     *  0.9627429677517041D-16,-0.2001467305704956D-17,
     *  0.2281135549255523D-19/
      DATA Q1(1),Q1(2),Q1(3)/
     *  0.1000000000000000D+01, 0.4718042809659422D-01,
     *  0.6506632750385951D-03/
      DATA P2(1),P2(2),P2(3),P2(4),P2(5),P2(6),P2(7),P2(8),P2(9),
     *  P2(10)/
     *  0.5090631412787398D-10,-0.3825365688580096D-10,
     *  0.1228604370321993D-10,-0.2268537220357605D-11,
     *  0.2713495394163037D-12,-0.2228922839346225D-13,
     *  0.1287159256741880D-14,-0.5176915417534199D-16,
     *  0.1360705933907432D-17,-0.1862932740017242D-19/
      DATA Q2(1),Q2(2),Q2(3),Q2(4)/
     *  0.1000000000000000D+01, 0.1007376163415838D+00,
```

```
     *   0.3879154735551808D-02, 0.5771243607943360D-04/
       DATA P3(1),P3(2),P3(3),P3(4),P3(5),P3(6),P3(7),P3(8),P3(9),
     *   P3(10),P3(11),P3(12)/
     *   0.1065510847621105D-11,-0.4329699038785094D-11,
     *   0.5339228259912200D-11,-0.3285578666112747D-11,
     *   0.1214258692405932D-11,-0.2956919071958690D-12,
     *   0.4989703866767861D-13,-0.5977039797271119D-14,
     *   0.5089576727127421D-15,-0.2997151277745661D-16,
     *   0.1123738382106986D-17,-0.2085754060951120D-19/
       DATA Q3(1),Q3(2),Q3(3),Q3(4),Q3(5),Q3(6)/
     *   0.1000000000000000D+01, 0.2819164150067740D+00,
     *   0.3553915544886188D-01, 0.2521054227091571D-02,
     *   0.1013954652969954D-03, 0.1865503374180758D-05/
       DATA P4(1),P4(2),P4(3),P4(4),P4(5),P4(6),P4(7),P4(8),P4(9),
     *   P4(10),P4(11),P4(12)/
     *   0.7685378555412953D-14, 0.7215207415691145D-13,
     *  -0.9915170840485140D-13, 0.2036598752990882D-13,
     *   0.2606319351448109D-13,-0.1997004498321409D-13,
     *   0.6809935791703558D-14,-0.1399032129805053D-14,
     *   0.1870692960142853D-15,-0.1636210402847266D-16,
     *   0.8767012277423275D-18,-0.2259777865901759D-19/
       DATA Q4(1),Q4(2),Q4(3),Q4(4),Q4(5),Q4(6)/
     *   0.1000000000000000D+01, 0.3636961325396057D+00,
     *   0.5968653281623899D-01, 0.5571448182260652D-02,
     *   0.2986926393636037D-03, 0.7441174275962927D-05/
       DATA P5(1),P5(2),P5(3),P5(4),P5(5),P5(6),P5(7),P5(8),P5(9),
     *   P5(10),P5(11),P5(12)/
     *  -0.3265191568559689D-14,-0.2673502283957873D-14,
     *   0.6982636990266155D-14,-0.1465510180585606D-14,
     *  -0.1298957833422274D-14, 0.6214669983266875D-15,
     *  -0.5445020844371717D-16,-0.2437659521403179D-16,
     *   0.8263191290877864D-17,-0.1172008800500900D-17,
     *   0.8710720003652106D-19,-0.2870191564326547D-20/
       DATA Q5(1),Q5(2),Q5(3),Q5(4),Q5(5),Q5(6)/
     *   0.1000000000000000D+01, 0.4024396699471615D+00,
     *   0.7418588417256544D-01, 0.7931767769889511D-02,
     *   0.5001434770891837D-03, 0.1525705081137765D-04/
