CTλ

$$A ::= \alpha \mid A \to A \mid N$$

$$t ::= x^A \mid \lambda x^A.t \mid tt \mid \mathrm{cond}_x(t,t) \mid 0 \mid St$$

$$\frac{}{\Gamma, x:A \vdash x:A} \qquad \frac{\Gamma \vdash t:A\to B \qquad \Gamma \vdash u:A}{\Gamma \vdash tu:B} \qquad \frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x^A.t : A\to B}$$

$$\frac{\Gamma \vdash f:A \qquad \Gamma, x:N \vdash g:A}{\Gamma \vdash \mathrm{cond}_x(f,g) : N\to A} \qquad \frac{}{\Gamma \vdash 0:N} \qquad \frac{\Gamma \vdash t:N}{\Gamma \vdash St:N}$$

$$(\eta) \qquad \frac{\Gamma \vdash t:N\to A}{\Gamma, x:N \vdash tx:A}$$

$$\mathrm{sum} = \lambda x.\,\mathrm{cond}_y(x, S(\mathrm{sum}(x)(y)))$$

$$\cfrac{\cfrac{\cfrac{}{x:N \vdash x:N} \qquad \cfrac{\cfrac{\cfrac{\vdash \mathrm{sum} : N \to N \to N}{x:N \vdash \mathrm{sum}(x) : N\to N}}{x:N,\, y:N \vdash \mathrm{sum}(x)(y) : N}}{x:N,\, y:N \vdash S(\mathrm{sum}(x)(y)) : N}}{x:N \vdash \mathrm{cond}_y(x, S(\mathrm{sum}(x)(y))) : N \to N}}{\cfrac{\vdash \lambda x.\,\mathrm{cond}_y(x, S(\mathrm{sum}(x)(y))) : N\to N\to N}{\vdash \mathrm{sum} : N \to N \to N}}$$

$$(\lambda x.t)u \longrightarrow t[u/x]$$

$$\mathrm{cond}_x(t,u)(0) \longrightarrow t$$

$$\mathrm{cond}_x(t,u)(Sv) \longrightarrow u[v/x]$$

global trace condition : Any infinite path contains a progressing trace

Prop Subject reduction

Def closed t^N is total $\overset{\text{def}}{\iff}$ $t \xrightarrow{*} n \in N$

closed $t^{A \to B}$ is total $\overset{\text{def}}{\iff}$ $\forall a^A$: closed total . $(ta)^B$: total

$t[\vec{x}]$ is total $\overset{\text{def}}{\iff}$ $\forall \vec{a} : \vec{A}$: closed total . $t[\vec{a}]$: total

$\vec{x} : \vec{A}$

$\vec{x} = FV(t)$

Def closed t^A is a value $\overset{\text{def}}{\iff}$ $(N \neq A \;\&\; t : \text{total})$ or $(N = A \;\&\; t \in N)$

Lem ① t^A : closed & $t \to u$: total $\Longrightarrow$ t : total

② $t^{N\to A}$: closed & $\forall n\in N$ tn : total $\Longrightarrow$ t : total

③ $\vec{x} = FV(t)$ & $\forall \vec{b}:\vec{B}$: closed value, ($t[\vec{b}]$: total) $\Longrightarrow$ $t[\vec{x}]$: total

④ $\vec{x} = FV(t)$ & $\vec{x}:\vec{B}$ & $\left.\begin{matrix}\forall \vec{u}:\vec{B}\\ \forall \vec{a}:\vec{A}\end{matrix}\right\}$ closed value, $t^{\vec{A}\to N}[\vec{u}]\,\vec{a}$: total $\Longrightarrow$ t : total

∵) ① Ind on A

- $A = N$ $\quad \underset{\text{closed}}{t^N} \longrightarrow \underset{\text{closed total}}{\overset{\text{SR}}{u^N}} \xrightarrow{\quad * \quad} n$ $\quad$ ∴ t^N : total

- $A = A_1 \to A_2$ $\quad$ Take $a : A_1$: closed total

$$\underset{\text{closed}}{(ta)^{A_2}} \longrightarrow \underbrace{(\overset{\text{SR}}{u^{A_1\to A_2}}\, a)^{A_2}}_{\text{total by } u:\text{total}}$$

∴ ta : total by I.H.

Then t : total.

