On the Proof Complexity of Cut-Free Bounded Deep Inference

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Abstract. It has recently been shown that cut-free deep inference systems exhibit an exponential speed-up over cut-free sequent systems, in terms of proof size. While this is good for proof complexity, there remains the problem of typically high proof search non-determinism induced by the deep inference methodology: the higher the depth of inference, the higher the non-determinism. In this work we improve on the proof search side by demonstrating that, for propositional logic, the same exponential speed-up in proof size can be obtained in bounded-depth cut-free systems. These systems retain the top-down symmetry of deep inference, but can otherwise be designed at the same depth level of sequent systems. As a result the non-determinism arising from the choice of rules at each stage of a proof is smaller than that of unbounded deep inference, while still giving access to the short proofs of deep inference.

1 Introduction

Deep inference is a proof methodology whose proof systems allow the application of inference rules on any connective appearing in a formula, in contrast to traditional proof systems whose inference rules only operate on the main connective of a formula. Within deep inference several formalisms have been defined, the most developed being the *Calculus of Structures* (CoS), and more recently an extension of it, *Open Deduction* [8]. Throughout this work we use the latter, but present complexity results for CoS so that they are more directly comparable to existing results. The two systems are polynomially equivalent and, for the reader familiar with CoS, the use of open deduction can be considered as just a convenient notation to present CoS proofs more clearly and with less syntax.

In this paper we consider "cut-free" or "analytic" deep inference systems as defined in [3]. For deep inference systems the "cut" rule is $\frac{A \wedge \bar{A}}{\mathsf{f}}$, which can be considered a generalized version of the cut from sequent calculi since it can be applied in any context. There are cut-elimination procedures for deep inference systems and they yield the results we would expect from such procedures, e.g. consistency of the system and Herbrand's Theorem. There is also a generalized version of the subformula property for cut-free deep inference systems: atoms appearing in a proof are just those that appear in its conclusion. This specializes to the traditional subformula property when restricted to the sequent calculus. A

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more detailed account of the cut rule in deep inference systems and the corollaries of cut-elimination can be found in [2].

Recently, in [3], Bruscoli and Guglielmi have shown that cut-free deep inference systems exhibit an exponential speedup over cut-free sequent systems in size of proofs. The Statman tautologies are shown to have polynomial-size proofs in the cut-free calculus of structures, while their proofs in cut-free sequent calculi have long been known to grow exponentially [6]. The first three Statman tautologies are shown below, from which the basic pattern should be apparent:

$$
\begin{aligned}
\mathsf{S}_1 &\equiv (c_1 \wedge d_1) \vee \left[\bar{c}_1 \vee \bar{d}_1\right] \quad , \\
\mathsf{S}_2 &\equiv (c_2 \wedge d_2) \vee \left[\left(\left(\left[\bar{c}_2 \vee \bar{d}_2\right] \wedge c_1\right) \wedge \left(\left[\bar{c}_2 \vee \bar{d}_2\right] \wedge d_1\right)\right) \vee \left[\bar{c}_1 \vee \bar{d}_1\right]\right] \quad , \\
\mathsf{S}_3 &\equiv (c_3 \wedge d_3) \vee \left(\left(\left[\bar{c}_3 \vee \bar{d}_3\right] \wedge c_2\right) \wedge \left(\left[\bar{c}_3 \vee \bar{d}_3\right] \wedge d_2\right)\right) \vee \\
&\qquad \left(\left(\left(\left[\bar{c}_3 \vee \bar{d}_3\right] \wedge \left[\bar{c}_2 \vee \bar{d}_2\right]\right) \wedge c_1\right) \wedge \left(\left(\left[\bar{c}_3 \vee \bar{d}_3\right] \wedge \left[\bar{c}_2 \vee \bar{d}_2\right]\right) \wedge d_1\right)\right) \vee \\
&\qquad \left[\bar{c}_1 \vee \bar{d}_1\right] \quad .
\end{aligned}
$$

It is not difficult to see that cut-free sequent calculus proofs of these tautologies are forced to create $O(2^n)$ branches, as demonstrated in [5]. However with deep inference systems it is possible to prove these formulae 'from the inside out' by copying the first disjunct into each following disjunct, reducing the formula to the previous tautology and repeating the process, yielding polynomial-size proofs.

It can be argued that the use of deep inference in this case is trivial as inference rules operate just beneath the surface of the formula; in particular the number of $\wedge$-$\vee$ alternations, or *depth*, of the Statman tautologies is constant. In this paper we introduce systems where the depth at which inference rules may apply is bounded. We refer to these as *bounded-depth* systems, although this should not be confused with bounded-depth Frege systems in which the depth of formulae appearing in a proof, rather than inference steps, is bounded.

In [3] it was conjectured that bounded-depth deep inference systems, while still giving polynomial-size proofs of the Statman tautologies, would result in an exponential blowup in the size of proofs for some other classes of tautologies. In Sect. 3 we prove this to be false; we construct a polynomial transformation of cut-free deep inference proofs to ones whose inference rules are not only bounded in depth but "shallow", in the sense that sequent calculus rules are shallow.

The result is possible because deep inference systems benefit over sequent systems not only in the depth of their inference steps, but also in the top-down symmetry they exhibit. A CoS derivation is a sequence of formulae in which the main connective may change many times, and so the system admits a notion of duality of inference rules. In contrast, sequent calculi have a strict tree structure with the implicit connective between branches being conjunction. For example consider the following rules from deep inference:

$$
\mathsf{c}{\uparrow}\,\frac{A}{A \wedge A} \qquad\qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}
$$

cocontraction *medial*

The first rule is an example of duality in deep inference systems: it is the dual rule for contraction. However neither rule can be fully captured by a sequent calculus; looking at the rules bottom-up, if the conjuncts in the conclusions are separate branches in a sequent calculus proof, the two branches would need to collapse into a single branch in order to obtain the premiss, which is not permitted. However this flexibility alone, it turns out, admits enough top-down symmetry to enjoy the same proof complexity as deep inference systems, and in Conclusion 5.1 we present a sequent-like system that exemplifies this.