       DATA P6(1),P6(2),P6(3),P6(4),P6(5),P6(6),P6(7),P6(8),P6(9),
     *   P6(10),P6(11),P6(12)/
     *   0.7562542232885177D-15, 0.5929037276410094D-15,
     *  -0.8587805112950065D-15,-0.2640252249506698D-16,
     *   0.1314965324525768D-15,-0.1705979530768308D-16,
     *  -0.4726668655225812D-17, 0.1220320645418400D-17,
     *  -0.3070675472170065D-19,-0.1760916292237040D-19,
     *   0.2259147563618146D-20,-0.9347401847168571D-22/
       DATA Q6(1),Q6(2),Q6(3),Q6(4),Q6(5),Q6(6)/
     *   0.1000000000000000D+01, 0.3971425753462018D+00,
     *   0.7230260152032716D-01, 0.7875074749927937D-02,
     *   0.5215024196414315D-03, 0.1879340869296128D-04/
       DATA P7(4),P7(3),P7(2),P7(1)/
     *  -0.1074988134249699D+00,-0.9784257666988060D-01,
     *   0.1849940252047095D-02,-0.1572345626964698D-04/
       DATA Q7(5),Q7(4),Q7(3),Q7(2),Q7(1)/
     *   0.1000000000000000D+01, 0.1976639631961117D+00,
```

```
     *  -0.8116606043381508D-02, 0.1125459366156353D-03,
     *  -0.6305145371285454D-06/
      DATA A0,A1,A2,A3,A4,B1,B2,B3,B4/
     *   0.2957068823498488D+02,-0.3610990629755978D+01,
     *   0.7101086382801846D-01,-0.5710280107822520D-03,
     *  -0.4249613732732543D-08,-0.8215705066334004D-01,
     *   0.1513233634668198D-02,-0.1144851750726675D-04,
     *  -0.8520528787437206D-10/
      DATA PI/3.141592653589793D+00/
      BES10 = 0.0D+00
      AX = DABS(X)
      IF(AX.LT.0.2D-2) GO TO 999
      X2 = X*X
      IF(AX.GT.6.0D+00) GO TO 10
      XX = 0.5555555555555556D-01*(X2-18.0D+00)
      BES10 = X2*X2*X2*X2*X2*RATEV(9,3,P1,Q1,XX)
      GO TO 999
 10   IF(AX.GT.10.0D+00) GO TO 20
      XX = 0.3125D-01*(X2-68.0D+00)
      BES10 = X2*X2*X2*X2*X2*RATEV(10,4,P2,Q2,XX)
      GO TO 999
 20   IF(AX.GT.16.0D+00) GO TO 30
      XX = 0.1282051282051282D-01*(X2-178.0D+00)
      BES10 = X2*X2*X2*X2*X2*RATEV(12,6,P3,Q3,XX)
      GO TO 999
 30   IF(AX.GT.21.0D+00) GO TO 40
      XX = 0.1081081081081081D-01*(X2-348.5D+00)
      BES10 = X2*X2*X2*X2*X2*RATEV(12,6,P4,Q4,XX)
      GO TO 999
 40   IF(AX.GT.25.0D+00) GO TO 50
      XX = 0.1086956521739130D-01*(X2-533.0D+00)
      BES10 = X2*X2*X2*X2*X2*RATEV(12,6,P5,Q5,XX)
      GO TO 999
 50   IF(AX.GT.28.0D+00) GO TO 60
      XX = 0.1257861635220126D-01*(X2-704.5D+00)
      BES10 = X2*X2*X2*X2*X2*RATEV(12,6,P6,Q6,XX)
      GO TO 999
 60   Z = 1.0D+00/X2
      PHI = (((((A0*Z+A1)*Z+A2)*Z+A3)*Z+A4)/
     *    ((((Z+B1)*Z+B2)*Z+B3)*Z+B4))/AX -4.75D+00*PI+AX
      BES10 = DSQRT(2.0D+00/(PI*AX))*DSIN(PHI)*
     *   (1.0D+00+Z*RATEV(4,5,P7,Q7,Z))
 999  RETURN
      END


      DOUBLE PRECISION FUNCTION RATEV(N,M,P,Q,X)
C
C  THIS ROUTINE IS CALLED BY BES9 AND BES10 . IT EVALUATES A