② Assume that $t^{N\to A}$: closed & $\forall n\in N$ tn : total

Take u^N : closed total. $\quad$ Then $u^N \xrightarrow{*} n$ for some $n\in N$

$$\underset{\text{closed}}{tu} \xrightarrow{*} \underset{\text{total}}{tn}$$

∴ tu : total by ①

∴ $t^{N\to A}$: total

③ Let $t = t[\vec{x}^{\vec{B}}, \vec{y}^{\vec{N}}]$ & $B_j \neq N$. $\quad$ By asmp, $\forall \vec{b}:\vec{B}$: closed total $\forall \vec{n}\in N$ ($t[\vec{b},\vec{n}]$: total)

Take $\vec{u}^{\vec{N}}$: closed total

∴ $\vec{u} \xrightarrow{*} \vec{n}$ for some $\vec{n} \in N$

$$t[\vec{b},\vec{u}] \xrightarrow{*} \underset{\text{closed total}}{t[\vec{b},\vec{n}]}$$

Hence $t[\vec{x},\vec{y}]$: total

④ Assume that $\vec{x} = FV(t)$ & $\vec{x}:\vec{B}$ & $\left.\begin{matrix}\forall \vec{u}:\vec{B}\\ \forall \vec{a}:\vec{A}\end{matrix}\right\}$ closed value, $t^{\vec{A}\to N}[\vec{u}]\,\vec{a}$: total

Ind on $|\vec{A}|$

- $|\vec{A}| = 0$: By ③ since $t^N[\vec{u}]$: total for all closed value $\vec{u}$
- $\vec{A} = A_0\vec{A}'$: Take $\vec{u}^{\vec{B}}$: closed values, $\vec{a}'^{\vec{A}'}$: closed values, $a_0^{A_0}$: closed value

By asmp. $(t[\vec{u}]\,a_0\,\vec{a}')^N$: total

By IH. $(t[\vec{u}]\,a_0)^{\vec{A}'\to N}$: total

∴ $t[\vec{u}]^{A_0\to\vec{A}'\to N}$: total $\quad$ by $\begin{cases} ② & \text{if } A_0 = N \\ \text{def} & \text{o.w.}\end{cases}$

By ③ $t[\vec{x}]$: total

Thm t: not total $\Rightarrow$ $\Gamma \nvdash t : A$ in $CT\lambda$

$\because$) Assume that t: not total & $\vec{x}:\vec{D} \vdash t : \vec{A} \to N$ has a $CT\lambda$-proof Π

By Lem ⊕, $\exists \vec{a}^{\vec{A}}, \vec{d}^{\vec{D}}$: closed values s.t. $t[\vec{d}]\vec{a}$: not total

We inductively construct $(e_i, \vec{d}_i, \vec{a}_i)$ for each $i \in \mathbb{N}$

s.t. $\begin{cases} e_i : \text{node of } \Pi \text{ where } \Pi e_i = (\vec{x}_i : \vec{D}_i \vdash t_i : \vec{A}_i \to N) \text{ \& } e_{i+1} \text{ is a child of } e_i \\ t_i : \text{not total} \\ \vec{d}_i^{\vec{D}_i}, \vec{a}_i^{\vec{A}_i} : \text{closed values s.t. } (t_i[\vec{d}_i]\vec{a}_i)^N : \text{not total} \text{ \& } \boxed{d_{i+1,k} \le d_{i,\ell}} \end{cases}$

- $(e_0, \vec{d}_0, \vec{a}_0) \overset{\text{def}}{=} (\varepsilon, \vec{d}, \vec{a})$ where $\Pi\varepsilon = (\vec{x}:\vec{D} \vdash t : \vec{A} \to N)$

- Assume that we already have $(e_i, \vec{d}_i, \vec{a}_i)$

Case (Ax): $\Pi e_i = \Gamma, x:A \vdash \underbrace{x}_{t_i} : A$ not the case (x : total)

Case (Zero): $\Pi e_i = \Gamma \vdash \underbrace{0}_{t_i} : N$ not the case (0 : total)

Case (Succ): $\dfrac{\Pi e_{i+1} \overset{\text{def}}{=} \Gamma \vdash t_{i+1} : N}{\Pi e_i = \Gamma \vdash \underbrace{S t_{i+1}}_{t_i} : N}$ By IH $(S t_{i+1})[\vec{d}_i]$: not total & $\vec{a}_i = \emptyset$

$\therefore$ $t_{i+1}[\vec{d}_i]$: not total

Take $\vec{d}_{i+1} \overset{\text{def}}{=} \vec{d}_i$, $\vec{a}_{i+1} \overset{\text{def}}{=} \emptyset$ ⌟

Case (Abs): $\dfrac{\Pi e_{i+1} \overset{\text{def}}{=} \Gamma, z:A \vdash t_{i+1} : \vec{A} \to N}{\Pi e_i = \Gamma \vdash \underbrace{\lambda z.t}_{t_i} : A \to \vec{A} \to N}$ By IH $\underbrace{(\lambda z.t)[\vec{d}_i]\underbrace{a\vec{a}}_{\vec{a}_i} : \text{not total}}$