In the literature there is generally a distinction made between systems containing cocontraction and ones that do not, as it is conjectured that cocontraction allows for an exponential speedup in size of proofs [3]. In this paper we consider only systems containing cocontraction and show that bounded-depth systems can polynomially simulate full deep inference. The analogous problem for systems without cocontraction remains open, although in Conclusion 5.2 we conjecture that an analogous result does not hold. Work in this area is ongoing, and may provide new directions for the wider problem of the effect of cocontraction on proof complexity in general (see Conclusion 5.3).

2 Preliminaries

Here we give only a brief account of the open deduction formalism and its usual proof systems for propositional logic, but a more comprehensive introduction can be found in [8].

Definition 1 (Formulae and Contexts). *The language of open deduction is a propositional language consisting of units* t, f, *countably many atoms which we denote* a, b, c, d, *possibly with subscripts and superscripts, two binary connectives* $\wedge$ *and* $\vee$ *and an involution* $a \mapsto \bar{a}$, *defined only on the set of atoms, representing negation, all with their usual classical interpretations.*

Formulae *are built freely in the usual way and we use* A, B, C, D *as metavariables ranging over formulae of the language. We extend negation to all formulae by identifying* $\bar{A}$ *with the negation normal form of* A. *For clarity we use parentheses for conjunctions and brackets for disjunction, and we sometimes omit external parentheses/brackets of a formula, and internal ones under associativity. For example the following are all formulae:*

$$a \wedge [\bar{b} \vee c] \qquad \mathsf{t} \wedge d \qquad \overline{\mathsf{f} \vee (a \wedge \bar{b})} \equiv \mathsf{t} \wedge [\bar{a} \vee b]$$

Definition 2 (Derivations). *All formulae are derivations, and we define, for a formula* A, *its premiss and conclusion* ($pr(A)$, $cn(A)$ *resp.) as* A. *If* Φ, Ψ *are derivations then* $\Phi \star \Psi$ *is a derivation for* $\star \in \{\wedge, \vee\}$, *with* $pr(\Phi \star \Psi) \equiv pr(\Phi) \star pr(\Psi)$ *and* $cn(\Phi \star \Psi) \equiv cn(\Phi) \star cn(\Psi)$. $\rho \frac{\Phi}{\Psi}$ *is a derivation just if* $\rho \frac{cn(\Phi)}{pr(\Psi)}$ *is an inference step associated with some rule* ρ, *and has premiss* $pr(\Phi)$ *and conclusion* $cn(\Psi)$. *Inference rules can operate anywhere in a formula, not just on the main connective. If* $pr(\Phi) \equiv \mathsf{t}$ *then we call* Φ *a* proof.

Rebracketing rules $\qquad\qquad\qquad$ *Unit* rules

$$
= \frac{A \vee B}{B \vee A} \qquad = \frac{[A \vee B] \vee C}{A \vee [B \vee C]} \qquad \mathsf{u}_1\!\uparrow \frac{A}{A \wedge \mathsf{t}} \qquad \mathsf{u}_2\!\uparrow \frac{A \wedge \mathsf{t}}{A} \qquad \mathsf{u}_3\!\uparrow \frac{\mathsf{f}}{\mathsf{f} \wedge \mathsf{f}} \qquad \mathsf{u}_4\!\uparrow \frac{\mathsf{f} \wedge \mathsf{f}}{\mathsf{f}}
$$

$$
= \frac{A \wedge B}{B \wedge A} \qquad = \frac{(A \wedge B) \wedge C}{A \wedge (B \wedge C)} \qquad \mathsf{u}_1\!\downarrow \frac{A \vee \mathsf{f}}{A} \qquad \mathsf{u}_2\!\downarrow \frac{A}{A \vee \mathsf{f}} \qquad \mathsf{u}_3\!\downarrow \frac{\mathsf{t} \vee \mathsf{t}}{\mathsf{t}} \qquad \mathsf{u}_4\!\downarrow \frac{\mathsf{t}}{\mathsf{t} \vee \mathsf{t}}
$$

commutativity $\qquad$ *associativity*

Fig. 1. Inference rules for equality

For a derivation Φ its size, $|\Phi|$, *is the number of unit and atom occurrences in it and its* length, $l(\Phi)$, *is the number of inference steps appearing in it.*

Definition 3 (Contexts). *A* context *is a formula with one hole appearing in place of a subformula, e.g.* $a \wedge \{\ \}$, $b \vee (a \wedge \{\ \}) \vee \mathsf{f}$, *and is denoted by* $\xi\{\ \}$. *The hole can be filled with any formula or derivation; we denote a context* $\xi\{\ \}$ *filled with a derivation* Φ *by* $\xi\{\Phi\}$.

Definition 4 (Systems). *A* system *is a set of inference rules, and if all inference rules appearing in a derivation (resp. proof)* Φ *belong to a system* $\mathcal{S}$ *then we say* Φ *is a* $\mathcal{S}$*-derivation (resp.* $\mathcal{S}$*-proof). If* Φ *is a* $\mathcal{S}$*-derivation with premiss* A *and conclusion* B *we write* $\begin{array}{c} A \\ \Phi \Vert \mathcal{S} \\ B \end{array}$. *If* $A \equiv \mathsf{t}$, *i.e.* Φ *is a proof, then we write* $\begin{array}{c} \Phi \Vert \mathcal{S} \\ B \end{array}$.

