UNIVERSITÀ DEGLI STUDI DI TORINO
DIPARTIMENTO DI INFORMATICA

Terminating via Ramsey's Theorem

Ph.D. Thesis

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DOTTORATO IN INFORMATICA - CICLO XXVIII
December 2015

Torino è una città che invita al rigore, alla linearità. Allo stile. Invita alla logica, e attraverso la logica apre la via alla follia.
(Italo Calvino)

Acknowledgements

First and foremost, I am very grateful to my supervisor Stefano Berardi, for his guidance, his precious suggestions and corrections, and all the time dedicated to me during these years. I would like to thank my co-supervisor Paulo Oliva for introducing me to termination theory in the first place and for the fruitful discussions. This work would not have been possible without their contribution.

I am thankful to Keita Yokoyama, for having introduced me to Reverse Mathematics. I acknowledge Alexander Kreuzer, Sylvain Schmitz, and the anonymous referees of my papers for their careful readings and their insightful remarks. For their useful comments and advice I thank Federico Aschieri, Giovanni Birolo, Emanuele Frittaion, Gerhard Jäger, Ulrich Kohlenbach, Alberto Marcone and Peter Schuster.

I thank the Logic and Set Theory Group of Torino. I am especially grateful to Alessandro Andretta, for introducing me to research in the first place, and to Matteo Viale, who spent a huge amount of time supporting my studies during these years. I thank Luca Motto Ros for the opportunity to attend and take part in the weekly seminars. I would like to thank Giorgio Audrito for all the time spent working together, it has been a pleasure. My deep gratitude goes to Raphaël Carroy, who has patiently guided me with both his invaluable advice and his frank scoldings.

For their friendship and their priceless support throughout any situation I am very thankful to Gemma Carotenuto and Fiorella Guichardaz.

I would like to thank colleagues and friends who contributed, each one in his own way, to make these years unforgettable and special: Cristina Battaglino, Livio Bioglio, Federico Capuzzimati, Paolo Castagno, Erika De Benedetti, Maurizio Drocco, Giulio Ferrero, Elena Gianaria, Pierluigi Grillo, Claudia Misale, Alan Perotti, Luca Piovesan, Manuela Sanguinetti, Fabio Tordini, Francesco Verdoja, Filippo Calderoni, Alessandro Vignati, Matteo Acerbi, Davide Rinaldi, Jules Hedges, Riccardo Maffucci, Alessandro Rubba, Emilio Zappa and Kevin Fournier.

Infine vorrei ringraziare la mia famiglia per tutto il loro affetto e il continuo supporto. Un grazie particolare ai miei genitori, per avermi insegnato due volte a camminare, ed a Camilla ed Edoardo, per avermi costantemente ricordardato di credere nelle favole!

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Introduction

The goal of this thesis is to investigate in suitable logical settings termination results, which make use of Ramsey's Theorem for pairs, in order to extract bounds for termination. Amongst the several applications of Ramsey's Theorem for pairs in Computer Science, we focus on the Termination Theorem by Podelski and Rybalchenko and the Size-Change Termination Theorem by Lee, Jones and Ben-Amram. This work is divided in three main parts. We start by isolating a fragment of Ramsey's Theorem for pairs which is provable in Second Order Intuitionistic Arithmetic; from this result we provide intuitionistic proofs and corresponding bounds for termination results. A second part is devoted to study bounds for the Termination Theorem by using a fundamental tool of Proof Theory: Spector's bar recursion. Eventually, we investigate termination analysis from the viewpoint of Reverse Mathematics.

0.1 Overview on motivation

Ramsey's Theorem for pairs [78] is a fundamental result in combinatorics. Ramsey's Theorem for pairs states that for any natural number k and for any coloring over the edges of the complete graph with countably many nodes in k-many colors there exists an infinite homogeneous set (i.e. an infinite subset of the nodes whose elements are all connected in the same color). As highlighted by Gasarch in [38], Ramsey's Theorem for pairs can be used to prove termination.

> Many programs allow the user to input data several times during its execution. If the program runs forever the user may input data infinitely often. A program terminates if it terminates no matter what the user does. [...] The methods employed [*to prove that a program terminates*] are well-founded orders, Ramsey's theorem, and matrices[1]. (Gasarch [38])

Moreover Gasarch pointed out that the methods described are used by real program checkers. Hence the importance of studying bounds for such methods.

In [77] Podelski and Rybalchenko expressed the termination of transition-based programs as a property of well-founded relations. The Termination Theorem by Podelski and Rybalchenko states that a transition-based program $\mathcal{R}$ is terminating if and only if there exist finitely many well-founded relations whose union contains the transitive

[1] Matrices methods in Gasarch paper express Size-Change Termination.

closure of the transition relation of the program. From the Termination Theorem, Cook, Podelski, and Rybalchenko [22] extracted an algorithm taking as input a transition-based program, and able to decide in some cases whether the program is terminating or not, and in some other cases leaving the question open. The termination problem is known to be undecidable in general.

According to the Π^0_2-conservativity of Classical Analysis with respect to Intuitionistic Analysis [37], the proof of the Termination Theorem hides some effective bounds for the transition-based programs which the theorem shows to terminate. Hence a natural question which arises is

Question 0.1.1. May we extract bounds from the proof of the Termination Theorem in order to know how many steps are required by some program to terminate?

The original proof of the Termination Theorem is based on Ramsey's Theorem for pairs. It is well-known that Ramsey's Theorem for pairs is a purely classical result. Indeed, there is one recursive coloring in two colors with no recursive infinite homogeneous sets (Speacker [84]). Besides, Jockusch proved that, for some recursive enumerable families of recursive colorings, it is not possible to recursively find a color for which there is an infinite homogeneous set [50]. In [10] Berardi and the author proved that the fragment of Ramsey's Theorem for pairs which can be expressed as a first order schema is equivalent in HA to Σ^0_3-LLPO, a classical principle strictly between the Excluded Middle for 3-quantifiers arithmetical formulas and the Excluded Middle for 2-quantifiers arithmetical formulas [2]. All these results say that extracting bounds from the original proof is a non-trivial task. Hence the need of using different perspectives. A result in this direction can be found in [93]. Vytiniotis, Coquand and Wahlsteldt provided an intuitionistic proof of the Termination Theorem by using the inductive definition of well-foundedness and Coquand's intuitionistic version of Ramsey's Theorem for pairs: Almost Full Theorem [23].[2] For the moment, no bound analysis based on the Almost Full Theorem has been conducted.

A second natural question which arises when analysing the Termination Theorem is:

Question 0.1.2. Is there a correspondence between the complexity of a primitive recursive transition relation and the number of well-founded relations which witness the termination?

For any natural number k, let $\mathcal{F}_k$ denote the level k of the Fast Growing Hierarchy [61]. By a deeper analysis of a miniaturization of the Dickson Lemma, Figueira et al. provided a partial answer to Question 0.1.2. Given a transition-based program such that

- its transition relation is the graph of a function in $\mathcal{F}_2$;
- there exist k-many relations whose union contains the transitive closure of the transition relation of the program, all having weight functions in $\mathcal{F}_1$

[2] More details are given in Section 2.5.

then the program computes a function in $\mathcal{F}_{k+1}$.[3] We obtain more results along this direction.

The Termination Theorem is not the only result which characterizes the termination of some class of programs by using Ramsey's Theorem for pairs. In 2001 Lee, Jones and Ben-Amram introduced the notion of size-change termination (SCT) for first order functional programs, a sufficient condition for termination. A first order functional program $\mathcal{P}$ is SCT if for any infinite sequence of calls which follows the control of $\mathcal{P}$ there exists a variable whose value has to decrease infinitely many times. If the domain of the values of $\mathcal{P}$ is well-founded this condition guarantees the termination. In [58] it is proven that any first order functional program is SCT if and only if it satisfies some combinatorial property which can be statically verified from the recursive definition of the program. This result is called the SCT Theorem. In order to prove the SCT Theorem the authors used Ramsey's Theorem for pairs.

Question 0.1.3. Which bounds may we get for SCT programs?

As for the Termination Theorem, Vytiniotis, Coquand and Wahlsteldt [93] intuitionistically proved a classical variant of the SCT Theorem by using the Almost Full Theorem instead of Ramsey's Theorem for pairs. Moreover in [6] Ben-Amram proved that SCT programs compute multiple recursive functions and that any tail-recursive SCT functional program is primitive recursive.

0.2 Formal Systems

Through this thesis we work in different suitable settings to face the questions above from several points of view. The main difference of perspective lies in the distinction between working in intuitionistic arithmetic or in classical arithmetic. As usual Peano Arithmetic consists of Robinson axioms (basic properties of zero, successor, sum and product) plus the first order induction scheme. Heyting Arithmetic (HA) adopts the axioms of Peano arithmetic, but uses intuitionistic logic as its rules of inference (in particular the Law of Excluded Middle does not hold). For the sake of completeness we briefly introduce all the systems we use.

Heyting Arithmetic in all finite types. In [40], Kurt Gödel interpreted HA in a quantifier free type theory with primitive recursion in all finite types, the so-called system T. This interpretation became known as "Dialectica" interpretation and it extends to typed Heyting arithmetic.

The system HA^ω (see [90]) can be seen as an expansion of HA with quantification over all finite types. For short we enrich the type system by finite sequences. The finite types are defined inductively where $\mathbb{N}$ is the basic finite type and, $\tau_0 \to \tau_1$ is the type of functions from τ_0 to τ_1, and τ_0^* is the type of finite sequences whose elements are of type τ_0.

[3] More details are given in Subsection 5.6.2.

The dialectica interpretation of arithmetic was extended by Spector to classical analysis in the system "$\mathsf{T}+$ bar recursion" [85].

Second order intuitionistic arithmetic. The language of second order arithmetic has two-sorted variables: number variables (denoted by lower case letters and usually interpreted as natural numbers) and set variables or predicates (denoted by upper case letters and usually interpreted as sets of natural numbers).

As defined in [89], second order intuitionistic arithmetic (HAS) consists of axioms of HA, the second order induction scheme, i.e.

for any formula $\varphi(n)$ of second order arithmetic with a free number variable n and possibly other free number and set variables ($\overline{m}$ and $\overline{Y}$)

$$\forall \overline{m} \forall \overline{Y}((\varphi(0) \wedge \forall n(\varphi(n) \implies \varphi(S(n)))) \implies \forall n \varphi(n)),$$

and the full second order comprehension scheme, i.e.

for any formula $\varphi(n)$ of second order arithmetic with a free number variable n and possibly other free number and set variables ($\overline{m}$ and $\overline{Y}$) for which X is not free

$$\forall \overline{m} \forall \overline{Y}(\exists X \forall n(n \in X \iff \varphi(n))).$$

The system HAS_0 is the subsystem of HAS (the second order intuitionistic arithmetic) with only arithmetical comprehension, which is a conservative extension of HA (see [89]).

Subsystems of second order classical arithmetic. Reverse Mathematics is a program of Mathematical Logic, introduced by Harvey Friedman in [35, 36], which arose from the following question. Given a theorem of ordinary mathematics, what is the weakest subsystem of second order arithmetic in which it is provable?

Second order arithmetic consists of basic axioms of arithmetic (Robinson axioms), the second order induction scheme and the full second order comprehension scheme. By restricting the induction and comprehension for formulas in some suitable class Γ we may define some of the several subsystems of second order arithmetic. Usually Γ is one of the classes of the standard arithmetical or analytical hierarchy of formulas (i.e. Σ^i_n, Π^i_n or Δ^i_n for some natural number n and $i < 2$).

Amongst the several subsystems of second order arithmetic, Simpson in [83] highlights the "Big Five", which occur more frequently in Reverse Mathematics. Let Γ-separation be the scheme: for any $\psi(x)$, $\varphi(x)$ in Γ which are exclusive and for which n, X are not free,

$$\exists X \forall n(n \in X \iff \psi(n) \wedge \neg\varphi(n)).$$

The Big Five are:

- Recursive Comprehension Axiom (RCA_0): axioms of arithmetic, Σ^0_1-induction, Δ^0_1-comprehension.

- Weak König Lemma[4] (WKL_0): RCA_0, Σ^0_1-separation.
- Arithmetical Comprehension Axiom (ACA_0): RCA_0, arithmetical comprehension.
- Arithmetical Transfinite Recursion (ATR_0): ACA_0, Σ^1_1-separation.
- Π^1_1 Comprehension Axiom (Π^1_1-CA_0): ACA_0, Π^1_1-comprehension.

0.3 Contributions and plan of the thesis

In this thesis we investigate Question 0.1.1 (extraction of bounds) from three different perspectives. First of all, we introduce a fragment of Ramsey's Theorem for pairs which is provable in HAS. From this result and by using the inductive definition of well-foundedness we provide intuitionistic proofs and corresponding bounds for termination results in Chapter 2 and Chapter 3. Then we study bounds for the Termination Theorem by using Spector's bar recursion, hence working in extensions of HA^ω we provide a bound by considering the classical definition of well-foundedness. Eventually, in Chapter 5 we investigate termination analysis from the point of view of Reverse Mathematics. In particular, the contributions and the structure of this thesis are the following.

Chapter 1. This chapter is devoted to the introduction of the main characters of this thesis. The aim of Section 1.1 is to present Ramsey's Theorem for pairs and the Termination Theorem. We recall also two other well-known combinatorial principles which are strictly related to Ramsey's Theorem for pairs: the Infinite Pigeonhole Principle and König's Lemma. The first one asserts that any partition in finitely many pieces of a countable set contains a piece with countably many elements. The second one states that any finitely branching tree with countably many nodes has an infinite branch.

In Section 1.2 we focus on the definition of well-foundedness, which is a crucial topic in this thesis. After investigating the relationship over HAS_0 of several classically equivalent definitions of well-foundedness, we summarize the main properties of inductive well-foundedness. Results in this section are a joint work with Stefano Berardi [11], except for results in Subsection 1.2.1 which are a joint work with Stefano Berardi and Paulo Oliva [9].

Chapter 2. In this chapter we provide an intuitionistic proof and corresponding bounds for the Termination Theorem by using the inductive definition of well-foundedness.

In Section 2.2 we intuitionistically prove the Termination Theorem by introducing a new intuitionistic version of Ramsey's Theorem for pairs, which we called H-closure Theorem. In Section 2.5, we highlight the main differences between H-closure Theorem and Almost Full Theorem [23]. Results in these sections are a joint work with Stefano Berardi [11].

[4] The name Weak König Lemma derives from the fact that it is equivalent to RCA_0 plus a weak form of König's Lemma, namely: every infinite subtree of the full binary tree has an infinite path [83, Lemma IV.4.4].

In Section 2.3 we consider the set of functions having at least one implementation in Podelski-Rybalchenko language for which there exist finitely many relations of height ω whose union contains the transitive closure of the transition relation of the program. We prove that this set is exactly the set of primitive recursive functions. Sylvain Schmitz has pointed out to us that Blass and Gurevich [15] and Delhommé [25] already observed that the computation of the ordinal height of a relation proven to be well-founded by the Termination Theorem is the natural product of the individual heights. Nonetheless our argument produces bounds by using the proof of the Termination Theorem based on H-closure, therefore this would hopefully provide a schema for other applications of H-closure. Results in this section are a joint work with Stefano Berardi and Paulo Oliva [8].

Finally, in Section 2.4, we show that the same result holds for the Terminator Algorithm based on the Termination Theorem, defined by Cook, Podelski and Rybalchenko [22]. A function has at least one implementation in Podelski-Rybalchenko's language which the Terminator Algorithm may catch terminating if and only if the function is primitive recursive.

Chapter 3. In this chapter we provide an intuitionistic proof of a classical variant of the SCT Theorem: our goal is to provide both a statement and a proof very similar to the original ones. This can be done by using the H-closure Theorem and inductive well-foundedness. Since we find no way to intuitionistically deduce the H-closure Theorem from the Almost Full Theorem, there are no apparent relationships between our proof and the classical variant of the SCT Theorem provided in [93].

As a side result we obtain another proof (completely different from the one by Ben-Amram [6]) of the characterization of functions computed by a tail-recursive SCT program. Our proof is based on the bounds found for the Termination Theorem in Section 2.3 and [34]. We can use these bounds since the SCT Theorem and the Termination Theorem are strictly related. Heizmann, Jones and Podelski proved that size-change termination is a property strictly stronger than termination [43]. By applying an argument similar to the one used in [43], we get a bound for a tail-recursive SCT program from the one for the Termination Theorem provided in [34]. Finally, we find a property in the "language" of size-change termination which is equivalent to Podelski and Rybalchenko's termination.

In this chapter we work in HAS; all the proofs are intuitionistic. Results in this chapter are published in [86].

Chapter 4. This chapter arose from an attempt to compare the bounds for the Termination Theorem obtained in constructive mathematics, proof theory and reverse mathematics (as in Chapter 2, Chapter 5 and [34, 93]) with the one we can obtain by using another fundamental tool of proof theory: Spector's bar recursion [85]. Spector's extension of the Dialectica interpretation assures us that the proof of the Termination Theorem can be interpreted using $\mathsf{T}+$ bar recursion. Following the approach of the proof mining program [54], we are applying the ideas from the bar-recursive interpretation of countable choice without following literally the functional interpretation of the proof, in order to obtain human-readable bounds.

The sub-recursive bounds we obtain make use of bar recursion, in the form of the product of selection functions [32], as this is used to interpret the Weak Ramsey Theorem for pairs.[5] The construction can be seen as calculating a modulus of well-foundedness for a given program given moduli of well-foundedness for the well-founded covering relations. When the input moduli are in system T, this modulus is also definable in system T by a result of Schwichtenberg on bar recursion [82]. Throughout this chapter we work in extensions of HA^{ω}. Results in this chapter are a joint work with Stefano Berardi and Paulo Oliva [9].

Chapter 5 The goal of this chapter is to study the H-closure Theorem and the Termination Theorem from the viewpoint of Reverse Mathematics. Results from Reverse Mathematics may be applied to several uses, in this thesis we apply them to the extraction of bounds. The first question is whether the H-closure Theorem and the Termination Theorem are equivalent over RCA_0 to Ramsey's Theorem for pairs. Due to our analysis we answer to [38, Open Problem 2] posed by Gasarch: finding a natural example showing that the Termination Theorem requires the full Ramsey Theorem for pairs. In this chapter we prove that such program cannot exist. We also answer negatively to [38, Open Problem 3] posed by Gasarch: is the Termination Theorem equivalent to Ramsey's Theorem for pairs?

In [34] Figueira et al. show that the Termination Theorem is a consequence of Dickson's Lemma by observing that any relation is well-founded if and only if it is embedded into a well-quasi-ordering. However this property of quasi-well-orderings is equivalent to ACA_0 over RCA_0 and therefore in order to analyse the strength of the Termination Theorem we need a different point of view. In Section 5.2 we prove that the Termination Theorem is equivalent over RCA_0 to a weak version of Ramsey's Theorem for pairs. As a corollary of this result we have that for any natural number k, CAC (the Chain-AntiChain principle) is stronger than the Termination Theorem for k many relations. The latter is the statement: given a binary relation R, if there exist k-many well-founded relations whose union contains the transitive closure of R, then R is well-founded.

From our results and by using [20, 33, 72] we obtain a different proof for the characterization of transition-based programs proved to be terminating by the Termination Theorem. In order to provide more precise termination bounds, in Section 5.5 we study the reverse mathematical strength of some bounded versions of both the H-closure Theorem and the Termination Theorem. These two results in the full case are not equivalent, in the restricted ones they turn out to be equivalent. Moreover we prove they are equivalent to a weaker version of the Paris Harrington Theorem [71]. Due to our analysis and by using the relationship between Paris Harrington Theorem and the Fast-Growing Hierarchy, in Section 5.6 we investigate Question 0.1.2. As a side result, in

[5] Weak Ramsey Theorem a is corollary of Ramsey's Theorem for pairs defined in [66].

Subsection 5.6.2, we provide a proof in the subsystem RCA_0^*[6] of the implication from $\mathrm{Tot}(\mathcal{F}_{k+1})$ to the Paris Harrington Theorem for pairs and k colors.[7]

In this chapter we work in subsystems of second order arithmetic. Results in this chapter (except those of Subsection 5.6.2) are a joint work with Keita Yokoyama [87].

Further remark In [7] Berardi presented an a posteriori justification of the existence of results presented in Section 2.2 and in Section 3.2. In the case of a classical proof of a statement of the form "if $R_0, \dots R_k$ are inductively well-founded then R is inductively well-founded", for $R_0, \dots, R_k, R$ primitive recursive, Berardi proved that constructivization is always possible. However, as claimed in [7], producing constructive proofs with this method is not feasible in practice. As shown in Section 2.2 and in Section 3.2, in the case of proofs which use Ramsey's Theorem, H-closure Theorem seems to be suitable for this purpose.

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[6]The system RCA_0^* is a subsystem of RCA_0, defined for the language of second order arithmetic enriched with an exponential operator. RCA_0^* consists of axioms of arithmetic, exponentiation axioms, Δ_0^0-induction and Δ_1^0-comprehension.

[7]In [51] Solovay and Ketonen prove that for any natural number k, $\mathrm{Tot}(\mathcal{F}_{k+5})$ implies the Paris Harrington Theorem for pairs and k colors. As far as we know our result is new.

Chapter 1

Starting to work

This chapter is devoted to the introduction of the main characters of this thesis. The aim of Section 1.1 is to present Ramsey's Theorem for pairs and the Termination Theorem. We recall also two other well-known combinatorial principles which are strictly related to Ramsey's Theorem for pairs: the Infinite Pigeonhole Principle and König's Lemma. The first one asserts that any partition in finitely many pieces of a countable set contains a piece with countably many elements. The second one states that any finitely branching tree with countably many nodes has an infinite branch.

In Section 1.2 we focus on the definition of well-foundedness, which is a crucial topic in this thesis. After investigating the relationship over HAS_0 of several classically equivalent definitions of well-foundedness, we summarize the main properties of inductive well-foundedness. Results in this section are a joint work with Stefano Berardi [11], except for results in Subsection 1.2.1 which are a joint work with Stefano Berardi and Paulo Oliva [9].

1.1 Introducing the Termination Theorem

Any natural number n is identified with the set $\{0, \ldots, n-1\}$ and we use $\mathbb{N}$ to denote the least infinite ordinal, which is identified with the set of natural numbers. For any set X and any natural number n, $[X]^n$ denotes the set of subsets of X of cardinality n: i.e.

$$[X]^n = \{Y \subseteq X \mid |Y| = n\}.$$

For $n = 1$, $[\mathbb{N}]^1$ is the set of the singletons of $\mathbb{N}$, therefore just another notation for $\mathbb{N}$. For $n = 2$, $[\mathbb{N}]^2$ is the complete graph on $\mathbb{N}$: any subset $\{x, y\}$ of $\mathbb{N}$ with $x \neq y$ is an edge of the graph. Observe that $\{x, y\}$ and $\{y, x\}$ are the same set, and so the same edge; conventionally for $x < y$ we denote the edge between them with $\{x, y\}$. Given a set X, the set $[X]^n$ is called complete n-regular *hypergraph* on X.

Let $n, k \in \mathbb{N}$, any map $c : [\mathbb{N}]^n \to k$ is called a *coloring* of $[\mathbb{N}]^n$ in k colors. If $n = 2$ and $c(\{x, y\}) = i < k$, then we say that the edge $\{x, y\}$ has color i. Given a coloring $c : [\mathbb{N}]^n \to k$ and given $X \subseteq \mathbb{N}$ we denote with $c''[X]^n$ the set of colors of the elements of

$[X]^n$, that is:

$$c''[X]^n = \{i < k \mid \exists e \in [X]^n \text{ such that } c(e) = i\}.$$

We say that $X \subseteq \mathbb{N}$ is *homogeneous* for c, or c is homogeneous on X, if all elements of $[X]^n$ have the same color, that is, there exists $i < k$ such that $c''[X]^n = \{i\}$. We also say that X is homogeneous for c in color i.

If $n = 1$, we can think of the function c as a partition of the natural numbers. In this case a homogeneous set X is any subset of $\mathbb{N}$ whose elements all have the same color (i.e. all are in the same class of the partition). If $n = 2$, we can think of the function c as a coloring over the edges of a complete graph having as its nodes the natural numbers. In this case a homogeneous set X is any set of points of $\mathbb{N}$ whose connecting edges all have the same color.

1.1.1 Infinite Pigeonhole Principle and König's Lemma

The Pigeonhole Principle was first formalized by Peter Gustav Lejeune Dirichlet in 1834 and states that for any natural numbers $k < n$, if we put n-many objects into k-many boxes, then at least one box must contain more than one object. We are interested in the infinite version of the Pigeonhole Principle, which is a standard combinatorial result (see e.g. [41]).

Theorem 1.1.1 (Infinite Pigeonhole Principle, ACA_0)**.** *Given a natural number k and a coloring $c : \mathbb{N} \to k$ there exists $X \subseteq \mathbb{N}$ such that it is infinite and $|c''X| = 1$.*

Proof. Suppose by contradiction that for all $i < k$, $c^{-1}(i) = \{x \in \mathbb{N} \mid c(x) = i\}$ is finite. Then

$$\mathbb{N} = \bigcup \left\{c^{-1}(i) \mid i < k\right\}$$

should be finite as well. Contradiction. □

Observe that the Infinite Pigeonhole Principle cannot be proved in intuitionistic logic since it implies the Law of Excluded Middle over HAS_0. For short we only prove that the Infinite Pigeonhole Principle implies the Law of Excluded Middle for Σ^0_1 predicates. The proof is well-known (see e.g. [2]).

Proposition 1.1.2 (HAS_0)**.** *Assume that for any natural number k and any coloring $c : \mathbb{N} \to k$ there exists an infinite subset $X \subseteq \mathbb{N}$ such that $|c''X| = 1$. Then for any Σ^0_1 predicate P, $P \vee \neg P$ holds.*

Proof. Suppose $P = \exists x Q(x)$ and define $c : \mathbb{N} \to 2$ such that $c(x) = 0$ if and only if $\forall y \leq x \neg Q(y)$. Since Q is Δ^0_1, c is a partition. By the Infinite Pigeonhole Principle, there exists an infinite subset $X \subseteq \mathbb{N}$ such that $|c''X| = 1$. If $c(\min(X)) = 0$ then by definition of c, $\forall y \neg Q(y)$ holds. On the other hand, if $c(\min(X)) = 1$, then $\exists y \leq \min(X) Q(y)$. □

The second standard combinatorial principle we need to recall is König's Lemma, due to Dénes König [53] in 1926.

We consider the notion of tree which is standard in Computer Science and Graph Theory. A tree may be defined as a partially ordered set $(\mathrm{Tr}, \prec)$ such that it has a $\prec$-minimal element of Tr, called root, and for each $t \in \mathrm{Tr}$, the set $\{s \in \mathrm{Tr} \mid s \prec t\}$ is finite and it is well-ordered by the relation $\prec$. We say that a node $x \in \mathrm{Tr}$ is the father of y (or y is a child of x) if

$$x \prec y \;\wedge\; \forall z(x \prec z \prec y \implies (z = x \vee z = y)).$$

Equivalently a tree can be defined as a directed acyclic connected graph with a designated root (see e.g. [41]). A branch of a tree is a maximal $\prec$-chain of Tr.

Theorem 1.1.3 (König's Lemma, ACA_0)**.** *Let* $(\mathrm{Tr}, \prec)$ *be a finitely branching tree with countably many nodes. Then there exists an infinite branch in* Tr*.*

Proof. For any $x \in \mathrm{Tr}$ define $S(x)$ to be the set of children of x: i.e.

$$S(x) = \{y \in \mathrm{Tr} \mid x \prec y \wedge \forall z(x \prec z \prec y \implies (z = x \vee z = y))\}.$$

Since Tr is finitely branching then for any $y \in \mathrm{Tr}$, $S(y)$ is finite. Let a_0 be the root of Tr. Since Tr has countably many nodes, a_0 has infinitely many descendants. We want to prove that there exists a child of a_0 which has infinitely many descendants. Since $S(a_0)$ is finite and a_0 has infinitely many descendants, thanks to the Infinite Pigeonhole Principle there exists $a_1 \in S(a_0)$ such that it has infinitely many descendants. By applying the same argument inductively we get $\langle a_i \mid i \in \mathbb{N}\rangle$ is an infinite branch of Tr. □

It is well-known that König's Lemma is a purely classical result. Indeed it implies the Infinite Pigeonhole Principle. Thus, thanks to Proposition 1.1.2, König's Lemma implies the Law of Excluded Middle.

Proposition 1.1.4 (HAS_0)**.** *König's Lemma implies the Infinite Pigeonhole Principle.*

Proof. Given $c : \mathbb{N} \to k$ define $\prec$ as follows:

$$x \prec y \iff (x < y \wedge c(x) = c(y)) \vee x = 0.$$

Hence $(\mathbb{N}, \prec)$ is a k-branching infinite tree with root 0. Therefore, by König's Lemma, there exists an infinite branch B. The set X of the elements greater than 0 in B is such that $|c''X| = 1$. □

1.1.2 Ramsey's Theorem

In 1930 Frank Plumpton Ramsey proved the combinatorial result which now bears his name [78]. Ramsey's Theorem states that for any n and k natural numbers and for any coloring over the edges of the complete n-regular hypergraph with k-many colors $c : [\mathbb{N}]^n \to k$, there exists an infinite homogeneous set, i.e. an infinite set $H \subseteq \mathbb{N}$ such that all elements in $[H]^n$ have the same color. In the case $n = 2$, this means that all edges whose nodes are all in H have the same color. Formally

Theorem 1.1.5 (Ramsey's Theorem)**.** *For any n, $k \in \mathbb{N}$ and for any coloring $c : [\mathbb{N}]^n \to k$ there exists an infinite homogeneous set.*

We are mainly interested in Ramsey's Theorem for pairs which is the case of Theorem 1.1.5 for $n = 2$. Since Ramsey's Theorem for pairs straightforwardly implies Ramsey's Theorem for singletons ($n = 1$, therefore the Infinite Pigeonhole Principle) and thanks to Proposition 1.1.2 we have that Ramsey's Theorem for pairs is a purely classical result. We may deduce this well-known fact in a new way: in [10] Berardi and the author proved in Intuitionistic Arithmetic that the first order fragment of Ramsey's Theorem is equivalent to the sub-classical principle Σ^0_3-LLPO, and Σ^0_3-LLPO is not intuitionistically derivable [2].

There are several proof of Ramsey's Theorem for pairs. Here we recall the one by Erdős-Rado [27], following the notation used by Kreuzer and Kohlenbach in [56]. It can be formalized in ACA_0.

Theorem 1.1.6 (Ramsey's Theorem for pairs, ACA_0)**.** *For any $k \in \mathbb{N}$ and for any coloring $c : [\mathbb{N}]^2 \to k$ there exists an infinite homogeneous set.*

Proof. Fix $k \in \mathbb{N}$. Our first goal is to define an infinite set $X \subseteq \mathbb{N}$ for which there exists $c^* : X \to k$ such that for any x, $y \in X$, if $x < y$ then $c(\{x,y\}) = c^*(x)$; i.e. for any edge which connects elements of X, its color depends only on the one of its minimum element.

Given $c : [\mathbb{N}]^2 \to k$, define for any $m \in \mathbb{N}$ the function c_m which associates to any $x \in m$ the color of the edge between x and m:

$$\begin{aligned} c_m : m &\longrightarrow k \\ x &\longmapsto c(\{x,m\}). \end{aligned}$$

Let now define a recursive partial order $\prec$ on $\mathbb{N}$ in such a way, we have $x \prec y$ if and only if for any $z \prec x$, the edges $\{z,x\}$ and $\{z,y\}$ have the same color. We define it by induction as follows:

- $0 \prec 1$.
- If $\prec$ is defined in m we extend it in $m+1$. For any $n \in m$ let

  $$P_n = \{x \in m \mid x \prec n\}.$$

  and we put

  $$n \prec m \Leftrightarrow c_{n \restriction P_n} = c_{m \restriction P_n};$$

  i.e. we require that the color of $\{x,n\}$ is the same as the color of $\{x,m\}$ for all $x \prec n$.

We prove that in any set which is totally ordered by $\prec$, the color of an edge depends only on its minimum. After that we prove that there exist infinite sets which are totally ordered by $\prec$.

Claim. *We have the following.*

1. $\prec \subseteq <_{\mathbb{N}}$, *in particular* $P_x = \mathrm{pd}(x)$, *where* $\mathrm{pd}(x) = \{y \in \mathbb{N} \mid y \prec x\}$.

2. $0 \prec x$ *for all* $x \in \mathbb{N} \setminus \{0\}$.

3. $\prec$ *is transitive.*

4. *On* $\mathrm{pd}(m)$ *the relations* $<_{\mathbb{N}}$ *and* $\prec$ *describe the same order; i.e. for any* $x, y \in \mathrm{pd}(m)$
$$x < y \iff x \prec y.$$

Proof. 1. By definition of $\prec$.

2. By definition of $\prec$ and due to the fact that any function restricted to the empty set is the empty set.

3. We prove
$$(x \prec y) \wedge (y \prec z) \implies x \prec z.$$
by induction on z. If $z = 0$ it is trivial. Assume that the transitivity holds for any $z' < z$, then:
$$\begin{aligned} x \prec y \wedge y \prec z &\implies c_{x \restriction P_x} = c_{y \restriction P_x} \wedge c_{y \restriction P_y} = c_{z \restriction P_y} \wedge P_x \subseteq P_y \\ &\implies c_{x \restriction P_x} = c_{y \restriction P_x} = c_{z \restriction P_x} \Rightarrow x \prec z, \end{aligned}$$
where $P_x \subseteq P_y$ follows from the inductive hypothesis on y.