$(\lambda z.t)[\vec{d}_i] a \vec{a} = (\lambda z. t[\vec{d}_i, z]) a \vec{a} \longrightarrow \underline{t[\vec{d}_i, a]\ \vec{a}}$

not total by & Lem ①

Take $\vec{d}_{i+1} \overset{\text{def}}{=} \vec{d}_i, a$, $\vec{a}_{i+1} \overset{\text{def}}{=} \vec{a}$ ⌟

Case (App): $\Pi e_i = \dfrac{\Gamma \vdash t : A \to \vec{A} \to N \quad \Gamma \vdash u : A}{\Gamma \vdash \underbrace{tu}_{t_i} : \vec{A} \to N}$ By IH. $(tu)[\vec{d}_i]\vec{a}_i$: not total

Subcase: $u[\vec{d}_i]\vec{a}_i$: not total

Take $\Pi e_{i+1} \overset{\text{def}}{=} (\Gamma \vdash u : A)$, $t_{i+1} \overset{\text{def}}{=} u$, $\vec{d}_{i+1} \overset{\text{def}}{=} \vec{d}_i$, $\vec{a}_{i+1} \overset{\text{def}}{=} \vec{a}_i$ ⌟

Subcase: $u[\vec{d}_i]\vec{a}_i$: total.

Take $\Pi e_{i+1} \overset{\text{def}}{=} (\Gamma \vdash t : A \to \vec{A} \to N)$, $t_{i+1} \overset{\text{def}}{=} t$, $\vec{d}_{i+1} \overset{\text{def}}{=} \vec{d}_i$, $\vec{a}_{i+1} \overset{\text{def}}{=} u[\vec{a}_i]\vec{a}_i$ ⌟

Case (η) : $\Pi e_{i+1} \stackrel{\text{def}}{=} \dfrac{\Gamma \vdash t : N \to A}{\Gamma, x : N \vdash \underbrace{t x}_{t_i} : A}$ (x:fresh) By IH. $(tx)[\underbrace{\vec{d} d}_{\vec{d_i}}]\vec{a_i}$: not total

$\Pi e_i =$

$\parallel$

$t[\vec{d}]\, d\, \vec{a_i}$

Take $t_{i+1} \stackrel{\text{def}}{=} t$ & $\vec{d_{i+1}} \stackrel{\text{def}}{=} \vec{d}$ & $\vec{a_{i+1}} \stackrel{\text{def}}{=} d\, \vec{a}$ $\lrcorner$

Case (Case):

$$\Pi e_i = \dfrac{\Gamma \vdash t : A \qquad \Gamma, z : N \vdash u : A}{\Gamma \vdash \mathrm{case}_{z.}(t, u) : N \to A}$$

By IH. $\mathrm{case}_{z.}(t,u)[\vec{d_i}]\, n\, \vec{a}$: not total

$\parallel$

$\mathrm{Case}_{z.}(t[\vec{d_i}], u[\vec{d_i}, z])\, n\, \vec{a}$

Subcase ($n = 0$) :

$\mathrm{Case}_{z.}(t[\vec{d_i}], u[\vec{d_i}, z])\, 0\, \vec{a} \longrightarrow \underline{t[\vec{d_i}]\, \vec{a}}$

not total by & Lem ①

Take $\Pi e_{i+1} \stackrel{\text{def}}{=} (\Gamma \vdash t : A)$

$t_{i+1} \stackrel{\text{def}}{=} t$

$\vec{d_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}$

$\vec{a_{i+1}} \stackrel{\text{def}}{=} \vec{a}$ $\lrcorner$

Subcase ($n = Sn'$) :

$\mathrm{Case}_{z.}(t[\vec{d_i}], u[\vec{d_i}, z])(Sn')\, \vec{a} \longrightarrow \underline{u[\vec{d_i}, n']\, \vec{a}}$

not total by & Lem ①

Take $\Pi e_{i+1} \stackrel{\text{def}}{=} (\Gamma, z : N \vdash u : A)$

$t_{i+1} \stackrel{\text{def}}{=} u$

$\vec{d_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}, n'$

$\vec{a_{i+1}} \stackrel{\text{def}}{=} \vec{a}$ $\lrcorner$

☆ $(e_i)_{i \in \omega}$ is an inf path in Π

By regularity, $(e_i)_{i \in \omega}$ contains progressing trace $(\tau_i)_{i \in \omega}$