Definition 5 (Sequential and Synchronal Forms). *We define two important forms of a derivation. The first is* sequential form, *where the derivation is just a sequence of formulae, and so also a CoS derivation. The second is* synchronal form, *where every inference step operates as shallow as possible, i.e. every inference step is just an instance of the inference rule itself with formulae substituted in for the metavariables. For every derivation both forms exist; synchronal form is unique while sequential form, in general, is not, and there is at most only a quadratic difference in the size of the two forms [4]. For example we present a derivation in synchronal and two sequential forms:*

$$
\frac{A}{C} \wedge \frac{B}{D} \qquad \begin{array}{c} A \wedge B \\ \hline A \wedge D \\ \hline C \wedge D \end{array} \qquad \begin{array}{c} A \wedge B \\ \hline C \wedge B \\ \hline C \wedge D \end{array}
$$

Note 6 (Equality Rules). We define the equality inference rules on formulae in Fig. 1. For the sake of clearer analysis of proof complexity and depth we consider them as real inference rules inducing actual inference steps (like in [9]), rather than a set of equations governing the sameness of formulae. When we introduce our notion of depth, and bounded-depth systems, the same restrictions we impose

$$
\begin{array}{c|cccc|c}
 & & \textit{Structural}\text{ rules} & & \textit{Logical}\text{ rules} & \\
\mathsf{KSg}^+ & & & \mathsf{c}\!\uparrow \dfrac{A}{A \wedge A} & \mathsf{m}\,\dfrac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]} & \\
 & & & \textit{cocontraction} & \textit{medial} & \\
 & \mathsf{i}\!\downarrow \dfrac{\mathsf{t}}{A \vee \bar{A}} & \mathsf{w}\!\downarrow \dfrac{\mathsf{f}}{A} & \mathsf{c}\!\downarrow \dfrac{A \vee A}{A} & \mathsf{s}\,\dfrac{A \wedge [B \vee C]}{(A \wedge B) \vee C} & \mathsf{KSg} \\
 & \textit{interaction}\text{ or }\textit{identity} & \textit{weakening} & \textit{contraction} & \textit{switch} &
\end{array}
$$

Fig. 2. Systems KSg and KSg^+

on the other inference rules also apply to the equality rules, so there is no doubt that the inference rules we apply really do have bounded depth.

When defining proof systems in this section, we implicitly assume that all the equality inference rules are also in that system. However we distinguish between the *rebracketing* rules and the *unit* rules in notation as we usually consider them separately. We in fact make little use of the unit rules and include them only as convention. In Sect. 4 we show that units and the unit rules can be dropped with no major effect on proof complexity.

Definition 7. *We define* $\mathsf{KSg} = \{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow, \mathsf{s}\}$, *and* $\mathsf{KSg}^+ = \mathsf{KSg} \cup \{\mathsf{c}\!\uparrow, \mathsf{m}\}$ *and these rules are defined in Fig. 2. As usual, both these systems also contain all equality inference rules. These systems are sound and complete [1].*

Definition 8 (Complexity). *We say that a system* $\mathcal{S}$ p-simulates *a system* $\mathcal{T}$ *if there is a polynomial* p *such that for every* $\mathcal{T}$*-proof* Ψ *there is a* $\mathcal{S}$*-proof* Φ *with the same conclusion such that* $|\Phi| \leq p(|\Psi|)$. *If the condition also holds for all derivations, preserving premises as well as conclusions, then we say that* $\mathcal{S}$ strongly p-simulates $\mathcal{T}$. *When two systems p-simulate (resp. strongly p-simulate) each other, we say they are* p-equivalent *(resp.* strongly p-equivalent*).*

Definition 9 (Depth). *For a formula* A *its depth,* $d(A)$, *is the maximum number of alternations of* $\wedge$ *and* $\vee$ *in its formula tree. The depth of a hole (resp. subformula) in a context,* $d(\{\ \}, \xi\{\ \})$, *is the number of alternations in the path to the hole (resp. subformula) in the context's formula tree. In a sequential derivation the depth of an inference step is the depth of the hole of the largest context common to both its premiss and conclusion. The depth of an inference step is invariant among its various sequential forms and so we extend uniquely this notion for all derivations. When calculating depth we adopt the convention that every formula or context has an outer* $\wedge$. *For example:*

$$d(a \vee (b \wedge c)) = d(\{\ \}, a \vee (b \wedge \{\ \})) = 2 \qquad d(a \wedge (b \wedge (c \wedge d))) = 0$$

Notation 10. *For a system $\mathcal{S}$ we write k-$\mathcal{S}$ to denote the system whose derivations are just $\mathcal{S}$-derivations where all inference steps have depth less than or equal to k. We call k-$\mathcal{S}$ a k-depth system. For an inference step ρ, we often indicate its depth in parentheses on the right, e.g.* $\rho(3)\,\dfrac{A}{B}$ *indicates that $d(\rho) = 3$, and we write $\mathcal{S} \cup \{\rho(k)\}$ to denote the system whose derivations are just $\mathcal{S} \cup \{\rho\}$-derivations with all ρ steps having depth k. For a context $\xi\{\ \}$, the depth of its hole may be indicated as a superscript,* e.g. *$\xi^2\{\ \}$ for a context with a hole at depth 2.*

3 The Depth-Change Trick

In this section we present our main result, that bounded-depth KSg^+ strongly p-simulates any cut-free deep inference system. The result also holds for $\mathsf{KSg} \cup \{\mathsf{c}\!\uparrow\}$, and the problem remains open for systems without cocontraction.