C  RATIONAL FUNCTION WITH NUMERATOR P(N)*X**(N-1)+P(N-1)*X**(N-2)+
C  +...+P(1) AND DENOMINATOR Q(M)*X**(M-1)+Q(M-1)*X**(M-2)+..+Q(1)
C
      DOUBLE PRECISION P,Q,S,T,X
```

```
      INTEGER I,K,M,N,NM
      DIMENSION P(N),Q(M)
      S = P(N)
      K = N
      NM = N-1
      DO 10 I=1,NM
        K = K-1
        S = S*X+P(K)
   10 CONTINUE
      T = Q(M)
      K = M
      NM = M-1
      DO 20 I=1,NM
        K = K-1
        T = T*X+Q(K)
   20 CONTINUE
      RATEV = S/T
      RETURN
      END


      DOUBLE PRECISION FUNCTION BESINT(X)
C
C  THIS FUNCTION EVALUATES THE INTEGRAL OVER (0,X) OF THE BESSEL
C  FUNCTION OF THE FIRST KIND AND ORDER ZERO (J0(X)).
C  THE APPROXIMATION IS A CHEBYSHEV SERIES APPROXIMATION, GIVEN BY
C  Y.L.LUKE, THE SPECIAL FUNCTIONS AND THEIR APPROXIMATIONS, VOL.2,
C  ACADEMIC PRESS, NEW YORK, 1969>
C
C
      DOUBLE PRECISION A,ABSX,B,BESINT,C,D,F,H,X,Y
      DOUBLE PRECISION DABS,DCOS,DSIN,DSQRT
      DIMENSION A(2),B(16),C(51)
      INTEGER IER,J,K,L,M,N
C
      DATA B(1),B(2),B(3),B(4),B(5),B(6),B(7),B(8),B(9),
     *     B(10),B(11),B(12),B(13),B(14),B(15),B(16)
     *                        /1.29671754121053D0, -.36520274074158 54D0
     *, .50821888566078930D0, -.30180691211699830D0,  .08576038744155830D0
     *,-.01444107253850050D0,  .00162455576482270D0, -.00013148973200730D0
     *, .00000805230017150D0, -.00000038695337760D0,  .00000001500207420D0
     *,-.00000000047960700D0,  .00000000001286890D0, -.00000000000029410D0
     *, .00000000000000580D0, -.00000000000000001D0/
      DATA C(1),C(2),C(3),C(4),C(5),C(6),C(7),C(8),C(9),C(10),
     *     C(11),C(12),C(13),C(14),C(15),C(16),C(17),C(18),C(19),
     *     C(20),C(21),C(22),C(23),C(24),C(25)
     *                        /1.97481523162976 9D0, -.01622955223898780D0
     *,-.00327411179733920D0,  .00036927699265510D0, -.00002083713476100D0
     *,-.00000682861017200D0,  .00000141140889470D0, -.00000010699959820D0
     *,-.00000002685706470D0,  .00000001285294900D0, -.00000000278292760D0
     *, .00000000023353690D0,  .00000000008916780D0, -.00000000005151790D0
     *, .00000000001490480D0, -.00000000000242800D0, -.00000000000015150D0
     *, .00000000000028870D0, -.00000000000012740D0,  .00000000000003610D0
```

```
     *,-.00000000000000057D0, -.00000000000000008D0,  .00000000000000011D0
     *,-.00000000000000005D0,  .00000000000000002D0/
      DATA C(26),C(27),C(28),C(29),C(30),C(31),C(32),C(33),C(34),C(35),
     *     C(36),C(37),C(38),C(39),C(40),C(41),C(42),C(43),C(44),C(45),
     *     C(46),C(47),C(48),C(49),C(50),C(51)
     *                       /-.11553334948198900D0, -.05561793742411520D0
     *, .00240404107087260D0,  .00019647777633030D0, -.00005462157649810D0