4. "$\Leftarrow$": it follows from (1). "$\Rightarrow$": We show it by induction on m. If $m = 0$ we are done. Hence assume that $m > 0$, $x, y \in \mathrm{pd}(m)$ with $x < y$ and assume that the statement is true for any $m' < m$. If $x = 0$ we are done again. Otherwise let i be the $<$-maximum natural number such that $i \prec x$ and $i \prec y$ (such number exists since $0 \prec x$ and $0 \prec y$). Let p be an immediate $\prec$-successor of i in the set $\mathrm{pd}(m)$: p exists since $i \prec x \prec m$. Due to the choice of p we have $p \leq x < y$. From $i \prec x$, p, $y \prec m$ we get, by definition of $\prec$, that the edges from i to x, y, p and m are in the same color: i.e.
$$c_x(i) = c_y(i) = c_m(i) = c_p(i).$$
By inductive hypothesis on $m' = p$, for any $z \prec p$ we have one of either $z \prec i$, $z = i$ or $i \prec z \prec p$. The third one is impossible since p is an immediate successor of i. Therefore for any $z \prec p$ we have either $z \prec i$ or $z = i$: i.e. $P_p = P_i \cup \{i\}$. Since $i \prec y$ and $c_y(i) = c_p(i)$ then for any $z \prec p$, the color of $\{z, y\}$ is the same of the color of $\{z, p\}$, since either $z \prec i \prec y$, p, or $z = i$. By $p < y$ we get $p \prec y$. Analogously, since $i \prec x$ and $c_x(i) = c_p(i)$, then for any $z \prec p$, the color of the edges $\{z, x\}$ and $\{x, p\}$ are the same, since either $z \prec i \prec x, p$, or $z = i$. Therefore by $p \leq x$, $p \prec x$ follows if $p < x$, otherwise $p = x$. Since i is maximal then we cannot have $p \prec x$, this guarantees that $p = x$ and, in particular $x \prec y$. $\square$

Let $\prec_1$ be the relation "immediate $\prec$-successor": i.e. $x \prec_1 y$ if and only if $x \prec y$ and there exists no z such that $x \prec z \prec y$. By (1), the relation $\prec$ defines a tree Tr on $\mathbb{N}$ with root 0 and whose father/child relation is $\prec_1$.

Let $0 = x_0 \prec_1 x_1 \prec_1 x_2 \prec_1 \dots$ be a branch of Tr. For any $i < j < k$ we have $x_i \prec x_j \prec x_k$ by transitivity, therefore the colors of the edges $\{x_i, x_j\}$ and $\{x_i, x_k\}$ are the same, the color depends only on x_i.

Moreover we can observe that this tree is k-branching, since any child has a color which differs from the one of its siblings and we have k-many colors. Indeed assume that $x_0 \prec_1 x_1 \prec_1 \cdots \prec_1 x_i \prec_1 x,\ y$, that the nodes x, y are siblings in Tr, and that $x < y$ in order to prove that $\{x_i, x\}$ and $\{x_i, y\}$ have different colors. By definition of $\prec$, the colors of $\{x_j, x\}$ and $\{x_j, y\}$ can differ only if $j = i$. Assume by contradiction that $\{x_i, x\}$ and $\{x_i, y\}$ are in the same color. Hence by $x < y$ we get $x \prec y$ since

$$c_{x \restriction P_i} = c_{y \restriction P_i} \wedge P_x = P_i \cup \{i\}.$$

And this is a contradiction with $x_i \prec_1 y$.

By applying König's Lemma there exists an infinite branch B. It has the property that any edge from a node x has the same color i. We define $c^*(x) = i$. The Infinite Pigeonhole Principle on c^* guarantees that there exists C which is an infinite subset of B such that any node of C has the same c^*-color. By the property of B, C is a homogeneous set. $\square$

1.1.3 Termination Theorem

In [77] Podelski and Rybalchenko expressed the termination of transition based program as a property of well-founded relations. Classically a relation R is well-founded if it has no infinite R-decreasing sequence, and ill-founded otherwise.

First of all we recall the definition of transition invariants and the Termination Theorem. For all details we refer to [77].

Definition 1.1.7.

- A *transition-based program* $\mathcal{R} = (S, I, R)$ consists of:
  - S: a set of states,
  - I: a set of initial states, such that $I \subseteq S$,
  - R: a transition relation, such that $R \subseteq S \times S$.
- A *computation* is a maximal sequence of states $s_0, s_1, \dots$ such that
  - $s_0 \in I$,
  - $(s_{i+1}, s_i) \in R$ for all $i \in \mathbb{N}$.[1]
- The set Acc of *accessible states* is the set of all states which appear in some computation.

[1] In the literature on transition-based programs, one usually considers $(s_i, s_{i+1}) \in R$. Here we choose the opposite to follow the standard notation of well-foundedness.

- The transition-based program $\mathcal{R}$ is *terminating* if and only if $R \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded.
- A *transition invariant* T is a superset of the transitive closure of the transition relation R restricted to the accessible states Acc. Formally,

$$T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}).$$

- A relation T is *disjunctively well-founded* if T is a finite union of well-founded relations; i.e. $T = R_0 \cup \cdots \cup R_{k-1}$ where for any $i \in k$, T_i is well-founded.

Being well-founded is not preserved under binary unions, therefore a disjunctively well-founded relation can be ill-founded. The main result by Podelski and Rybalchenko is the following.

Theorem 1.1.8 (Podelski and Rybalchenko, ACA_0)**.** *The transition-based program $\mathcal{R}$ is terminating if and only if there exists a disjunctively well-founded transition invariant for $\mathcal{R}$.*

Proof. "$\Leftarrow$": Suppose by contradiction that $T = R_0 \cup \cdots \cup R_{k-1}$ is a disjunctively well-founded transition invariant for $\mathcal{R}$ and that $\mathcal{R}$ is non terminating. Let R be the transition relation of $\mathcal{R}$ and let $\{x_n \mid n \in \mathbb{N}\}$ be a infinite decreasing R-sequence of accessible states. Define the following coloring:

$$\begin{aligned} c : [\mathbb{N}]^2 &\longrightarrow k \\ (n,m) &\longmapsto \mu i (x_m R_i x_n) \end{aligned}$$

Observe that c is defined for any pair in $[\mathbb{N}]^2$ since, for all $n < m$, $x_m (R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})) x_n$ and $T \supseteq (R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}))$. By Ramsey's Theorem for pairs in k colors, there exists an infinite homogeneous set X in color i for some $i < k$. Therefore X is an infinite decreasing R_i-sequence. Since R_i is well-founded, we get a contradiction.

"$\Rightarrow$": Assume that $\mathcal{R}$ is terminating. Define $T = R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$. Suppose by contradiction that there exists an infinite decreasing T-sequence $\{x_n \mid n \in \mathbb{N}\}$. Since $x_0 \in \mathrm{Acc}$ and $(x_{n+1}, x_n) \in R^+$ this means that there exists an infinite decreasing R-sequence of accessible states. Contradiction. □

By unfolding definitions Theorem 1.1.8 states that a binary relation R is well-founded if and only if there exist a natural number k and k-many well-founded relations $R_0, \ldots, R_{k-1}$ whose union contains the transitive closure of R. This is non-trivial since in general a disjunctively well-founded relation can be ill-founded. The fact that a transitive binary relation which is the union of two well-founded relations is well-founded has been remarked before Podelski and Rybalchenko, for instance see [39, page 31] and [21, 26, 58] (see [15, page 2] for details).

Let us see as example one simple application of the Termination Theorem. We represent each state as a finite map s (or equivalently a finite sequence) which provides the values of the variables and the location of the state (for an introduction to these concepts see [43, page 8]). Given a state s and a variable x we write $s(x)$ to mean the value of x in the state s, while $s(\mathrm{pc})$ is the current location of s. For short when not needed we omit the location.

Example 1.1.9. Consider the following program:

```
while (x > 0 AND y > 0)
    (x,y) = (y+1, x-2)  (1)
      OR
    (x,y) = (x+2, y-2)  (2)
```

where x, y have domain all integers, and OR represents a non-deterministic choice. A transition invariant for this program is $R_1 \cup R_2$, where

$$\begin{array}{l} R_1 = \left\{ (\langle x', y' \rangle, \langle x, y \rangle) \mid x + y > 0 \wedge x' + y' < x + y \right\} \\ R_2 = \left\{ (\langle x', y' \rangle, \langle x, y \rangle) \mid y > 0 \wedge y' < y \right\}. \end{array}.$$

Indeed if $\langle x', y' \rangle R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \langle x, y \rangle$, by definition of transitive closure there exists a finite number of R-steps between them. At each step the sum $x + y$ weakly decreases. If one of these steps is a (1)-step then $(\langle x', y' \rangle, \langle x, y \rangle) \in R_1$. Otherwise any step is a (2)-step, then the second variable decreases any time and so $(\langle x', y' \rangle, \langle x, y \rangle) \in R_2$. Since each R_i is well-founded and thanks to the Termination Theorem, the program terminates.

1.1.4 Erdős' Trees from some particular computations

Erdős' trees associated to a given coloring are inspired by the trees used first by Erdős then by Jockusch [50] in their proofs of Ramsey's Theorem for pairs, hence the name. Fix any k-colored graph G on some n or on $\mathbb{N}$. A subset G' of G is 1-colored if any two edges in G' from the same node have the same color. A tree Tr included in G is an Erdős' tree if all branches of Tr are 1-colored, and whenever x has children y_1, y_2 in Tr in the same color, then $y_1 = y_2$. Here we recall the definition of Erdős' tree associated to some particular computations, while in Subsection 2.1.3 we will provide a more general definition.

Given binary relations R, R_0 and R_1 such that $R_0 \cup R_1 \supseteq R^+$, let σ be a (finite or infinite) computation which follows R, that is, some decreasing R-sequence. Define $X = \{\sigma(i) \mid i \in |\sigma|\}$[2] and $c : [X]^2 \to 2$ such that $c(\{\sigma(i), \sigma(j)\}) = 0 \iff i < j \wedge \sigma(j) R_0 \sigma(i)$. We say that a finite sequence t is 1-colored if for any $i < j < k < |t|$ we have $c(\{t(i), t(j)\}) = c(\{t(i), t(k)\})$. For each $n \in |\sigma|$ we define the finite binary tree Tr^σ_n in two colors, with $n+1$-nodes. Here we represent a tree as a set of finite sequences which is closed for initial segments.[3] Informally we add $\sigma(n+1)$ at the end of the longest sequence in Tr^σ_n such that its extension with $\sigma(n+1)$ is a 1-coloring. Formally, we define the tree Tr^σ_n by induction over n:

[2] If σ is infinite put $|\sigma| = \mathbb{N}$.

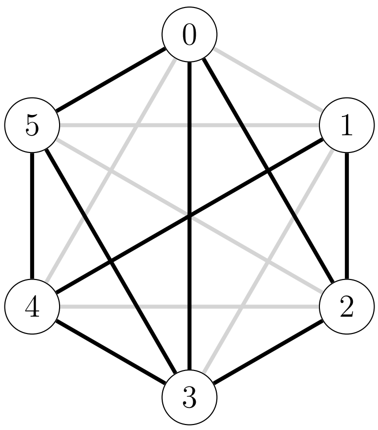
[3] This definition is equivalent to the one given in Subsection 1.1.1.

- $\mathrm{Tr}_0^\sigma = \{\langle\rangle, \langle\sigma(0)\rangle\}$.
- $\mathrm{Tr}_{n+1}^\sigma = \mathrm{Tr}_n^\sigma \cup \{\langle\sigma(b(0)),\ldots,\sigma(b(|b|-1)),\sigma(n+1)\rangle\}$, where b is the minimal sequence with respect to the lexicographic order of the set $A_{n+1} \subseteq n^{<\omega}$ defined as $t \in A_{n+1}$ if
  - Tr_n^σ has a branch r along t, i.e. $r = \langle\sigma(t(0)),\ldots,\sigma(t(|t|-1))\rangle \in \mathrm{Tr}_n^\sigma$;
  - $r * \langle\sigma(n+1)\rangle = \langle\sigma(t(0)),\ldots,\sigma(t(|t|-1)),\sigma(n+1)\rangle$ is a 1-coloring;
  - r is a maximal branch satisfying the latter property, i.e. for all $z \in n+1$

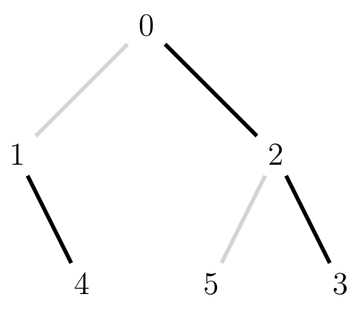
$$r * \langle\sigma(z)\rangle \notin \mathrm{Tr}_n^\sigma \vee c(\{\sigma(t(|t|-1)),\sigma(z)\}) \neq c(\{\sigma(t(|t|-1)),\sigma(n+1)\})$$

Observe that for any $n \in |\sigma|$, $\langle\rangle \in \mathrm{Tr}_{n-1}^\sigma$ and $\langle\sigma(n)\rangle$ is 1-colored, because it has a unique element. Therefore the set A_n is not empty, it is totally ordered lexicographically, and it is finite because it is a subset of Tr_n^σ. Hence there exists a minimal sequence of A_n with respect the lexicographical order. Note that by construction Tr_n^σ is binary, since any node of Tr_n^σ has at most one descendant for any color.

The Erdős' tree associated to the (finite or infinite) computation σ is $\bigcup\{\mathrm{Tr}_n^\sigma \mid n \in |\sigma|\}$. Note that by construction any node in the Erdős' tree is connected with its descendants in the same color. For instance, assume given a sequence $\sigma = \langle 0,\ldots,5\rangle$ such that the coloring between its elements is as follows.



Then the Erdős' tree associated to σ and this coloring is:



1.2 Well-foundedness

Since we mostly do not work in classical logic, we have to be careful by using "well-foudedness" in the definition of terminating program. In fact there are several definitions of well-foundedness which are classically, but not intuitionistically, equivalent. Throughout this work we use some of them. In this section we recall the classically equivalent, but intuitionistically different, formulations of well-foundedness we will deal with and we compare them. After that we focus on one of them, the inductive definition [1, 3], in order to present its main properties.

1.2.1 Comparing well-foundedness definitions

First of all we list some classically equivalent but intuitionistically different formulations of well-foundedness. Given a set S, let ${}^{\mathbb{N}}S$ be the set of infinite sequences $\sigma : \mathbb{N} \to S$. We define $\sigma_n = \sigma(n)$.

Definition 1.2.1. Let $R \subseteq S \times S$ be a relation on a set S.

(1) A sequence $\sigma \in {}^{\mathbb{N}}S$ is an infinite decreasing R-sequence if $\forall i \in \mathbb{N}(\sigma_{i+1} R \sigma_i)$. R is *weakly well-founded* if there are no infinite decreasing R-sequences.

(2) R is *classically well-founded* if every decreasing R-sequence is finite; i.e.

$$\forall \sigma \in {}^{\mathbb{N}}S \exists i \in \mathbb{N} \neg(\sigma_{i+1} R \sigma_i).$$

(3) We say that $X \subseteq S$ is R-inductive if X includes y in S whenever X includes all R-predecessors of y. Formally, we require that $\mathrm{IND}^R(X)$ holds, where $\mathrm{IND}^R(X)$ is

$$\forall y \in S[(\forall z \in S\ (zRy \implies z \in X)) \implies y \in X].$$

R is *inductively well-founded* if every R-inductive set contains all the elements in S, i.e.

$$\forall X \subseteq S(\mathrm{IND}^R(X) \implies S \subseteq X).$$

(4) R is *strongly well-founded* if every inhabited subset $X \subseteq S$ has a "minimal" R-element; i.e.

$$\forall X \subseteq S(\exists x(x \in X) \implies \exists x \in X \forall y \in X \neg(yRx)).$$

Classically, all four definitions above are (easily shown to be) equivalent. Observe that if S is uncountable to prove this equivalence we need also some form of choice; anyway from now on we assume that S is countable. When convenient we feel free to assume that $S \subseteq \mathbb{N}$. Intuitionistically, however, the four definitions are pairwise distinct. The strong notion (4) is too strong if we want to work intuitionistically, since the existence of a strong well-founded inhabited relation implies the Law of Excluded Middle.

Proposition 1.2.2 (HAS_0)**.** *Let $R \subseteq S \times S$ be a strongly well-founded inhabited relation on S. Then for any predicate P, $P \vee \neg P$ holds.*

Proof. Since R is inhabited we have bRa for some $a, b \in S$. As R is strongly well-founded it follows that $a \neq b$. Let P be any proposition. Define the set $X = \{a\} \cup \{x \in S \mid x = b \wedge P\}$. We may prove in HAS_0 that:

(i) $a \in X$;

(ii) $P \iff b \in X$;

(iii) $c \in X \implies c = a \vee c = b$.

By (i) the set X is inhabited. By the assumption that R is strongly well-founded it follows that there exists a $c \in X$ such that

(iv) $\forall y \in X \neg(yRc)$

By (iii) we have that either $c = a$ or $c = b$.

- Assume that $c = a$. Then, by (iv) we have $\forall y \in X \neg(yRa)$. Instantiating y with b we get $b \in X \implies \neg(bRa)$. Assume also P, so that $b \in X$, and hence $\neg(bRa)$. By our assumption bRa this is a contradiction and hence we have $\neg P$.

- Assume that $c = b$. Then $b \in X$, and by (ii) we have that P holds. □

Since (4) implies the Law of Excluded Middle and since all the definitions of well-foundedness are classically equivalent, we have that in HAS_0 any strongly well-foundedness relation is also classically, inductively and weakly well-founded.

As a corollary, we can also observe that in HAS_0 (3) does not imply (4).

Proposition 1.2.3 (HAS_0)**.** *If every inductively well-founded binary relation is strongly well-founded, then for any predicate P, $P \vee \neg P$ holds.*

Proof. The usual ordering of $\mathbb{N}$ is inductively well-founded. Therefore there exists a inhabited inductively well-founded relation. By hypothesis there exists an inhabited strongly well-founded relation and thanks to Proposition 1.2.2 the thesis follows. □

On the other hand the weaker notion of well-foundedness (1) is too weak since it is a negation, and negations are of little use in intuitionistic proofs. It is easy to show that (2) implies (1) in HAS_0. As we will see (3) implies (2) in HAS_0, and hence (3) implies (1). Clearly, (1) does not implies (2); in fact, as shown in the following proposition, "(1) $\implies$ (2)" implies the double-negation elimination for formulas in HA, and therefore the Law of Excluded Middle (EM) because $\neg\neg$EM is intuitionistically provable. This means that it does not hold in HAS_0, since HAS_0 is conservative over HA and the double-negation elimination is unprovable in HA.

Proposition 1.2.4 (HAS_0)**.** *Assume that every weakly well-founded binary relation is classically well-founded. Then for any predicate P in* HA*, $\neg(\neg P) \implies P$ holds.*

Proof. Assuming the premise let us first prove the double-negation elimination principle for Σ^0_1 formulas. Suppose $P \in \Delta^0_1$ and assume that $\neg\forall n \neg P(n)$. We will show $\exists n P(n)$. Let us define the following binary relation on $\mathbb{N}$:

$$mRn \iff (m = n+1 \wedge \neg(P(m))).$$

R is weakly well-founded since the existence of an infinite decreasing sequence would prove $\forall n \neg P(n)$. By hypothesis R is classically well-founded. Hence by considering the sequence $\sigma_n = n$, we deduce $\exists n \neg((n+1)Rn)$. This implies $\exists n \neg(\neg P(n+1))$, which is intuitionistically equivalent to $\exists n P(n+1)$, since $P \in \Delta^0_1$. Then this implies the double-negation elimination for Σ^0_1 formulas. Now the same argument and the double-negation for Π^0_k provide the double-negation elimination for Σ^0_{k+1} predicates. Double-negation elimination for Π^0_k, in turn, follows from double-negation elimination for Σ^0_{k-1}. By induction on k we conclude the argument. □

In order to complete the picture we need to compare notions (2) and (3). The following proposition is well-known.

Proposition 1.2.5 (HAS$_0$)**.** *Let $R \subseteq S \times S$ be an inductively well-founded binary relation. If R is decidable (i.e., $\forall x, y((xRy) \vee \neg(xRy))$) then it is classically well-founded.*

Proof. Assume that R is inductively well-founded. First of all we prove that

(*) for any $S' \subseteq S$, $R' = R \cap S'^2$ is inductively well-founded. Let X be inductive with respect to R', we claim that $Y = \{x \in S \mid x \in S' \implies x \in X\}$[4] is inductive with respect to R. Indeed given $x \in S$, assume that $\forall y(yRx \implies y \in Y)$ in order to prove that $x \in Y$. Assume that $x \in S'$ then we get $x \in X$ by $R \supseteq R'$, $\forall y \in S'(yRx \implies y \in S)$ and X is inductive.

Given an infinite sequence σ we want to prove that $\exists i \in \mathbb{N} \neg(\sigma_{i+1} R \sigma_i)$. Put $S' = \{\sigma_i \mid i \in \mathbb{N}\}$ and define $R' = R \cap S'^2$. We prove that

$$X = \{\sigma_i \mid \exists j \geq i \neg(\sigma_{j+1} R \sigma_j)\}$$

is inductive with respect to R'. Fix $\sigma_i \in S'$. Assume that for any z such that $zR'\sigma_i$ then $z \in X$. We want to prove that $\sigma_i \in X$. If $\sigma_{i+1} R \sigma_i$ then $\sigma_{i+1} R' \sigma_i$ and thus $\sigma_{i+1} \in X$. Therefore there exists $j \geq i+1$ such that $\neg(\sigma_{j+1} R \sigma_j)$. Such j witnesses that $\sigma_i \in X$. Otherwise, by decidability of R we have $\neg(\sigma_{i+1} R \sigma_i)$, i witnesses $\sigma_i \in X$. Hence X is inductive with respect to R'. Since by (*) R' is inductively well-founded, we have that $X = S'$, which implies $\sigma_0 \in X$ and so $\exists i \in \mathbb{N} \neg(\sigma_{i+1} R \sigma_i)$, as wished. □

Note that the proof above can be carried out in HAS$_0$. In order to have the converse implication, however, we have to use an axiom not provable in HAS$_0$, bar induction [52]. We say that P is a bar if P is downward closed for extension and it includes some

[4] Classically, $Y = X \cup (S \setminus S')$

finite prefix of every sequence. Q is bar inductive if Q includes some bar P and Q is inductive. Bar induction says that all bar inductive predicates are always true. Let $s*t$ be the concatenation of a finite sequence s and a finite or infinite sequence t, and let $\mathrm{last}(s)$ denote the last element of a finite sequence s. Formally we define bar induction as follows:

Definition 1.2.6 (Bar Induction BI)**.** Let P, Q be subsets of $\mathbb{N}^*$ such that

1. P is downward closed for extension, i.e.
$$\forall s \in \mathbb{N}^* \forall m \in \mathbb{N}(P(s) \implies P(s * \langle m \rangle));$$
2. P contains a finite prefix of every infinite sequence, i.e.
$$\forall \sigma \in {}^{\mathbb{N}}\mathbb{N} \exists n \in \mathbb{N}(P(\langle \sigma(0), \ldots, \sigma(n-1) \rangle));$$
3. Q includes P, i.e. $\forall s \in \mathbb{N}^*(P(s) \implies Q(s))$;
4. Q is inductive, i.e. $\forall s \in \mathbb{N}^*[(\forall n \in \mathbb{N}^* Q(s * \langle n \rangle)) \implies Q(s)]$.

Then $Q(\langle\rangle)$.

Proposition 1.2.7 ($\mathrm{HAS}_0 + \mathrm{BI}$)**.** *Any classically well-founded binary relation is inductively well-founded.*

Proof. Let X be a R-inductive set. We want to prove that $X = S$. Fix $x \in S$. Then define

$$P(s) = \langle x \rangle * s \text{ is a decreasing } R\text{-sequence} \implies \forall n \in \mathbb{N}(\langle x \rangle * s * \langle n \rangle \text{ is not}).$$
$$Q(s) = \langle x \rangle * s \text{ is a decreasing } R\text{-sequence} \implies \mathrm{last}(\langle x \rangle * s) \in X.$$

Observe that since R is classically well-founded P and Q satisfy the hypothesis of BI. Then $Q(\langle\rangle)$ holds, and this implies $x \in X$. $\square$

Thus, summing up the results presented in this section, for any relation R for which $\forall x, y((xRy) \vee \neg(xRy))$, we can draw the following diagram; where the full arrows denote provability in HAS_0, and the dashed arrow denotes provability in $\mathrm{HAS}_0 + \mathrm{BI}$.

$$\begin{tikzcd}
4 \arrow[r, "\text{Prop. 1.2.2}", "\text{+ classical logic}"'] & 3 \arrow[r, shift left, "\text{Prop. 1.2.5}"] & 2 \arrow[l, dashed, shift left, "\text{Prop. 1.2.7}"] \arrow[r, "\text{“Finite” implies}", "\text{“non-infinite”}"'] & 1
\end{tikzcd}$$

Remark 1.2.8. Quantification over sets can be expressed in HA^ω as quantification over characteristic functions. We observe that although the definitions of well-foundedness can be stated in HA^ω, the proofs of Proposition 1.2.2 and of Proposition 1.2.5 can not be done in HA^ω since we need arithmetical comprehension CA. The system $\mathrm{HA}^\omega + \mathrm{CA}$, however, already proves the Law of Excluded Middle.

Corresponding definitions of termination. As introduced in Subsection 1.1.3 a transition-based program $\mathcal{R} = (S, I, R)$ is terminating if and only if $R \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded. Throughout this work we deal with different notions of well-foundedness, and hence of termination.

In Chapter 2 and Chapter 3 we focus on the inductive definition; thus $\mathcal{R}$ is terminating if and only if $R \cap (\mathrm{Acc} \times \mathrm{Acc})$ is inductively well-founded. In Chapter 4 we consider the classical definition of well-foundedness; hence $\mathcal{R}$ is terminating if and only if

$$\forall \sigma \in {}^{\mathbb{N}}S(\sigma_0 \in I \implies \exists n \in \mathbb{N} \neg(\sigma_{n+1} R \sigma_n)).$$

Notice that this is a slight reformulation of the original definition which uses classical well-foundedness, namely:

$$\forall \sigma \in {}^{\mathbb{N}}S(\sigma_0 \in \mathrm{Acc} \implies \exists n \in \mathbb{N} \neg(\sigma_{n+1} R \sigma_n))$$

where Acc is formally defined as the set of states for which there is a transition sequence from some initial state:

$$a \in \mathrm{Acc} \iff \exists \sigma \in {}^{\mathbb{N}}S \exists k \in \mathbb{N}(\sigma_0 \in \mathrm{I} \wedge \sigma_k = a \wedge \forall i < k(\sigma_{i+1} R \sigma_i)).$$

Anyway we can easily prove these two definitions are equivalent over HAS_0.

Lemma 1.2.9 (HAS_0). *Given a transition based program $\mathcal{R} = (S, I, R)$ the following are equivalent:*

1. $\forall \sigma \in {}^{\mathbb{N}}S(\sigma_0 \in I \implies \exists n \in \mathbb{N} \neg(\sigma_{n+1} R \sigma_n))$;
2. $\forall \sigma \in {}^{\mathbb{N}}S(\sigma_0 \in \mathrm{Acc} \implies \exists n \in \mathbb{N} \neg(\sigma_{n+1} R \sigma_n))$.

Proof. "$1 \Rightarrow 2$": Since $\sigma_0 \in \mathrm{I}$ implies $\sigma_0 \in \mathrm{Acc}$ we are done.

"$2 \Rightarrow 1$": Assume σ is such that $\sigma_0 \in \mathrm{Acc}$. By definition of Acc there exists a sequence σ' and a natural number k such that

$$\sigma'_0 \in \mathrm{I} \ \wedge \ \sigma'_k = \sigma_0 \ \wedge \ \forall i < k(\sigma'_{i+1} R \sigma'_i).$$

Hence we can define the sequence $\sigma^* = \langle \sigma'_0 \dots \sigma'_{k-1} \rangle * \sigma$ and we can find the witness n of (1). Observe that n is greater than $k-1$ since $\forall i < k-1(\sigma'_{i+1} R \sigma'_i)$ and $\sigma_0 R \sigma'_{k-1}$. Then $n-k$ is the witness of (2) for the given σ. $\square$

In Chapter 5 we work with the weakly definition of well-foundedness. Hence $\mathcal{R}$ is terminating if there are no infinite decreasing $R \cap (\mathrm{Acc} \times \mathrm{Acc})$-sequences. For short we assume that any state is accessible.

1.2.2 Inductive well-foundedness

Here we work within HAS, and for sake of simplicity we only consider relations R over some set S of natural numbers. The results of this section could be proved for any set S, but for our purposes the case $S \subseteq \mathbb{N}$ is enough.

As we said in the previous subsection, the intuitionistic definition of well-founded relation [1, 3] is based on the definition of inductive property, where a property is R-inductive if whenever it is true for all R-predecessors of a point it is true also for the point. For the remainder of this chapter we write "well-foundedness" instead of "inductive well-foundedness" for short. So given a binary relation R on S, $x \in S$ is R-well-founded if and only if it belongs to every R-inductive property; and R is well-founded if every x in S is R-well-founded.

A binary structure, just a *structure* for short, is a pair (S, R), where R is a binary relation on S. We say that (S, R) is well-founded if R is well-founded.

We need also the notion of co-inductive property. A property X is R-co-inductive if it satisfies the inverse property of R-inductive: if the property X holds for a point, then it holds also for all its R-predecessors. Formally:

Definition 1.2.10. Let R be a binary relation on S.

- A property X is *R-co-inductive in $y \in S$* if and only if

$$\forall z \in S\,(zRy \implies z \in X)\,.$$

- A property X is *R-co-inductive* if and only if $\mathrm{CoIND}^R(X)$; where $\mathrm{CoIND}^R(X)$ is

$$\forall y \in S[y \in X \implies \forall z \in S\,(zRy \implies z \in X)].$$

The following well-known proposition connects inductiveness and co-inductiveness.

Proposition 1.2.11 (HAS)**.** *Assume that R is a binary relation on $S \subseteq \mathbb{N}$. Then the set of R-well-founded elements of S is both inductive and co-inductive. That is, for all $x \in S$: x is R-well-founded if and only if all $y \in S$ such that yRx are R-well-founded.*

Proof.
- *Well-foundedness is inductive.* Let $x \in S$. Assume that any $y \in S$ such that yRx is R-well-founded, in order to prove that x is R-well-founded. To this aim, assume that X is any R-inductive property, we claim that $x \in X$. Since X is inductive, our thesis follows by proving that for all $z \in S$ such that zRx we have $z \in X$. By hypothesis from zRx we deduce that z is R-well-founded, hence by definition z belongs to all inductive sets. Thus, by X inductive, we conclude that $z \in X$, as wished.

- *Well-foundedness is co-inductive.* It suffices to show that the set

$$X = \{y \mid \forall w \in S(wRy \implies w \text{ is } R\text{-well-founded})\}$$

is R-inductive. In fact if we prove it then we know the property X holds for any x R-well-founded, therefore for any x R-well-founded and any wRx we have w R-well-founded, as wished.

In order to show that X is R-inductive, assume that for all $z \in S$ such that zRy we have $z \in X$, in order to prove that $y \in X$. By the previous point, for any

$z \in X$ we have z R-well-founded, therefore for all $z \in S$ such that zRy we have z R-well-founded. By definition of X this means that $y \in X$. □

Some examples of intuitionistic proofs of well-foundedness

The simplest non-trivial example is the intuitionistic proof that $(\mathbb{N}, <)$ is well-founded, where $<$ is the classical order of the natural numbers. Recall that, throughout this section, "well-founded" is short for "inductive well-founded".

Example 1.2.12 (HAS)**.** $(\mathbb{N}, <)$ is well-founded.

Proof. In order to prove the thesis we need to show that for any $x \in \mathbb{N}$ and for every $<$-inductive property X, $x \in X$. By definition the following holds:

$$\forall y\, (\forall z(z < y \implies z \in X)) \implies y \in X.$$

So it is sufficient to show that $[0,x] \subseteq X$, for every $x \in \mathbb{N}$. We prove it by Peano induction:

$$(0 \in X \wedge \forall x([0,x] \subseteq X \implies [0,x+1] \subseteq X)) \implies \forall z([0,z] \subseteq X).$$

We have $0 \in X$, since 0 has no predecessor and X is $<$-inductive. Moreover if $[0,x] \subseteq X$, then for every $y < x+1$ we have $y \in X$. Since X is $<$-inductive, $x+1 \in X$. So $[0,x+1] \subseteq X$. □

Now let $<$ be the classical order in $\mathbb{Z}$ and let now consider the following set:

$$\mathbb{Z}^- = \{z \in \mathbb{Z} \mid z \leq 0\}.$$

Example 1.2.13 (HAS)**.** Every $z \in \mathbb{Z}^-$ is not $<$-well-founded in $\mathbb{Z}^-$.

Proof. Let $X = \emptyset$. Then X is $<$-inductive, since for every $z \in \mathbb{Z}^-$ the inductive hypothesis

$$\forall y\, (y < z \implies y \in X)$$

is false for $y = z - 1$. So $z \in \mathbb{Z}^-$ is not $<$-well-founded since X is $<$-inductive and $z \notin X$. □

In general we will intuitionistically prove that if there exists an infinite decreasing R-sequence from x then x is not R-well-founded. Classically, x is R-well-founded if and only if there are no infinite decreasing R-sequences from x, and R is well-founded if and only if there are no infinite decreasing R-sequences in S. As shown in Subsection 1.2.1, this result is not intuitionistically provable and we will not use it through this section.

1.2.3 Proving inductive well-founded relations

There are several folklore methods to prove that a binary relation R is well-founded by using the well-foundedness of another binary relation Q. The most commonly used are the following ones:

- a subset of a well-founded relation is well-founded;
- if there exists a morphism from a relation R to a relation Q and Q is well-founded then R is well-founded;
- if there exists a "simulation" relation from R to Q, then each point "simulable" in a Q-well-founded point is R-well-founded.

The goal of this section is to recall the proofs of these results. All intuitionistic proofs are folklore, except in the case of simulation relations, in which as far as we know, our proofs are the first intuitionistic versions of well-known classical results (see for instance [73]).