Throughout this section we present derivations in sequential form, i.e. CoS derivations, both for clarity and to establish complexity results directly comparable to existing ones. Derivations are often presented with long sequences of commutativity and associativity steps, and sometimes brackets (resp. parantheses) are omitted in large disjunctions (resp. conjunctions). From the point of view of complexity this shortens proofs by at most cubic degree and so preserves p-simulation. A proof of the following lemma can be found in [3]:

Lemma 11. *Every rebracketing* $\begin{array}{c} A \\ \Phi \| \{=\} \\ B \end{array}$ *can be achieved with quadratic length, i.e. there exists* $\begin{array}{c} A \\ \Psi \| \{=\} \\ B' \end{array}$ *such that $l(\Psi) = O(B^2)$, and so $|\Psi| = O(B^3)$.*

In this section we work in KSg^+, the system of all the usual "analytic" rules for propositional logic. The medial rule is, in fact, derivable from the other rules and so does not have a major effect the complexity of the system:

$$\mathsf{c}\!\downarrow \dfrac{4\cdot\mathsf{w}\!\downarrow \dfrac{4\cdot\mathsf{u}_2\!\downarrow \dfrac{(A \wedge B) \vee (C \wedge D)}{([A \vee \mathsf{f}] \wedge [B \vee \mathsf{f}]) \vee ([A \vee \mathsf{f}] \wedge [B \vee \mathsf{f}])}}{([A \vee C] \wedge [B \vee D]) \vee ([A \vee C] \wedge [B \vee D])}}{[A \vee C] \wedge [B \vee D]}$$

In particular the main result presented here, that bounded-depth KSg^+ strongly p-simulates unbounded-depth KSg^+, also holds for $\mathsf{KSg} \cup \{\mathsf{c}\!\uparrow\}$, i.e. without medial. However since the derivation of medial contains depth 3 rule applications (the weakening steps), the result would only hold for systems of depth greater than or equal to 3, which is somewhat less clean than the result for KSg^+.

Definition 12. *Observe the following derivations in* $1\text{-}\mathsf{KSg}^+$*:*

$$
\dfrac{\dfrac{A \vee (B \wedge C)}{(A \wedge A) \vee (B \wedge C)}\,\mathsf{c}{\uparrow}(1)}{[A \vee B] \wedge [A \vee C]}\,\mathsf{m}(0)
\qquad
\dfrac{\dfrac{(A \wedge B) \vee (A \wedge C)}{[A \vee A] \wedge [B \vee C]}\,\mathsf{m}(0)}{A \wedge [B \vee C]}\,\mathsf{c}{\downarrow}(0)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[A \vee B] \wedge [A \vee C]}{[A \vee B] \wedge [C \vee A]}\,{=}(0)}{A \vee (B \wedge C) \vee A}\,\mathsf{s}(1),\mathsf{s}(0)}{A \vee A \vee (B \wedge C)}\,{=}(1)}{A \vee (B \wedge C)}\,\mathsf{c}{\downarrow}(1)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{A \wedge [B \vee C]}{A \wedge A \wedge [B \vee C]}\,\mathsf{c}{\uparrow}(0)}{A \wedge [B \vee C] \wedge A}\,{=}(0)}{(A \wedge B) \vee (C \wedge A)}\,2{\cdot}\mathsf{s}(0)}{(A \wedge B) \vee (A \wedge C)}\,{=}(1)
$$

From these we define four macro-rules, collectively known as the distributivity laws, which should be understood as abbreviations for the above derivations:

$$
\mathsf{d}_2{\uparrow}\,\dfrac{A \vee (B \wedge C)}{[A \vee B] \wedge [A \vee C]}
\qquad
\mathsf{d}_2{\downarrow}\,\dfrac{(A \wedge B) \vee (A \wedge C)}{A \wedge [B \vee C]}
$$

$$
\mathsf{d}_1{\downarrow}\,\dfrac{[A \vee B] \wedge [A \vee C]}{A \vee (B \wedge C)}
\qquad
\mathsf{d}_1{\uparrow}\,\dfrac{A \wedge [B \vee C]}{(A \wedge B) \vee (A \wedge C)}
$$

Like switch and medial, these rules can increase or decrease the depth of a formula. However, unlike switch and medial, all the above rules are invertible, indeed rules in the same column are inverse to each other, and rules in the same row are dual to each other. This invertibility allows us to "unfold" formulae at will, bringing subformulae out to whatever depth we wish, and then pushing them back down again. For example the following transformation decreases the depth of an inference step by 1, where D is a formula at depth 2 containing its premiss and ξ^2 is the context of D:

$$
\rho(k{+}1)\,\dfrac{\xi^2\{D\}}{\xi^2\{D'\}}
\quad\rightsquigarrow\quad
\rho' :
\begin{array}{c}
\xi^2\{D\} \\
\Big\Vert\ \{=(k)\} \\
\dfrac{\dfrac{\dfrac{A \wedge [B \vee (C \wedge D)]}{A \wedge [B \vee C] \wedge [B \vee D]}\,\mathsf{d}_2{\uparrow}}{A \wedge [B \vee C] \wedge [B \vee D']}\,\rho(k)}{A \wedge [B \vee (C \wedge D')]}\,\mathsf{d}_1{\downarrow} \\
\Big\Vert\ \{=(k)\} \\
\xi^2\{D'\}
\end{array}
\tag{1}
$$

The transformation is local and so preserves derivability. We extend this trick to show that a 1-depth system strongly p-simulates unbounded depth systems.