     *, .00000496153395630D0,  .00000043762392900D0, -.00000026476639700D0
     *, .00000005344509820D0, -.00000000378906540D0, -.00000000153672500D0
     *, .00000000075741250D0, -.00000000018616120D0,  .00000000002163340D0
     *, .00000000000480020D0, -.00000000000382170D0,  .00000000000133840D0
     *,-.00000000000029300D0,  .00000000000001800D0,  .00000000000001980D0
     *,-.00000000000001220D0,  .00000000000000440D0, -.00000000000000110D0
     *, .00000000000000001D0,  .00000000000000001D0, -.00000000000000001D0
C
      BESINT = 0.0D+00
      H = 0.0D+00
      ABSX = DABS(X)
      IF(X.EQ.0.0D+00) GO TO 999
      IF (ABSX.GT.8.0D+00) GO TO 20
      Y = 0.625D-01*ABSX*ABSX-2.0D+00
      F = B(16)
      DO 10 K=1,15
         D = BESINT
         BESINT = F
         L=16-K
         F = B(L)+Y*BESINT-D
   10 CONTINUE
      BESINT = 0.125D+00*X*(F-BESINT)
      GO TO 999
   20 Y = 2.0D+01/ABSX-2.0D+00
      M = 25
      N = 24
      DO 40 J=1,2
         F = C(M)
         DO 30 K=1,N
            D = H
            H = F
            L = M-K
            F = C(L)+Y*H-D
   30    CONTINUE
         A(J) = 0.5D+00*(F-D)
         M = 51
         N = 25
         H = 0.0D+00
   40 CONTINUE
      H = ABSX+0.785398163397448 3D+00
      BESINT = 1.0D+00-0.797884560802865 4D+00*(DCOS(H)*A(1)-
     * DSIN(H)*A(2))/DSQRT(ABSX)
      IF (X.LT.0.0D+00) BESINT = -BESINT
  999 RETURN
      END
```

```
      SUBROUTINE CHEBY(F,A,B,N,C,IER)
C  THIS IS AN EXPERIMENTAL SUBROUTINE (NOT WRITTEN IN STANDARD
C  FORTRAN, SINCE DOUBLE PRECISION COMPLEX ARITHMETIC IS USED),
C  WHICH COMPUTES APPROXIMATIONS C(I),I=1,2,...,N+1 TO THE
C  COEFFICIENTS OF THE TRUNCATED CHEBYSHEV SERIES  OF DEGREE N
C  OF THE FUNCTION F, OVER THE INTERVAL (A,B).
C
C  THE METHOD OF COMPUTATION IS BASED ON THE FFT-ALGORITHM AS
C  DESCRIBED BY J.STOER, EINFUEHRUNG IN DIE NUMERISCHE MATHEMATIK I
C  SPRINGER VERLAG 1972,P.66
C
C  THIS ROUTINE CALLS A SUBROUTINE FOR COMPUTING THE FFT. WE HAVE
C  USED SUBROUTINE FFT, WRITTEN BY C.DE BOOR, FFT AS NESTED
C  MULTIPLICATION WITH A TWIST, SIAM J.SCIENT.STAT.COMP.1 (1980),
C  PP.171-178
C
C  INPUT PARAMETERS
C     F      - DOUBLE PRECISION FUNCTION
C              THIS FUNCTION HAS TO BE APPROXIMATED BY A TRUNCATED
C              CHEBYSHEV SERIES EXPANSION
C     A,B    - DOUBLE PRECISION VARIABLES
C              LIMITS OF THE INTERVAL OF APPROXIMATION
C     N      - INTEGER
C              DEGREE OF THE APPROXIMATION (N.GT.2 .AND. N.LE.50)
C
C  OUTPUT PARAMETERS
C     C      - ONE-DIMENSIONAL ARRAY OF DOUBLE PRECISION VARIABLES
C              CONTAINING THE APPROXIMATIONS TO THE CHEBYSHEV SERIES
C              COEFFICIENTS. THE FIRST COEFFICIENT (C(1)) HAS TO BE
C              HALVED.