Simulation relations

An important tool for proving that a relation is well-founded are simulations [73]. A simulation relation is a binary relation which connects two other binary relations. Intuitively a simulation of a binary relation $R \subseteq S^2$ into a binary relation $Q \subseteq S'^2$ is a way of associating, step by step, any R-decreasing sequence to some Q-decreasing sequence.

Definition 1.2.14. Let R be a binary relation on S and Q be a binary relation on S'. Let T be a binary relation on $S \times S'$.

- $\mathrm{dom}(T) = \{x \in S \mid \exists y \in S'(xTy)\}$.
- $f : (S,R) \to (S',Q)$ is a *morphism* if f is a function such that $\forall x, y \in S(xRy \implies f(x)Qf(y))$.
- T is a *simulation* of R in Q if and only if it is a relation and

$$\forall x, z \in S \forall y \in S' \Big((xTy \wedge zRx) \implies \exists t \in S' \,(tQy \wedge zTt)\Big)$$

- A simulation relation T of R in Q is *total* if $\mathrm{dom}(T) = S$.
- R is *simulated* in Q if there exists a total simulation relation T of R in Q.

We may describe the behaviour of a simulation T of R in Q by filling the lower right angle of the following diagram.

$$\begin{tikzcd}
S & S' \\[-2em]
x \arrow[d, "R"'] & y \arrow[l, "T"'] \arrow[d, "Q"] \\
z & t \arrow[l, "T"]
\end{tikzcd}$$

If we have a simulation T of R in Q and xTy holds, we can transform each finite decreasing R-sequence in S from x in a finite decreasing Q-sequence in S' from y.

In fact it suffices to complete the lower right angle by following the order y', y'', ...

$$
\begin{tikzcd}
S & S' \\[-2em]
x \arrow[d, "R"'] & y \arrow[l, "T"'] \arrow[d, "Q"] \\
x' \arrow[d, "R"'] & y' \arrow[l, "T"'] \arrow[d, "Q"] \\
x'' \arrow[d, dashed, dash] & y'' \arrow[l, "T"'] \arrow[d, dashed, dash] \\
{} \arrow[r, dashed, dash] & {}
\end{tikzcd}
$$

By using the Axiom of Choice this result holds also for infinite decreasing R-sequences from a point in dom(T). Then if there are no infinite decreasing Q-sequences in S' there are no infinite decreasing R-sequences in dom(T). If, furthermore, the simulation is total there are no infinite decreasing R-sequences in S. By using classical logic and the Axiom of Choice we may conclude that if Q is well-founded and T is a total simulation relation of R in Q then R is well-founded. In the last subsection of this section we will present an intuitionistic proof of this result which does not use the Axiom of Choice.

We recall some basic examples of simulation relations.

Example 1.2.15. Let R be a binary relation on S, and let Q be a binary relation on S'. If there exists a morphism $f : (S,R) \to (S',Q)$, then $T = \{(x, f(x)) \mid x \in S\}$ is a total simulation of R in Q.

Indeed if zRx, then $f(z)Qf(x)$ due to the definition of morphism. This guarantees that we may complete the diagram by choosing $t = f(z)$.

Example 1.2.16. Let R, Q be binary relations on S such that $R \subseteq Q$. Then the relation $T = \{(x,x) \mid x \in S\}$ is a total simulation of R in Q.

In this case it suffices to complete the diagram by putting $t = y = x$.

We may see binary relations as abstract reduction relations. From now on, by an abstract reduction relation we simply mean a binary relation (for example a rewriting relation). Classically, a reduction relation R is said to be terminating or strongly normalizing if and only if there are no infinite R-sequences [42]. Intuitionistically, we require that R is (inductively) well-founded. Observe that we use simulation to prove well-foundedness and this is the same method used for labelled state transition systems [73], except that, for us, the set of labels is always a singleton.

Some operations on binary structures

In this subsection we introduce some operations mapping binary structures into binary structures. In the next one we prove that these operations preserve well-foundedness.

The first operation is the successor operation (adding a top element).

Definition 1.2.17 (The successor relation $R+1$)**.** Let R be a relation on S and let $\top$ be an element not in S. We define the relation $R+1 = R \cup \{(x,\top) \mid x \in S\}$ on $S+1 = S \cup \{\top\}$. We define the *successor structure* of (S,R) as $(S,R)+1 = (S+1, R+1)$.

Remark that we may always rename the elements of $S \subseteq \mathbb{N}$ in order to have some $\top \in \mathbb{N} \setminus S$. Another operation on binary relations is the relation $R \otimes S$ defined by components, inspired by the componentwise order.

Definition 1.2.18 (The componentwise relation $R \otimes S$)**.** Let $S, S' \subseteq \mathbb{N}$, R be a binary relation on S, and Q be a binary relation on S'. The relation $R \otimes Q$ of components R, Q is defined as below:

$$R \otimes Q = (R \times \mathrm{Diag}(S')) \cup (\mathrm{Diag}(S) \times Q) \cup (R \times Q),$$

where $\mathrm{Diag}(X) = \{(x,x) \mid x \in X\}$.

Equivalently $R \otimes Q$ is defined for all $x, x' \in S$ and for all $y, y' \in S'$ by:

$$(x,y) R \otimes Q (x',y') \iff$$
$$\Big((xRx') \wedge (y = y')\Big) \vee \Big((x = x') \wedge (yQy')\Big) \vee \Big((xRx') \wedge (yQy')\Big).$$

If R, Q are weak orderings (both R, Q are reflexive), then $R \otimes Q$ is the componentwise ordering, also called the product ordering. In this case $R \otimes Q = R \times Q$, while in general we may have $R \otimes Q \supset R \times Q$.

Properties of intuitionistic well-foundedness

Now we may list the main intuitionistic properties of well-founded relations. Intuitionistic proofs are folklore, except in the case of simulation relation. In this case, as far as we know they are new. It is a kind of generalization of the preservation of well-foundedness by inverse image [75], but it is more than a direct reformulation. In fact a relation R can be simulated in a relation Q even if R is not embeddable in Q. For instance, if R is the strict order relation on natural numbers, and

$$Q = \{(\langle x,y \rangle, \langle y,z \rangle) : x < y < z\}.$$

Then R may be simulated in Q via

$$T = \{(y, \langle x,y \rangle) : x < y\},$$

but R cannot be embedded in Q.

Proposition 1.2.19 (HAS)**.** *Let R be a binary relation on S, and let Q be a binary relation on S'.*

1. *If R, Q are well-founded, then $R \otimes Q$ is well-founded.*

2. *If T is a simulation of R in Q and if xTy and y is Q-well-founded, then x is R-well-founded.*

3. *If T is a simulation of R in Q and Q is well-founded, then* $\mathrm{dom}(T)$ *is R-well-founded.*

4. *If R is simulated in Q and Q is well-founded, then R is well-founded.*

5. *Assume that $f:(S,R)\to(S',Q)$ is a morphism. If $x\in S$ and $f(x)$ is Q-well-founded, then x is R-well-founded. If Q is well-founded, then R is well-founded.*

6. *If R is included in Q and Q is well-founded then R is well-founded.*

Proof. 1. Define X as the set of $x\in S$ such that every pair (x,y) is $R\otimes Q$-well-founded, i.e.

$$X=\left\{x\in S \mid \forall y\in S'((x,y)\text{ is } R\otimes Q\text{-well-founded})\right\}.$$

In order to prove that $R\otimes Q$ is well-founded, it is enough to prove that X is R-inductive. It will follow $X=S$, that is, that all pairs (x,y) are $R\otimes Q$-well-founded. In order to prove that X is R-inductive, we assume that $\forall z(zRx \implies z\in X)$, in order to show that $x\in X$, that is, that (x,y) is $R\otimes Q$-well-founded for all $y\in S'$. To prove this latter, we are going to verify that Y_x is Q-inductive; where

$$Y_x=\left\{y\in S' \mid (x,y)\text{ is } R\otimes Q\text{-well-founded}\right\}.$$

From Y_x Q-inductive it will follow that $Y_x=S'$, that is, that (x,y) is $R\otimes Q$-well-founded for all $y\in S'$. We prove that Y_x is Q-inductive. Assume that $\forall w(wQy \implies w\in Y_x)$, in order to show that $y\in Y_x$, that is, that (x,y) is $R\otimes Q$-well-founded. By Proposition 1.2.11, this is equivalent to show:

$$\forall(z,w)((z,w)R\otimes Q(x,y) \implies (z,w)\text{ is } R\otimes Q\text{-well-founded}).$$

By definition of $\otimes$ we have:

$$(z,w)R\otimes Q(x,y) \iff (zRx\wedge(w=y\vee wQy))\vee(z=x\wedge wQy).$$

We reason by cases. If zRx then $z\in X$ by inductive hypothesis on X. By definition of X, (z,w) is $R\otimes Q$-well-founded for any $w\in S'$. If $z=x\wedge wQy$ then $w\in Y_x$, by hypothesis on Y_x. This implies that (z,w) is $R\otimes Q$-well-founded and we are done.

2. Assume that T is a simulation of R in Q and that $x\in S$, $y\in S'$, xTy and y is Q-well-founded, in order to prove that x is R-well-founded. It is enough to prove that

$$Y=\left\{y\in S' \mid \forall x\in S(xTy \implies x\text{ is } R\text{-well-founded})\right\}$$

is Q-inductive. From y Q-well-founded it will follow that $y\in Y$, that is, that if $x\in S$ and xTy then x is R-well-founded. In order to prove that Y is Q-inductive, we assume that $v\in S'$ and $\forall w\in S'(wQv \implies w\in Y)$, and we have to prove that

$v \in Y$. By definition of Y, we choose any $u \in S$ such that uTv, and we need to show that u is R-well-founded. Thanks to Proposition 1.2.11, it suffices to verify that for all $z \in S$, if zRu then z is R-well-founded. If $zRu \wedge uTv$, then, by definition of simulation, there is some $t \in S'$ such that tQv and zTt. By hypothesis on v, from tQv we deduce that $t \in Y$. By definition of Y, from zTt we deduce that z is R-well-founded, as we wished to show.

3. If T is a simulation of R in Q and Q is well-founded, we want to prove that $\mathrm{dom}(T)$ is R-well-founded. If $x \in \mathrm{dom}(T)$, then by definition of $\mathrm{dom}(T)$ there exists $y \in S'$ such that xTy. Moreover y has to be Q-well-founded, since Q is well-founded. By point (2) above x is R-well-founded.

4. By definition of simulated, there exists a simulation T of R in Q such that $\mathrm{dom}(T) = S$. Due to point (3) above, each $x \in S$ is R-well-founded.

5. Assume that $f : (S,R) \to (S',Q)$ is a morphism, $x \in S$ and $f(x)$ is Q-well-founded, in order to prove that x is R-well-founded. $T = \{(x, f(x)) \mid x \in S\}$ is a simulation of R in Q, and $xTf(x)$. By point (2) above, from $f(x)$ Q-well-founded x is R-well-founded.

6. Assume that $R \subseteq Q$ and Q is well-founded. Then $T = \{(x,x) \mid x \in S\}$ is a total simulation of R in Q. By point (4) above we deduce that R is well-founded. □

The next remark requires the notion of R-minimal.

Definition 1.2.20. Let R be a binary relation on S. An element $x \in S$ is R-*minimal* if and only if there are no y such that yRx.

We may observe that if x is R-minimal then x is R-well-founded by Proposition 1.2.11: trivially, since x has no R-predecessors.

Example 1.2.21. The empty relation V is well-founded, since every element is V-minimal.

Recall that, as usual, given a binary relation R over a set S,

- for any $x \in S$, the height of x with respect to R is:

$$\mathrm{ht}_R(x) = \sup\{\mathrm{ht}_R(y) + 1 \mid yRx\}.$$

- the height of R is:

$$\mathrm{ht}(R) = \sup\{\mathrm{ht}_R(x) \mid x \in S\}.$$

Definition 1.2.22. Let R be a binary relation on S, let $x \in S$ and let $n \in \mathbb{N}$. We say that x has R-height n if the longest decreasing R-sequence from x has $n+1$ points.

Corollary 1.2.23 (HAS)**.** *Let R be a binary relation on S, $n \in \mathbb{N}$, and $x \in S$. If x has R-height n then it is R-well-founded.*

Proof. By induction on n. Assume that $n=0$. Then x is R-minimal, so it is R-well-founded. Assume that the thesis holds for any n, we prove it for $n+1$. If x has R-height $n+1$, then every y such that yRx has R-height $\leq n$, so y is R-well-founded by inductive hypothesis. By applying Proposition 1.2.11, x is R-well-founded. $\square$

Corollary 1.2.24 (HAS). *Let R be a binary relation on S. If (S,R) is well-founded then so is $(S,R)+1$.*

Proof. $T=\{(x,x) \mid x \in S\}$ is simulation of $(S,R)+1$ in (S,R). Indeed if $x \in S$ and $y(R+1)x$, then yRx by definition of $R+1$. So if (S,R) is well-founded, then any $x \in S$ is also $(R+1)$-well-founded by Proposition 1.2.19.4. Moreover $\top$ is well-founded in $(S,R)+1$ by Proposition 1.2.11; indeed, $x(R+1)\top$ implies that $x \in S$, therefore x is $(R+1)$-well-founded, as we just proved. $\square$

Corollary 1.2.25 (HAS). *Let R be a binary relation on S and $x \in S$. If there exists an infinite decreasing R-sequence from x, then x is not R-well-founded.*

Proof. Assume that there exists an infinite decreasing R-sequence from x:

$$\dots Rx_2Rx_1Rx_0 = x,$$

then there exists a morphism

$$\begin{aligned} f:(\mathbb{Z}^-,<) &\longrightarrow (S,R) \\ -n &\longmapsto x_n. \end{aligned}$$

Suppose by contradiction that x is R-well-founded, then (by Proposition 1.2.19.5) the element 0 should be $<$-well-founded. Contradiction (see Example 1.2.13). $\square$

So the inductive definition of well-founded intuitionistically implies the classical definition of well-founded; while the converse implication is purely classical.

In this work we often deal with relations of height ω.

Definition 1.2.26. We say that a relation R over a set S has *height* ω if there exists a *weight function*; i.e. a function $f: S \to \mathbb{N}$ such that for any $x,\ y \in S$

$$xRy \implies f(x) < f(y).$$

By using the total simulation $T=\{(x,f(x)) \mid x \in \mathrm{dom}(R)\}$ we can easily prove that if R has height ω then it is inductively well-founded. To conclude we may provide a further example of non inductive well-founded set.

Example 1.2.27. Each element of $(\mathbb{R},<)$ is not well-founded, since there exists an infinite decreasing $<$-sequence from any real.

Since $(\mathbb{N},<)$ is well-founded, we may observe that well-foundedness is not preserved by adding elements. Well-foundedness is not preserved also by adding relations over the existing elements. Trivially, $(\mathbb{R},\emptyset)$, where $\emptyset$ is the empty binary relation, is well-founded, while $(\mathbb{R},<)$ is not, and $\emptyset \subseteq <$.

Chapter 2

An intuitionistic analysis of the Termination Theorem

In this chapter we provide an intuitionistic proof and corresponding bounds for the Termination Theorem by using the inductive definition of well-foundedness.

In Section 2.2 we intuitionistically prove the Termination Theorem by introducing a new intuitionistic version of Ramsey's Theorem for pairs, which we called H-closure Theorem. In Section 2.5, we highlight the main differences between H-closure Theorem and Almost Full Theorem [23]. Results in these sections are a joint work with Stefano Berardi [11].

In Section 2.3 we consider the set of functions having at least one implementation in Podelski-Rybalchenko language which admits a disjunctively well-founded transition invariant where each relation has height ω. We prove that this set is exactly the set of primitive recursive functions. Sylvain Schmitz has pointed out to us that Blass and Gurevich [15] and Delhommé [25] already observed that the computation of the ordinal height of a relation proven to be well-founded by the Termination Theorem is the natural product of the individual heights. Nonetheless our argument produces bounds by using the proof of the Termination Theorem based on H-closure, therefore this would hopefully provide a schema for other applications of H-closure. Results in this section are a joint work with Stefano Berardi and Paulo Oliva [8].

Finally, in Section 2.4, we show that the same result holds for the Terminator Algorithm based on the Termination Theorem, defined by Cook, Podelski and Rybalchenko [22]. A function has at least one implementation in Podelski-Rybalchenko's language which the Terminator Algorithm may catch terminating if and only if the function is primitive recursive.

2.1 The ordinal height of k-ary trees

In Section 1.1 we introduced trees as partially ordered sets with a minimum element. In this section we focus on k-ary trees, which are trees whose nodes have at most k-many children.

In the first subsection we present the formal definition of k-ary tree we deal with in this chapter; in the second one we study the ordinal height of the set of k-ary trees labelled on some ordinal α; in the last subsection we provide the formal definition of Erdős' trees we introduced in Section 1.1.

2.1.1 k-ary trees

A finite k-ary tree may be defined in several ways, the most common runs as follows.

Definition 2.1.1. A *finite k-tree* on S is defined inductively as an empty tree, called Nil, or a $(k+1)$-tuple composed by one element of S and k-many trees, called immediate subtrees: so we have either $\mathrm{Tr} = \mathrm{Nil}$ or $\mathrm{Tr} = \langle x, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_{k-1} \rangle$.

$$k\text{-Tr} = \{\mathrm{Tr} \mid \mathrm{Tr} \text{ is a } k\text{-ary tree }\}.$$

Let $\mathrm{Tr} = \langle x, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_{k-1} \rangle$, then we say that

- Tr is a tree with *root* x;
- if $\mathrm{Tr}_0 = \cdots = \mathrm{Tr}_{k-1} = \mathrm{Nil}$, Tr is a *leaf-tree*;

Moreover, if $k = 2$, so if Tr is binary,

- if $\mathrm{Tr}_0 \neq \mathrm{Nil}$ and $\mathrm{Tr}_1 = \mathrm{Nil}$, Tr has exactly one left child;
- if $\mathrm{Tr}_0 = \mathrm{Nil}$ and $\mathrm{Tr}_1 \neq \mathrm{Nil}$, Tr has exactly one right child;
- if $\mathrm{Tr}_0 \neq \mathrm{Nil}$ and $\mathrm{Tr}_1 \neq \mathrm{Nil}$, Tr has two children: one right child and one left child.

The universe $|\,\mathrm{Tr}\,|$ of a k-ary tree on S is the set of the elements of S in Tr, formally:

Definition 2.1.2. Let Tr be a k-ary tree. The *universe* $|\,\mathrm{Tr}\,|$ is defined by induction on k-Tr:

- $|\,\mathrm{Nil}\,| = \emptyset$;
- $|\langle x, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_{k-1} \rangle| = \{x\} \cup |\,\mathrm{Tr}_0\,| \cup \cdots \cup |\,\mathrm{Tr}_{k-1}\,|$.

We denote a list on S with $\langle x_0, \ldots, x_{n-1} \rangle$ where $n \in \mathbb{N}$ and $x_i \in S$ for any $i < n$; $\langle\rangle$ is the empty list. We define the operation of concatenation of two lists on S in the natural way as follows:

$$\langle x_0, \ldots, x_{n-1} \rangle * \langle y_0, \ldots, y_{m-1} \rangle = \langle x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1} \rangle.$$

We define the relation of one-step expansion $\succ$ between two lists L, M on the same S, as $L \succ M \iff L = M * \langle y \rangle$, for some y. If $L = \langle x_0, \ldots, x_{n-1} \rangle$ is a list on S, we define the universe of L as $|L| = \{x_0, \ldots, x_{n-1}\}$.

Definition 2.1.3. Let L be a list on S and Tr be a k-ary tree on S. L is *covered* by Tr if and only if $|L| = |\mathrm{Tr}|$.

The covering relation will be useful in order to simulate (see Section 1.2) a set of lists in a set of trees. Each list will be associated with a tree with the same universe.

A k-ary tree may also be defined as a "combinatorial tree"; i.e. a labelled oriented graph on S, empty (if $\mathrm{Tr} = \mathrm{Nil}$) or with a special element, called root, which has exactly one path from the root to any node. Each edge is labelled with a color $c \in k$ in such a way that from each node there is at most one edge in each color.

Equivalently we may define firstly colored lists and then the binary trees as particular sets of colored lists.

Definition 2.1.4. A *colored list* (L, f) on a set S is a pair, where $L = \langle x_0, \ldots, x_{n-1}\rangle$ is a list on S equipped with a list $f = \langle c_0, \ldots, c_{n-2}\rangle$ on k. The colored list $\mathrm{nil} = (\langle\rangle, \langle\rangle)$ is the empty colored list and $\mathrm{CList}(k)$ is the set of all colored lists with colors in k.

We should imagine that the list L is drawn as a sequence of its elements and that for each $i < n-1$ the segment (x_i, x_{i+1}) has color c_i. Observe that if $L = \langle\rangle$ or if $L = \langle x\rangle$, then $f = \langle\rangle$: if there are no edges in L, then there are no colors in (L, f).

Throughout this chapter we use $\lambda, \mu, \ldots$ to denote colored lists in $\mathrm{CList}(k)$. Let $h \in k$. We define the composition of color h of two colored lists by connecting the last element of the first list (if any) with the first of the second list (if any) with an edge in color h. Formally we set $\mathrm{nil} *_h \lambda = \lambda *_h \mathrm{nil} = \lambda$, and $(L, f) *_h (M, g) = (L*M, f*\langle h\rangle*g)$ whenever $L, M \neq \mathrm{nil}$.

We can define the relation *one-step extension on colored lists*: $\succ_h$ is the one-step extension of color h and $\succ_{\mathrm{col}}$ is the one-step extension of any color. Assume that $x \in S$ and $\lambda, \mu \in \mathrm{CList}(k)$. Then we set:

- $\lambda *_h (\langle x\rangle, \langle\rangle) \succ_h \lambda$.
- $\lambda \succ_{\mathrm{col}} \mu$ if $\lambda \succ_h \mu$ for some $h \in k$.

Now we can equivalently define a k-ary tree on S as a particular set of some colored lists.

Definition 2.1.5. A k-ary tree Tr is a set of colored lists on S, such that:

1. nil is in Tr;
2. if $\lambda \in \mathrm{Tr}$ and $\lambda \succ_{\mathrm{col}} \mu$, then $\mu \in \mathrm{Tr}$;
3. each list in Tr has at most one one-step extension for each color $h \in k$: if $\lambda_0, \lambda_1, \lambda \in \mathrm{Tr}$ and $\lambda_0, \lambda_1 \succ_h \lambda$, then $\lambda_0 = \lambda_1$.

For all sets $\mathcal{L} \subseteq \mathrm{CList}(k)$ of colored lists, k-$\mathrm{Tr}(\mathcal{L})$ is the set of k-ary trees whose branches are all in $\mathcal{L}$. For short, let k-$\mathrm{Tr} = k$-$\mathrm{Tr}(\mathrm{CList}(k))$.

For instance the empty tree is the set $\mathrm{Nil} = \{\mathrm{nil}\}$. From $(\langle x\rangle, \langle\rangle) \succ_h \mathrm{nil}$ we deduce that there is at most one $(\langle x\rangle, \langle\rangle) \in \mathrm{Tr}$: x is root of Tr. The leaf-tree of root x may be represented as $\{(\langle x\rangle, \langle\rangle), \mathrm{nil}\}$. The tree with only one root x and two children y, z may be represented as

$$\{(\langle x, y\rangle, \langle 0\rangle), (\langle x, z\rangle, \langle 1\rangle), (\langle x\rangle, \langle\rangle), \mathrm{nil}\}.$$

The last definition we need is the one-step extension $\succ_1$ between k-ary tree; $\mathrm{Tr}' \succ_1 \mathrm{Tr}$ if Tr' has one leaf more than Tr.

Definition 2.1.6 (One-step extension for k-ary trees)**.** If Tr is a k-ary tree and $\lambda \in \mathrm{Tr}$ and $\mu \succ_h \lambda$ and $\lambda' \succ_h \lambda$ for no $\lambda' \in \mathrm{Tr}$ (λ has no extensions in color h in Tr), then

$$\mathrm{Tr} \cup \{\mu\} \succ_1 \mathrm{Tr}$$

2.1.2 A formula to compute the ordinal height

Here we use $\alpha, \beta, \gamma, \delta$ to denote ordinals. In particular, we use ω to denote the least infinite ordinal, which is identified with the set of natural numbers.

In this subsection we consider k-branching trees, ordered by reverse-inclusion and whose nodes are labelled by decreasing sequences of ordinal less than an ordinal α. We prove that they form a well-order, that the ordinal height of each element is computable, and we include a formula to compute it. We need this formula in our study about the computational complexity of Podelski and Rybalchenko's Termination Theorem in Section 2.3.

First of all we need to recall some well-known facts about the natural sum (also known as Hessenberg sum [18]). $\alpha \oplus \beta$ is defined as

$$\sup\left\{\alpha' \oplus \beta + 1, \alpha \oplus \beta' + 1 \mid \alpha' < \alpha, \beta' < \beta\right\}.$$

By the Cantor Normal Form Theorem, each pair of ordinals α, β may be written as

$$\alpha = \omega^{\gamma_1} \cdot n_1 + \cdots + \omega^{\gamma_p} \cdot n_p$$
$$\beta = \omega^{\gamma_1} \cdot m_1 + \cdots + \omega^{\gamma_p} \cdot m_p$$

for some $\gamma_1 > \gamma_2 > \cdots > \gamma_p$ and some $n_1, \ldots, n_p, m_1, \ldots, m_p < \omega$. By principal induction over α and secondary induction over β

$$\alpha \oplus \beta = \omega^{\gamma_1} \cdot (n_1 + m_1) \oplus \cdots \oplus \omega^{\gamma_p} \cdot (n_p + m_p)$$

can be proved. As a corollary we deduce that natural sum is commutative and associative.

Definition 2.1.7. Let α be any ordinal. We define $\mathcal{L}_\alpha$ to be the set of $(L, f) \in \mathrm{CList}(k)$ such that L is a decreasing sequence of ordinals in α.

Hence k-$\mathrm{Tr}(\mathcal{L}_\alpha)$ is the set of all finite k-branching trees labelled with decreasing ordinals in α. As usual, we use Nil to denote the k-branching empty tree.

The goal of this subsection it to define a map $h_k(\cdot,\alpha)$ computing the ordinal height of the tree T in k-$\mathrm{Tr}(\mathcal{L}_\alpha)$ with respect to the binary relation $\succ_1$ on k-ary trees. Nil has the highest ordinal height with respect to $\succ_1$, hence $h_k(\mathrm{Nil},\alpha)$ computes the ordinal height of the entire set of such trees. For the results of this chapter we only need to know the values of $h_k(\cdot,\alpha)$ for $\alpha<\omega^2$. For sake of completeness, however, we include a study of $h_k(\cdot,\alpha)$ for all α.

For any Tr in k-$\mathrm{Tr}(\mathcal{L}_\alpha)$ we define a map $h_k(\mathrm{Tr},\alpha)$, then we prove that it computes the ordinal height of Tr in k-$\mathrm{Tr}(\mathcal{L}_\alpha)$.

Definition 2.1.8. Let α be an ordinal and let $\mathrm{Tr}\in k$-$\mathrm{Tr}(\mathcal{L}_\alpha)$. Then we define by principal induction on α and secondary induction over $\succ_1$.

$$h_k(\mathrm{Nil},\alpha)=\sup\{h_k(\mathrm{Tr},\alpha)+1 \mid \mathrm{Tr}\in k\text{-}\mathrm{Tr}(\mathcal{L}_\alpha),\mathrm{Tr}\succ_1\mathrm{Nil}\}.$$

And, if $\mathrm{Tr}\neq\mathrm{Nil}$, β is the label of the root of Tr, and Tr_i is the immediate subtree of Tr in color i:

$$h_k(\mathrm{Tr},\alpha)=\bigoplus_{i=0}^{k-1}h_k(\mathrm{Tr}_i,\alpha_i).$$

Where α_i is either β if $\mathrm{Tr}_i=\mathrm{Nil}$ or the successor of the label of the root of Tr_i otherwise.

We have to prove that $h_k(\mathrm{Tr},\alpha)$ computes the ordinal height of Tr in k-$\mathrm{Tr}(\mathcal{L}_\alpha)$. First we observe that there is an equivalent but simpler description of $h_k(\mathrm{Nil},\cdot)$.

Given an ordinal α and a natural number k we define the natural product [18] as usual:

$$\alpha*k=\alpha\oplus\cdots\oplus\alpha$$

where there are k-many α. With $\alpha\cdot k$, instead, we denote the standard product of ordinals.

Lemma 2.1.9. $h_k(\mathrm{Nil},\cdot)$ *is such that for all* α

$$h_k(\mathrm{Nil},\alpha)=\sup\{h_k(\mathrm{Nil},\beta)*k+1 \mid \beta<\alpha\}.$$

Proof. Fix an ordinal α. We need to prove that

$$\sup\{h_k(\mathrm{Tr},\alpha)+1 \mid \mathrm{Tr}\in k\text{-}\mathrm{Tr}(\mathcal{L}_\alpha),\mathrm{Tr}\succ_1\mathrm{Nil}\}=\sup\{h_k(\mathrm{Nil},\beta)*k+1 \mid \beta<\alpha\}.$$

Let $\mathrm{Tr}\in k$-$\mathrm{Tr}(\mathcal{L}_\alpha)$ such that $\mathrm{Tr}\succ_1\mathrm{Nil}$, then Tr is a root-tree. Let β be the label of the root of Tr. Then, by definition $h_k(\mathrm{Tr},\alpha)=h_k(\mathrm{Nil},\beta)*k$. Hence

$$\sup\{h_k(\mathrm{Tr},\alpha)+1 \mid \mathrm{Tr}\in k\text{-}\mathrm{Tr}(\mathcal{L}_\alpha),\mathrm{Tr}\succ_1\mathrm{Nil}\}\leq\sup\{h_k(\mathrm{Nil},\beta)*k+1 \mid \beta<\alpha\}.$$

Vice versa, given $\beta\in\alpha$ let $\mathrm{Tr}\succ_1\mathrm{Nil}$ be the leaf-tree where the root's label is β. Then again $h_k(\mathrm{Tr},\alpha)=h_k(\mathrm{Nil},\beta)*k$, therefore

$$h_k(\mathrm{Nil},\beta)*k+1=h_k(\mathrm{Tr},\alpha)+1.$$ □

Now we prove our thesis about $h_k(\cdot,\cdot)$.

Proposition 2.1.10. *Let α be an ordinal.*

- *If* $\mathrm{Tr}',\mathrm{Tr} \in k\text{-}\mathrm{Tr}(\mathcal{L}_\alpha)$ *and* $\mathrm{Tr}' \succ_1 \mathrm{Tr}$ *then* $h_k(\mathrm{Tr}',\alpha) < h_k(\mathrm{Tr},\alpha)$.
- *Let* $\mathrm{Tr} \in k\text{-}\mathrm{Tr}(\mathcal{L}_\alpha)$*, then* $h_k(\mathrm{Tr},\alpha)$ *is the ordinal height of* Tr *in* $k\text{-}\mathrm{Tr}(\mathcal{L}_\alpha)$*:*

$$h_k(\mathrm{Tr},\alpha) = \sup\left\{h_k(\mathrm{Tr}',\alpha)+1 \mid \mathrm{Tr}' \succ_1 \mathrm{Tr}\right\}.$$

Proof.

- Given $\mathrm{Tr}' \succ_1 \mathrm{Tr}$, let γ be the label of the father of the new node Tr' and β be the label of the new node of Tr'. Hence $h_k(\mathrm{Tr}',\alpha)$ has as addends k-many $h_k(\mathrm{Nil},\beta)$ instead of one $h_k(\mathrm{Nil},\gamma)$. Since the labelling is decreasing we have that $\beta < \gamma$. By definition

$$h_k(\mathrm{Nil},\gamma) = \sup\{h_k(\mathrm{Nil},\beta) * k + 1 \mid \beta < \gamma\}$$

  then $h_k(\mathrm{Nil},\gamma) > h_k(\mathrm{Nil},\beta) * k$. Since the natural sum is increasing in each argument, $h_k(\mathrm{Tr}',\alpha) < h_k(\mathrm{Tr},\alpha)$ holds.

- If $\mathrm{Tr} = \mathrm{Nil}$ then the thesis follows by definition of $h_k(\mathrm{Nil},\alpha)$. Hence assume that $\mathrm{Tr} \neq \mathrm{Nil}$. Due to the previous point we have

$$h_k(\mathrm{Tr},\alpha) \geq \sup\left\{h_k(\mathrm{Tr}',\alpha)+1 \mid \mathrm{Tr}' \succ_1 \mathrm{Tr}\right\}.$$

  We prove the other inequality by induction over α.