$$
\rho(2r)\,\frac{\xi^{2r}\{D\}}{\xi^{2r}\{D'\}} \quad \leadsto \quad \rho' : \quad
\begin{array}{c}
\xi^{2r}\{D\} \\
\Psi_1^{-1} \Big\Vert \{=(1)\} \\
\mathsf{d}_2{\uparrow}\,\dfrac{A \wedge \left[B_1 \vee \left(C_1 \wedge \xi_1^{2(r-1)}\{D\}\right)\right]}{A \wedge \left([B_1 \vee C_1] \wedge \left[B_1 \vee \xi_1^{2(r-1)}\{D\}\right]\right)} \\
\Psi_2^{-1} \Big\Vert \{=(1)\} \\
\vdots \\
\Psi_r^{-1} \Big\Vert \{=(1)\} \\
\mathsf{d}_2{\uparrow}\,\dfrac{A \wedge [B_1 \vee C_1] \wedge \cdots \wedge [B_r \vee (C_r \wedge D)]}{A \wedge [B_1 \vee C_1] \wedge \cdots \wedge ([B_r \vee C_r] \wedge [B_r \vee D])} \\
\rho(1)\,\dfrac{}{A \wedge [B_1 \vee C_1] \wedge \cdots \wedge ([B_r \vee C_r] \wedge [B_r \vee D'])} \\
\mathsf{d}_1{\downarrow}\,\dfrac{}{A \wedge [B_1 \vee C_1] \wedge \cdots \wedge [B_r \vee (C_r \wedge D')]} \\
\Psi_r \Big\Vert \{=(1)\} \\
\vdots \\
\Psi_3 \Big\Vert \{=(1)\} \\
\mathsf{d}_1{\downarrow}\,\dfrac{(A \wedge [B_1 \vee C_1]) \wedge \left([B_2 \vee C_2] \wedge \left[B_2 \vee \xi_2^{2(r-2)}\{D'\}\right]\right)}{(A \wedge [B_1 \vee C_1]) \wedge \left[B_2 \vee \left(C_2 \wedge \xi_2^{2(r-2)}\{D'\}\right)\right]} \\
\Psi_2 \Big\Vert \{=(1)\} \\
\mathsf{d}_1{\downarrow}\,\dfrac{A \wedge \left([B_1 \vee C_1] \wedge \left[B_1 \vee \xi_1^{2(r-1)}\{D'\}\right]\right)}{A \wedge \left[B_1 \vee \left(C_1 \wedge \xi_1^{2(r-1)}\{D'\}\right)\right]} \\
\Psi_1 \Big\Vert \{=(1)\} \\
\xi^{2r}\{D'\}
\end{array}
$$

Fig. 3. Full depth-decreasing transformation of an inference step

Theorem 13 (The Depth-Change Trick). 1-KSg^+ *strongly p-simulates* KSg^+.

Proof. Suppose ρ is an inference rule with depth $2r \geq 2$ [1]. Let D be its premiss, D' its conclusion and ξ its context. Let $\xi_i\{D\}$ be the smallest subformula of $\xi\{D\}$ at depth $2i$ containing D, so that $d(\xi_i\{\ \}) = 2(r-i)$. In Fig. 3 we define a transformation of ρ to a derivation ρ' that has the same premiss and conclusion but contains only at most depth 1 inference steps.

When $d(\rho) \leq 1$ we define $\rho' = \rho$, with ρ construed as a length 1 derivation. We can now extend the transformation to whole derivations as follows:

[1] If $d(\rho)$ is odd then the derivation is the same, but in the middle D would now appear in a disjunction, still at depth 1.

$$\rho\,\frac{A \wedge [B \vee [C \vee D]]}{A \wedge [B \vee [C \vee D']]} \quad \rightsquigarrow \quad \rho' : \begin{array}{rc} & A \wedge [B \vee [C \vee D]] \\ = & \overline{A \wedge [[B \vee C] \vee D]} \\ \rho & \overline{A \wedge [[B \vee C] \vee D']} \\ = & \overline{A \wedge [B \vee [C \vee D']]} \end{array}$$

$$\rho\,\frac{A \wedge [B \vee (C \wedge D)]}{A \wedge [B \vee (C \wedge D')]} \quad \rightsquigarrow \quad \rho' : \begin{array}{rc} & A \wedge [B \vee (C \wedge D')] \\ \mathsf{d}_2\uparrow & \overline{A \wedge ([B \vee C] \wedge [B \vee D'])} \\ = & \overline{(A \wedge [B \vee C]) \wedge [B \vee D']} \\ \rho & \overline{(A \wedge [B \vee C]) \wedge [B \vee D]} \\ = & \overline{A \wedge ([B \vee C] \wedge [B \vee D])} \\ \mathsf{d}_1\downarrow & \overline{A \wedge [B \vee (C \wedge D)]} \end{array}$$

Fig. 4. Interweaving rebracketing and distributivity steps

$$\Phi : \begin{array}{rc} & A_0 \\ \rho_1 & \overline{} \\ & \vdots \\ \rho_n & \overline{} \\ & A_n \end{array} \quad \rightsquigarrow \quad \Phi' : \begin{array}{rc} & A_0 \\ \rho_1' & = \\ & \vdots \\ \rho_n' & = \\ & A_n \end{array}$$

The depth of a formula is less than or equal to its size so we have $r_i \leq |A_i|$, where $2r_i$ is the depth of ρ_i. Each Ψ_k is a rebracketing step which has cubic complexity in the size of its conclusion, by Lemma 11, and the conclusions of Ψ_k have size at most $k \cdot |A_i|$. From this we calculate $|\rho_i'| \leq 2 \cdot \sum_{k=1}^{r} |\Psi_k| \leq \sum_{k=1}^{r} (k \cdot |A_i|)^3 \leq |A_i|^3 \sum_{k=1}^{r} k^3 = O(|A_i|^3 \cdot r^4) = O(|A_i|^7)$, and so $|\Phi'| = O(|\Phi|^7)$. □

Note 14 (The Complexity of the Depth-Change Trick). The upper bound of the polynomial degree of inefficiency estimated above, 7, can be improved upon vastly, for example by interweaving the rebracketing and distributivity steps.

Consider the transformation of an inference rule in Fig. 4. When $d(\rho) \leq 1$ we define $\rho' = \rho$, with ρ construed as a length 1 derivation. If the size of the conclusion is n, then the transformation only needs to be applied at most $n-1$ times (i.e. the number of connectives) to guarantee that ρ applies at depth less than or equal to 1. Each application of the transformation at most adds n then multiplies by 6, and so applying it $n-1$ times gives $|\rho'| \leq \sum_{i=1}^{n-1} 6 \cdot i \cdot n = O(n^3)$. So the depth-change trick can be achieved with at most a cubic loss in efficiency. In comparison, *local* systems in CoS, ones that operate only on atoms, suffer a quadratic loss in efficiency, as shown in [3].