C     IER    - INTEGER
C               =0  NORMAL COMPUTATION
C               =1  INVALID N
C
      DOUBLE PRECISION A,B,BMA,BPA,C,DC,DI,DN,F,PI
      DOUBLE PRECISION DCOS,DREAL
      INTEGER I,IER,INZEE,J,L,N,ND2,NP1,N2
      DOUBLE COMPLEX AA,BB,DCMPLX
      DIMENSION AA(100),BB(100),C(51)
      IER = 1
      IF (N.LT.3.OR.N.GT.50) GO TO 999
      IER = 0
      NP1 = N+1
      N2 = N*2
      PI = 3.141592653589793D+00
      DN = N
      BMA = (B-A)/2.0D+00
      BPA = (B+A)/2.0D+00
      AA(1) = DCMPLX(F(B),0.0D+00)
      AA(NP1) = DCMPLX(F(A),0.0D+00)
      ND2 = (N+1)/2
      DO 10 I=2,ND2
        DI = I-1
        DC = DCOS(PI*DI/DN)
```

```
        AA(I) = DCMPLX(F(DC*BMA+BPA),0.0D+00)
        L = N+2-I
        AA(L) = DCMPLX(F(-DC*BMA+BPA),0.0D+00)
   10 CONTINUE
      IF(2*(N/2).EQ.N) AA(ND2+1) = DCMPLX(F(BPA),0.0D+00)
      DO 20 I=2,N
        J = N2-I+2
        AA(J) = AA(I)
   20 CONTINUE
      INZEE = 1
      CALL FFT(AA,BB,N2,INZEE)
      IF(INZEE.EQ.2) GO TO 40
      DO 30 I=1,NP1
        C(I) = DREAL(AA(I))/DN
   30 CONTINUE
      GO TO 60
   40 DO 50 I=1,NP1
        C(I) = DREAL(BB(I))/DN
   50 CONTINUE
   60 C(NP1) = C(NP1)*0.5D+00
  999 RETURN
      END
C     DRIVER PROGRAM FOR SUBROUTINE BESCHE
C     COMPUTATION OF THE INTEGRAL OF EXP(-2*X)*J(ALFA(I)*X,NU)
C     OVER (0,30)
C     ALFA(I)=1,10,100,1000,10000,100000
C     NU=0,1,...,10
      DOUBLE PRECISION AB,ALFA,DIF,EXACT,FUN,RESULT
      DOUBLE PRECISION DSQRT
      INTEGER I,INFO,K,N,NU,NUM,NUU
      DIMENSION ALFA(6),INFO(6),RESULT(6)
      EXTERNAL FUN
      NUM = 6
      N = 30
      ALFA(1) = 1.0D+00
      DO 10 K=2,NUM
        ALFA(K) = 1.0D+01*ALFA(K-1)
   10 CONTINUE
      WRITE(6;900)
      DO 30 NUU=1,11
        NU = NUU-1
C     COMPUTATION OF THE INTEGRALS
        CALL BESCHE(FUN,3.0D+01,ALFA,NUM,NU,N,RESULT,INFO)
C     COMPUTATION OF THE EXACT VALUE OF THE INTEGRALS AND OF
C     THE ERROR
        DO 20 I=1,NUM
          AB = DSQRT(4.0D+00+ALFA(I)**2)
          EXACT = ((AB-2.0D+00)/ALFA(I))**NU/AB
          DIF = EXACT-RESULT(I)
          WRITE(6,901) NU,ALFA(I),EXACT,DIF,IER,N
   20   CONTINUE
   30 CONTINUE
  900 FORMAT(2HNU,6X,7HALFA(I),6X,14HEXACT INTEGRAL,8X,