  - Assume that $\alpha = 0$. Then $h_k(\mathrm{Tr},0) = \sup\emptyset = 0$.
  - Assume that $\alpha = \beta+1$. If the root of Tr has label less than β then by inductive hypothesis we are done, since $h_k(\mathrm{Tr},\beta+1) = h_k(\mathrm{Tr},\beta)$ and for each $\mathrm{Tr}' \succ_1 \mathrm{Tr}$, $h_k(\mathrm{Tr}',\beta+1) = h_k(\mathrm{Tr}',\beta)$. So assume that the root of Tr has label β. Let $\mathrm{Tr}_0,\ldots,\mathrm{Tr}_{k-1}$ be the immediate subtrees of Tr. By definition and since β is the root of Tr

$$h_k(\mathrm{Tr},\beta+1) = \bigoplus_{i=0}^{k-1} h_k(\mathrm{Tr}_i,\beta).$$

    We want to prove that for any $\gamma < h_k(\mathrm{Tr},\beta+1)$ there exists $\mathrm{Tr}' \succ_1 \mathrm{Tr}$ such that $\gamma < h_k(\mathrm{Tr}',\beta+1)+1$. Since

$$\gamma < h_k(\mathrm{Tr},\beta+1) = \bigoplus_{i=0}^{k-1} h_k(\mathrm{Tr}_i,\beta),$$

    by definition of natural sum there exist $\gamma_0, \ldots, \gamma_{k-1}$ such that

    * there exists one $j \in k$ such that $\gamma_j < h_k(\mathrm{Tr}_j,\beta)$;
    * for any $i \in k$, if $i \neq j$ then $\gamma_i = h_k(\mathrm{Tr}_i,\beta)$;

  * $\gamma \leq \gamma_0 \oplus \cdots \oplus \gamma_{k-1}$;

  By induction hypothesis we have that

$$h_k(\mathrm{Tr}_j, \beta) = \sup\left\{h_k(\mathrm{Tr}'', \beta) \mid \mathrm{Tr}'' \succ_1 T_j\right\}.$$

  Then, since $\gamma_j < h_k(\mathrm{Tr}_j, \beta)$, there exists $\mathrm{Tr}'' \succ_1 \mathrm{Tr}_j$ such that $h_k(\mathrm{Tr}'', \beta) > \gamma_j$. Let $\mathrm{Tr}'_j = \mathrm{Tr}''$. For any $i \in k$ if $i \neq j$ we define $\mathrm{Tr}'_i = \mathrm{Tr}_i$. Let Tr' be the tree whose root has label β and immediate subtrees $\mathrm{Tr}'_0, \ldots, \mathrm{Tr}'_{k-1}$. By construction $\mathrm{Tr}' \succ_1 \mathrm{Tr}$. Moreover

$$\gamma < \bigoplus_{i=0}^{k-1} h_k(\mathrm{Tr}'_i, \beta) = h_k(\mathrm{Tr}', \beta + 1).$$

  Then

$$h_k(\mathrm{Tr}, \beta + 1) \leq \sup\left\{h_k(\mathrm{Tr}', \beta + 1) + 1 \mid \mathrm{Tr}' \succ_1 \mathrm{Tr}\right\}.$$

- Assume that α is limit. Then the root of Tr has label $\gamma < \alpha$. Hence $\gamma + 1 < \alpha$, and by inductive hypothesis on $\gamma + 1$ we are done, since $h_k(\mathrm{Tr}, \alpha) = h_k(\mathrm{Tr}, \gamma + 1)$ and for each $\mathrm{Tr}' \succ_1 \mathrm{Tr}$ we have $h_k(\mathrm{Tr}', \alpha) = h_k(\mathrm{Tr}', \gamma + 1)$. □

Thanks to Lemma 2.1.9 we may define $h_k(\mathrm{Nil}, \cdot)$ as follows:

$$h_k(\mathrm{Nil}, \alpha) = \begin{cases} 0 & \text{if } \alpha = 0; \\ h_k(\mathrm{Nil}, \beta) * k + 1 & \text{if } \alpha = \beta + 1; \\ \sup\{h_k(\mathrm{Nil}, \gamma) \mid \gamma < \alpha\} & \text{if } \alpha \text{ is limit.} \end{cases}$$

If we may compute $h_k(\mathrm{Nil}, \alpha)$ then we may compute $h_k(\mathrm{Tr}, \alpha)$, that is, by Proposition 2.1.10, the ordinal height of any $\mathrm{Tr} \in k\text{-}\mathrm{Tr}(\mathcal{L}_\alpha)$. We may easily compute $h_k(\mathrm{Nil}, \alpha)$ if either $k = 1$ or $\alpha < \omega^2$, by induction over α.

Lemma 2.1.11. • *If $k = 1$, $h_1(\mathrm{Nil}, \alpha) = \alpha$.*

- *For any $m \in \omega$*

$$h_k(\mathrm{Nil}, m) = \sum_{i=0}^{m-1} k^i = \frac{k^m - 1}{k - 1}.$$

- *For any $m, n \in \omega$, $n \neq 0$*

$$h_k(\mathrm{Nil}, \omega \cdot n + m) = \omega^n \cdot k^m + \sum_{i=0}^{m-1} k^i.$$

Now we want to derive what is the value of $h_k(\mathrm{Nil}, \alpha)$ for any α. This analysis is only added for completeness and is not used to derive the results of this chapter.

Lemma 2.1.12. *Let $k \geq 2$ and let $\alpha = \gamma + n$ where γ is either 0 or a limit ordinal and n is a natural number, then*

- *if* $\gamma = 0$*:*
$$h_k(\mathrm{Nil}, n) = \frac{k^n - 1}{k - 1};$$
- *otherwise*
$$h_k(\mathrm{Nil}, \alpha) = k^\alpha + \frac{k^n - 1}{k - 1}.$$

Proof. By induction on α. By Lemma 2.1.9,

$$h_k(\mathrm{Nil}, \alpha) = \sup\{h_k(\mathrm{Nil}, \beta) * k + 1 \mid \beta < \alpha\}.$$

Thus we have three cases

- $\alpha = 0$. Then $h_k(\mathrm{Nil}, 0) = 0$.
- $\alpha = \beta + 1$. Then $\alpha = \gamma + (n+1)$, $\beta = \gamma + n$. Note that $h_k(\mathrm{Nil}, \cdot)$ is monotone in the second argument. Indeed if $\delta < \beta$ any $\mathrm{Tr} \in k\text{-}\mathrm{Tr}(\mathcal{L}_\delta)$ belongs to $k\text{-}\mathrm{Tr}(\mathcal{L}_\beta)$. Hence:
$$h_k(\mathrm{Nil}, \alpha) = \sup\{h_k(\mathrm{Nil}, \delta) * k + 1 \mid \delta < \alpha\} = h_k(\mathrm{Nil}, \beta) * k + 1.$$
  - If α is finite, also β is; then
$$h_k(\mathrm{Nil}, \alpha) = h_k(\mathrm{Nil}, \beta) * k + 1 = \frac{k^n - 1}{k - 1} * k + 1 = \frac{k^{n+1} - k + k - 1}{k - 1} = \frac{k^{n+1} - 1}{k - 1}.$$
  - If α is infinite, also β is; then
$$\begin{aligned} h_k(\mathrm{Nil}, \alpha) &= h_k(\mathrm{Nil}, \beta) * k + 1 = (k^\beta + \frac{k^n - 1}{k - 1}) * k + 1 \\ &= k^{\beta+1} + \frac{k^{n+1} - 1}{k - 1} = k^\alpha + \frac{k^{n+1} - 1}{k - 1}. \end{aligned}$$
- α limit. Then
$$h_k(\mathrm{Nil}, \alpha) = \sup\{h_k(\mathrm{Nil}, \beta) * k + 1 \mid \beta < \omega\} \vee \sup\{h_k(\mathrm{Nil}, \beta) * k + 1 \mid \omega \le \beta < \alpha\}.$$

  Hence there are two different cases:

  - If $\alpha = \omega$. Then only the first set is not empty. Moreover it is cofinal in ω. Then $h_k(\mathrm{Nil}, \omega) = \omega = k^\omega$
  - If $\alpha > \omega$. Then the first set is cofinal in ω, while the second set is cofinal in $[\omega, \alpha)$. Then, since α is limit:
$$h_k(\mathrm{Nil}, \alpha) = \omega \vee \sup\left\{k^\beta + \frac{k^{n+1} - 1}{k - 1} + 1 \mid \omega \le \beta < \alpha\right\} = k^\alpha. \qquad \square$$

Since if α is a limit ordinal, then $2^\alpha = k^\alpha$ for any $k \geq 2$, it follows that if α is limit then $h_k(\mathrm{Nil}, \alpha) = k^\alpha = 2^\alpha$. Moreover if $\alpha = \omega \cdot k$, we have that $2^\alpha = \omega^k$, since by definition of ordinal exponentiation $2^\omega = \sup\{2^n \mid n \in \omega\}$.

2.1.3 Erdős' trees

In this subsection we recall the definitions of Erdős' trees. In Section 1.1 we informally introduced the construction of an Erdős' trees for some particular computations. Here we formally define Erdős' tree in a bit more general case: for k-many binary relations $R_0, \dots, R_{k-1}$ on a set S. If $R_0, \dots, R_{k-1}$ set up a partition of $S^2 \setminus \{(x,x) \mid x \in S\}$ then we obtain a coloring of $[S]^2$.

Given k many relations $R_0, \dots, R_{k-1}$ we may think of each branch of an Erdős' tree on $R_0, \dots, R_{k-1}$ as a simultaneous construction of all R_i-decreasing transitive lists for all $i \in k$.

In order to formally define an Erdős' tree we need the definition of special colored lists. A 1-colored list with respect to $R_0, \dots, R_{k-1}$ is a colored list $(\langle x_0, \dots, x_{n-1}\rangle, \langle c_0, \dots, c_{n-2}\rangle)$ with the following property: the node x_i is related by R_{c_i} to any node x_j with $j > i$ in the same list. In the particular case that $R_0, \dots, R_{k-1}$ provide a coloring of $[S]^2$, then a 1-colored list is a colored subgraph of particular kind, in which the color of any edge $\{x_i, x_{i+1}\}$ depends only on the first node x_i of the edge. In a sense, in a 1-colored list the coloring is assigned to nodes, not to edges. We may give an alternative interpretation of 1-colored lists in term of R_h-decreasing sequences: a 1-colored list is an attempt to build simultaneously one R_h-decreasing chain for each $h \in k$. For short we write $(R_0, \dots, R_{k-1})$-colored to mean 1-colored list with respect to $R_0, \dots, R_{k-1}$.

Definition 2.1.13. $(L, f) \in \mathrm{CList}(k)$ is a $(R_0, \dots, R_{k-1})$-*colored list* on a set S if $L = \langle x_0, \dots, x_{n-1}\rangle$ is a list on S, $f = \langle c_0, \dots, c_{n-2}\rangle$, and

$$\forall i \in n-1 (c_i = h \implies (\forall j \in n (i < j \implies (x_j R_h x_i)))).$$

1-$\mathrm{CList}(R_0, \dots, R_{k-1}) \subseteq \mathrm{CList}(k)$ is the set of $(R_0 \dots, R_{k-1})$-colored lists.

Take any $(R_0, \dots, R_{k-1})$-colored list (L, f) on S. Let L' be the sublist of $L = \langle x_0, \dots, x_{n-1}\rangle$ consisting of all elements x_i such that either $c_i = h$ or $i = n-1$. Then L' is a R_h-decreasing list, because by definition each element of L' is connected by R_h to each element of L after it, hence, and with more reason, is connected by R_h to each element of L' after it. Thus, as we anticipated, any $(R_0, \dots, R_{k-1})$-colored list defines simultaneously one R_h-decreasing list for each $h \in k$.

We call *Erdős' tree* over $R_0, \dots, R_{k-1}$ any k-ary tree whose branches are all in 1-$\mathrm{CList}(R_0, \dots, R_{k-1})$.

2.2 The H-closure Theorem

In this section we present an intuitionistic proof of the Termination Theorem. Our first step is to formulate a version of Ramsey's Theorem for pairs which has a proof in intuitionistic second order logic plus basic arithmetical axioms, that is, has a proof in ordinary mathematics using only intuitionistic logic, as it is the case for proofs in Bishop's book [13]. To put otherwise, we do not assume the Excluded Middle, and we do not assume other principles which are often added to Intuitionistic logic, like the Choice Axiom, Brouwer's Thesis or the Fan Theorem. Our version of Ramsey's Theorem for pairs is informative, in the sense that it has no negation, while it has a disjunction. We say that a relation R is homogeneous-well-founded, or H-well-founded for short, if the tree of all R-decreasing transitive sequences is well-founded with respect to the inductive definition of well-foundedness (see Section 1.2). In our version of Ramsey's Theorem for pairs, R-decreasing transitive sequences take the place of homogeneous subsets of colored graphs, and the relation R between two elements takes the place of a color for the edge connecting the two elements. We express Ramsey's Theorem for pairs as a property of well-founded relations, saying that H-well-founded relations are closed under finite unions. For short we call this statement *H-closure Theorem*. We are able to split the proof of Ramsey's Theorem for pairs into two parts: the intuitionistic proof of H-closure, followed by some "simple" (in the sense of the Reverse Mathematics, see Chapter 5) classical proof of the equivalence between Ramsey's Theorem for pairs and H-closure.

The result closest to the H-closure Theorem we could find is the Almost Full Theorem by Coquand [23]. Coquand, as Veldman and Bezem did before him [92], considers *almost full relations* (a kind of dual of H-closed relations) and proves that they are closed under finite *intersections*. Veldman and Bezem use the Choice Axiom of type 0 (if $\forall x \in \mathbb{N} \exists y \in \mathbb{N} C(x,y)$, then $\exists f : \mathbb{N} \to \mathbb{N} \forall x \in \mathbb{N} C(x, f(x))$) and Brouwer's thesis. Coquand's proof, instead, is purely intuitionistic, and it may be used to give a purely intuitionistic proof of the Termination Theorem [93]. However, it is not evident what are the effective bounds hidden in Coquand's proof of the Termination Theorem. If we compare the H-closure Theorem with the Almost Full Theorem, in the most recent version by Coquand [23], we find no easy way to intuitionistically deduce one from the other, due to the use of de Morgan's Law to move from the definition of almost full to the definition of H-well-founded. H-closure is in a sense more similar to the original Ramsey's Theorem for pairs, because it was obtained from it with just one classical step, a contrapositive, while in order to derive Ramsey's Theorem from almost fullness we need one application of de Morgan's Law, followed by a contrapositive. We expect that H-closure, hiding one application less of de Morgan's Law, should be a version of Ramsey's Theorem for pairs simpler to use in intuitionistic proofs and for extracting bounds.

2.2.1 From Ramsey's Theorem to the H-closure Theorem

In this subsection we argue classically. We introduce and discuss a property of well-founded relations we call the H-closure Theorem, or just *H-closure* for short. Classically, H-closure is but a variant of Ramsey's Theorem for pairs, and therefore it is classically provable. In the rest of this section we show that H-closure has an intuitionistic proof, and that by using H-closure instead Ramsey's Theorem for pairs, the Termination Theorem by Podelski and Rybalchenko turns out to be intuitionistically provable (provided we use the inductive definition of well-foundedness).

Recall that Ramsey's Theorem for pairs guarantees that given a coloring over the edges of a complete graph with countably many nodes in k-many colors, there is an infinite homogeneous set. Assume that $\{x_i \mid i \in \mathbb{N}\}$ is an injective enumeration of the nodes. If $j < i$, we arbitrarily represent a non-oriented edge between two nodes x_i, x_j by the pair (i,j). Edges are not oriented, therefore the opposite edge from x_j to x_i is the same edge, and it is again represented by the pair (i,j). Thus, a partition of edges in k sets $S_0, \ldots, S_{k-1}$ may be represented by a partition of the set $\{(x_i,x_j) \mid j < i\}$ into k binary relations $S_0, \ldots, S_{k-1}$. Therefore one possible formalization of Ramsey's Theorem for pairs is the following.

Theorem 2.2.1 (Ramsey's Theorem for pairs)**.** *Assume that S is a set having some injective enumeration $S = \langle x_i \mid i \in \mathbb{N}\rangle$. Assume that $R_0, \ldots, R_{k-1}$ are binary relations on S which are a partition of $\{(x_i,x_j) \in S \times S \mid j < i\}$, namely:*

1. $R_0 \cup \cdots \cup R_{k-1} = \{(x_i,x_j) \in S \times S \mid j < i\}$
2. *for all $h < h' < k$:* $R_h \cap R'_h = \emptyset$.

Then for some $h < k$ there exists a set $Y \subseteq \mathbb{N}$, such that: $\forall i,j \in Y(j < i \implies x_i R_h x_j)$.

Then the set $X = \{x_i \mid i \in Y\}$ is the infinite homogeneous set for the graph. In the statement above three assumptions may be dropped.

1. First of all, we may drop the assumption that $R_0, \ldots, R_{k-1}$ are pairwise disjoint. Suppose we do. Then, if we put $R'_h = R_h \setminus \bigcup\left\{R'_j \mid j < h\right\}$ for any $h < k$ we obtain a partition $R'_0, \ldots, R'_{k-1}$ of $\{(x_i,x_j) \mid j < i\}$. Therefore by applying Theorem 2.2.1 to the coloring given by the relations R'_h for $h < n$, there exists a $h < n$ and an infinite $Y \subseteq \mathbb{N}$, such that $\forall i,j \in Y(j < i \implies x_i R'_h x_j)$, and with more reason, $\forall i,j \in Y(j < i \implies x_i R_h x_j)$.

2. Second, we may drop the assumption "the enumeration is injective" (in this case, $X = \{x_i \mid i \in Y\}$, may be a finite set). Assume we do. Then, if we set $R'_h = \{(i,j) \mid x_i R_h x_j\}$ for any $h < k$, we obtain k relations $R'_0, \ldots, R'_{k-1}$ on $\mathbb{N}$, whose union is the set $\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid j < i\}$. Therefore, thanks to Theorem 2.2.1, there exists a $h < k$ and an infinite $Y \subseteq \mathbb{N}$, such that $\forall i,j \in Y(j < i \implies i R'_h j)$, and with more reason, $\forall i,j \in Y(j < i \implies x_i R_h x_j)$.

3. Third, we may drop the assumption that $\langle x_i \mid i \in \mathbb{N}\rangle$ is an enumeration of S. Suppose we do. Then, if we restrict $R_0, \ldots, R_{k-1}$ to $S' = \{x_i \mid i \in \mathbb{N}\}$, we obtain some binary relations $R'_0, \ldots, R'_{k-1}$ on S' such that $R'_0 \cup \ldots \cup R'_{k-1} = \{(x_i,x_j) \in S' \times S' \mid j < i\}$. Again, we conclude by Theorem 2.2.1 that there exists some $h < k$ and some infinite $Y \subseteq \mathbb{N}$ such that $\forall i,j \in Y(j < i \implies x_i R'_h x_j)$, and with more reason, $\forall i,j \in Y(j < i \implies x_i R_h x_j)$.

Summing up, we showed that, classically, we may restate Ramsey's Theorem for pairs as follows:

For any infinite sequence $\langle x_i \mid i \in \mathbb{N}\rangle$ *of elements of* S*, if* $\forall i,j \in \mathbb{N}(j < i \implies x_i(R_0 \cup \ldots \cup R_{k-1})x_j)$*, then for some* $h < k$ *there is some infinite* $Y \subseteq \mathbb{N}$*, such that* $\forall i,j \in Y(j < i \implies x_i R_h x_j)$*.*

It is likely that even this statement cannot be intuitionistically proved, because Y is akin to a homogeneous set, and there is no effective way to produce homogeneous sets [50, 84]. By taking the contrapositive and using the classical fact "any non-infinite set is finite", we may obtain the following corollary:

If for all $h < k$*, all the sequences* $\langle y_i \mid i \in \mathbb{N}\rangle$ *such that* $\forall i,j \in \mathbb{N}(j < i \implies y_i R_h y_j)$ *are finite, then all sequences* $\langle x_i \mid i \in \mathbb{N}\rangle$ *such that* $\forall i,j \in \mathbb{N}(j < i \implies x_i(R_0 \cup \ldots \cup R_{k-1})x_j)$ *are finite.*

We call this property *classical* H*-closure*. It is immediate to check that, classically, this is yet another version of Ramsey's Theorem for pairs. The finite sequences $\langle y_i \mid i \in \mathbb{N}\rangle$ such that $\forall i,j \in \mathbb{N}(j < i \implies y_i R_k y_j)$ are the corresponding, in the statement of classical H-closure, of a finite homogeneous subset of color k in the statement of Ramsey's Theorem for pairs. For this reason we call any sequence $\langle y_i \mid i \in \mathbb{N}\rangle$ such that $\forall i,j \in \mathbb{N}(j < i \implies y_i R_k y_j)$ a homogeneous sequence with respect to R_k.

Given a relation R over S, let $H(R)$ be the set of all finite homogeneous sequences with respect to R. Then classical H-closure may be restated as follows: if R_0, ..., R_{k-1} are binary relations over some set S, and $H(R_0)$, ..., $H(R_{k-1})$ are sets of lists well-founded by one-step extension, then $H(R_0 \cup \ldots \cup R_{k-1})$ is a set of lists well-founded by one-step extension as well. We say that a relation R is H-well-founded if $H(R)$ is well-founded by extension. Then classical H-closure states that H-well-founded relations are closed under finite unions. Classical H-closure is a property classically equivalent to Ramsey's Theorem for pairs, but which deals with well-founded relations. In Proof Theory, there is plenty of examples of classical proofs of well-foundedness which are turned into intuitionistic proofs[1], and indeed from H-closure we obtain an intuitionistic version of Ramsey's Theorem for pairs.

If we want a purely intuitionistic proof, there is a last step to be done. We call *intuitionistic* H*-closure*, or just H-closure for short, the statement obtained by replacing, in classical H-closure, the classical definition of well-foundedness with the inductive definition of well-foundedness (see Section 1.2).

For us, the interest of an intuitionistic proof of H-closure lies in the fact that it is the combinatorial fragment of Ramsey's Theorem for pairs required in order to intuitionistically prove some results about termination. In the proof of the Termination Theorem by Podelski and Rybalchenko [77], the part of Ramsey's Theorem which is actually used is H-closure. Here, by replacing Ramsey's Theorem with H-closure, we obtain an intuitionistic proof of the Termination Theorem. Moreover in Chapter 3, by

[1]Berardi in [7] proved that given a classical proof of a statement of the form "if R_0, $\ldots R_k$ are inductively well-founded then R is inductively well-founded", for $R_0, \ldots, R_k, R$ primitive recursive, a constructivization is always possible.

using H-closure, we can intuitionistically prove the Size-Change Termination Theorem by Lee, Jones and Ben-Amram [58].

2.2.2 H-well-founded relations

Here we present the formal definition of H-well-foundedness, which is fundamental to state the new intuitionistic form of Ramsey's Theorem for pairs, as seen in the previous subsection. In this chapter we use the inductive definition of well-foundedness, therefore for short we write well-founded instead of inductively well-founded. From now on, throughout this section we work in HAS. In particular we do not use the Choice Axiom, Brouwer's Thesis, the Fan Theorem, and we do not use Bar-Induction. In fact, the H-closure Theorem and the Termination Theorem could be proved using even less, intuitionistic inductive definition [63], a fragment of second order intuitionistic arithmetic.

We call a R-homogeneous sequence any finite transitive decreasing R-sequence. We say that R is homogeneous-well-founded, just H-well-founded for short, if the one-step expansion $\succ$ on the set of R-homogeneous sequences is well-founded. Being H-well-founded is weaker than being well-founded, as we will see in a moment. Formally, the definition runs as follows.

Definition 2.2.2. Let R be a binary relation on S.

- a *R-homogeneous list* on S is any R-decreasing transitive finite sequence on S:

$$\langle x_0, \ldots, x_{n-1} \rangle \in H(R) \iff \forall i, j < n\, (i < j \implies x_j R x_i)\,.$$

- $H(R)$ is the set of R-homogeneous lists on S
- R is homogeneous well-founded, *H-well-founded* for short, if $H(R)$ is $\succ$-well-founded.

From the previous definition follows that R is classically H-well-founded (analogously to Definition 1.2.1) if and only if there are no infinite decreasing transitive R-sequences. It also follows that if R is decidable then also $H(R)$ is.

Let us provide an example of a H-well-founded relation which is not well-founded. Consider $R = \{(n+1, n) \mid n \in \mathbb{N}\}$. R is not well-founded because $0, 1, 2, \ldots$ is an infinite *R-decreasing* sequence. Moreover

$$H(R) = \{\langle\rangle\} \cup \{\langle n \rangle \mid n \in \mathbb{N}\} \cup \{\langle n, n+1 \rangle \mid n \in \mathbb{N}\}\,,$$

and $(H(R), \succ)$ is inductively well-founded, by Corollary 1.2.23.

We prove now, in HAS, that well-founded relations are H-well-founded relations, and the converse holds for transitive relations.

Proposition 2.2.3 (Well-founded and H-well-founded relations)**.** *Assume that R is a binary relation over some set $S \subseteq \mathbb{N}$.*

1. *If R is well-founded then R is H-well-founded.*
2. *If R is H-well-founded and R is transitive then R is well-founded.*

Proof. 1. Let X be the set of elements $y \in S$ such that all homogeneous lists of the form $L*\langle y\rangle$ are well-founded by extension in $H(R)$: i.e.

$$X = \{y \in S \mid \text{ each } L*\langle y\rangle \in H(R) \text{ is } \succ\text{-well-founded in } H(R)\}$$

Assume that X is R-inductive. From the assumption that R is well-founded and definition of well-foundedness we deduce that $\forall x \in S(x \in X)$, that is, that all homogeneous lists of the form $L*\langle x\rangle$ for some $x \in S$ are $\succ$-well-founded in $H(R)$. By Proposition 1.2.11 the empty list is $\succ$-well founded. Hence $H(R)$ is $\succ$-well-founded.

Thus, we prove that X is R-inductive. Assume that $\forall z \in S(zRy \implies z \in X)$, that is, that for all $z \in S$ such that zRy, all homogeneous lists $M*\langle z\rangle \in H(R)$ are well-founded. We have to prove that $y \in X$, that is, that all homogeneous lists $L*\langle y\rangle \in H(R)$ are $\succ$-well-founded. By unfolding definition, this means that $L*\langle y\rangle$ belongs to all properties Y which are inductive with respect to $\succ$. Assume that $L*\langle y\rangle \in H(R)$ and that Y is $\succ$-inductive, in order to prove that $L*\langle y\rangle \in Y$. Since Y is inductive, in order to prove that $L*\langle y\rangle \in Y$ it is enough to prove that: $\forall M \in H(R)(M \succ L*\langle y\rangle \implies M \in Y)$. Assume that $M \in H(R)$, $M \succ L*\langle y\rangle$ in order to prove that $M \in Y$. By assumption on M we have $M = L*\langle y\rangle*\langle z\rangle$ for some $z \in S$ such that zRy. By assumption on y we have $z \in X$. By definition of X, this guarantees that M is $\succ$-well-founded. This means that M belongs to any $\succ$-inductive property of $H(R)$, and in particular that $M \in Y$, as we wished to show.

2. Assume that $H(R)$ is $\succ$-well-founded and R is transitive. We have to show that any $x \in S$ is well-founded with respect to R, that is, that x belongs to any R-inductive property X. Assume that $x \in S$ and X is R-inductive, in order to prove that $x \in X$. We define the set Y of homogeneous lists which are either empty, or of the form $L*\langle z\rangle$ for some $z \in X$: $Y = \{\langle\rangle\} \cup \{L*\langle z\rangle \in H(R) \mid z \in X\}$. If we may prove that Y is $\succ$-inductive in $H(R)$, then we will deduce that $Y = H(R)$ by $H(R)$ $\succ$-well-founded. In particular the R-homogeneous list $L = \langle x\rangle$ is in Y, hence $x \in X$ by definition of Y. Therefore it is enough to show that Y is $\succ$-inductive in $H(R)$. To this aim, we assume that $\forall M \in H(R)(M \succ L \implies M \in Y)$, in order to prove that $L \in Y$. We prove $L \in Y$ by cases on L. If $L = \langle\rangle$ then $L \in Y$ by definition of Y. If $L = L'*\langle y\rangle$ for some $y \in S$, then $L \in Y$ if and only if $y \in X$. We assumed that X is R-inductive, that is: $\forall z \in S((zRy \implies z \in X) \implies y \in X)$. Thus, in order to prove that $y \in X$ we have to prove that for all $z \in S$ such that zRy we have $z \in X$. If $z \in S$ is such that zRy, then, by y last element of L and transitivity of R, the list $L*\langle z\rangle$ is R-decreasing and *transitive*, that is, it is homogeneous: we have $L*\langle z\rangle \in H(R)$, and $L*\langle z\rangle \succ L$. We assumed that $\forall M \in H(R)(M \succ L \implies M \in Y)$, thus we deduce

that $L*\langle z\rangle \in Y$. Since z is the last element of $L*\langle z\rangle$ and by definition of Y we have $z \in X$, for any zRy, $z \in S$, as we wished to show. □

Observe that whether S and R are finite, we may characterize the well-foundedness and the H-well-foundedness in an elementary way.

Definition 2.2.4. Let R be a binary relation on S and $x \in S$. A finite sequence $\langle x_0, \ldots, x_n\rangle$ is a R-cycle from x if $n > 0$ and

$$x = x_n R x_{n-1} R x_{n-2} R \ldots R x_0 = x.$$

If $n = 1$ (that is, if xRx), we call the R-cycle a R-loop.

Proposition 2.2.5. *Assume that $S = \{x_0, \ldots, x_{k-1}\} \subseteq \mathbb{N}$ for some $k \in \mathbb{N}$. Let R be any binary relation on S (we do* not *assume the decidability of R).*

1. *R is well-founded if and only if there are no R-cycles.*
2. *R is H-well-founded if and only if there are no R-loops.*

Proof. 1. Suppose there are no R-cycles, in order to prove that R is well-founded. In no R-sequence $\{y_0, \ldots, y_{m-1}\}$ we may have $i < j < m$ and $y_i = y_j$ otherwise there exists a R-cycle in S. By the Finite Pigeonhole Principle (which is intuitionistically derivable), if it were $m > k$ we would have $i < j < m$ and $y_i = y_j$ contradiction. We deduce that every R-sequence has at most k-many elements. Then each $x \in S$ has height less than k. Thanks to Corollary 1.2.23 R is well-founded.

Now suppose that R is well-founded, in order to prove that there exists no R-cycle from x. If there were a R-cycle from x, there would exist an infinite decreasing R-sequence from x, hence, by Corollary 1.2.25, x would not be well-founded. Contradiction.

2. Suppose that there exist no R-loops, in order to prove that R is H-well-founded, that is, that the set $H(R)$ of R-homogeneous sequences is well-founded by one-step extension. There is no decreasing transitive R-sequence $\langle x_0, \ldots, x_n\rangle$ such that $x_0 = x_n$ and $n > 0$, since this would imply that $x_n R x_0 = x_n$: $\langle x_n\rangle$ would be an R-loop. Then any $L \in H(R)$ has at most k elements, the cardinality of S. Thanks to Corollary 1.2.23, we conclude that $(H(R), \succ)$ is well-founded, since any list $L \in H(R)$ has height at most k.

On the other hand, assume that $H(R)$ is well-founded, in order to prove that there exists no R-loop for x. If there were a R-loop from x, then for every $n \in \mathbb{N}$, the list composed by x repeated n times would be transitive and R-decreasing. Hence $H(R)$ would be ill founded, contradiction. □

Thanks to Proposition 2.2.5 we may prove the H-closure Theorem if $R_0, \ldots, R_{k-1}$ are relations over a finite set S. In fact $R = (R_0 \cup \cdots \cup R_{k-1})$ is H-well-founded if and only if there are no R-loops. xRx is equivalent to xR_ix for some $i \in k$, therefore there are no

R-loops if and only if there are no R_i-loops for any $i \in k$. Hence R is H-well-founded if and only if for each $i \in k$, R_i is H-well-founded. We now want to prove the H-closure Theorem for any set $S \subseteq \mathbb{N}$.

2.2.3 An intuitionistic version of König's Lemma

In this subsection we deal with binary trees as defined in Section 2.1. We use binary trees to prove an intuitionistic version of König's Lemma for nested binary trees (binary trees whose nodes are themselves binary trees), which we call the *Intuitionistic Nested Fan Theorem*. As in the classical case, there is a strong link between intuitionistic Ramsey's Theorem for pairs and the Intuitionistic Nested Fan Theorem.

As shown in Section 1.1, König's Lemma is a result of classical logic which guarantees that any infinite binary tree has some infinite branch. By taking the contrapositive, this result has the classically equivalent form: if every branch of a binary tree is finite then the tree is finite. This version is called the Fan Theorem and was included in intuitionistic mathematics by Brouwer [2]. As we explained in the introduction, in this chapter we do not assume it. We only use intuitionistic logic with basic arithmetical axioms, and intuitionistic logic with basic arithmetical axioms is unable to prove the Fan Theorem, because this theorem fails in the model in which all maps are recursive. Indeed, Kleene in 1952 shown there is an infinite recursive binary tree without infinite recursive branches (e.g. see [67, Proposition V.5.25]).

There exists a result weaker than Fan Theorem, which can be derived only using intuitionistic logic with basic arithmetic. This result is stated as follows.

Lemma 2.2.6 (Fan Theorem for Intuitionistic Logic)**.** *Each (inductively) well-founded binary tree is finite.*

For instance, the Fan Theorem is not derivable in Bishop's constructive mathematics [13], while the Fan Theorem for Intuitionistic Logic is [24]. Here we are interested to an intuitionistic version of the Fan Theorem for nested trees (trees whose nodes are trees), that we call the Intuitionistic Nested Fan Theorem.

Let consider a tree Tr' whose nodes are finite binary trees, and whose father/child relation between nodes is the one-step extension $\succ_1$. Classically we may prove what follows. Assume that for each branch $b = \ldots \mathrm{Tr}_2 \succ_1 \mathrm{Tr}_1 \succ_1 \mathrm{Tr}_0$ of Tr', the union $\mathrm{Tr} = \bigcup\{\mathrm{Tr}_i \mid \mathrm{Tr}_i \in b\}$ is a binary tree having only finite branches. Then Tr is finite by König's Lemma, therefore b is finite. Thus, the tree of trees Tr' has only finite branches, and it is classically well-founded.

In the intuitionistic proof of the intuitionistic Ramsey's Theorem for pairs we use an intuitionistic version of this statement, in which the finiteness of the branches is replaced by inductive well-foundedness of branches. The Intuitionistic Nested Fan Theorem says: if a set of colored lists $\mathcal{L}$ is well-founded then the set 2-Tr($\mathcal{L}$), of all binary trees whose branches are all in $\mathcal{L}$, is well-founded with respect to the tree one-step extension $\succ_1$.

[2]As remarked by Kohlenbach, van Dalen in [91] points out that the fan theorem was formulated by Brouwer earlier than König's lemma was designed by König.