4 Reducing Non-determinism

Having shown that bounded-depth systems have similar size proofs as deep inference, we can now construct systems that have far less non-determinism than those in deep inference: at each stage of a proof there are fewer choices available.

$$
\begin{array}{c|cccc|c}
\textit{Axiom} & \multicolumn{3}{c}{\textit{Structural}\text{ rules}} & \textit{Logical}\text{ rules} & \\
 & & & \mathsf{c}{\uparrow}\,\dfrac{A}{A \wedge A} & \mathsf{m}\,\dfrac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]} & \\
 & & & \textit{cocontraction} & \textit{medial} & \\
\mathsf{i}{\downarrow}\,\dfrac{}{A \vee \bar{A}} & \mathsf{i}{\downarrow}\,\dfrac{A}{A \wedge [B \vee \bar{B}]} & \mathsf{w}{\downarrow}\,\dfrac{A}{A \vee B} & \mathsf{c}{\downarrow}\,\dfrac{A \vee A}{A} & \mathsf{s}\,\dfrac{A \wedge [B \vee C]}{(A \wedge B) \vee C} & \\
 & \textit{identity} & \textit{weakening} & \textit{contraction} & \textit{switch} &
\end{array}
$$

$\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g}^+$ = all rules above; $\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g}$ = bottom row.

Fig. 5. Unit-free systems $\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g}$ and $\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g}^+$

In this section we restrict our bounded-depth system to further reduce this non-determinism. Proofs and derivations are given in synchronal form for convenience and we introduce a new measure, known as the *height* of a derivation, for inductions. We give only proof sketches of theorems for the sake of brevity.

Definition 15 (Height of a Derivation). *The height of a proof/derivation is defined inductively as follows:*

$$
\begin{aligned}
h(a) &= 0 \\
h(\Phi \wedge \Psi) &= \max(h(\Phi), h(\Psi)) \\
h(\Phi \vee \Psi) &= \max(h(\Phi), h(\Psi))
\end{aligned}
\qquad
h\left(\frac{\Phi}{\Psi}\right) = 1 + h(\Phi) + h(\Psi)
$$

Remark 16. The height of a derivation is not invariant among its various forms. Height is equal to length for sequential derivations but this is not, in general, true for derivations in synchronal form. For example, the three forms of a derivation given in Definition 5 have heights 1, 2, 2 respectively.

Units and unit rules, it turns out, do not play a major role in proof complexity; in most circumstances we can polynomially transform a proof with units and unit rules to one without. The system we present in Fig. 5 is similar to the unit-free system appearing in [10]. From the point of view of proof search, the advantage of this is clear: fewer rules and no constants means less non-determinism.

Definition 17 (Unit-Free Systems). *The language of unit-free open deduction is just the language of open deduction with all units removed. Unit-free derivations are derivations containing no units (and so no instances of the unit rules), and a unit-free proof is a unit-free derivation with empty premiss. With inference rules defined as in Fig. 5 define* $\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g} = \{\mathsf{i}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{c}{\downarrow}, \mathsf{s}\}$ *and* $\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g}^+ = \{\mathsf{i}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{c}{\downarrow}, \mathsf{c}{\uparrow}, \mathsf{s}, \mathsf{m}\}$ *respectively, along with all rebracketing rules but not the unit rules.*

The special cases when a proof-with-units cannot be transformed to a unit-free proof is when the conclusion "reduces", in some sense, to just a unit. Otherwise we reduce the conclusion to some unique unit-free formula and construct a proof of that. This reduction is captured by the following equivalence relation:

Definition 18 (Unitary Equivalence on Formulae). *We define unitary equivalence, $\cong$, on formulae by closing the following equations by reflexivity, symmetry, transitivity and by applying context closure.*

$$\begin{array}{rcccl} A \vee \mathsf{f} & \cong & A & \cong & \mathsf{f} \vee A \\ A \wedge \mathsf{t} & \cong & A & \cong & \mathsf{t} \wedge A \\ A \vee \mathsf{t} & \cong & \mathsf{t} & \cong & \mathsf{t} \vee A \\ A \wedge \mathsf{f} & \cong & \mathsf{f} & \cong & \mathsf{f} \wedge A \end{array} \qquad \begin{array}{c} \text{Context Closure:} \\ \\ \textit{if } A \cong B \textit{ then } \xi\{A\} \cong \xi\{B\} \end{array}$$

Remark 19. $\cong$ is an equivalence relation on formulae, and each formula's equivalence class contains either t, f or a unique unit-free formula, which we call its *reduction*. The reduction of a formula can be calculated in polynomial time [3].

Theorem 20 (Dropping Units). *A* KSg^+*-proof whose conclusion reduces to a unit-free formula A can be polynomially transformed to a unit-free proof of A.*

Proof. Replace instances of units and unit rules with the appropriate unit-free rules and formulae. The exceptional cases are when t appears in a disjunction or f appears in a conjunction in the conclusion. For example:

$$\left(\mathsf{w}\!\downarrow \frac{\overset{\Vert}{A \wedge B}}{A \wedge [B \vee \mathsf{t}]} \right) \qquad \left(\mathsf{s}\, \frac{\mathsf{u}_2\!\downarrow \dfrac{\overset{\Vert}{A \wedge B}}{[A \vee \mathsf{f}] \wedge B}}{A \vee (B \wedge \mathsf{f})} \right)$$

In both examples the conclusion reduces to A from premiss $A \wedge B$, which is an instance of *coweakening* (see [3]). We show the admissibility of coweakening by transforming the proof of $A \wedge B$ to a proof, of equal length, of a formula that reduces to A as follows: replace each atom of B in the conclusion with t, then mimic the proof of $A \wedge B$ upwards. Identity steps affected by this substitution are replaced by weakening to give a valid proof. □

Corollary 21. 1-$\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g}^+$ *p-simulates* KSg^+ *over unit-free formulae.*

Proof. The proof of the depth-change trick (Theorem 13) makes no use of units and so can be replicated for the unit-free system in the obvious manner. □

Henceforth rules and systems are assumed to be unit-free if not already specified, implicitly containing the rebracketing rules but not the unit rules.