     *  10HABS. ERROR,3X,4HINFO,4X,1HN/1X)
```

```
  901 FORMAT(I2,3X,F10.1,3X,D23.16,2X,D9.2,2X,I5,2X,I5)
      STOP
      END
```

```
      DOUBLE PRECISION FUNCTION FUN(X)
      DOUBLE PRECISION  DEXP,X
      FUN=DEXP(-2.0D+00*X)
      RETURN
      END
```

| NU | ALFA(I) | EXACT INTEGRAL | ABS. ERROR | INFO | N |
|---|---|---|---|---|---|
| 0 | 1.0 | 0.4472135954999579D+00 | 0.25D-09 | 0 | 30 |
| 0 | 10.0 | 0.9805806756909202D-01 | -0.28D-09 | 0 | 30 |
| 0 | 100.0 | 0.9998000599800070D-02 | -0.17D-10 | 0 | 30 |
| 0 | 1000.0 | 0.9999980000060000D-03 | -0.17D-13 | 0 | 30 |
| 0 | 10000.0 | 0.9999999800000006D-04 | -0.17D-16 | 0 | 30 |
| 0 | 100000.0 | 0.9999999998000000D-05 | -0.19D-19 | 0 | 30 |
| 1 | 1.0 | 0.1055728090000841D+00 | -0.68D-11 | 0 | 30 |
| 1 | 10.0 | 0.8038838648618160D-01 | 0.56D-08 | 0 | 30 |
| 1 | 100.0 | 0.9800039988003999D-02 | -0.84D-10 | 0 | 30 |
| 1 | 1000.0 | 0.9980000399998800D-03 | -0.87D-12 | 0 | 30 |
| 1 | 10000.0 | 0.9998000000040000D-04 | -0.87D-14 | 0 | 30 |
| 1 | 100000.0 | 0.9999800000000040D-05 | -0.87D-16 | 0 | 30 |
| 2 | 1.0 | 0.2492235949962145D-01 | -0.22D-11 | 0 | 30 |
| 2 | 10.0 | 0.6590271297461938D-01 | 0.38D-08 | 0 | 30 |
| 2 | 100.0 | 0.9605999000279910D-02 | -0.12D-09 | 0 | 30 |
| 2 | 1000.0 | 0.9960059999900001D-03 | -0.17D-11 | 0 | 30 |
| 2 | 10000.0 | 0.9996000599999990D-04 | -0.17D-13 | 0 | 30 |
| 2 | 100000.0 | 0.9999600006000000D-05 | -0.17D-15 | 0 | 30 |
| 3 | 1.0 | 0.5883371001598307D-02 | -0.61D-11 | 0 | 30 |
| 3 | 10.0 | 0.5402730129633384D-01 | -0.16D-08 | 0 | 30 |
| 3 | 100.0 | 0.9415800027992802D-02 | -0.14D-09 | 0 | 30 |
| 3 | 1000.0 | 0.9940159800000280D-03 | -0.25D-11 | 0 | 30 |
| 3 | 10000.0 | 0.9994001599800000D-04 | -0.26D-13 | 0 | 30 |
| 3 | 100000.0 | 0.9999400015999800D-05 | -0.26D-15 | 0 | 30 |
| 4 | 1.0 | 0.1388875493228224D-02 | 0.17D-11 | 0 | 30 |
| 4 | 10.0 | 0.4429179245608584D-01 | -0.60D-08 | 0 | 30 |
| 4 | 100.0 | 0.9229366999160198D-02 | -0.13D-09 | 0 | 30 |
| 4 | 1000.0 | 0.9920299360699999D-03 | -0.32D-11 | 0 | 30 |
| 4 | 10000.0 | 0.9992002999360070D-04 | -0.35D-13 | 0 | 30 |
| 4 | 100000.0 | 0.9999200029999360D-05 | -0.35D-15 | 0 | 30 |
| 5 | 1.0 | 0.3278690286854098D-03 | 0.68D-11 | 0 | 30 |