In this subsection $\leq$ denotes the usual relation of prefix between lists on S:

$$L \leq M \iff \exists N \in \{\text{lists on } S\}\,(L{*}N = M).$$

Lemma 2.2.7 (Intuitionistic Nested Fan Theorem)**.** *Let $\mathcal{L} \subseteq \mathrm{CList}(2)$ be any set of colored lists with all colors in 2. Then*

$$(\mathcal{L}, \succ_{col}) \textit{ is well-founded} \implies (2\text{-}\mathrm{Tr}(\mathcal{L}), \succ_1) \textit{ is well-founded.}$$

Proof. Assume that Tr is any binary tree with root $x \in S$. Let $i \in 2$. We define $\pi_i(\mathrm{Tr})$ as the immediate subtree number i of Tr (the unique subtree connected to the root of Tr by an edge in color i). Formally we set $\pi_i(\mathrm{Tr}) = \{\lambda \in \mathrm{CList}(2) \mid (\langle x\rangle, \langle\rangle){*_i}\lambda \in \mathrm{Tr}\}$. $\pi_i(\mathrm{Tr})$ is undefined when $\mathrm{Tr} = \mathrm{Nil}$.

If $i \in 2$, $\lambda \in \mathrm{CList}(2)$ and $\mathrm{Tr} \in 2\text{-}\mathrm{Tr}$, we denote with $\lambda{*_i}\mathrm{Tr}$ the set $\{\lambda{*_i}\mu \mid \mu \in \mathrm{Tr}\}$. For instance $\mathrm{nil}{*_i}\mathrm{Tr} = \mathrm{Tr}$.

Let $i \in 2$, $\lambda \in \mathrm{CList}(2)$. We define $2\text{-}\mathrm{Tr}(\mathcal{L}, \lambda, i)$ as the set of the binary trees occurring in some tree of $2\text{-}\mathrm{Tr}(\mathcal{L})$, as immediate subtree number i of the last node of the branch λ. Formally we set $2\text{-}\mathrm{Tr}(\mathcal{L}, \lambda, i) = \{\mathrm{Tr} \in 2\text{-}\mathrm{Tr} \mid \lambda{*_i}\mathrm{Tr} \subseteq \mathcal{L}\}$. For instance, $2\text{-}\mathrm{Tr}(\mathcal{L}, \mathrm{nil}, i) = 2\text{-}\mathrm{Tr}(\mathcal{L})$, for any $i \in 2$.

We prove that $(2\text{-}\mathrm{Tr}(\mathcal{L}, \lambda, i), \succ_1)$ is well-founded for all $\lambda \in \mathcal{L}$. The thesis follows if we set $\lambda = \mathrm{nil}$, $i = 1$ (a dummy value). Since $\mathcal{L}$ is well-founded, we argue by induction over λ with respect to $\succ_{\mathrm{col}}$.

Let us abbreviate with $B = 2\text{-}\mathrm{Tr}(\mathcal{L}, \lambda, i)$ the set of binary trees which we may append to the branch λ in color i obtaining some binary tree in $2\text{-}\mathrm{Tr}(\mathcal{L})$. Assume that $\mathrm{Tr} \in B$. We have to prove that Tr is well-founded in $(B, \succ_1)$. We distinguish two cases.

1. Assume that Tr has root some $x \in S$. Let us abbreviate with $\lambda_x = \lambda{*_i}(\langle x\rangle, \langle\rangle)$ the branch ending in x and with $B_j = 2\text{-}\mathrm{Tr}(\mathcal{L}, \lambda_x, j)$ the set of binary trees which we may append to the branch λ_x in color j obtaining some binary tree in $2\text{-}\mathrm{Tr}(\mathcal{L})$. We define a simulation T from $(B, \succ_1)$ to $(B_0, \succ_1) \otimes (B_1, \succ_1)$ such that $\mathrm{Tr} \in \mathrm{dom}(T)$. Since by inductive hypothesis on λ_x ($\lambda_x \succ_{\mathrm{col}} \lambda$), both $(B_0, \succ_1)$ and $(B_1, \succ_1)$ are well-founded, the thesis follows by Proposition 1.2.19.1 and Proposition 1.2.19.2. Informally, we simulate a tree by the pair of its immediate subtrees. Formally, the simulation T is defined by $\mathrm{Tr}'\ T\ (\pi_0(\mathrm{Tr}'), \pi_1(\mathrm{Tr}'))$ whenever Tr' has root x. T is well-defined because $\mathrm{Tr}' \neq \mathrm{Nil}$, hence $\pi_0(\mathrm{Tr}'), \pi_1(\mathrm{Tr}')$ are well-defined. We have $\mathrm{Tr} \in \mathrm{dom}(T)$ because Tr has root x. We have $\pi_i(\mathrm{Tr}') \in B_i$ by definition. Whenever $\mathrm{Tr}' \succ_1 \mathrm{Tr}''$, then Tr'' is obtained from Tr' adding one node either in the first or in the second immediate subtree of Tr'. In the first case we have $\pi_0(\mathrm{Tr}') \succ_1 \pi_0(\mathrm{Tr}'')$ and $\pi_1(\mathrm{Tr}') = \pi_1(\mathrm{Tr}'')$, in the second case we have $\pi_0(\mathrm{Tr}') = \pi_0(\mathrm{Tr}'')$ and $\pi_1(\mathrm{Tr}') \succ_1 \pi_1(\mathrm{Tr}'')$. In both cases, the pair $(\pi_0(\mathrm{Tr}'), \pi_1(\mathrm{Tr}'))$ is related to $(\pi_0(\mathrm{Tr}''), \pi_1(\mathrm{Tr}''))$ by $\succ_1 \otimes \succ_1$. Thus, T is a simulation such that $\mathrm{Tr} \in \mathrm{dom}(T)$, as we wished to show.

2. Assume that $\mathrm{Tr} = \mathrm{Nil}$. Then all one-step extensions of Tr in B are not empty, therefore they are well-founded by point 1 above. Thus, Tr is well-founded by Proposition 1.2.11. $\square$

2.2.4 An intuitionistic form of Ramsey's Theorem

Here we prove the H-closure Theorem. As shown in Section 1.1, Ramsey's Theorem for pairs is not an intuitionistic result. The H-closure Theorem, instead, is a new intuitionistically valid version of Ramsey's Theorem for pairs, which may replace the original in order to obtain intuitionistic proofs of many interesting corollaries.

The H-closure Theorem states that H-well-founded relations are closed under finite unions. It is classically true, because there exists a simple classical proof of the equivalence between Ramsey's Theorem for pairs and the H-closure Theorem. This is one reason for finding an intuitionistic proof of the H-closure Theorem: it splits the proof of Ramsey's Theorem for pairs into two parts, one intuitionistic and the other classical but simple, where simple means it could be proved in RCA_0 (see Chapter 5) and it could be proved using the sub-classical principle Σ^0_3-LLPO [2].

Recall that an Erdős' tree over R_0, R_1, a (R_0, R_1)-tree for short, is any binary tree whose branches $x_0, x_1, x_2, \ldots$ are 1-colored with respect to R_0, R_1, that is: if $x_{p+1} R_i x_p$ then $x_q R_i x_p$ for any $q > p \geq 0$ in the branch. Formally, we express this request by asking that all branches of the tree are in $1\text{-CList}(R_0, R_1)$ (see Subsection 2.1.3). We may think of a (R_0, R_1)-tree as a simultaneous construction of one R_0-decreasing transitive list and one R_1-decreasing transitive list. Let $2\text{-Tr}(1\text{-CList}(R_0, R_1))$ be the set of all (R_0, R_1)-trees. We consider the one-step extension $\succ_{\text{col}}$ on colored lists in $1\text{-CList}(R_0, R_1)$, and the one-step extension $\succ_1$ on binary trees in $2\text{-Tr}(1\text{-CList}(R_0, R_1))$.

Let us assume that R_0 and R_1 are H-well founded, that is, that the set of R_0-decreasing transitive lists and the set of R_1-decreasing transitive lists are well-founded. Our goal is to prove that $(R_0 \cup R_1)$ is H-well-founded, that is, the set of $(R_0 \cup R_1)$-decreasing transitive lists is well-founded. We will simulate the set of $(R_0 \cup R_1)$-decreasing transitive lists into the set of Erdős' trees over R_0, R_1 with one-step extension: by Proposition 1.2.19 it will be enough to prove that this latter is well-founded. The proof that the set of Erdős' trees with one-step extension is well-founded, in turn, will use Lemma 2.2.7, the fact that the set of branches of Erdős' trees is the set of (R_0, R_1)-colored lists and that this set is well-founded. So, eventually, we reduce ourselves to prove that the set of all (R_0, R_1)-colored lists is well-founded. We start proving this last statement by simulating the set of (R_0, R_1)-colored lists in the set $(H(R_0) \times H(R_1)) + 1$, then proving that $(H(R_0) \times H(R_1)) + 1$ is well-founded. From this statement we will move back in our line of reasoning, and eventually we will prove that $(R_0 \cup R_1)$ is H-well-founded.

Lemma 2.2.8 (Simulation of $H(R_0 \cup R_1)$ in the set of Erdős' trees)**.** *Let R_0, R_1 be binary relations on a set S.*

1. *$(1\text{-CList}(R_0, R_1), \succ_{col})$ is simulated in $(H(R_0) \times H(R_1), \succ \otimes \succ) + 1$.*
2. *$(H(R_0 \cup R_1), \succ)$ is simulated in $(2\text{-Tr}(1\text{-CList}(R_0, R_1)), \succ_1)$.*

Proof. 1. We have to simulate the set of (R_0, R_1)-colored lists in the set $(H(R_0) \times H(R_1), \succ \otimes \succ) + 1$. We define a simulation relation T between $1\text{-CList}(R_0, R_1)$ and $(H(R_0) \times H(R_1)) + 1$ as follows:

- $(\langle\rangle, \langle\rangle) T \top$;
- If $L = \langle x_0, \ldots, x_n \rangle$ and $f = \langle c_0, \ldots, c_{n-1} \rangle$, then $(L, f) T (L_0, L_1)$ if for any $i < 2$, L_i is the sublist of L composed by all $x_j \in L$ such that $j < n$ and $c_j = i$.

Note that $(\langle x_0 \rangle, \langle\rangle) T (\langle\rangle, \langle\rangle)$, for any x_0. We need to prove that T is indeed a simulation, and in fact is a total simulation.

- T is a relation between 1-CList(R_0, R_1) and $(H(R_0) \times H(R_1)) + 1$. We have to prove that if $L = \langle x_0, \ldots, x_n \rangle$ and $f = \langle c_0, \ldots, c_{n-1} \rangle$, then $L_i \in H(R_i)$ for $i \in 2$. In fact, since (L, f) is a (R_0, R_1)-colored list, if $j < j'$ are indexes of L_i, then $c_j = i$, and then $x_{j'} R_i x_j$ by definition of (R_0, R_1)-colored list. Thus, L_i is a R_i-decreasing transitive list, that is, $L_i \in H(R_i)$, as wished.
- T is a total relation by definition.
- T is a simulation relation. Assume that we have $(L, f), (L', f') \in$ 1-CList(R_0, R_1), $(L', f') \succ_{\text{col}} (L, f)$ and $(L, f) T \theta$ for some $\theta \in (H(R_0) \times H(R_1)) + 1$, in order to prove that there exists $\theta' \in (H(R_1) \times H(R_2)) + 1$ such that $(L', f') T \theta'$ and θ' is related to θ by $(\succ \otimes \succ) + 1$.
  - Assume that $L = \langle\rangle$. Then $f = \langle\rangle$, $(L', f') \succ_{\text{col}} (\langle\rangle, \langle\rangle)$ and $(\langle\rangle, \langle\rangle) T \theta$, hence $\theta = \top$. Since $L' \succ \langle\rangle$, then L' has exactly one element and $f' = \langle\rangle$. Hence we choose $\theta' = (\langle\rangle, \langle\rangle)$. In fact $(L', f') T \theta'$ holds by definition of T. Moreover, by definition of $\top$ and of $(\succ \otimes \succ) + 1$, the pair $(\langle\rangle, \langle\rangle)$ is related to $\top$ by $(\succ \otimes \succ) + 1$.
  - Assume that $L \neq \langle\rangle$. Since $L' \succ L \succ \langle\rangle$, then L' has two or more elements. By definition of $\succ_{\text{col}}$ and by $(L', f') \succ_{\text{col}} (L, f)$ we get $(L', f') = (L * \langle y \rangle, f * \langle c \rangle)$ for some $y \in S$ and some $c \in 2$. Moreover $\theta = (L_0, L_1)$, with L_0, L_1 defined as above (observe that if $f = \langle\rangle$ both of them are empty). Due to the definition of T there exist L_0' and L_1' such that $(L * \langle y \rangle, f * \langle c \rangle) T (L_0', L_1')$. By definition of L_0' and L_1', if $c = 0$ then $L_0' = L_0 * \langle y \rangle$ and $L_1' = L_1$, otherwise (if $c = 1$) $L_0' = L_0$ and $L_1' = L_1 * \langle y \rangle$. In both the cases $(L_0', L_1')(\succ \otimes \succ)(L_0, L_1)$, therefore $(L_0', L_1')(\succ \otimes \succ) + 1(L_0, L_1)$. We choose $\theta' = (L_0', L_1')$.

2. We simulate $H(R_0 \cup R_1, \succ)$ in the set $(\text{2-Tr(1-CList}(R_0, R_1)), \succ_1)$ of Erdős' trees. We define a simulation relation T between $H(R_0 \cup R_1)$ and 2-Tr(1-CList(R_0, R_1)) by: $L \; T \, \text{Tr}$ if and only if $|L| = |\,\text{Tr}\,|$. We need to prove that T is a total simulation. We show it by induction on the length of the list. First, observe that $\langle\rangle T \, \text{nil}$, since $|\langle\rangle| = \emptyset = |\,\text{nil}\,|$. Now assume that $L = \langle x_0, \ldots, x_{n-1} \rangle$, $L * \langle y \rangle \in H(R_0 \cup R_1)$, and that there exists $\text{Tr} \in$ 2-Tr(1-CList(R_0, R_1)) such that $|L| = |\,\text{Tr}\,|$ (induction hypothesis). We want to prove that there exists $\text{Tr}' \succ_1 \text{Tr}$ such that $|L * \langle y \rangle| = |\,\text{Tr}'\,| = |\,\text{Tr} \cup \{y\}|$. We reason by cases.

Assume that $\text{Tr} = \text{nil}$. Then Tr' shall be the leaf-tree with root y. Otherwise, if $\text{Tr} \neq \text{nil}$, then $n \geq 1$. Let x_0 be the root of Tr. Observe that, since L is a

(R_0,R_1)-colored list, then by definition we have $\forall j<n\exists i<2(yR_ix_j)$. Hence there exists a map $h:\{0,\dots,n-1\}\to\{0,1\}$ such that $\forall j<n(yR_{h(j)}x_j)$

Now we define a set $J\subseteq$ 1-CList(R_0,R_1) which contains some lists with last element y. We will prove that there is some list in J which we may add to Tr in order to obtain some $\mathrm{Tr}'\succ_1\mathrm{Tr}$ as required. J consists of all colored lists of the form $(\langle x_{j_0},\dots,x_{j_m-1},y\rangle,\langle h(j_0),\dots,h(j_{m-1})\rangle)$ for some $j_0,\dots,j_{m-1}<n$ (even with repetitions) and such that

$$(\langle x_{j_0},\dots,x_{j_{m-1}}\rangle,\langle h(j_0),\dots,h(j_{m-2})\rangle)\in\mathrm{Tr}\,.$$

Observe that

- J is not empty since $(\langle x_0,y\rangle,\langle h(x_0)\rangle)\in J$.
- J is some set of (R_0,R_1)-colored lists. Let $(M*\langle y\rangle,g*\langle h(j_{m-1})\rangle)\in J$, then $(M,g)\in\mathrm{Tr}$ and so $(M,g)\in$ 1-CList(R_0,R_1). Moreover $\forall k<m(yR_{h(j_k)}x_{j_k})$, by the choice of h, hence $(M*\langle y\rangle,g*\langle h(j_{m-1})\rangle)$ is a (R_0,R_1)-colored list.
- Given any $(M*\langle y\rangle,g*\langle h(j_{m-1})\rangle)\in J$, then we may find some

  $$(M'*\langle y\rangle,g'*\langle h(j_l)\rangle)\in J$$

  such that
  $$\mathrm{Tr}'=\mathrm{Tr}\cup\left\{(M'*\langle y\rangle,g'*\langle h(j_l)\rangle)\right\}\succ_1\mathrm{Tr}$$
  and $\mathrm{Tr}'\in$ 2-Tr(1-CList(R_0,R_1)). We prove it by inverse induction on the length of $(M,g)\in\mathrm{Tr}$. In fact Tr is finite then there is a finite maximal length. Let $(M*\langle y\rangle,g*\langle h(j_{m-1})\rangle)\in J$. If (M,g) has no extension in color $h(j_{m-1})$ in Tr, then we may extend Tr with $(M*\langle y\rangle,g*\langle h(j_{m-1})\rangle)$. Otherwise, (M,g) has some extension $(M*\langle x_{j_m}\rangle,g*\langle h(j_{m-1})\rangle)$ in Tr. In this case J contains

  $$(M'*\langle y\rangle,g'*\langle h(j_m)\rangle)=(M*\langle x_{j_m},y\rangle,f*\langle h(j_{m-1}),h(j_m)\rangle).$$

  Since $(M'*\langle y\rangle,g'*\langle h(j_m)\rangle)$ is associated to the list

  $$(M*\langle x_{j_m}\rangle,g*\langle h(j_{m-1})\rangle)\in\mathrm{Tr}$$

  that is longer than (M,g), we may apply the inductive hypothesis and we are done.

By the first observation there is some element in J; from it we find, due to the last observation, some
$$\mathrm{Tr}'\in\text{2-Tr(1-CList}(R_0,R_1))$$
such that $\mathrm{Tr}'\succ_1\mathrm{Tr}$ and $|\mathrm{Tr}'|=|\mathrm{Tr}|\cup\{y\}$, as we wished to show. □

As a corollary, if R_0,R_1 are H-well-founded then the set of Erdős' trees over R_0,R_1 is.

Corollary 2.2.9 (Erdős' trees are well-founded). *Let R_0, R_1 be binary H-well-founded relations on a set S.*

1. *The set $(\text{1-CList}(R_0, R_1), \succ_{col})$ of (R_0, R_1)-colored lists is well-founded.*

2. *The set $(\text{2-Tr}(\text{1-CList}(R_0, R_1)), \succ_1)$ of Erdős' trees over R_0, R_1 is well-founded.*

Proof. 1. $(H(R_0) \times H(R_1), \succ \otimes \succ)$ is well-founded by Proposition 1.2.19.1, since its components are. By Corollary 1.2.24, $(H(R_0) \times H(R_1), \succ \otimes \succ) + 1$ is well-founded. Since $(\text{1-CList}(R_0, R_1), \succ_{\text{col}})$ is simulated in $(H(R_0) \times H(R_1), \succ \otimes \succ) + 1$ by Lemma 2.2.8, then it is well-founded by Proposition 1.2.19.4.

2. By the previous point, $(\text{1-CList}(R_0, R_1), \succ_{\text{col}})$ is well-founded. We deduce that $(\text{2-Tr}(\text{1-CList}(R_0, R_1)), \succ_1)$ is well-founded by Lemma 2.2.7.

□

We are now ready to prove that H-well-founded relations are closed under finite unions. We first prove that $\emptyset$, the empty binary relation on S, is H-well-founded. $H(\emptyset)$ does not contain lists of length greater or equal than 2. Hence $H(\emptyset) = \{\langle x \rangle \mid x \in I\} \cup \{\langle\rangle\}$. $H(\emptyset)$ is $\succ$-well-founded since each $\langle x \rangle$ is $\succ$-minimal, and $\langle\rangle$ has height less or equal than 1. Thus, the empty relation is H-well-founded.

Theorem 2.2.10 (H-closure Theorem). *Let $k \in \mathbb{N}$. If $R_0, \ldots, R_{k-1}$ H-well-founded then $(R_0 \cup \cdots \cup R_{k-1})$ is H-well-founded.*

Proof. By induction on k. If $k = 0$, we have already proved that the empty relation is H-well-founded. Note also that the case $k = 1$ is trivial. Assume that $k > 0$, and that the thesis holds for any $h < k$. Then $R_0 \cup \cdots \cup R_{k-2}$ is H-well-founded. Thus, in order to prove that $(R_0 \cup \cdots \cup R_{k-1})$ is H-well-founded, it is enough to consider the case $k = 2$.

By Corollary 2.2.9.2, the set $(\text{2-Tr}(\text{1-CList}(R_0, R_1)), \succ_1)$ of the Erdős' trees over R_0, R_1 is well-founded. By Lemma 2.2.8, the set $(H(R_0 \cup R_1), \succ)$ is simulated in the set $(\text{2-Tr}(\text{1-CList}(R_0, R_1)), \succ_1)$ of Erdős' trees over R_0, R_1. Thus, $(H(R_0 \cup R_1), \succ)$ is well-founded by Proposition 1.2.19.3. □

Corollary 2.2.11. *Let $k \in \mathbb{N}$. $R_0, \ldots, R_{k-1}$ are H-well-founded if and only if $(R_0 \cup \cdots \cup R_{k-1})$ is H-well-founded.*

Proof. "⇒": Theorem 2.2.10.

"⇐": If R and R' are binary relations such that $R \subseteq R'$, then R' is H-well-founded implies that R is H-well-founded. In fact we have $H(R) \subseteq H(R')$; so by Proposition 1.2.19.6, if $(H(R'), \succ)$ is well-founded we get $(H(R), \succ)$ is well-founded. Since $\forall i < k(R_i \subseteq R_0 \cup \cdots \cup R_{k-1})$, then the thesis follows. □

2.2.5 Podelski and Rybalchenko's Termination Theorem

The H-closure Theorem is useful in order to intuitionistically prove some results about termination, since it contains the combinatorial fragment of Ramsey's Theorem for pairs required to prove them. In this last section we prove that the Termination Theorem recalled in Section 1.1 is intuitionistically valid. In order to do that we need the following basic lemma.

Lemma 2.2.12. *Given transition-based program with transition relation R. If $R \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded then $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded.*

Proof. For short put $R_1 = R \cap (\mathrm{Acc} \times \mathrm{Acc})$ and $R_2 = R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ Assume that R_1 is well-founded, we prove that

$$(y \text{ is } R_1\text{-well-founded}) \implies (y \text{ is } R_2\text{-well-founded})$$

by induction on y.

Recall that by Proposition 1.2.11

$$(y \text{ is } R_2\text{-well-founded}) \iff \forall z(zR_2y \implies z \text{ is } R_2\text{-well-founded}).$$

Assume that zR_2y, then we have two possibilities:

- zR_1y, then z is R_1-well-founded, hence by inductive hypothesis z is R_2-well-founded.
- $zR_2x \wedge xR_1y$ for some $x \in \mathrm{Acc}$. In fact x has to be in Acc since it can be reached after some R-steps from $y \in \mathrm{Acc}$. So x is R_1-well-founded and, by inductive hypothesis, it is R_2-well-founded. This implies that (since zR_2x) also z is R_2-well-founded.

So for each zR_2y , z is R_2-well-founded. This implies that y is R_2-well-founded. □

Intuitionistic proof of the Termination Theorem (Theorem 1.1.8). "$\Leftarrow$": Given a transition-based program with transition relation R, assume that $T = R_0 \cup \cdots \cup R_{k-1} \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ with $R_0, \ldots, R_{k-1}$ well-founded. Then R_i is H-well-founded by Proposition 2.2.3 and by the H-closure Theorem 2.2.10 also T is H-well-founded. Therefore, since H-well-foundedness is preserved between subsets, $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is H-well-founded. Moreover $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is transitive then, thanks to Proposition 2.2.3, we obtain $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded, so $\mathcal{R}$ is terminating.

"$\Rightarrow$": Let $\mathcal{R}$ be terminating then $R \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded. By Lemma 2.2.12 $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded. We choose $T = R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ and we are done. □

In Chapter 3 we will present another termination result which turns out to be provable by using H-closure, namely the SCT Theorem by Lee, Jones and Ben-Amram [58].

2.3 Proving termination with transition invariants of height ω

As shown in the previous section the H-closure Theorem intuitionistically derives the Termination Theorem. The goal of this section is to characterize the Termination Theorem in the case of relations with height ω. In order to do that we first strengthen H-closure as follows. If each R_i has ordinal height less or equal than α_i, then the $(R_0 \cup \cdots \cup R_{k-1})$-homogeneous sequences have ordinal height less or equal than $2^{\alpha_0 \oplus \cdots \oplus \alpha_{k-1}}$, where $\oplus$ is the natural sum of ordinals as defined in [18]. The proof uses a simulation of the ordering of $H(R_0 \cup \cdots \cup R_{k-1})$ in the inclusion ordering over the set of k-branching trees, whose branches are decreasing sequences in $R_0 \cup \cdots \cup R_{k-1}$.

As second step, we prove that given a transition relation which is the graph of a partial recursive map restricted to a primitive recursive domain, and given a disjunctively well-founded transition invariant whose relations are primitive recursive and have height ω, we may compute the number of its steps and the final state by primitive recursive functions.

For short we say that a transition invariant has height ω if it is composed of primitive recursive relations with height ω. Moreover we call *Termination Theorem for height ω* the statement “a binary relation for which there exists a transition invariant of height ω is well-founded”.

2.3.1 Labelling an Erdős’ tree

In Section 2.2.1 we proved that each one-step extension in a $H(R_0 \cup \cdots \cup R_{k-1})$ may be simulated as an one-step extension of some Erdős’ tree on $(R_0, \ldots, R_{k-1})$, that is, as adding a child to some $(R_0, \ldots, R_{k-1})$-colored list of the tree. From the well-foundedness of the set k-Tr(1-CList$(R_0, \ldots, R_{k-1})$) of Erdős’ trees we derived the H-closure Theorem. Now we want to strengthen this analysis by using the ordinal bound for k-ary trees defined in Section 2.1 to derive an ordinal bound for $H(R_0 \cup \cdots \cup R_{k-1})$, in the case $R_0, \ldots, R_{k-1}$ all have height ω.

So let now consider only the Erdős’ trees (in k-Tr(1-CList$(R_0, \ldots, R_{k-1})$)), following the notation of Subsection 2.1.3. From now on we assume that $R_0, \ldots, R_{k-1}$ have height ω. We may associate to each node the k-tuple $(y_0, \ldots, y_{k-1})$ of integer heights of the node with respect to the relations $R_0, \ldots, R_{k-1}$. Thus, it is enough to compute an upper bound to the ordinal height of an Erdős’ tree Tr with respect to $\succ_1$ in the following case: the set of nodes of Tr is $S = \omega \times \omega \cdots \times \omega$ (k-many times), and for all $h \in k$ $(y_0, \ldots, y_{k-1}) R_h (y'_0, \ldots, y'_{k-1})$ is equivalent to $y_h < y'_h$. If we are able to give an upper bound in this case, we are able to give an upper bound whenever $R_0, \ldots, R_{k-1}$ have height ω. There is no obvious guess about such property: if $(y_0, \ldots, y_{k-1})$ is a node and $(y'_0, \ldots, y'_{k-1})$ is the child number h of the node, all we do know is that $y_h > y'_h$. The remaining components of $(y_0, \ldots, y_{k-1})$ and $(y'_0, \ldots, y'_{k-1})$ may be in any relation. In fact, it is not even evident that all branches are finite: this result requires, and is immediately equivalent to, Ramsey’s Theorem for pairs and for the particular $R_0, \ldots, R_{k-1}$ we chose.

Our first task is to label the nodes of any $(R_0,\dots,R_{k-1})$-list in a decreasing way, by ordinals $< (\omega\cdot k)$. To this aim, we first introduce the notion of i-node. In order to follow the next definition, recall that each node y, including the root, is some k-tuple $y=(y_0,\dots y_{k-1})$ of natural numbers.

Definition 2.3.1. Let Tr be in k-Tr(1-CList$(R_0,\dots,R_{k-1})$), $y=(y_0,\dots,y_{k-1})$ be a node of Tr, $i\in\mathbb{N}$ and $h_0,\dots,h_{i-1}\in k$.

1. $(y_0,\dots,y_{k-1})$ is an i-node of Tr with respect to $h_0,\dots,h_{i-1}$ if the branch from the root to y has exactly i-many distinct colors $h_0,\dots,h_{i-1}\in k$.

2. Assume that y is an i-node with respect to $h_0,\dots,h_{i-1}$. For any $j\in i$, we denote by $y^{h_j}=(p_0^{h_j},\dots,p_{k-1}^{h_j})$ the lowest (the furthest to the root) proper ancestor of $y=(y_0,\dots,y_{k-1})$ in the branch from the root to the node, which is followed by an edge in color h_j.

Every node is an i-node for some $i\in k+1$, and $i>0$ if and only if the node is not the root. By definition, if a node z is followed by an edge of color h then all descendants of z are smaller with respect to R_h. Thus, if $y=(y_0,\dots,y_{k-1})$ is an i-node of Tr with respect to $h_0,\dots,h_{i-1}$, then:

- for any proper ancestor z of y we have $yR_{h_j}z$, for some $j\in i$;
- for any $j\in i$, there exists an ancestor z of y such that $yR_{h_j}z$.

Then we may label the nodes in a decreasing way with ordinals $<\omega\cdot k$, as follows.

Definition 2.3.2. (Labelling α). Let $\mathrm{Tr}\in k$-Tr(1-CList$(R_0,\dots,R_{k-1})$) and $(z_0,\dots,z_{k-1})$ be a node of Tr:

- if $(z_0,\dots,z_{k-1})$ is the root of the tree, then
$$\alpha((z_0,\dots,z_{k-1}))=\max_{i\in k}\{z_i+1\}\oplus\omega*(k-1);$$
- if, for some $j>0$, $(z_0,\dots,z_{k-1})$ is a j-node with respect to $h_0,\dots h_{j-1}$
$$\alpha((z_0,\dots,z_{k-1}))=p_{h_0}^{h_0}\oplus\dots\oplus p_{h_{j-1}}^{h_{j-1}}\oplus\omega*(k-j).$$

We may observe that each node has label less than the one of its father.

Lemma 2.3.3. *Let* $\mathrm{Tr}\in k$-Tr(1-CList$(R_0,\dots,R_{k-1})$), *then the labelling* α *is decreasing with respect to the father/child relation.*

Proof. Let $(z_0,\dots,z_{k-1})$ be a node of the tree and assume that $(y_0,\dots,y_{k-1})$ is its father, then we have three possibilities.

- If the father is the root then there exists $j \in k$ such that:

$$\begin{aligned}\alpha((z_0,\dots,z_{k-1})) &= y_j \oplus \omega * (k-1) \\ &< \max_{i\in k}\{y_i+1\} \oplus \omega * (k-1) = \alpha((y_0,\dots,y_{k-1}))\end{aligned}$$

- If, for some $j > 0$, the father is a j-node and the child is still a j-node, then the child is connected to its father with the relation R_{h_i} for some $i \in j$. Hence the lowest h_i-ancestor of the child is its father (whose h_i component is less than the one of its h_i-ancestor). Then the label decreases.

- If, for some $j > 0$, the father is a j-node and the child is a $j+1$-node then the labels decreases since we have an "infinite component" which becomes finite.

□

Hence, for any $\mathrm{Tr} \in k\text{-Tr}(1\text{-CList}(R_0,\dots,R_{k-1}))$, $\alpha : \mathrm{Tr} \to \omega \cdot k$ is a decreasing labelling. Thus, following the notation of Subsection 2.1.2, we can think to Tr as an element in $k\text{-Tr}(\mathcal{L}_{\omega\cdot k})$[3]. By Proposition 2.1.10 we get the following.

Lemma 2.3.4. *If* Tr', Tr *are Erdős' trees and* $\mathrm{Tr}' \succ_1 \mathrm{Tr}$ *then* $h_k(\mathrm{Tr}',\omega\cdot k) < h_k(\mathrm{Tr},\omega\cdot k)$.

Moreover if each relation has height ω we have

$$h_k(\cdot,\omega\cdot k) : k\text{-Tr}(1\text{-CList}(R_0,\dots,R_{k-1})) \to \omega^k + 1;$$

where $h_k(\mathrm{Nil},\omega\cdot k) = \omega^k$ and for each $\mathrm{Tr} \neq \mathrm{Nil}$ $h_k(\mathrm{Tr},\omega\cdot k) < \omega^k$.

Therefore we have a primitive recursive function from the set of Erdős' trees over $R_0,\dots,R_{k-1}$ in ω^k+1 such that if $\mathrm{Tr}' \succ_1 \mathrm{Tr}$ then $h_k(\mathrm{Tr}',\omega\cdot k) < h_k(\mathrm{Tr},\omega\cdot k)$. From the fact that each primitive recursive decreasing sequence of ordinals of ω^k has some primitive recursive bound, and the fact that we may embed any transitive subset of $R_0 \cup \dots \cup R_{k-1}$ in the set of Erdős' trees over $R_0,\dots,R_{k-1}$, we derive our primitive recursive bounds about the Termination Theorem. For short, from now on we use $h_k(\mathrm{Tr})$ instead of $h_k(\mathrm{Tr},\omega\cdot k)$.

2.3.2 Bounding the Termination Theorem for height ω

In order to state our result about primitive recursive bounds we need the following definition.

Definition 2.3.5. Let D be any subset of S and R any binary relation on S. R is the graph of a primitive recursive function restricted to a primitive recursive domain D if

1. D is primitive recursive and

[3] In fact Tr is isomorphic to a tree in $\mathrm{Tr}' \in k\text{-Tr}(\mathcal{L}_{\omega\cdot k})$. By using α we can straightforwardly define a bijection $\phi : \mathrm{Tr} \to \mathrm{Tr}'$ such that for any $\lambda,\mu \in \mathrm{Tr}$: $\lambda \succ_1 \mu \iff \phi(\lambda) \succ_1 \phi(\mu)$.

2. R is the graph of a primitive recursive function $f : S \to S$ restricted to D: i.e.

$$R = \{(x, f(x)) \mid x \in D\}.$$

We may formally state our main result as follows: given a transition relation which is the graph of a primitive recursive function restricted to a primitive recursive domain such that there exists a disjunctively well-founded transition invariant whose relations are primitive recursive and have height ω, there exists a primitive recursive bound for the length of the computations.