In what follows we drop the structural rules contraction, identity and weakening from the system; again, this provides a clear advantage from the point of view of proof search. It is known that these rules can be dropped in both sequent and deep inference systems and dropping these rules for bounded-depth systems does not have a major effect on proof complexity.

Before we can drop the structural rules, we must replace switch and medial with distributivity rules $\mathsf{d}_1\!\downarrow$ and $\mathsf{d}_2\!\downarrow$. Whether these rules are better or worse for proof search is debatable: on one hand they can blow up a formula exponentially large, while the former rules are linear, but on the other hand, since they

$$\mathsf{d_1}\!\downarrow \dfrac{\mathsf{w}\!\downarrow\dfrac{A}{A\vee C}\wedge[B\vee C]}{(A\wedge B)\vee C}$$

switch(0)

$$\mathsf{d_2}\!\downarrow \dfrac{\mathsf{d_2}\!\downarrow \dfrac{\mathsf{w}\!\downarrow\dfrac{A\wedge B}{(A\wedge B)\vee(A\wedge D)}}{A\wedge[B\vee D]} \vee \mathsf{d_2}\!\downarrow \dfrac{\mathsf{w}\!\downarrow\dfrac{C\wedge D}{(C\wedge B)\vee(C\wedge D)}}{C\wedge[B\vee D]}}{[A\vee C]\wedge[B\vee D]}$$

medial(0, 1)

$$\mathsf{d_2}\!\downarrow \dfrac{\mathsf{w}\!\downarrow \dfrac{A\wedge[B\vee C]}{\mathsf{d_2}\!\downarrow\dfrac{(A\wedge[B\vee C])\vee(D\wedge[B\vee C])}{[A\vee D]\wedge[B\vee C]}} \vee \mathsf{d_2}\!\downarrow \dfrac{D}{\mathsf{w}\!\downarrow\dfrac{(A\wedge D)\vee \mathsf{c}\!\uparrow\dfrac{D}{D\wedge D}}{[A\vee D]\wedge D}}}{\mathsf{d_1}\!\downarrow\dfrac{\mathsf{w}\!\downarrow\dfrac{A\vee D}{A\vee C\vee D}\wedge[B\vee C\vee D]}{(A\wedge B)\vee C\vee D}}$$

switch(1)

Fig. 6. Derivations of switch and medial in 1-$\{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow, \mathsf{c}\!\uparrow, \mathsf{d_1}\!\downarrow, \mathsf{d_2}\!\downarrow\}$

are invertible, there is no need to check the validity of the inference, which is beneficial from the point of view of automated deduction. In actuality the two sets of rules are easily derivable from each other in the presence of contraction and cocontraction, and so the first point is irrelevant.

Theorem 22. 1-$\{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow, \mathsf{c}\!\uparrow, \mathsf{d_1}\!\downarrow, \mathsf{d_2}\!\downarrow\}$ *p-simulates* KSg^+.

Proof. See Fig. 6 for derivations of switch and medial in the former system. □

Corollary 23. $\mathsf{d_1}\!\uparrow(0)$,$\mathsf{d_2}\!\uparrow(0)$ *are derivable in* 1-$\{\mathsf{w}\!\downarrow, \mathsf{c}\!\uparrow, \mathsf{d_1}\!\downarrow, \mathsf{d_2}\!\downarrow\}$.

Proof. Immediate from Definition 12 and derivations in Fig. 6. □

Remark 24. Since we can derive the inverse distributivity rules without identity or contraction, we can also use the depth-change trick after absorbing these rules.

Lemma 25. $\mathsf{c}\!\downarrow(2)$ *is derivable in* 2-$\{\mathsf{w}\!\downarrow, \mathsf{c}\!\uparrow, \mathsf{d_1}\!\downarrow, \mathsf{d_2}\!\downarrow\}$.

Proof. See Fig. 7. □

Theorem 26 (Dropping Contraction). 1-$\{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\uparrow, \mathsf{d_1}\!\downarrow, \mathsf{d_2}\!\downarrow\}$ *p-simulates* KSg^+.

Proof. Call the former system 1-$\mathcal{S}$ for convenience. We work in 2-$(\mathcal{S}\cup\{\mathsf{c}\!\downarrow\})$, which contains 1-$(\mathcal{S}\cup\{\mathsf{c}\!\downarrow\})$ and so p-simulates KSg by Theorem 22, and observe that every sound instance of contraction can either be pushed to depth 2 using distributivity or otherwise trivially eliminated. By Lemma 25 it follows that 2-$\mathcal{S}$ p-simulates KSg^+. Finally 1-$\mathcal{S}$ p-simulates 2-$\mathcal{S}$ by Remark 24. □

$$
=\dfrac{\left[\mathsf{c}\!\uparrow\dfrac{B}{B\wedge B}\vee\mathsf{d_1}\!\downarrow\dfrac{\mathsf{d_2}\!\downarrow\dfrac{\left(C\wedge\mathsf{d_2}\!\downarrow\dfrac{\mathsf{d_1}\!\downarrow\dfrac{\mathsf{c}\!\uparrow\dfrac{\mathsf{d_1}\!\downarrow\dfrac{\mathsf{c}\!\uparrow\dfrac{D\vee D}{[D\vee D]\wedge[D\vee D]}}{D\vee(D\wedge D)}}{[D\vee(D\wedge D)]\wedge[D\vee(D\wedge D)]}}{(D\wedge D)\vee(D\wedge D)}}{\mathsf{w}\!\downarrow\dfrac{\mathsf{c}\!\uparrow\dfrac{D}{D\wedge D}}{(D\wedge D)\vee(D\wedge C)}\wedge[D\vee D]}\right)}{\mathsf{w}\!\downarrow\dfrac{C\wedge D}{B\vee(C\wedge B)\vee(C\wedge D)}\wedge\mathsf{d_1}\!\downarrow\dfrac{[D\vee C]\wedge[D\vee D]}{\mathsf{w}\!\downarrow\dfrac{D}{D\vee(C\wedge B)}\vee(C\wedge D)}}}{(B\wedge D)\vee(C\wedge B)\vee(C\wedge D)}\right]}{\mathsf{d_1}\!\downarrow\dfrac{\mathsf{d_2}\!\downarrow\dfrac{\mathsf{d_2}\!\downarrow\dfrac{(B\wedge B)\vee(B\wedge D)}{B\wedge[B\vee D]}\vee\mathsf{d_2}\!\downarrow\dfrac{(C\wedge B)\vee(C\wedge D)}{C\wedge[B\vee D]}}{[B\vee C]\wedge[B\vee D]}}{B\vee(C\wedge D)}}
$$