| 5 | 10.0 | 0.3631058431389951D-01 | -0.65D-08 | 0 | 30 |
| 5 | 100.0 | 0.9046625348026394D-02 | -0.12D-09 | 0 | 30 |
| 5 | 1000.0 | 0.9900478602557480D-03 | -0.40D-11 | 0 | 30 |
| 5 | 10000.0 | 0.9990004798600257D-04 | -0.43D-13 | 0 | 30 |
| 5 | 100000.0 | 0.9999000047998600D-05 | -0.44D-15 | 0 | 30 |
| 6 | 1.0 | 0.7739937848658522D-04 | -0.77D-12 | 0 | 30 |
| 6 | 10.0 | 0.2976755873052603D-01 | -0.35D-08 | 0 | 30 |

| | | | | | |
|---|---|---|---|---|---|
| 6 | 100.0 | 0.886750198523914 2D-02 | -0.87D-10 | 0 | 30 |
| 6 | 1000.0 | 0.988069744628977 0D-03 | -0.46D-11 | 0 | 30 |
| 6 | 10000.0 | 0.998800699744063 1D-04 | -0.52D-13 | 0 | 30 |
| 6 | 100000.0 | 0.999880006999744 1D-05 | -0.52D-15 | 0 | 30 |
| 7 | 1.0 | 0.182715147390689 1D-04 | -0.52D-12 | 0 | 30 |
| 7 | 10.0 | 0.244035608216891 0D-01 | 0.11D-08 | 0 | 30 |
| 7 | 100.0 | 0.869192526861682 8D-02 | -0.49D-10 | 0 | 30 |
| 7 | 1000.0 | 0.986095581277232 1D-03 | -0.53D-11 | 0 | 30 |
| 7 | 10000.0 | 0.998600959580128 1D-04 | -0.60D-13 | 0 | 30 |
| 7 | 100000.0 | 0.999860009599580 1D-05 | -0.61D-15 | 0 | 30 |
| 8 | 1.0 | 0.431331953030959 4D-05 | -0.23D-10 | 0 | 30 |
| 8 | 10.0 | 0.200061344018504 0D-01 | 0.51D-08 | 0 | 30 |
| 8 | 100.0 | 0.851982497449446 9D-02 | -0.76D-11 | 0 | 30 |
| 8 | 1000.0 | 0.984125362303868 0D-03 | -0.59D-11 | 0 | 30 |
| 8 | 10000.0 | 0.998401259360231 1D-04 | -0.69D-13 | 0 | 30 |
| 8 | 100000.0 | 0.999840012599360 1D-05 | -0.70D-15 | 0 | 30 |
| 9 | 1.0 | 0.101823661783052 9D-05 | 0.87D-10 | 0 | 30 |
| 9 | 10.0 | 0.164011070609489 4D-01 | 0.74D-08 | 0 | 30 |
| 9 | 100.0 | 0.835113226963704 9D-02 | 0.37D-10 | 0 | 30 |
| 9 | 1000.0 | 0.982159079828016 6D-03 | -0.65D-11 | 0 | 30 |
| 9 | 10000.0 | 0.998201599076384 0D-04 | -0.77D-13 | 0 | 30 |
| 9 | 100000.0 | 0.999820015999076 1D-05 | -0.78D-15 | 0 | 30 |
| 10 | 1.0 | 0.240373058987479 2D-06 | -0.38D-09 | 0 | 30 |
| 10 | 10.0 | 0.134456915774708 2D-01 | 0.75D-08 | 0 | 30 |
| 10 | 100.0 | 0.818577968370898 7D-02 | 0.82D-10 | 0 | 30 |
| 10 | 1000.0 | 0.980196725984555 9D-03 | -0.71D-11 | 0 | 30 |
| 10 | 10000.0 | 0.998001978720600 6D-04 | -0.86D-13 | 0 | 30 |
| 10 | 100000.0 | 0.999800019798720 1D-05 | -0.87D-15 | 0 | 30 |