Finding a primitive recursive bound with the lexicographic order

To reach our goal we need an auxiliary result about the lexicographical order of $\mathbb{N}^k$. We prove that each primitive recursive decreasing sequence of ordinals of ω^k has some primitive recursive bound. This fact is well-known (e.g. see [81]), but for the sake of completeness we include a proof. We prove first the existence of a primitive recursive bound in the case $k = 1$, then in the general case.

Lemma 2.3.6. *If $\phi : \mathbb{N} \to \mathbb{N}$ is primitive recursive and there exist $m, n \in \mathbb{N}$ such that $m < n$ and $\phi(m) < \phi(n)$ then $\exists p \in [m, n-1](\phi(p) < \phi(p+1))$.*

Proof. Since the statement is decidable we may argue by contradiction and de Morgan's Law. Assume that $\forall p \in [m, n-1](\phi(p) \geq \phi(p+1))$, then $\phi(m) \geq \phi(n)$. Contradiction. □

We denote with $\preccurlyeq_k$ the lexicographic order of $\mathbb{N}^k$. Given a function g, define $g^n(x) = g \circ g^{n-1}(x)$. We may observe that if g is primitive recursive, also $H(n,x) = g^{n+1}(x)$ is. In fact:

$$H(n,x) = \begin{cases} x & n = 0 \\ g(H(n-1,x)) & \text{otherwise.} \end{cases}$$

Lemma 2.3.7. *For each $\phi : \mathbb{N} \to \mathbb{N}^k$ primitive recursive, there exists $g : \mathbb{N} \to \mathbb{N}$ primitive recursive such that*

$$\forall n \exists m \in [n, g(n)](\phi(m) \preccurlyeq_k \phi(m+1)).$$

Proof. By induction on k. If $k = 1$ we put $g(n) = n + \phi(n) + 1$. Let $n \in \mathbb{N}$, we want to prove that there exists $m \in [n, n + \phi(n)]$ such that $\phi(m) \leq \phi(m+1)$. Suppose, by contradiction, $\forall m \in [n, n + \phi(n) + 1]\phi(m) > \phi(m+1)$. Then we obtain a sequence of $(\phi(n)+2)$-many decreasing natural numbers from $\phi(n)$. Contradiction.

Assume that the thesis holds for k. We prove it for $k+1$. Let

$$\begin{aligned} \phi : \mathbb{N} &\longrightarrow \mathbb{N}^{k+1} \\ n &\longmapsto (\phi_1(n), \phi_k(n)) \end{aligned}$$

be primitive recursive, where $\phi_1 : \mathbb{N} \to \mathbb{N}$ and $\phi_k : \mathbb{N} \to \mathbb{N}^k$. Then also ϕ_k is primitive recursive, hence by inductive hypothesis there exists g_k such that

$$\forall n \exists m \in [n, g_k(n)](\phi_k(m) \preccurlyeq_k \phi_k(m+1)).$$

Put $H(0,x) = x$ and, for any $n > 0$ $H(n,x) = g_k(H(n-1,x)+1)$. For any $n \in \mathbb{N}$, define $g = H(\phi_1(n)+2, n)$. We want to prove that

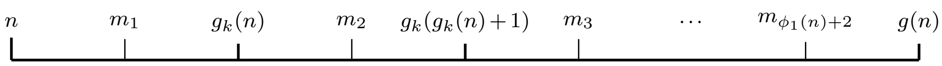
$$\forall n \exists m \in [n, g_k^{\phi_1(n)+2}(n)](\phi(m) \preccurlyeq_{k+1} \phi(m+1)).$$

If there exist $i < j \in [n, g_k^{\phi_1(n)+2}(n)]$ such that $\phi_1(i) < \phi_1(j)$ then by Lemma 2.3.6 we obtain $\exists p \in [i, j-1]$ $\phi_1(p) < \phi_1(p+1)$. It follows that $\phi(p) \preccurlyeq \phi(p+1)$ in the lexicographic order and we are done. Otherwise assume that

$$\forall i,j \in [n, g_k^{\phi_1(n)+2}(n)](\phi_1(i) \geq \phi_1(j)).$$

By applying the inductive hypothesis over ϕ_k and the disjoint intervals

$$([n, g_k(n)], [g_k(n)+1, g_k(g_k(n)+1)], \dots),$$



We obtain that there are some $m_1 < m_2 < \dots < m_{\phi_1(n)+2}$ such that $\phi_k(m_i) \preccurlyeq_k \phi_k(m_i+1)$. Moreover, by assumption on $[n, g_k^{\phi_1(n)+2}(n)]$ we have $\phi_1(n) \geq \phi_1(m_1) \geq \phi_1(m_2) \geq \dots \geq \phi_1(m_{\phi_1(n)+2})$. Then there exists $i \in [1, \phi_1(n)+2]$ such that $\phi_1(m_i) = \phi_1(m_{i+1})$. Hence

$$\phi_1(m_i) = \phi_1(m_i+1) = \phi_1(m_{i+1}).$$

Since, by inductive hypothesis $\phi_k(m_i) \preccurlyeq_k \phi_k(m_i+1)$, we get $\phi(m_i) \preccurlyeq_{k+1} \phi(m_i+1)$. $\square$

From $H(R_0 \cup \dots \cup R_{k-1})$ to Erdős' trees

We want to define a primitive recursive function from $H(R_0 \cup \dots \cup R_{k-1})$ to the Erdős' trees, as a preliminary step in order to find a primitive recursive bound for the number of steps of a sequence in $H(R_0 \cup \dots \cup R_{k-1})$.

Let $\mathrm{clist}(x) = \langle \langle x \rangle, \langle \rangle \rangle$ be the colored list including only x. The first step is to define a primitive recursive function l, taking an Erdős' tree Tr, some y not in Tr, and returning some branch $b * \langle y \rangle$ which we may add to Tr so as to obtain an Erdős' tree.

$$l(\mathrm{Tr}, y) = \begin{cases} \mathrm{clist}(y) & \text{if } \mathrm{Tr} = \mathrm{Nil} \\ \mathrm{clist}(r) *_i l(\mathrm{Tr}_i, y) & \text{if } r \text{ is the root of } \mathrm{Tr},\ i = \mu j \in k(y R_j r). \end{cases}$$

Let define a primitive recursive function E from $H(R_0 \cup \cdots \cup R_{k-1})$ to the set of the Erdős' trees as follows.

$$E(\langle x_0, \ldots, x_{n-1} \rangle) = \begin{cases} \text{nil} & n = 0 \\ E(\langle x_0, \ldots, x_{n-2} \rangle) \cup \{l(E(\langle x_0, \ldots, x_{n-2} \rangle), x_{n-1})\} & n > 0; \end{cases}$$

Lemma 2.3.8. *If $L' \succ L$ then $E(L') \succ_1 E(L)$.*

Proof. If $L' \succ L$ then $L' = L * \langle y \rangle$ for some y. If follows that $E(L') = l(E(L), y)$, i.e. we add a leaf to a leaf of $E(L)$. Then $E(L') \succ_1 E(L)$. □

A primitive recursive bound

We define a primitive recursive decreasing map $f^* : H(R_0 \cup \cdots \cup R_{k-1}) \to \omega^k + 1$, from $(R_0 \cup \cdots \cup R_{k-1})$-homogeneous sequences to ordinals, by $f^*(s) = h_k(E(s))$.

Lemma 2.3.9. *(f^* is decreasing) If $L' \succ L$ then $f^*(L') < f^*(L)$.*

Proof. By applying Lemma 2.3.8 $E(L') \succ_1 E(L)$. By Lemma 2.3.4

$$f^*(L') = h_k(E(L')) < h_k(E(L)) = f^*(L).$$ □

Let $\mathcal{R}$ be a transition-based program with transition relation R. Let t be a computation which behaves like R until it reaches a final state and then it repeats this state, i.e. if x is a final state $t(x) = x$.

Theorem 2.3.10. *Assume that $\mathcal{R}$ is such that $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) = R_0 \cup \cdots \cup R_{k-1}$, where*

1. *the complement of the set of the final states of R ($S \setminus F$) is a primitive recursive set;*

2. *R is the graph of a primitive recursive function $f : S \setminus F \to S$: i.e.*

$$R = \{(x, f(x)) \mid x \in S \setminus F\}.$$

3. *$R_0, \ldots, R_{k-1}$ are primitive recursive relations and have height ω.*

Then there exists $g' : S \to \mathbb{N}$ such that $t^{g'(s)}(s) = t^{g'(s)+1}(s)$.

Proof. Observe that due to hypotheses 1 and 2 we have that t is primitive recursive, while due to the third one f^* is primitive recursive.

Let $\phi(x) = f^*(\langle s_0, t^1(s_0), \ldots, t^x(s_0) \rangle)$. Then $\phi : \mathbb{N} \to \mathbb{N}^k$, since the input list for f^* cannot be the empty list. Moreover $\phi(x)$ is primitive recursive. In fact let

$$\theta(x) = \begin{cases} \langle s_0 \rangle & \text{if } x = 0; \\ \theta(x-1) * t^x(s_0) & \text{otherwise}. \end{cases}$$

It follows that ϕ is a composition of primitive recursive functions, so is primitive recursive: $\phi(x) = f^*(\theta(x))$.

Thanks to Lemma 2.3.7 there exists g primitive recursive such that

$$\forall n \exists m \in [n, g(n)](\phi(m) \preccurlyeq_k \phi(m+1)).$$

Put $n = 0$. Then there exists $m \in [0, g(0)]$ such that

$$f^*(\theta(m)) = \phi(m) \preccurlyeq_k \phi(m+1) = f^*(\theta(m+1)).$$

Observe that, since $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) = R_0 \cup \cdots \cup R_{k-1}$ and $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is transitive, for any $m \in \mathbb{N}$ if $t^{m-1}(s_0)$ is not a final state then $\theta(m) \in H(R_0 \cup \cdots \cup R_{k-1})$, and if $t^m(s_0)$ is not a final state then $\theta(m+1) \succ_{\mathrm{col}} \theta(m)$.

By Lemma 2.3.9 we obtain that if $t^m(s_0)$ is not a final state, then

$$f^*(\theta(m)) \succ_k f^*(\theta(m+1)),$$

contradicting

$$f^*(\theta(m)) \preccurlyeq_k f^*(\theta(m+1)).$$

Thus, $t^m(s_0)$ is a final state. □

2.3.3 Vice versa

The vice versa of Theorem 2.3.10 would be: for all primitive recursive maps f, any program with primitive recursive transition relation which computes f may be proved to be terminating using primitive recursive transition invariants of height ω. We suspect that this is false: the program could contain some part computing some non-primitive recursive function g, then could erase the result of g. However, we may prove a weak form of vice versa of Theorem 2.3.10: for any primitive recursive map f, there is some program with primitive recursive transition relation which computes f and for which there exists a primitive recursive transition invariant of height ω.

Theorem 2.3.11. *Let f be a primitive recursive function, then there exists a program $\mathcal{R}$ which evaluates f such that:*

- *its transition relation is a graph of a primitive recursive function restricted to a primitive recursive domain;*
- *$\mathcal{R}$ has a disjunctively well-founded transition invariant made of primitive recursive relations of height ω.*

Proof. By induction on primitive recursive functions. In every step we define a transition-based program $\mathcal{R} = (S, I, R)$ which satisfies the thesis of the theorem and such that each state consists of a sequence whose last element belongs to $\mathbb{N} \cup \{\star\}$. Moreover if the last element of a state s is some $n \in \mathbb{N}$, then s is a final state and n is the value of the function we are computing.

- Constant function 0. If $\forall x(f(x)=0)$, define $S_f=\{\langle x,y\rangle \mid x\in\mathbb{N}, y\in\{0\}\cup\{\star\}\}$, $I_f=\{\langle x,\star\rangle \mid x\in\mathbb{N}\}$ and

$$\langle x',y'\rangle R\langle x,y\rangle \iff y=\star\wedge y'=0\wedge x'=x.$$

  The program $\mathcal{R}=(S_f,I_f,R_f)$ computes f. Since R has height ω, the thesis follows.

- Successor function. If $\forall x(f(x)=x+1)$, define $S_f=\{\langle x,y\rangle \mid x\in\mathbb{N}, y\in\mathbb{N}\cup\{\star\}\}$, $I_f=\{\langle x,\star\rangle \mid x\in\mathbb{N}\}$ and

$$\langle x',y'\rangle R\langle x,y\rangle \iff y=\star\wedge y'=x+1\wedge x'=x.$$

  The program $\mathcal{R}=(S_f,I_f,R_f)$ computes f and the thesis follows.

- Projection function. Let $i\in n$. If $\forall x_0,\dots,x_{n-1}(f(x_0,\dots,x_{n-1})=x_i)$, define $S_f=\{\langle\bar{x},y\rangle \mid \bar{x}\in\mathbb{N}^n, y\in\mathbb{N}\cup\{\star\}\}$, $I_f=\{\langle\bar{x},\star\rangle \mid \bar{x}\in\mathbb{N}^n\}$ and

$$\langle\bar{x}',y'\rangle R\langle\bar{x},y\rangle \iff y=\star\wedge y'=x_i\wedge\bar{x}'=\bar{x}.$$

  Once again the program $\mathcal{R}=(S_f,I_f,R_f)$ computes f and the thesis follows.

- Composition. Let

$$\forall x_0,\dots,x_{n-1}(f(x_0,\dots x_{n-1})=g_0(g_1(x_0,\dots,x_{n-1}),\dots g_k(x_0,\dots,x_{n-1}))).$$

  By induction hypothesis assume that for any $i\in k+1$ there exists a program $\mathcal{R}_i=(S_i,I_i,R_i)$ which computes g_i and for which there exists a transition invariant disjunctively well-founded T_i of height ω. Then we define $\mathcal{R}_f$ as follows. Without loss of generality for any $i\in k+1$ we represent a state in S_i as $\langle s,r\rangle$, where $r\in\mathbb{N}\cup\{\star\}$ is the last element. Moreover let $\mathrm{In}_i(\bar{x})$ be the initial state in I_i which corresponds to the input $\bar{x}$. Set

$$S_f=\{\langle a,\langle s_1,r_1\rangle,\dots,\langle s_k,r_k\rangle,s_0,r_0\rangle \mid a\in k+1,\forall i\in k+1(s_i,r_i)\in S_i\},$$

$$I_f=\{\langle 0,\mathrm{In}_1(\bar{x}),\dots,\mathrm{In}_k(\bar{x}),s_0,\star\rangle \mid \bar{x}\in\mathbb{N}^n,\langle s_0,\star\rangle\in I_0\}\,[4].$$

  And define $\langle a',\langle s_1',r_1'\rangle,\dots,\langle s_k',r_k'\rangle,s_0',r_0'\rangle R_f\langle a,\langle s_1,r_1\rangle,\dots,\langle s_k,r_k\rangle,s_0,r_0\rangle$ if either:

  - $a'=a<k$ and for any $i\neq a+1$ $\langle s_i',r_i'\rangle=\langle s_i,r_i\rangle$ and

$$\langle s_{a+1}',r_{a+1}'\rangle R_{a+1}\langle s_{a+1},r_{a+1}\rangle;$$

  - $a<k-1$ and $a'=a+1$ and for any $i\le k$ $\langle s_i',r_i'\rangle=\langle s_i,r_i\rangle$ and $r_{a+1}\neq\star$;
  - $a=k-1$ and $a'=k$ and for any $i\in[1,k]$ $\langle s_i',r_i'\rangle=\langle s_i,r_i\rangle$ and $r_k\neq\star$ and $\langle s_0',r_0'\rangle=\mathrm{In}_0(\langle r_1,\dots,r_k\rangle)$;

[4] Actually s_0 can assume any value at the beginning.

– $a = a' = k$ and for any $i \in [1,k]$ $\langle s'_i, r'_i\rangle = \langle s_i, r_i\rangle$ and $\langle s'_0, r'_0\rangle R_0 \langle s_0, r_0\rangle$.

It is straightforward to prove that $\mathcal{R} = (S_f, I_f, R_f)$ computes f. Define

$$T = \Big\{(\langle a, \langle s_1, r_1\rangle, \dots, \langle s_k, r_k\rangle, s_0, r_0\rangle, \langle a', \langle s'_1, r'_1\rangle, \dots, \langle s'_k, r'_k\rangle, s'_0, r'_0\rangle) \\ \mid a < a' < k+1\Big\}$$

and for any $i \in [1,k]$

$$T_i^\star = \Big\{(\langle i-1, \langle s_1, r_1\rangle, \dots, \langle s_k, r_k\rangle, s_0, \star\rangle, \langle i-1, \langle s'_1, r'_1\rangle, \dots, \langle s'_k, r'_k\rangle, s_0, \star\rangle) \\ \mid \forall j \neq i \langle s'_j, r'_j\rangle = \langle s_j, r_j\rangle,\ \langle s'_i, r'_i\rangle T_i \langle s_i, r_i\rangle\Big\}$$

and

$$T_0^\star = \Big\{(\langle k, \langle s_1, r_1\rangle, \dots, \langle s_k, r_k\rangle, s_0, r_0\rangle, \langle k, \langle s_1, r_1\rangle, \dots, \langle s_k, r_k\rangle, s'_0 r'_0\rangle) \\ \mid \langle s'_0, r'_0\rangle T_0 \langle s_0, r_0\rangle\Big\}.$$

Hence $T_f = T \cup \bigcup\{T_i^\star \mid i \in k+1\}$ is a transition invariant disjunctively well-founded of height ω for $\mathcal{R}$. In fact if

$$(\langle a, \bar{s}\rangle, \langle a', \bar{s}'\rangle) \in R_f^+ \cap (\mathrm{Acc} \times \mathrm{Acc}),$$

we have one of the following possibilities:

– $a = a'$. Then, by unfolding definitions, $(\langle a, \bar{s}\rangle, \langle a, \bar{s}'\rangle) \in T_a^\star$.
– $a < a'$. Hence $(\langle a, \bar{s}\rangle, \langle a', \bar{s}'\rangle) \in T$.

This proves that it is a transition invariant. It has height ω since T_i has height ω for any $i < k+1$ and T has height ω as well. Moreover it is straightforward to prove directly that R_f is a graph of a primitive recursive function restricted to a primitive recursive domain.

- Primitive Recursion. For any $x_0, \dots, x_{k-1}$, let

$$\begin{cases} f(0, x_0, \dots, x_{k-1}) = g_0(x_0, \dots, x_{k-1}), \\ f(S(y), x_0, \dots, x_{k-1}) = g_1(y, f(y, x_0, \dots, x_{k-1}), x_0, \dots x_{k-1}). \end{cases}$$

By induction hypothesis for g_0 and g_1 there exist two programs $\mathcal{R}_0 = (S_0, I_0, R_0)$ and $\mathcal{R}_1 = (S_1, I_1, R_1)$ equipped with disjunctively well-founded transition invariants T_0, T_1 of height ω. As above assume that for any $i \in 2$, any state in S_i is represented as a pair $\langle s, r\rangle$ whose last element belongs to $\mathbb{N} \cup \{\star\}$ and that $\mathrm{In}_i(\bar{x})$ is the initial state in I_i which corresponds to input $\bar{x}$. Set

$$S_f = \Big\{\langle \bar{x}, z, y, s_0, r_0, s_1, r_1, r\rangle \mid z, y \in \mathbb{N}, \bar{x} \in \mathbb{N}^k, \forall i \in 2 \langle s_i, r_i\rangle \in S_i, r \in \mathbb{N} \cup \{\star\}\Big\},$$

$$I_f = \left\{ \langle \bar{x}, 0, y, s_0, r_0, s_1, r_1, \star \rangle \mid y \in \mathbb{N}, \bar{x} \in \mathbb{N}^k, \forall i \in 2 \langle s_i, r_i \rangle \in I_i \right\}^{5}.$$

The state has a counter z. When $z < y$, we already applied g_1 for z times to g_0. When $z = y$ the computation is over. We define

$$\langle \bar{x}, z', y, s_0', r_0', s_1', r_1', r' \rangle R_f \langle \bar{x}, z, y, s_0, r_0, s_1, r_1, r \rangle$$

if either:

- Computation step of g_0. $z' = z = 0, \langle s_0', r_0' \rangle R_0 \langle s_0, r_0 \rangle$, $\langle s_1', r_1' \rangle = \langle s_1, r_1 \rangle$ and $r' = r = \star$;
- Initial step of the first computation of g_1. $z' = z + 1 = 1$, $r_0 \in \mathbb{N}$, $\langle s_1', r_1' \rangle = \mathrm{In}_1(\langle z, r_0, \bar{x} \rangle)$, $\langle s_0', r_0' \rangle = \langle s_0, r_0 \rangle$ and $r' = r = \star$.
- Computation step of g_1. $z' = z > 0$, $\langle s_1', r_1' \rangle R_1 \langle s_1, r_1 \rangle$, $\langle s_0', r_0' \rangle = \langle s_0, r_0 \rangle$ and $r' = r = \star$;
- Initial step of the any computation of g_1 but the first one. $1 < z' = z + 1 < y$, $r_1 \in \mathbb{N}$, $\langle s_1', r_1' \rangle = \mathrm{In}_1(\langle z, r_1, \bar{x} \rangle)$, $\langle s_0', r_0' \rangle = \langle s_0, r_0 \rangle$ and $r' = r = \star$.
- End. $z' = z + 1 = y$, $r_1 \in \mathbb{N}$, for all $i \in 2$ $\langle s_i', r_i' \rangle = \langle s_1, r_i \rangle$, $r = \star$ and $r' = r_1$.

It is straightforward to prove that $\mathcal{R} = (S_f, I_f, R_f)$ computes f. Define

$$T = \Big\{ (\langle \bar{x}, z, y, s_0, r_0, s_1, r_1, r \rangle, \langle \bar{x}, z+1, y, s_0', r_0', s_1', r_1', r' \rangle) \\ \mid y \in \mathbb{N}, z \leq y, \forall i \in 2 (\langle s_i, r_i \rangle \in S_i), r, r' \in \mathbb{N} \cup \{\star\} \Big\}.$$

$$T_0^\star = \left\{ (\langle \bar{x}, 0, y, s_0, r_0, s_1, r_1, \star \rangle, \langle \bar{x}, 0, y, s_0', r_0', s_1, r_1, \star \rangle) \mid \langle s_0', r_0' \rangle T_0 \langle s_0, r_0 \rangle \right\},$$

$$T_1^\star = \Big\{ (\langle \bar{x}, z, y, s_0, r_0, s_1, r_1, \star \rangle, \langle \bar{x}, z, y, s_0, r_0, s_1', r_1', \star \rangle) \\ \mid \langle s_1', r_1' \rangle T_1 \langle s_1, r_1 \rangle, z < y \in \mathbb{N} \Big\}.$$

We claim that $T_0^\star \cup T_1^\star \cup T$ is a transition invariant. In fact if

$$(\langle \bar{x}, z, \bar{s} \rangle, \langle \bar{x}, z', \bar{s}' \rangle) \in R_f^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$$

then we have one of the following possibilities.

- If $z = z' = 0$, then by unfolding definitions $(\langle \bar{x}, z, \bar{s} \rangle, \langle \bar{x}, z', \bar{s}' \rangle) \in T_0^\star$.
- If $z = z' > 0$, then by unfolding definitions $(\langle \bar{x}, z, \bar{s} \rangle, \langle \bar{x}, z', \bar{s}' \rangle) \in T_1^\star$.
- If $y \geq z' > z$, then $(\langle \bar{x}, z, \bar{s} \rangle, \langle \bar{x}, z', \bar{s}' \rangle) \in T$.

We conclude it is a transition invariant. $T_0^* \cup T_1^* \cup T$ has height ω since T_0 has, $T_1^\star$ is the union of relations of height ω since T_1 has, and T has height ω as well.

[5] Also in this case s_1 can assume any value at the beginning.

Moreover, as above, it is straightforward to prove directly that R_f is a graph of a primitive recursive function restricted to a primitive recursive domain. □

2.4 Which are the functions proved to be terminating by Terminator?

In the previous section we proved that the functions having at least one implementation which may be proved terminating by using a disjunctively well-founded transition invariant whose relations are primitive recursive and have height ω are exactly the primitive recursive functions. This result, however, does not apply to the Termination algorithm directly. In this section we consider the algorithm and we prove that the set of functions having at least one implementation proved to be terminating by Terminator's algorithm presented in [22] is exactly the set of primitive recursive functions, as expected.

Throughout this section we use the notation adopted by Cook, Podelski and Rybalchenko in [22].

2.4.1 Overview on Terminator Algorithm

In order to reach our goal, we need to recall how Terminator is defined. The algorithm takes as input a transition-based program $\mathcal{R}$ and works on the control-flow graph representation of $\mathcal{R}$, i.e. a finite state automata associated to the program. For each cycle π in this representation, which is a semantic cycle of $\mathcal{R}$, Terminator checks whether there exists a linear ranking function for π, by using the method presented in [76]. If there is not such function then Terminator stops and returns π as counterexample. Otherwise we add the ranking relation obtained from the linear ranking function to the partial transition invariant (which at the first step is empty) and we go on with another cycle until we build a transition invariant. However, as noticed in [22], the number of semantic cycles of $\mathcal{R}$ is infinite. Cook, Podelski and Rybalchenko provided an argument which guarantees to build a full transition invariant by considering only a finite number of semantic cycles. In order to do that, they introduced the abstraction function. Given a set of transition predicates (i.e. atomic conditions over the states) P and a cycle $\pi = \tau_0 \dots \tau_{n-1}$, define

$$\alpha_P(\pi) = \bigwedge \{p \in P \mid p \supseteq \rho_\pi\},$$

where $\rho_\pi = \rho_{\tau_0} \circ \dots \circ \rho_{\tau_{n-1}}$ and ρ_τ is the transition relation associated to τ. Observe that $\rho_\pi \subseteq \alpha_P(\pi)$ for all π, therefore, as shown in [22], by combining with the Termination Theorem we have the following.

Theorem 2.4.1. *The program is terminating if there exists a finite set of well-founded relations $T = \{R_i \mid i \in k\}$ such that for any cyclic path π $\exists i \in k(\alpha_P(\pi) \subseteq R_i)$.*

We present the Terminator algorithm as defined in [22]. We consider only binary relations defined as conjunction of a set of atomic formula in the language considered in

[22]. Given a binary relation R, let $\mathrm{Preds}(R)$ be the set of atomic formulas defining R. The Terminator algorithm is:

$T = \emptyset$
$P = \emptyset$
repeat
if exists $\pi = \tau_0, \ldots, \tau_{n-1}$ s.t. $\alpha_P(\pi) \nsubseteq R$ for any $R \in T$ **then**
  if exists $R \in T$ such that $\rho_\pi \subseteq R$ **then** [refinement step]
      $P_p = \bigcup \{ \mathrm{Preds}(\rho_{\tau_i} \circ \cdots \circ \rho_{\tau_n}) \mid i \in n\}$
      $P_l = \mathrm{Preds}(R) \cup \bigcup \{ \mathrm{Preds}(\rho_{\tau_i} \circ \cdots \circ \rho_{\tau_n} \circ R) \mid i \in n\}$
      $P = P \cup P_p \cup P_l$
  else
      if π is well-founded by the ranking relation R **then**
          $T = T \cup \{R\}$ [weakening step]
      else
          return "Counterexample cyclic path $\tau_0, \ldots, \tau_{n-1}$"
else
      return "Program $\mathcal{R}$ terminates"
end

For details about the definition of the abstraction function and about the refinement step see [22]. Here we resume only the main properties:

- each path which is not a cycle (so which starts and ends in different locations l and l') is contained in the well-founded relation

$$pc = l \ \wedge \ pc' = l',$$

  with pc and pc' the values of the program counter (the next instruction to execute) in the first and last step of the path. For any fixed l, l', this relation has height 1 and therefore it is well-founded. Moreover the possible different locations are finite, so in order to obtain a disjunctively well-founded transition invariant we have just to add finitely many well-founded relations (of height ω) to T provided by Terminator. From now on, we refer to T as the transition invariant without considering these trivial relations.

- Although the number of cyclic paths is infinite, the search converges because the range of the abstraction function α_P is finite.

- The refinement step has the following property, stated in [22, Theorem 3]: if $\pi \subseteq R$ and $\alpha_P(\pi) \nsubseteq R$ then none of the cyclic paths obtained by concatenating π with itself repeatedly is such that $\alpha_P(\pi \ldots \pi) \nsubseteq R$, after one or two iterations of the algorithm. This implies that in one or two iterations the algorithm may treat an infinite numbers of computation paths.

In order to prove that all functions proved to be terminating by Terminator are primitive recursive we need to study how T is built, and so how Terminator finds the ranking relations. In Subsection 2.4.2 we prove that if Terminator proves the termination of a program $\mathcal{R}$, then the relations which compose the transition invariant T are primitive recursive and with height ω. By using the results presented in Section 2.3, this suffice to prove that $\mathcal{R}$ evaluates a primitive recursive function. On the other hand we want to prove that there exists an implementation for each primitive recursive function which is proved to be terminating by Terminator. We do it in Subsection 2.4.3. We provide, by induction over the primitive recursive functions, a program which is proved to be terminating by Terminator: so it is such that for each cycle there exists a ranking function and there exists a finite set of transition predicates P^* such that

- P at each step of the algorithm is included in P^*;
- when the algorithm reaches P^*, it terminates without any refinement steps.

2.4.2 Getting primitive recursive bounds

In this section we want to prove that if $\mathcal{R}$ is proved to be terminating by Terminator's algorithm then $\mathcal{R}$ is the code of some primitive recursive function.

First of all when Terminator proves that a program $\mathcal{R}$ terminates, it provides a transition invariant whose relations have height ω, (since each relation is a ranking relation induced by a linear ranking function whose codomain is $\mathbb{N}$) and are primitive recursive. Observe that T is a transition invariant (by adding a finite number of relations of height ω corresponding to the paths which are not cycles) since we have $\rho_\pi \subseteq \alpha_P(\pi)$ for any π.

Proposition 2.4.2. *Assume Terminator proves the termination of $\mathcal{R}$. The transition invariant provided by Terminator, is composed of primitive recursive relations with height ω.*

Proof Sketch. Looking at the algorithm, we just need to study how R is defined when a semantic cycle π is well-founded. In this step Terminator uses the method explained in [76] in order to obtain a linear ranking function and then it defines the ranking relation from this function. Just observe that the linear ranking function provided by this method is of the form:

$$\rho(x) = \begin{cases} a(x) & \text{if COND}(x) \\ b & \text{otherwise,} \end{cases}$$

where $\text{COND}(x)$ is the boolean condition of the semantic cycle, namely a conjunction of atomic propositions. Moreover $a : \mathbb{N}^n \to \mathbb{N}$ is primitive recursive function and b is a natural number which bounds any computation in which $\text{COND}(x)$ is false.

Let R be the ranking relation: i.e.

$$R = \Big\{(x, x') \mid x, x' \in \mathbb{N} \wedge \rho(x) > \rho(x') \geq 0\Big\}.$$

By construction it is primitive recursive and has height ω. □

For more details about the construction of this linear ranking function see [76]. In the previous section we proved that if a function has an implementation in Podelski and Rybalchencko's language with a disjunctively well-founded transition invariant of height ω, then the function is primitive recursive. So, thanks to this result and to Proposition 2.4.2, we are done.[6]

2.4.3 Vice Versa

By using the programs presented in Subsection 2.3.3, we may prove a weak vice versa.

Theorem 2.4.3. *Let f be a primitive recursive function, then there exists a program $\mathcal{R}$ which evaluates f such that $\mathcal{R}$ is proved to be terminating by the Terminator's algorithm.*

Proof. First we need to remark that the method defined in [76] is complete [76, Theorem 2]: if there exists a linear ranking function this method finds it. Moreover the algorithm defined in [22] proves the termination of $\mathcal{R}$ if for each semantic cycle in $\mathcal{R}$ it can find a linear ranking function. Then we have only to prove that for each f primitive recursive there exists a program $\mathcal{R}$ which evaluates f such that for each semantic cycle there is a linear ranking function. We prove it by induction on primitive recursive functions by using the same programs used in the proof of Theorem 2.3.11.

- Constant, successor and projection functions. The thesis holds since there are no semantic cycles.

- Composition. Let

  $$\forall x_0,\ldots,x_{n-1}(f(x_0,\ldots x_{n-1}) = g_0(g_1(x_0,\ldots,x_{n-1}),\ldots g_k(x_0,\ldots,x_{n-1}))).$$

  By induction hypothesis for any $i < k+1$ there exists a transition-based program $\mathcal{R}_i = (S_i, I_i, R_i)$ such that it is proved to be terminating by the Terminator algorithm. Let $\mathcal{R}_f$ defined as in the proof of Theorem 2.3.11(composition).

  Assume that π is a semantic cycle of $\mathcal{R}_f$, then, by definition of R_f it is a semantic cycle of $\mathcal{R}_i$ for some $i < k+1$. Let $l : S_i \to \mathbb{N}$ be the linear ranking function found by Terminator for π in the proof of the termination of $\mathcal{R}_i$. Then the extension of l to S_f, i.e. $l^* : S_f \to \mathbb{N}$ defined as

  $$l^*(\langle a,\ldots,\langle s_1,r_1\rangle,\ldots,\langle s_k,r_k\rangle,s_0,r_0\rangle) = l(\langle s_i,r_i\rangle),$$

  is a linear ranking function for π.

  Moreover we shall verify that the Terminator algorithm terminates when it runs with this program in input. At each cycle of Terminator it produce P in such a way it is a finite expansion of the previous one and this expansion depends only

[6] The example presented in [22], is a good example also for our goal. In fact the transition invariant obtained by the algorithm, $T = R_1 \cup R_2 \cup R_3 \cup R_4$, is such that R_i is primitive recursive, is well-founded and has height less or equal than ω for each $i \in [1,4]$.

on the relation which witnesses the well-foundedness of some semantic cycle. In addition if $P' \supseteq P$ then $\alpha_{P'}(\pi) < \alpha_P(\pi)$ for each cycle π. So in order to prove that Terminator ends we prove that P at each step of the program is included in $P_0 \cup \dots \cup P_k$, where P_i is the witnesses produced by Terminator running on the programs $\mathcal{R}_i$. Since each cycle in $\mathcal{R}_f$ is a cycle of some $\mathcal{R}_i$, then it involves only the variables used in the program $\mathcal{R}_i$. Since, by hypothesis P_p and P_l defined by the Terminator Algorithm (see Subsection 2.4.1) for this cycle are included in P_i, we are done.