Fig. 7. Derivation of $\mathsf{c}\!\downarrow(2)$ in 2-$\{\mathsf{w}\!\downarrow,\mathsf{c}\!\uparrow,\mathsf{d_1}\!\downarrow,\mathsf{d_2}\!\downarrow\}$

Identity can be dropped since all instances can be pushed to the top of a proof and incorporated within an axiom. Weakening, on the other hand plays an essential role in the depth-change trick and dropping identity, but can nonetheless be proved from scratch. The following lemma is proved in [1]:

Lemma 27. *A $\dot{\mathsf{K}}\dot{\mathsf{S}}\mathsf{g}^+$-derivation can be polynomially transformed into one where all identity steps appear at the top and have depth 0.*

Definition 28. Generalized identity *is the axiom* $\mathsf{id}\,\dfrac{}{\bigwedge_i B_i\vee A_i\vee\bar{A}_i}$.

Theorem 29 (Dropping Identity). 1-$\{\mathsf{id},\mathsf{w}\!\downarrow,\mathsf{c}\!\uparrow,\mathsf{d_1}\!\downarrow,\mathsf{d_2}\!\downarrow\}$ *p-simulates* KSg^+.

Proof. Identity is not used in the depth-change trick so just drop identity first, by Lemma 27, then apply the depth-change trick (Theorem 13). □

Theorem 30 (Dropping Weakening). 1-$\{\mathsf{id},\mathsf{c}\!\uparrow,\mathsf{d_1}\!\downarrow,\mathsf{d_2}\!\downarrow\}$ *p-simulates* KSg^+.

Proof. We notice that any sound instance of weakening in a proof can be transformed to depth 1 instances, of the same height, using distributivity. Then observe that any depth 1 instance of weakening in a proof can always be "moved" up above another rule, possibly reducing depth and using distributivity if necessary, thereby reducing the height of its application. The theorem follows by induction on the height of an instance of weakening. □

$$\mathsf{c}_2 \frac{\Gamma}{\Gamma \quad \Gamma} \qquad \vee_2 \frac{\Gamma, A \vee B}{\Gamma, A, B} \qquad \wedge_2 \frac{\Gamma, A \wedge B, C \wedge D}{\Gamma, A, C \quad \Gamma, B, D}$$

$$\underset{\textit{identity}}{\mathsf{id}\,\frac{}{A, \bar{A}}} \qquad \underset{\textit{weakening}}{\mathsf{w}\,\frac{\Gamma}{\Gamma, A}} \qquad \underset{\textit{contraction}}{\mathsf{c}_1 \frac{\Gamma, A, A}{\Gamma, A}} \qquad \underset{\textit{disjunction}}{\vee_1 \frac{\Gamma, A, B}{\Gamma, A \vee B}} \qquad \underset{\textit{conjunction}}{\wedge_1 \frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta}}$$

Fig. 8. System cut-free $\mathsf{GS1p}^+$

5 Conclusions

In this paper we showed that cut-free bounded-depth systems containing cocontraction can polynomially simulate their unbounded-depth counterparts, and have discussed their complexity. We argued the favorability of such systems for proof-search and further improved the situation by showing the admissibility of units, unit rules and certain structural rules with at most polynomial increase in proof size. Now we present some directions in which research in this area may continue, and for which the results presented here may be beneficial.

5.1 Applications to Sequent Calculi

Sequent calculi can essentially be considered depth 1 systems, since the relation between branches is conjunction and the comma is interpreted as disjunction. It is therefore possible to embed our systems into a sequent-like system, augmented slightly to give it top-down symmetry. We present an example of such a system in Fig. 8, based on the one-sided calculus called $\mathsf{GS1p}$ in [11], although less non-deterministic systems can be designed by making use of the results in Sect. 4. While this system is less deterministic than standard sequent calculi it is still a vast improvement to the non-determinism present in unbounded deep inference.

5.2 Bounded-Depth Systems Not Containing Cocontraction

While we have proved that bounded-depth $\mathsf{KSg} \cup \{\mathsf{c}\uparrow\}$ polynomially simulates unbounded-depth $\mathsf{KSg} \cup \{\mathsf{c}\uparrow\}$, it remains open whether a similar result can be obtained for KSg. We think that this is unlikely as the depth-change trick is reliant on cocontraction to compress proofs. It is simple to observe that the inverse distributivity rules, $\mathsf{d}_1\uparrow$ and $\mathsf{d}_2\uparrow$, cannot be derived in a system not containing cocontraction. We make the following conjecture:

Conjecture 31. No bounded-depth KSg system polynomially simulates KSg.

If the conjecture is true, then it would be interesting to see how the efficiency of bounded-depth systems without cocontraction change as the bound on depth is increased. One might intuitively expect a hierarchy of systems, each unable to p-simulate its successor, however it is also possible that all bounded-depth systems are p-equivalent but still not p-equivalent to the unbounded depth system.

5.3 The Effect of Cocontraction on Proof Complexity

It is currently an open problem as to whether KSg can p-simulate $\mathsf{KSg} \cup \{\mathsf{c}\uparrow\}$ but this is thought unlikely to be the case [3]. It is believed that cocontraction compresses proofs by sometimes an exponential factor and this is supported by observational evidence, as well as research in "atomic flows" [7].

An answer to the previous question, on bounded-depth KSg, is probably easier, but may shed some light on this situation.

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