- Primitive Recursion. For any $x_0, \dots, x_{k-1}$, let

$$\begin{cases} f(0, x_0, \dots, x_{k-1}) = g_0(x_0, \dots, x_{k-1}), \\ f(S(y), x_0, \dots, x_{k-1}) = g_1(y, f(y, x_0, \dots, x_{k-1}), x_0, \dots x_{k-1}). \end{cases}$$

By induction hypothesis for g_0 and g_1 there exist two programs $\mathcal{R}_0, \mathcal{R}_1$ which are proved to be terminating by Terminator algorithm. Let $\mathcal{R}_f$ defined as in the proof of Theorem 2.3.11(primitive recursion).

Let π be a semantic cycle of $\mathcal{R}_f$, then we have two possibilities:

  - it is a semantic cycle either of g_0 or of g_1 and so, as we did for the composition step, we are done;
  - it is part of the "new" while, then we can define a linear ranking function $l^* : S_f \to \mathbb{N}$ such that $l^*(\langle \bar{x}, z, y, s_0, r_0, s_1, r_1, r \rangle) = y - z$. Since z increases between two rounds of the "new" while, l^* is a ranking function.

In order to verify that the algorithm terminates, we can note again that there exists P^* such that it is finite and P after each refinement step is contained in it. Put $P^* = P_0 \cup P_1 \cup P_\pi$, where P_π is the union of the P_p and P_l, defined by the Terminator Algorithm (see Subsection 2.4.1) and produced by the ranking relation which derives from the ranking function l^*. As we did in the previous case if we consider a cycle in g_0 or g_1 we are done. Otherwise we are considering the new cycle and so the refinement step adds P_π. $\square$

2.5 Related and further works

In Section 2.2 we introduced a new intuitionistic version of Ramsey's Theorem for pairs. We could not find any evident connection with the intuitionistic interpretations by Bellin and by Oliva and Powell. Bellin in [5] applied the no-counterexample interpretation to Ramsey's Theorem, while Oliva and Powell in [69] used the Dialectica interpretation. They approximated the homogeneous set by a set which may stand any test checking whether some initial segment is homogeneous. The approximation they provide depends on the test itself.

Instead, we found interesting connections with the intuitionistic interpretations expressing Ramsey's Theorem as a property of well-founded relations. This research line started in 1974: the very first intuitionistic proof used Bar Induction. We refer to [92, section 10] for an account of this earlier stage of the research. Until 1990, all intuitionistic versions of Ramsey's Theorem for pairs were negated formulas, hence non-informative. In 1990 [92] Veldman and Bezem proved, using the Choice Axiom and Brouwer's thesis, the first intuitionistic negation-free version of Ramsey's Theorem: *almost full relations are closed under finite intersections*, from now on the *Almost Full Theorem*.

We explain the Almost Full theorem. In [92], a binary relation R over a set is *almost full* if for all infinite sequences $x_0, x_1, x_2, \ldots, x_n, \ldots$ on I there are *some* $i < j$ such that $x_i R x_j$. The Almost Full Theorem says that almost full relations are closed under finite intersections.

Veldman and Bezem used *Brouwer's thesis*, which says: a relation R is inductively well-founded if and only if all R-decreasing sequences are finite. Brouwer's thesis is classically true and it is often associated with intuitionistic reasoning. Yet, Brouwer's thesis is not provable using the rules of intuitionistic natural deduction, and we did not use it while proving our intuitionistic version of Ramsey's Theorem.

In [23] (first published in 1994, updated in 2011) Coquand showed that we may bypass the need of the Choice Axiom and Brouwer's thesis in the Almost Full Theorem, provided we take as definition of almost full relation an inductive definition (as we do in this chapter for the definition of well-founded).

We claim that, classically, the set of almost full relations R is the set of relations such that *the complement of the inverse of* R is H-well-founded. Indeed, let $\neg R^{-1}$ be the complement of the inverse of R: then, classically, $\neg R^{-1}$ almost full means that in all infinite sequences we have $x_i \neg R^{-1} x_j$ for some $i < j$, that is, $x_j \neg R x_i$ for some $i < j$, that is, all sequences such that $x_j R x_i$ for all $i < j$ (all R-homogeneous sequences) are finite. Classically, this is equivalent to: R is H-well-founded. The fact that the relationship between H-well-founded and almost full requires a complement explains why we prove closure under finite unions, while Veldman, Bezem and Coquand proved the closure under finite *intersections*.

For the moment, no bound analysis based on the Almost Full Theorem is known, and the only analysis based on a constructive proof are those presented in this chapter and in Chapter 3. For the future, a possible challenge is to extract the bounds implicit in the intuitionistic proof [93], which, as we said, uses Ramsey's Theorem for pairs in the form: "almost full relations are closed under finite intersection", and to compare them with the bound we obtained in Section 2.3.

Chapter 3

An intuitionistic analysis of Size-Change Termination

In this chapter we provide an intuitionistic proof of a classical variant of the SCT Theorem: our goal is to provide both a statement and a proof very similar to the original ones. This can be done by using the H-closure Theorem and inductive well-foundedness. Since we find no way to intuitionistically deduce the H-closure Theorem from the Almost Full Theorem, there are no apparent relationships between our proof and the classical variant of the SCT Theorem provided in [93].

As a side result we obtain another proof (completely different from the one by Ben-Amram [6]) of the characterization of functions computed by a tail-recursive SCT program. Our proof is based on the bounds found for the Termination Theorem in Section 2.3 and [34]. We can use these bounds since the SCT Theorem and the Termination Theorem are strictly related. Heizmann, Jones and Podelski proved that size-change termination is a property strictly stronger than termination [43]. By applying an argument similar to the one used in [43] we get a bound for a tail-recursive SCT program from the one for the Termination Theorem provided in [34]. Finally, we find a property in the "language" of size-change termination which is equivalent to Podelski and Rybalchenko's termination.

In this chapter we work in HAS; all the proofs are intuitionistic. Results in this chapter are published in [86].

3.1 From SCT to SCT*

In this section after a brief summary on the original definition of SCT presented in [43], we introduce a variant of SCT, which we call SCT*, and which is classically equivalent to SCT. Due to this definition we can intuitionistically prove the SCT Theorem. This is similar to what we did in the previous chapter: in order to intuitionistically prove the Termination Theorem we had to consider a classical equivalent of the termination property which is intuitionistically easier to prove. We obtain SCT* from SCT by taking the contrapositive and by considering the inductive well-foundedness instead of the classical one.

From now on we deal with a language for functions on $\mathbb{N}$ with call-by-value semantics considered in [58]. We use the recursive definitions and notations for maps $f : \mathbb{N}^n \to \mathbb{N}$ which Heizmann, Jones and Podelski present in their paper, for details see [43, pages 2-4]. Another useful reference is [6].

3.1.1 Size-change Termination

Here we recall the definition of SCT. Informally, a recursive definition of a function has the SCT property if in every infinite sequence of function calls there is some infinite sequence of values of arguments which is weakly decreasing, and strictly decreasing infinitely many times. In the case the domain of the function is $\mathbb{N}$, there is no such sequence of values, thus SCT is a sufficient condition for termination. In order to formally express SCT, first of all we need the definition of size-change graph. From now on we fix a recursive definition for a functional program $\mathcal{P}$ characterized as above. Let f be defined in $\mathcal{P}$ as follows:

$$\begin{aligned} f(x_0,\dots,x_{n-1}) := x_i \mid x_i+1 \mid x_i-1 \mid f(f(x_0,\dots,x_{n-1}),\dots,f(x_0,\dots,x_{n-1})) \\ \mid \text{if}(B(x_0,\dots,x_{n-1})) \text{ then } f(x_0,\dots,x_{n-1}) \text{ else } f(x_0,\dots,x_{n-1}), \end{aligned}$$

where

$$\begin{aligned} B(x_0,\dots,x_{n-1}) := x_i = 0 \mid x_i = 1 \mid x_i < x_j \mid x_i \le x_j \mid B(x_0,\dots,x_{n-1}) \wedge B(x_0,\dots,x_{n-1}) \\ \mid B(x_0,\dots,x_{n-1}) \vee B(x_0,\dots,x_{n-1}) \mid \neg B(x_0,\dots,x_{n-1})|. \end{aligned}$$

Then we denote the set $\{x_0,\dots,x_{n-1}\}$ by $\mathrm{Var}(f)$. Given such a function f, a state is a pair $(f,\mathbf{v})$ where $\mathbf{v}$ is a finite sequence of natural numbers whose length is n. If in the definition of f there is a call

$$\dots\tau : g(e_0,\dots,e_{m-1})$$

we define a state transition $(f,\mathbf{v}) \xrightarrow{\tau} (g,\mathbf{u})$ to be a pair of states such that $\mathbf{u}$ is the sequence of values obtained by the expressions $(e_0,\dots,e_{m-1})$ when f is evaluated with $\mathbf{v}$.

Definition 3.1.1 (Size-change graph)**.** Let f, g be defined in $\mathcal{P}$. A *size-change graph* $G : f \to g$ for $\mathcal{P}$ is a bipartite directed graph on $(\mathrm{Var}(f),\mathrm{Var}(g))$. The set of edges is a subset of $\mathrm{Var}(f) \times \{\downarrow,\Downarrow\} \times \mathrm{Var}(g)$ such that there is at most one edge for any $x \in \mathrm{Var}(f)$, $y \in \mathrm{Var}(g)$. We say that f is the source function of G and g is the target function of G.

We stress that we may have no edge from a variable x to a variable y in the graph G. We will represent in this way the fact that we know nothing at all about the relationship between x and y. We call $(x,\downarrow,y)$ the decreasing edge, and we denote it with $x \xrightarrow{\downarrow} y$. We call $(x,\Downarrow,y)$ the weakly-decreasing edge, and we denote it with $x \xrightarrow{\Downarrow} y$.

Definition 3.1.2. Let $f(x_0,\dots,x_{n-1})$ be defined with a call $\tau : g(e_0,\dots,e_{m-1})$ (where $\mathrm{Var}(g) = \{y_0,\dots,y_{m-1}\}$).

- The edge $x_i \xrightarrow{r} y_j$ *safely describes* the $x_i - y_j$ relation in the call τ, if for any $\mathbf{v} \in \mathbb{N}^n$ and $\mathbf{u} \in \mathbb{N}^m$ such that $(f, \mathbf{v}) \xrightarrow{\tau} (g, \mathbf{u})$, then $r = \downarrow$ implies that $u_j < v_i$ and $r = \Downarrow$ implies that $u_j \leq v_i$.

- The size-change graph G_τ is *safe* for the call τ if every edge in G_τ is a safe description.

- Set $\mathcal{G}_\mathcal{P} = \{G_\tau \mid \tau \text{ is a call in } \mathcal{P}\}$. We say that $\mathcal{G}_\mathcal{P}$ is a *safe description* of $\mathcal{P}$ if for any call τ, G_τ is safe.

Note that the absence of edges between two variables x and y in the size-change graph G_τ which is safe for τ indicates either an unknown or an increasing relation in the call τ.

Example 3.1.3. Let us consider the following basic functional program which computes $f(x, y) = x + y$.

$$\begin{aligned} f(x, y) :=& \text{if } (y = 0) \quad x \\ & \text{else } \tau : f(x+1, y-1) \end{aligned}$$

The unique size-change graph $G : f \to f$ which is safe for the call τ is

$$\begin{tikzcd} x & x \\ y \arrow[r, "\downarrow"] & y \end{tikzcd}$$

Definition 3.1.4. A *multipath* $\mathcal{M}$ is a graph sequence $G_0, \ldots, G_n, \ldots$ such that the target function of G_i is the source function of G_{i+1} for any i. A *thread* is a connected path of edges in $\mathcal{M}$ that starts at some G_t, where $t \in \mathbb{N}$. A multipath $\mathcal{M}$ has *infinite descent* if some thread in $\mathcal{M}$ contains infinitely many decreasing edges.

Definition 3.1.5 (SCT program)**.** Let $\mathcal{T}$ be the set of calls in program $\mathcal{P}$. Suppose that each size-change graph $G_\tau : f \to g$ is safe for every call τ in

$$\mathcal{G}_\mathcal{P} = \{G_\tau \mid \tau \in \mathcal{T}\}.$$

$\mathcal{P}$ is *size-change terminating* (SCT) if, for any infinite call sequence $\pi = \tau_1, \ldots, \tau_n, \ldots$ that follows $\mathcal{P}$'s control flow, the multipath $M_\pi = G_{\tau_1}, \ldots, G_{\tau_n}, \ldots$ has an infinite descent.

3.1.2 Composing size-change graphs

As in [43], given two size-change graphs $G_0 : f \to g$ and $G_1 : g \to h$ we define their *composition* $G_0; G_1 : f \to h$. The composition of two edges $x \xrightarrow{\Downarrow} y$ and $y \xrightarrow{\Downarrow} z$ is one edge $x \xrightarrow{\Downarrow} z$. In all other cases the composition of two edges from x to y and from y to z is the edge $x \xrightarrow{\downarrow} z$. Formally, $G_0;G_1$ is the size-change graph with the following set of edges:

$$\begin{aligned} E = \{x \xrightarrow{\downarrow} z \mid\ & \exists y \in \mathrm{Var}(g)\ \exists r \in \{\downarrow,\Downarrow\}\,((x \xrightarrow{\downarrow} y \in G_0 \wedge y \xrightarrow{r} z \in G_1) \\ & \vee (x \xrightarrow{r} y \in G_0 \wedge y \xrightarrow{\downarrow} z \in G_1))\} \\ \cup\{x \xrightarrow{\Downarrow} z \mid\ & \exists y \in \mathrm{Var}(g)(x \xrightarrow{\Downarrow} y \in G_0 \wedge y \xrightarrow{\Downarrow} z \in G_1) \wedge \forall y \in \mathrm{Var}(g) \\ & \forall r,r' \in \{\downarrow,\Downarrow\}\,((x \xrightarrow{r} y \in G_0 \wedge y \xrightarrow{r'} z \in G_1) \implies r = r' = \Downarrow)\}. \end{aligned}$$

Observe that the composition operator ";" is associative. Given a finite call sequence $\pi = \tau_0,\dots,\tau_{n-1}$ we define $G_\pi = G_{\tau_0};\dots;G_{\tau_{n-1}}$. Moreover we say that the size-change graph G is *idempotent* if $G;G = G$.

Given a finite set of size-change graphs $\mathcal{G}$, $\mathsf{cl}(\mathcal{G})$ is the smallest set which contains $\mathcal{G}$ and is closed by composition. Formally $\mathsf{cl}(\mathcal{G})$ is the smallest set such that

- $\mathcal{G} \subseteq \mathsf{cl}(\mathcal{G})$;
- If $G_0 : f \to g$ and $G_1 : g \to h$ are in $\mathsf{cl}(\mathcal{G})$, then $G_0;G_1 \in \mathsf{cl}(\mathcal{G})$.

Once we fixed the number of variables, there are only finitely many bipartite graphs with two labels for the edges (even if their number is huge), therefore classically $\mathsf{cl}(\mathcal{G})$ is finite. Moreover we can intuitionistically prove that it is finite due to the following proposition.

Proposition 3.1.6. *Assume that S is a finite set where the equality is decidable and that op: $S \times S \to S$ is a computable map. Then the closure of any finite subset of S is intuitionistically finite.*

In fact if $I \subseteq S$, we can define $I_0 = I$, $I_{k+1} = \{\mathrm{op}(a,b) \mid a,b \in \bigcup\{I_h \mid h \leq k\}\} \setminus \bigcup\{I_h \mid h \leq k\}$. By decidability of equality, we may effectively compute $A \setminus B$ for any A, B finite subsets of S. Therefore we may intuitionistically prove by induction over $S \setminus \bigcup\{I_h \mid h \leq k\}$ that there is a $k \leq |S|$ such that $I_{k+1} = \emptyset$. Thus k defines the closure of I.

3.1.3 Definition of SCT*

As seen above, $\mathcal{P}$ is SCT if and only if

for any infinite call sequence π that follows $\mathcal{P}$, M_π has an infinite descent.

Now we want to apply some classical step in order to obtain a statement SCT* classically equivalent to SCT but intuitionistically easier to prove. From the definition of SCT, by taking a contrapositive, we obtain

for any call sequence π which follows $\mathcal{P}$, M_π has no infinite descents implies that π is not infinite.

This is the sentence from which we will obtain our definition. We introduce a symbol $\emptyset$ denoting the empty call. Formally a call sequence which follows $\mathcal{P}$ is a function $\pi : \mathbb{N} \longrightarrow \{\tau \mid \tau \text{ is a call in } \mathcal{P}\} \cup \{\emptyset\}$ such that

- if $\pi(n) = \emptyset$ for some $n \in \mathbb{N}$, then $\forall m > n(\pi(m) = \emptyset)$;
- if $\pi(n+1) = \tau$, then τ is a call which appears in the definition corresponding to the call $\pi(n)$.

Observe that π being infinite in this notation means that $\forall n(\pi(n) \neq \emptyset)$. In order to keep the notation of [58] for any natural number n we denote $\tau_n = \pi(n)$.

We introduce two binary relations, π^+ on $\mathbb{N}$ and R_π on $\mathbb{N} \times \mathrm{Var}$. Then, by following the idea of [1, 3, 7, 23], we translate "M_π has no infinite descents" with "R_π is inductively well-founded" and "π is not infinite" with "π^+ is inductively well-founded". Classically, π^+ is inductively well-founded is equivalent to $\pi(n) = \emptyset$ for some n, but intuitionistically expresses the fact that we know that n exists without knowing its value.

Let $\mathcal{P}$ be a program and let π be a call sequence which follows $\mathcal{P}$. We define a binary relation π^+ on $\mathbb{N}$ by:

$$m\pi^+n \iff m > n \wedge \tau_m \neq \emptyset.$$

Observe that if π is infinite then $m\pi^+n$ holds if and only if $m > n$, while if l is the minimum number such that $\pi(l) = \emptyset$ then $m\pi^+n$ holds if and only if $l > m > n$.

Now, we define a binary relation R_π on $\mathbb{N} \times \mathrm{Var}$. Here $(m,y)R_\pi(n,x)$ holds if and only if y becomes strictly smaller than x when we step from τ_n to τ_{m-1} along the call sequence π.

Definition 3.1.7. Given a sequence π that follows $\mathcal{P}$, R_π is defined as:

$$(m,y)R_\pi(n,x) \iff m\pi^+n \wedge x \xrightarrow{\downarrow} y \in G_{\tau_n};\ldots;G_{\tau_{m-1}},$$

where G_τ is the size-change graph associated to τ.

From R_π and π^+ we define SCT*.

Definition 3.1.8 (SCT* program)**.** $\mathcal{P}$ is SCT* if and only if for any call sequence π which follows $\mathcal{P}$: R_π is (inductively) well-founded implies that π^+ is (inductively) well-founded.

We highlighted the use of the inductive definition of well-foundedness instead of the classical one, since as seen it is crucial in order to give an intuitionistic proof of the SCT Theorem. However for short, from now on we write well-foundedness instead of inductive well-foundedness.

3.2 Proving the SCT* Theorem

The goal of this section is to give an intuitionistic proof of the SCT* Theorem, which is the contrapositive of the SCT Theorem by Lee et al. expressed by using the inductive definition of well-foundedness. The SCT Theorem states that a program $\mathcal{P}$ is SCT if and only if for any idempotent $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$ there exists a variable x in the source of G such that $x \xrightarrow{\downarrow} x \in G$. Recall that in the classical proof of the SCT Theorem the main ingredient is Ramsey's Theorem for pairs. In the classical proof the authors suppose by contradiction that there exists an infinite call sequence π which follows $\mathcal{P}$. Then they define a coloring $c : [\mathbb{N}]^2 \to \mathsf{cl}(\mathcal{G}_\mathcal{P})$ which associates to any pair of natural numbers $n < m$ the size-change graph which corresponds to $G_{\tau_n};\ldots;G_{\tau_{m-1}}$. Since $\mathsf{cl}(\mathcal{G}_\mathcal{P})$ if finite and by applying Ramsey's Theorem for pairs they obtain an infinite homogeneous sequence and therefore a contradiction.

Also in this case we can use the H-closure Theorem instead of Ramsey's Theorem for pairs. To this aim we introduce a binary relation π^+_G for any $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$ and for any call sequence π which follows $\mathcal{P}$. We have that $m\pi^+_G n$ holds if and only if the size-change graph associated to $\tau_n,\ldots,\tau_{m-1}$ is G.

Definition 3.2.1. Let π be a call sequence which follows $\mathcal{P}$ and let $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$. Define $\pi^+_G \subseteq \mathbb{N}^2$ as:

$$m\pi^+_G n \iff m\pi^+ n \wedge G_{\tau_n};\ldots;G_{\tau_{m-1}} = G.$$

Observe that π^+_G is decidable since G_{τ_i} is finite and π^+ is decidable. Moreover, as remarked in Subsection 2.2.2, if R is decidable then $H(R)$ is.

Lemma 3.2.2. *Let π be a call sequence which follows $\mathcal{P}$ and let $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$. Then both π^+_G and $H(\pi^+_G)$ are decidable.*

By applying the H-closure Theorem to relations π^+_G, for $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$, we can intuitionistically prove the SCT* Theorem.

Theorem 3.2.3 (SCT* Theorem)**.** *Every idempotent graph in $\mathsf{cl}(\mathcal{G}_\mathcal{P})$ has an edge $x \xrightarrow{\downarrow} x$ if and only if $\mathcal{P}$ is* SCT*.

Proof. "$\Rightarrow$": Assume that any idempotent graph in $\mathsf{cl}(\mathcal{G}_\mathcal{P})$ has an edge $x \xrightarrow{\downarrow} x$. Let π be a call sequence which follows $\mathcal{P}$ such that R_π is well-founded. By unfolding the definition of SCT* (Definition 3.1.8), we have to prove that π^+ is well-founded. We prove the following Claim first.

Claim. *For any G in $\mathsf{cl}(\mathcal{G}_\mathcal{P})$, π^+_G is H-well-founded.*

Proof of the Claim. Since "G is idempotent" is a decidable statement we can consider two cases.

- If G is not idempotent, then each $L \in H(\pi^+_G)$ has length at most 2. Otherwise assume that $\langle n,m,l\rangle \in H(\pi^+_G)$ for some $n < m < l$. By definition, this would imply that $m\pi^+_G n$, $l\pi^+_G m$ and $l\pi^+_G n$, therefore we would have

$$G;G = G_{\tau_n};\ldots;G_{\tau_{m-1}};G_{\tau_m};\ldots;G_{\tau_{l-1}} = G.$$

This means that G is idempotent. Contradiction. Hence we have

$$\neg\exists n,m,l(\langle n,m,l\rangle \in H(\pi^+_G)),$$

Hence π^+_G is H-well-founded.

- If G is idempotent, then there exists $x \xrightarrow{\downarrow} x \in G$ by hypothesis. Define the following binary relation
$$T_x = \{(n,(n,x)) \mid n \in \mathbb{N}\}.$$
We claim that T_x is a simulation of π^+_G in R_π. In fact if $m\pi^+_G n \wedge nT_x(n,x)$, then, since $x \xrightarrow{\downarrow} x \in G = G_{\tau_n};\dots;G_{\tau_{m-1}}$, by definition of R_π (Definition 3.1.7) and of T_x we have $mT_x(m,x) \wedge (m,x)R_\pi(n,x)$. Since by hypothesis R_π is well-founded, then by Proposition 1.2.19 also π^+_G is well-founded. By Proposition 2.2.3 π^+_G is H-well-founded. □

We prove that Theorem from the Claim. Observe that

$$\pi^+ = \bigcup\left\{\pi^+_G \mid G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})\right\},$$

since every $G_{\tau_n};\dots;G_{\tau_{m-1}}$ equates some $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$ by definition of $\mathsf{cl}(\mathcal{G}_\mathcal{P})$. Hence by applying both the H-closure Theorem (Theorem 2.2.10) and finiteness of $\mathsf{cl}(\mathcal{G}_\mathcal{P})$ we obtain that π^+ is H-well-founded. Moreover π^+ is transitive by definition, then π^+ is well-founded by Proposition 2.2.3 and we are done.

"$\Leftarrow$": Suppose that $\mathcal{P}$ is SCT* and let G_τ be an idempotent size-change graph. By idempotency if we define the call sequence π such that $\pi(n) = \tau$ for any $n \in \mathbb{N}$, then π is an infinite call sequence which follows $\mathcal{P}$. In particular π^+ is not well-founded. Since $\mathcal{P}$ is SCT*, R_π is not well-founded. In this case, we may observe that:

$$(m,y)R_\pi(n,x) \iff x \xrightarrow{\downarrow} y \in G_{\tau_n};\dots;G_{\tau_{m-1}} = G_\tau;\dots;G_\tau = G_\tau$$

Then $(m,y)R_\pi(n,x) \iff x \xrightarrow{\downarrow} y \in G_\tau$. Define

$$y\tilde{R}_\pi x \iff x \xrightarrow{\downarrow} y \in G_\tau,$$

It is immediate that if $\tilde{R}_\pi$ is well-founded, then also R_π is. We can prove it by using the simulation

$$T = \{((n,x),x) \mid n \in \mathbb{N}, x \text{ a variable in the source of } G_\tau\}.$$

Hence $\tilde{R}_\pi$ is not well-founded. Since $\tilde{R}_\pi$ is not well-founded and it is finite, thanks to Proposition 2.2.5, it has a cycle for some variable z:

$$z = z_n\tilde{R}_\pi z_{n-1}\tilde{R}_\pi \dots \tilde{R}_\pi z_0 = z.$$

Moreover $\tilde{R}_\pi$ is transitive: in fact if $x \xrightarrow{\downarrow} y \in G_\tau$ and $y \xrightarrow{\downarrow} z \in G_\tau$, then $x \xrightarrow{\downarrow} z \in G_\tau; G_\tau = G_\tau$. Since $\tilde{R}_\pi$ is transitive, we get $z\tilde{R}_\pi z$. Therefore $z \xrightarrow{\downarrow} z \in G_\tau$. We have proved that if G_τ is idempotent, then $z \xrightarrow{\downarrow} z \in G_\tau$. □

By comparing the classical proofs of the termination theorems, the version of Ramsey's Theorem for pairs used in the proof of the Termination Theorem is weaker then the one used in the proof of the SCT Theorem. In fact in order to prove the Termination Theorem it is sufficient to have an infinite homogeneous sequence (i.e. an ordered set $\{x_i \mid i \in \mathbb{N}\}$ such that each pair of consecutive elements has the same color) instead of an infinite homogeneous set (in which also non-consecutive pairs have the same color). This result which is an infinite version of Erdős–Szekeres's Theorem [29] is called also Weak Ramsey's Theorem [66] for pairs and it is strictly weaker than Ramsey's Theorem for pairs in two colors in RCA_0 (see Chapter 5). We can stress this difference also in the intuitionistic proofs. In fact the proof of the Termination Theorem in [11] uses:

if $R_0, \ldots, R_{n-1}$ are well-founded then $\bigcup\{R_i \mid i < n\}$ is H-well-founded.

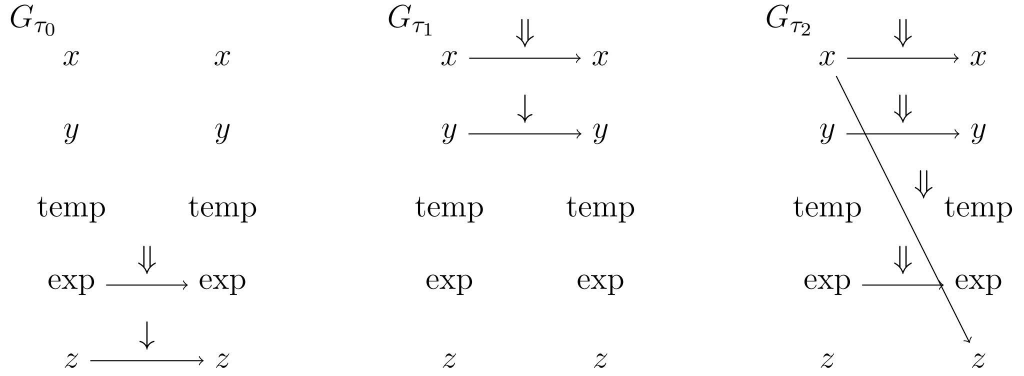
where the hypothesis is stronger than in the H-closure Theorem, by Proposition 2.2.3. On the other hand the proof of the SCT* Theorem above uses the whole H-closure Theorem.

Let us conclude this section with an example of a SCT* program.

Example 3.2.4. Let us consider the following functional program, where $*$ denotes any value.

$$
\begin{aligned}
g(x, y, \text{temp}, \exp, z) :=& \text{if } (z = 0) \quad 0 \\
& \text{else if } (z = 1) \quad \text{temp} \\
& \text{else } \tau_0 : g(*, *, \text{temp} + \exp, \exp, z - 1) \\
f(x, y, \text{temp}, \exp, z) :=& \text{if } (y = 0) \quad 1 \\
& \text{else if } (y = 1) \quad \exp \\
& \text{else } \tau_1 : f(x, y - 1, *, \tau_2 : g(x, y, 0, \exp, x), *)
\end{aligned}
$$

Note that $f(x, y, 0, 1, z)$ computes x^y. Every size-change graph corresponds to some composition of $G_{\tau_0} : g \to g$, $G_{\tau_1} : f \to f$ and $G_{\tau_2} : f \to g$.



The idempotent graphs in $\mathsf{cl}(\mathcal{G})$ are $G_{\tau_0}: g \to g$ and $G_{\tau_1}: f \to f$ (since the source and the target of the other size-change graphs are different). Hence this program is SCT* since it satisfies the condition of the SCT* Theorem.

3.3 Tail-recursive SCT* programs

In this section we compare size-change termination and transition invariant termination. As Heizmann, Jones and Podelski did, we restrict the domain of the programs we consider in order to match transition invariants termination and SCT*. In fact SCT (and so SCT*) is defined for functional programs, while Podelski and Rybalchenko's termination is defined for transition-based programs. As they did from now on we consider just tail-recursive functional programs (where all functions use the same variables), for which there exists a direct transition-based translation into transition-based programs. We refer to [43] for details. The reader has to keep in mind that along all this section we have at the same time a functional program, which we denote by $\mathcal{P}$, and its translation as a transition-based program $\mathcal{R}_{\mathcal{P}}$. The only property we use of the translation $\mathcal{R}_{\mathcal{P}}$ is that $\mathcal{R}_{\mathcal{P}}$ consists of: while, if, a program counter and the values of the variables of $\mathcal{P}$. If $\mathcal{P}$ were recursive but not tail-recursive, $\mathcal{R}_{\mathcal{P}}$ should include also a stack, but we explicitly assume that this is not the case. We derive a characterization of $\mathcal{P}$ from a characterization of $\mathcal{R}_{\mathcal{P}}$.

The goal of this section is to prove, by using the result obtained in Section 2.3, that the functional programs which are tail-recursive and SCT* compute exactly the primitive recursive functions. Ben-Amram in [6] has already proved that tail-recursive SCT programs compute primitive recursive functions, however we present a completely different proof which uses an analysis of the intuitionistic proof of the Termination Theorem, in the case of a transition invariant of height ω. In fact we intuitionistically prove that if a program $\mathcal{P}$ is SCT* then $\mathcal{R}_{\mathcal{P}}$ has a transition invariant of height ω, and by results of Section 2.3 computes a primitive recursive function.

First of all we recall some definitions and results useful to compare size-change termination and Podelski and Rybalchenko's termination. Each state in the transition-based program $\mathcal{R}_{\mathcal{P}}$ which corresponds to the tail-recursive functional program $\mathcal{P}$ is a tuple s composed of the location $s(\mathrm{pc})$ of the program instruction (called "program counter") and a value $s(x)$ for any variable x. We define a relation $\Phi(G)$ on states saying that whenever G includes a decreasing edge $x \xrightarrow{\downarrow} y$ then the value of y in the second state is smaller than the value of x in the first state, and similarly for any weakly-decreasing edge. Formally, as in [43], the definition runs as follows.

Definition 3.3.1 (Transition relation of a size-change graph)**.** Given a size-change graph $G : f \to g$, define the binary relation over states $\Phi(G) \subseteq S \times S$ by: $s'\Phi(G)s$ if and only if $s(\mathrm{pc}) = f$, $s'(\mathrm{pc}) = g$ and

$$\bigwedge \left\{ s(z_i) \geq s'(z_j) \mid (z_i \xrightarrow{\Downarrow} z_j) \in G \right\} \wedge \bigwedge \left\{ s(z_i) > s'(z_j) \mid (z_i \xrightarrow{\downarrow} z_j) \in G \right\}.$$

The transition relation ρ_τ associated to the transition

$$f(x_0, \ldots, x_{n-1}) = \ldots \tau : g(e_0, \ldots, e_{n-1}), \ldots,$$

is defined by:

$$\rho_\tau = \left\{((f, \mathbf{v}), (g, \mathbf{u})) \mid (f, \mathbf{v}) \xrightarrow{\tau} (g, \mathbf{u})\right\}.$$

Observe that if G_τ is the size-change graph assigned to the call τ of program $\mathcal{P}$, G_τ is safe for τ if and only if the inclusion $\rho_\tau \subseteq \Phi(G_\tau)$ holds. Moreover, as shown in [43, Lemma 29]:

Lemma 3.3.2. *The composition of the two size-change graphs $G_1 : f \to g$ and $G_2 : g \to h$ overapproximates the composition of the relations they define, i.e.*

$$\Phi(G_1) \circ \Phi(G_2) \subseteq \Phi(G_1; G_2).$$

Heizmann, Jones and Podelski proved the following lemma about the connection between G and $\Phi(G)$.

Lemma 3.3.3. *Let G be a size-change graph such that source and target of G coincide. If G has an edge of form $x \xrightarrow{\downarrow} x$ then the relation $\Phi(G)$ is well-founded.*

They also noticed that the vice versa does not hold in the general case, however we prove that if G is idempotent this equivalence holds.

Lemma 3.3.4. *Let G be an idempotent size-change graph. Then G has an edge of form $x \xrightarrow{\downarrow} x$ if and only if the relation $\Phi(G)$ is well-founded.*

Proof. "$\Rightarrow$": Assume that G has an edge of the form $x \xrightarrow{\downarrow} x$. It is straightforward to prove directly that every state s is $\Phi(G)$-well-founded by induction over $s(x)$.

"$\Leftarrow$": Observe that "there exists a variable x in the source of G, $x \xrightarrow{\downarrow} x \in G$" is a decidable statement, since G is finite. Therefore either G has some edge of the form $x \xrightarrow{\downarrow} x$, or G has no edge of this form. In the first case we are done, in the second one we prove a contradiction, namely $\Phi(G)$ is not well-founded. This argument intuitionistically shows the thesis, because from a contradiction we may derive everything. Let V be the set of the variables in the source of G. Define the following preorder on V:

$$x \precsim y \iff x = y \vee y \xrightarrow{\downarrow} x \vee y \xrightarrow{\Downarrow} x.$$

It is reflexive by definition and it is transitive since G is idempotent: if $y \xrightarrow{r} x \in G$ and $z \xrightarrow{r'} y \in G$ then $z \xrightarrow{r''} x \in G$ for some $r'' \in \{\downarrow, \Downarrow\}$.

Let us consider the quotient X of V with respect to the equivalence relation

$$x \sim y \iff x \precsim y \wedge y \precsim x.$$

In X, $\precsim$ is an order since it is also antisymmetric. Moreover X is finite then, by Proposition 2.2.5, $\precsim$ is well-founded on X. Hence for any equivalence class $[x] \in X$ we can define $h([x])$ to be the number of elements $[y] \in X$ such that $[y] \precsim [x]$.

Then we can define a state s as follows: for any $x \in V$, $s(x) = h([x])$. We claim that $s\Phi(G)s$. In fact

- if $y \xrightarrow{\Downarrow} x \in G$ then $x \precsim y$ and we have two possibilities: if $[x] = [y]$ in X then $s(x) = s(y)$, otherwise $h([x]) < h([y])$ and so $s(x) < s(y)$;
- if $y \xrightarrow{\downarrow} x \in G$ then $x \precsim y$. Moreover $y \precsim x$ is false, otherwise by case analysis from $y \xrightarrow{\downarrow} x \in G$ and $x = y \vee x \xrightarrow{\downarrow} y \in G \vee x \xrightarrow{\Downarrow} y \in G$ we deduce in all cases $x \xrightarrow{\downarrow} x \in G$, contradicting the hypothesis. Hence $h([x]) < h([y])$ and so $s(x) < s(y)$.

Then $s\Phi(G)s$ and so $\Phi(G)$ is ill-founded. Contradiction. $\square$

Thanks to the SCT* Theorem and the lemma above, we may observe that if $\mathcal{P}$ is tail-recursive, $\mathcal{P}$ is SCT* if and only if for every $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$ idempotent $\Phi(G)$ is well-founded.

Our next goal is to prove that any tail-recursive program which is SCT* computes a primitive recursive function. In order to do that we modify the proof by Heizmann, Jones and Podelski of "$\bigcup\{\Phi(G) \mid G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})\}$ is a transition invariant". We prove that it is a transition invariant of height ω. In order to do this we need the following lemmas. The first one is well-known.

Lemma 3.3.5. *Every finite semigroup G has an idempotent element.*

Proof. Let $x \in G$ and consider the following sequence

$$x \mapsto x^2 \mapsto (x^2)^2 = x^4 \mapsto \ldots \mapsto x^n \mapsto (x^n)^2 \mapsto \ldots.$$

Since G is finite, there exists y in the previous sequence such that $y^k = y$, for some $k \geq 2$. Put $z = y^{k-1}$, then

$$z \cdot z = y^{k-1} \cdot y^{k-1} = y^k \cdot y^{k-2} = y \cdot y^{k-2} = y^{k-1} = z. \qquad \square$$

Even if the previous definitions and results can be stated for any functional program, we highlight now that we need tail-recursive functional programs in order to have the translation $\mathcal{R}_\mathcal{P}$ of $\mathcal{P}$. In this case each state of $\mathcal{R}_\mathcal{P}$ is composed of the location of the program and the values in $\mathbb{N}$ of the variables.

Lemma 3.3.6. *Let G be a size-change graph. Let k be a positive natural number. If G^k is such that $x \xrightarrow{\downarrow} x$ for some x then $\Phi(G)$ has height ω.*

Proof. Assume that $x \xrightarrow{\downarrow} x$ in G^k. We distinguish the cases $k = 1$ and $k \geq 2$.

If $k = 1$, let $f : \mathrm{dom}(\Phi(G)) \to \mathbb{N}$ be such that $f(s) = s(x)$. If $s'\Phi(G)s$, then by $x \xrightarrow{\downarrow} x \in G$ we deduce $s'(x) < s(x)$. Hence $f(s') < f(s)$. Thus f is a weight function for $\Phi(G)$.

Assume now that $k \geq 2$, since $x \xrightarrow{\downarrow} x$ in G^k, then there exist $y_0, \dots, y_{k-2}$ such that

$$x \to y_0 \to \cdots \to y_{k-2} \to x$$

where at least one of these arrows is strictly decreasing. Then define a function $f : \mathrm{dom}(\Phi(G)) \to \mathbb{N}$ by

$$f(s) = \sum_{i=0}^{k-2} s(y_i) + s(x).$$

Hence if $s'\Phi(G)s$ then each of the $s'(y_i)$ and $s'(x)$ is less or equal to the ones of s. Moreover one of these is strictly less, since at least one of the edges of G is strictly decreasing. So $f(s') < f(s)$ and this means that f witnesses that $\Phi(G)$ has height ω. $\square$

By using the results above we can modify the Theorem Idempotence and well-foundedness [43, Theorem 32] which states that if for any $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$ idempotent $\Phi(G)$ is well-founded, then $\Phi(G)$ is well-founded for any $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$.

Theorem 3.3.7. *If*

$$\forall G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})(G;G = G \implies \Phi(G) \textit{ is well-founded})$$

then for every graph in $\mathsf{cl}(\mathcal{G}_\mathcal{P})$*,* $\Phi(G)$ *has height* ω*.*

Proof. Let $G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})$ be a size-change graph. There are two possibilities. On the one hand, if the source and target of G do not coincide, then $\Phi(G) \circ \Phi(G) = \emptyset$, therefore $\Phi(G)$ has height ω. On the other hand, assume that source and target of G coincide. Then G^n is defined for any $n \in \mathbb{N}$. Since the semigroup $(\{G^n \mid n > 0\}, ;)$ is finite, by Lemma 3.3.5 it has an idempotent element G^k. Since $\Phi(G^k)$ is well-founded by hypothesis we obtain, by applying Lemma 3.3.4, that $x \xrightarrow{\downarrow} x \in G^k$ for some x. By Lemma 3.3.6 $\Phi(G)$ has height ω and we are done. $\square$

Corollary 3.3.8 (Corollary 33 [43])**.** *If the program* $\mathcal{P}$ *is size-change terminating for a set of size-change graphs* $\mathcal{G}_\mathcal{P}$ *that is a safe description of* $\mathcal{P}$*, then the relation defined by its closure* $\mathsf{cl}(\mathcal{G}_\mathcal{P})$

$$\bigcup \{\Phi(G) \mid G \in \mathsf{cl}(\mathcal{G}_\mathcal{P})\}$$

is a disjunctively well-founded transition invariant for $\mathcal{R}_\mathcal{P}$*.*

Therefore, thanks to Theorem 3.3.7, all $\Phi(G)$ have height ω: by definition, the transition invariant has height ω. In Section 2.3 we proved that if $\mathcal{R}$ has a transition invariant of height ω then it computes a primitive recursive function.

We apply this result to the transition-based program $\mathcal{R}_\mathcal{P}$ which translates the tail-recursive program P. We obtain:

Proposition 3.3.9. *Each tail-recursive program which is* SCT* *is primitive recursive.*

This result was already proved by Ben-Amram in [6] using the classical definition of SCT. He proved that in general SCT programs compute multiple recursive functions. As a corollary, by observing that if you do not use nested recursive calls a multiple recursive function is primitive recursive, he obtained that any tail-recursive SCT program computes a primitive recursive function.

Let $\mathcal{F}_k$ denote the usual k-class of the Fast Growing Hierarchy [61]. Define

$$F_0(x) = x + 1$$
$$F_{k+1}(x) = F_k^{(x+1)}(x).$$

Then $\mathcal{F}_k$ is the closure under limited recursion and substitution of the set of functions composed of constant, projections, sum and F_h for any $h \leq k$. As shown in [34] if $\mathcal{R}$ is a program such that

- its transition relation R is the graph of a function in $\mathcal{F}_2$;
- there exists a transition invariant for $\mathcal{R}$ which is the union of k relations, all having weight functions in $\mathcal{F}_1$.

Then the function computed by $\mathcal{R}$ is in $\mathcal{F}_{k+1}$.

By using both our proof and the bound provided in [34] we can easily obtain a bound whose class is given by the number of relations of the transition invariant. In fact the weight function provided in Lemma 3.3.6 is in $\mathcal{F}_1$, since $f(s) < |s| \cdot F_0(\max(s))$ and for any program $|s|$ is fixed. Therefore by applying the bound provided in [34] we have that if the transition relation of $\mathcal{R}_{\mathcal{P}}$ is the graph of a function in $\mathcal{F}_n$, there is a bound in $\mathcal{F}_{k+n-1}$ where k is the number of relations which compose the transition invariant whose weight functions are in $\mathcal{F}_1$. Therefore by Corollary 3.3.8, we can conclude that if the transition relation is in $\mathcal{F}_2$, the function is in $\mathcal{F}_{|\mathsf{cl}(\mathcal{G}_{\mathcal{P}})|+1}$. Unfortunately $\mathsf{cl}(\mathcal{G}_{\mathcal{P}})$ is exponential in $\mathcal{G}_{\mathcal{P}}$, so this bound is huge. We may produce also another bound on the number of variables. In fact in the proof of Theorem 3.3.7 we saw that for any G there exists $k > 0$ such that G^k is idempotent. Let us consider the minimum such k. Then, by following the proof of Lemma 3.3.6 there exists either a $x \xrightarrow{\downarrow} x$ for some x or a sequence

$$x \to y_0 \to \cdots \to y_{k-2} \to x$$

for some variables, where at least one arrow is strict. Observe that all variables in the sequence are different: there is not a path which connect some y_i to itself, by minimality of k. The weight function we built for $\Phi(G)$ is given by the sum of the values which corresponds to these variables. This means that if we have n-many variables, the number of possible weight functions f of this kind is

$$\sum_{i=k}^{n} \binom{n}{k} = 2^n - 1.$$

Since if R and R' have the same weight function, then also $R \cup R'$ have this weight function, we can merge the relations in the transition invariant found in such a way their number is less or equal to the number of the possible weight functions. Unfortunately also this bound is exponential (in the number of variables), so it is huge too.

Example 3.3.10. The program considered in Example 3.2.4 is tail-recursive. Observe that

$$\begin{aligned}
\Phi(G_{\tau_0}) &= \Big\{(s', s) \mid s'(\mathrm{pc}) = s(\mathrm{pc}) = g \wedge s'(z) < s(z) \wedge s'(\exp) \le s(\exp)\Big\};\\
\Phi(G_{\tau_1}) &= \Big\{(s', s) \mid s'(\mathrm{pc}) = s(\mathrm{pc}) = f \wedge s'(y) < s(y) \wedge s'(x) \le s(x)\Big\};\\
\Phi(G_{\tau_2}) &= \Big\{(s', s) \mid s'(\mathrm{pc}) = f \wedge s(\mathrm{pc}) = g \wedge s'(x) \le s(x)\\
&\qquad \wedge\ s'(y) \le s(y) \wedge s'(z) \le s(x) \wedge s'(\exp) \le s(\exp)\Big\};\\
\Phi(G_{\tau_1\tau_2\tau_0}) &= \Big\{(s', s) \mid s'(\mathrm{pc}) = f \wedge s'(z) < s(x)\Big\}.
\end{aligned}$$

Then $\Phi(G_{\tau_0}) \cup \Phi(G_{\tau_1}) \cup \Phi(G_{\tau_2}) \cup \Phi(G_{\tau_1\tau_2\tau_0})$ is already a transition invariant of height ω for the transition-based program which corresponds to it.

```
while  ( y > 0 )
      g: temp = 0; z = x;
      while ( z > 0 )
           temp = temp +  esp; z = z - 1;
      f: y = y - 1; esp = temp;
```

Trivially, also $\bigcup\{\Phi(G) \mid G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})\}$ is a transition invariant and the function computed is primitive recursive.

Furthermore we can observe that each primitive recursive function has a tail-recursive implementation which is SCT*.

Proposition 3.3.11. *Each primitive recursive function has an implementation which is* SCT**.*

Proof. By induction on the primitive recursive functions.

- For the constant function, successor function and the projection function it is trivial since we can write tail-recursive first order functional programs which have no idempotent size-change graphs.

- Assume that the primitive recursive functions $g_0, \dots, g_{n-1}$ and f have an implementation which is SCT*. By using them it is straightforward to show that the standard program which computes their composition

  $$h(x_0, \dots, x_{k-1}) = f(g_0(x_0, \dots, x_{k-1}), \dots, g_{n-1}(x_0, \dots, x_{k-1}))$$

  is SCT*. In fact each idempotent size-change graph corresponds to some call in the definitions either of g_i for some $i < n$ or of f.

- Assume that the primitive recursive functions f and g have a SCT* program which computes it. Then, by using these programs we can define a tail-recursive SCT* program which computes

$$h(x_0,\dots,x_{k-1},y) = \begin{cases} f(x_0,\dots,x_{k-1}) & \text{if } y=0 \\ g(h(x_0,\dots,x_{k-1},y-1),y) & \text{otherwise.} \end{cases}$$

An example of such a program is:

$$\begin{aligned} f(x_0,\dots,x_{n-1}) &:= \dots; \\ g(y_0,y_1) &:= \dots; \\ h(r,x_0,\dots,x_{n-1},y) &:= \text{if } (y=0) \quad r; \\ &\qquad \text{else} \quad h(g(r,y),x_0,\dots,x_{n-1},y-1); \\ h(f(x_0,\dots,x_{n-1}),&x_0,\dots,x_{n-1},y); \end{aligned}$$

As observed in the previous point each size-change graph which corresponds to some call either in f or in g has the desired property. There is only one new size-change graph $G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})$ derived from the definition of h. Since $y \xrightarrow{\downarrow} y \in G$, we are done. □

3.4 Transition Invariants Termination as a property of size-change graphs

In the previous section we proved that if a tail-recursive program is SCT* then it has a transition invariant of height ω. In this section we provide a statement on functional programs strictly weaker than SCT* which is equivalent to the definition of termination by Podelski and Rybalchenko, so it is equivalent to have a transition invariant of general height.

Thanks to Lemma 3.3.4, we saw that if $\mathcal{P}$ is tail-recursive, $\mathcal{P}$ is SCT* if and only if for every $G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})$ idempotent $\Phi(G)$ is well-founded. Recall that Podelski and Rybalchenko analyse the termination of transition-based programs, and in order to have a simple relationship with transition-based programs we restrict the functional programs to be tail-recursive. Here we prove that a tail-recursive program $\mathcal{P}$ has a disjunctively well-founded transition invariant if and only if for any $G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})$ idempotent $\Phi(G) \cap R^+$ is well-founded, where R is the transition relation of $\mathcal{R}_{\mathcal{P}}$. The proof of "for any G idempotent $\Phi(G) \cap R^+$ is well-founded implies termination" follows, with some little changes, the proof of Corollary 3.3.8 studied in [43, Corollary 33].

The first step is to intuitionistically prove another version of the Theorem Idempotence and well-foundedness [43, Theorem 32]. In order to do that we need the following lemma.

Lemma 3.4.1. *Let R be a binary relation on I, $k \in \mathbb{N}$ and T be a transitive binary relation. If $R^k \cap T$ is well-founded then $R \cap T$ is well-founded.*

Proof.

- Let $k=2$. Induction on x with respect to R^2. Assume that

$$\forall z(z(R^2\cap T)x \implies z \text{ is } (R\cap T)\text{-well-founded }).$$

By two applications of Proposition 1.2.11,

$$\begin{aligned} x \text{ is } (R\cap T)\text{-well-founded} &\iff \forall y(y(R\cap T)x \implies y \text{ is } (R\cap T)\text{-well-founded}) \\ &\iff \forall y(y(R\cap T)x \implies (\forall z(z(R\cap T)y \implies z \text{ is } (R\cap T)\text{-well-founded}))). \end{aligned}$$

Observe that since $z(R\cap T)y$ and $y(R\cap T)x$ then $z(R^2\cap T)x$. This implies by inductive hypothesis that z is $(R\cap T)$-well-founded. So for every $x\in I$, x is $(R\cap T)$-well-founded.

- The idea of the proof for $k>2$ is to prove it by induction on x with respect to R^k and to repeat the same argument provided in the case above, by using k-many steps following of Proposition 1.2.11 in order to get $z(R^k\cap T)x$. By applying the inductive hypothesis we obtain our thesis. □

Theorem 3.4.2. *If*

$$\forall G\in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})(G;G=G \implies \Phi(G)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc}) \text{ well-founded })$$

then $\Phi(G)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc})$ *is well-founded for every graph in* $\mathsf{cl}(\mathcal{G}_{\mathcal{P}})$.

Proof. Let $G\in\mathsf{cl}(\mathcal{G}_{\mathcal{P}})$ be a size-change graph. We have two cases. On the one hand, if the source and target of G do not coincide, then $\Phi(G)\circ\Phi(G)=\emptyset$, therefore $\Phi(G)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc})$ is well-founded. On the other hand, assume that source and target of G coincide. In this case G^n is defined for all $n\in\mathbb{N}$. Since the semigroup $(\{G^n \mid n\in\mathbb{N}\},;)$ is finite, by Lemma 3.3.5 it has an idempotent element G^k. By Lemma 3.3.2 the inclusion $\Phi(G)^k\subseteq\Phi(G^k)$ holds. Then

$$\Phi(G)^k\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc})\subseteq\Phi(G^k)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc}).$$

By hypothesis, since G^k is idempotent we have: $\Phi(G^k)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc})$ is well-founded. Then $\Phi(G)^k\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc})$ is well-founded by Lemma 3.3.2 and therefore by Lemma 3.4.1 also $\Phi(G)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc})$ is well-founded. □

Therefore we obtain the corresponding version of Corollary 3.3.8. Note that the number of relations in the provided transition invariant is again quite big.

Corollary 3.4.3. *Let $\mathcal{P}$ be a program and let $\mathcal{G}_{\mathcal{P}}$ be a set of size-change graphs that is a safe description of $\mathcal{P}$. If for every $G\in\mathsf{cl}(\mathcal{G}_{\mathcal{P}})$ idempotent, $\Phi(G)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc})$ is well-founded, then the relation defined by its closure $\mathsf{cl}(\mathcal{G}_{\mathcal{P}})$*

$$\bigcup\left\{\Phi(G)\cap R^+\cap(\mathrm{Acc}\times\mathrm{Acc}) \mid G\in\mathsf{cl}(\mathcal{G}_{\mathcal{P}})\right\}$$

is a disjunctively well-founded transition invariant for $\mathcal{R}_{\mathcal{P}}$.

Proof. Due to the proof of Corollary 3.3.8 in [43, Corollary 33]

$$R^+ \subseteq \bigcup \{\Phi(G) \mid G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})\}.$$

Then

$$R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \subseteq \bigcup \left\{\Phi(G) \cap R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \mid G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})\right\}.$$

Moreover, by hypothesis and thanks to Lemma 3.4.2 the relation $\Phi(G) \cap R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded. Hence $\bigcup \left\{\Phi(G) \cap R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \mid G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})\right\}$ is a disjunctively well-founded transition invariant for $\mathcal{R}_{\mathcal{P}}$. □

Finally we prove the equivalence, the other implication is trivial.

Theorem 3.4.4. *Given a program $\mathcal{P}$ the followings are equivalent:*

1. *for every $G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})$ idempotent, $\Phi(G) \cap R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded;*
2. *there is a disjunctively well-founded transition invariant for $\mathcal{R}_{\mathcal{P}}$.*

Proof. "$\Uparrow$": If (2) holds, then $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded. Then for every $G \in \mathsf{cl}(\mathcal{G}_{\mathcal{P}})$

$$\Phi(G) \cap R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \subseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}),$$

is well-founded.

"$\Downarrow$": If (1) holds, then by Corollary 3.4.3 we obtain the thesis. □

To conclude observe that the condition (1) in the previous theorem is strictly weaker than being SCT. To put otherwise: using a transition invariant, we may prove terminating some programs $\mathcal{R}_{\mathcal{P}}$ which are transition-based translations of non-SCT tail-recursive programs $\mathcal{P}$. The following basic example explains why.

Example 3.4.5. Let us consider the following functional program:

$$\begin{aligned} f(x,y) := \quad & \text{if } (x > y) \quad x \\ & \text{else } \tau : f(x+1, y). \end{aligned}$$

It is not SCT since G_τ is idempotent and has no decreasing edges. In particular $\Phi(G)$ is not well-founded. However $\Phi(G) \cap R^+$ is and therefore this program satisfies the condition (1) of Theorem 3.4.4, thus in particular it is terminating.

3.5 Related and further works

In Section 2.2 we showed that we may provide an intuitionistic proof of the Termination Theorem by replacing the use of Ramsey's Theorem for pairs with the use of the H-closure Theorem. In this chapter we did the same for the SCT Theorem. This is not the first intuitionistic proof of the SCT Theorem. Vytiniotis, Coquand and Wahlsteldt in [93] intuitionistically proved it by using Almost Full relations. The proof of the SCT Theorem in [93] uses the following facts:

- almost full relations are closed under finite intersections;
- if R and T are two binary relations such that $T \cap R^{-1} = \emptyset$ and R is almost full, then T is well-founded.

In our intuitionistic proof, instead, we use H-well-founded relations and:

- H-well-founded relations are closed under finite unions;
- if a binary relation R is H-well-founded and transitive then it is well-founded.

Since Ramsey's Theorem for pairs is used in many branches of mathematics, in future works we hope to apply this method to other classical results based on it, in order to obtain intuitionistic proofs.

We proved that the functions which are computed by a tail-recursive SCT* program are exactly the primitive recursive functions. This result fits in with the one by Ben-Amram [6]: he proved that tail-recursive SCT programs compute primitive recursive functions. More in details, for any tail-recursive SCT program we provided a primitive recursive bound for the number of computation steps given an input. However as discussed in Subsection 3.3 the bound obtained in this way is large. An open question is whether we may extract from the intuitionistic proof of the SCT* Theorem a bound tighter than this one.

Chapter 4

Termination Theorem via Bar Recursion

This chapter arose from an attempt to compare the bounds for the Termination Theorem obtained in constructive mathematics, proof theory and reverse mathematics (as in Chapter 2, Chapter 5 and [34, 93]) with the one we can obtain by using another fundamental tool of proof theory: Spector's bar recursion [85]. Spector's extension of the Dialectica interpretation assures us that the proof of the Termination Theorem can be interpreted using $\mathsf{T}+$ bar recursion. Following the approach of the proof mining program [54], we are applying the ideas from the bar-recursive interpretation of countable choice without following literally the functional interpretation of the proof, in order to obtain human-readable bounds.

The sub-recursive bounds we obtain make use of bar recursion, in the form of the product of selection functions [32], as this is used to interpret the Weak Ramsey Theorem for pairs.[1] The construction can be seen as calculating a modulus of well-foundedness for a given program given moduli of well-foundedness for the disjunctively well-founded finite set of covering relations. When the input moduli are in system T, this modulus is also definable in system T by a result of Schwichtenberg on bar recursion [82]. Throughout this chapter we work in extensions of HA^{ω}. Results in this chapter are a joint work with Stefano Berardi and Paulo Oliva [9].

4.1 Bounds with classical well-foundedness

In the two intuitionistic versions of the Termination Theorem proposed in [93] and in Section 2.2 both the hypothesis and the thesis are intuitionistically stronger than the ones in Podelski and Rybalchenko's Termination Theorem (see Section 1.2). In fact the theorem proved is the following:

[1] Weak Ramsey Theorem is a corollary of Ramsey's Theorem for pairs defined in [66].

Theorem 4.1.1 (Intuitionistic Termination Theorem)**.** *A program $\mathcal{R}$ has a inductively well-founded transition relation if and only if there exists a disjunctively inductively well-founded transition invariant for $\mathcal{R}$.*

In this chapter we give an semi-intuitionistic proof of the Termination Theorem by considering the classical definition of well-foundedness, as introduced in Section 1.2. Hence here we say that a transition-based program $\mathcal{R} = (S, I, R)$ is terminating if

$$\forall \sigma \in {}^{\mathbb{N}}S(\sigma_0 \in I \implies \exists n \neg(\sigma_{n+1} R \sigma_n)).$$

In Chapter 2 and [34] it has been proved that if the transition invariant is simple enough (for example if the height of the relations which composed the transition invariant is ω), then we have a primitive recursive bound for the transition relation of the program. We want to study a more general situation. We assume that any functional in this chapter is continuous. Brouwer's definition of continuous functional is inductive. A constant functional is continuous. Given a countable family of continuous functional $\{F_m : m \in \mathbb{N}\}$ and a natural number n, then the functional F defined as: $F(\alpha) = F_{\alpha(n)}(\alpha)$ is continuous.[2] Assume that R is a binary relation on $\mathbb{N}$, and let us call a *modulus of well-foundedness* for R any map[3] $\omega : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ providing a bound for the length of any R-sequence. That is, we assume that for all sequences $\sigma \in {}^{\mathbb{N}}\mathbb{N}$ there is some $j < \omega(\sigma)$ such that $\neg(\sigma_{j+1} R \sigma_j)$. Our main theorem is:

Theorem 4.1.2. *Let $\mathcal{R}$ be a given program with transition relation R. There exists a construction Φ, definable in $\mathsf{T}+$ bar recursion, such that for all $n \in \mathbb{N}$ and all continuous $\omega_0, \ldots, \omega_{n-1} : {}^{\mathbb{N}}\mathbb{N} \to \mathbb{N}$ and $R_0, \ldots, R_{n-1}$, if*

$$R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \subseteq R_0 \cup \cdots \cup R_{n-1} \ \wedge \ \forall i < n \ \forall \sigma \ \exists j < \omega_i(\sigma) \ \neg(\sigma_{j+1} R_i \sigma_j)$$

then, for all σ such that $\sigma_0 \in \mathrm{I}$

$$\exists m < \Phi(R, \omega_0, \ldots, \omega_{n-1}, R_0, \ldots, R_{n-1}, \sigma) \ \neg(\sigma_{m+1} R \sigma_m).$$

For short when the parameters are clear from the context we write $\Phi(\sigma)$ instead of $\Phi(R, \omega_0, \ldots, \omega_{n-1}, R_0, \ldots, R_{n-1}, \sigma)$. As stated, the functional Φ is definable in $\mathsf{T}+$ bar recursion. We can then study the complexity of the bound Φ relative to the complexity of the relations R_i and the functions ω_i. For instance, assuming that all relations $R_0, \ldots, R_{k-1}$ have modulus of well-foundedness in system T [85, Section 5], we obtain that $\Phi(\sigma)$ is also T-definable, due to the result by Schwichtenberg [82] on the closure of Spector's bar recursion of types $\mathbb{N}$ and $\mathbb{N} \to \mathbb{N}$ over system T.

The basic ideas behind the proof are those presented by Oliva and Powell in [69] for the classical Ramsey's Theorem, which in turn is based on [56]. Our proof of Theorem

[2] By assuming Brouwer's Thesis a functional F is continuous if and only if $\forall \alpha \exists n \forall \beta([\alpha](n) = [\beta](n) \implies F(\alpha) = \omega(\beta))$ (e.g. see [89]).

[3] Note that ω does not indicate the least infinite ordinal. Throughout this chapter we overlay ω in the attempt to follow the standard notation introduced in previous works on selection functions, e.g. [32].

4.1.2 requires the Weak Ramsey Theorem for pairs, a (strictly weaker) corollary of the Ramsey's Theorem for pairs [66]. In this corollary we state the existence of an infinite sequence with all edges between two consecutive elements of the same color. For a complete discussion about this version of Ramsey's Theorem for pairs we remand to Chapter 5.

Theorem 4.1.3 (Weak Ramsey Theorem for Pairs)**.** *For any coloring $c : [\mathbb{N}]^2 \to n$ there exists an infinite homogeneous sequence, i.e. there exists a $k < n$ and a $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall i(\sigma_i < \sigma_{i+1} \wedge c(\sigma_i, \sigma_{i+1}) = k).$$

Theorem 4.1.3 will be fully proved in Section 4.3 and it is the only part of Ramsey's Theorem required in order to follow this chapter.

The structure of the proof of our main result, Theorem 4.1.2, is the following. Let us consider the case $n = 2$ for simplicity:

- Given a computation σ, which we wish to show is well-founded, we consider all the finite prefixes of σ.

- As in Blackwell's proof of Erdős-Szerkeres Theorem (see [14] and Theorem 4.3.2) we associate to each prefix of σ a finite sequence of finite sequences. These are lists of sequences connected by edges in one of the colors, whose last elements "behaves well" with respect to the other color.

- The moduli of well-foundedness ω_0 and ω_1 are used to control the size of this sequence of sequences.

- Bar recursion is used to compute how far into σ we need to look to be sure that the well-foundedness of σ can be shown.

4.2 Bar Recursion

In order to obtain the witness of Theorem 4.1.2, we will make use of Spector's bar recursion in the guise of the product of selection functions, cf. [32]. Although the two recursion schemata are computationally equivalent (over system T), the product of selection functions seems to have a more intuitive reading given its connection to calculations of optimal strategies in higher-order games [30, 31].

In this section we summarize the definitions and the main theorem we are going to use. First of all we need to introduce some notation we use in this chapter. We use $x : \tau$ and x^τ to say "x has type τ".

Notation 4.2.1. For any $s : \tau_0^*$, $\eta : \mathbb{N} \to \tau_0$, $q : (\mathbb{N} \to \tau_0) \to \tau_1$, $p : \tau_0 \to \tau_1$ and $\delta : (\tau_0 \to \tau_1) \to \tau_0$ we will use the following abbreviations.

- $s * t$, of type τ_0^* or $\mathbb{N} \to \tau_0^*$, is the concatenation of a finite sequence $s : \tau_0^*$ and a finite $t : \tau_0^*$ or infinite sequence $t : \mathbb{N} \to \tau_0$.

- $\hat{s} = s * 0^{\mathbb{N}\to\tau_0} : \mathbb{N} \to \tau_0$, is the canonical infinite extension of s. We extend s with infinitely many $0 : \tau_0$.
- $[\eta^{\mathbb{N}\to\tau_0}](n) = \langle \eta(0), \dots, \eta(n-1)\rangle : \tau_0^*$, initial segment of η of length n.
- $q_x^{(\mathbb{N}\to\tau_0)\to\tau_1}(\eta) = q(\langle x\rangle * \eta) : \tau_1$, where x is of type τ_0.
- $\overline{\delta}(p) = p(\delta(p)) : \tau_1$.

A selection function is any element of type $(\tau_0 \to \tau_1) \to \tau_0$. In this chapter we introduce selection functions in general but we only use them in the case $\tau_0 = \mathbb{N}^*$ and $\tau_1 = \mathbb{N}$. In [32], Escardó and Oliva defined a binary product of selection functions and then the iterated product. However, for sake of brevity, here we define the product of selection functions with an equivalent definition provided in [69].

Definition 4.2.2 (EPS [69]). Let $\{\varepsilon_s : (\tau_0 \to \tau_1) \to \tau_0 \mid s : \tau_0^*\}$ be a family of selection functions. Let $\omega : (\mathbb{N} \to \tau_0) \to \mathbb{N}$ and $q : (\mathbb{N} \to \tau_0) \to \tau_1$. The *explicitly controlled product of the selection functions* EPS is defined as follows:

$$\mathrm{EPS}_s^\omega(\varepsilon)(q) \stackrel{\mathbb{N}\to\tau_0}{=} \begin{cases} 0^{\mathbb{N}\to\tau_0} & \text{if } \omega(\hat{s}) < |s| \\ \langle e_s\rangle * \mathrm{EPS}^\omega_{s*e_s}(\varepsilon)(q_{e_s}) & \text{otherwise} \end{cases} \tag{4.1}$$

where $e_s \stackrel{\tau_0}{=} \varepsilon_s(\lambda x.\overline{\mathrm{EPS}^\omega_{s*x}(\varepsilon)}(q_x))$.

Given ω, $\{\varepsilon_s \mid s : \tau_0^*\}$, q and any finite sequence s, $\mathrm{EPS}_s^\omega(\varepsilon)(q) : \mathbb{N} \to \tau_0$ is an infinite sequence over τ_0, which we use to define some infinite extension $s * \mathrm{EPS}_s^\omega(\varepsilon)(q)$ of the finite sequence $s : \tau_0^*$. The infinite sequence $s * \mathrm{EPS}_s^\omega(\varepsilon)(q)$ actually consists of a finite sequence t followed by zeros of type τ_0, where t is the first prefix which satisfies $\omega(\hat{t}) < |t|$. In order to guarantee that this stopping condition is eventually reached one normally assumes continuity or majorizability of ω [12, 80].

By T+EPS we mean we have constants EPS with axioms (4.1) for each τ_0, τ_1. The main theorem of Spector's bar recursion which provides a solution to the Dialectica interpretation of double negation shift (DNS) can be stated by using EPS as in [69].

Theorem 4.2.3. *Let $q : (\mathbb{N} \to \tau_0) \to \tau_1$ and $\omega : (\mathbb{N} \to \tau_0) \to \mathbb{N}$ and $\{\varepsilon_s \mid s : \tau_0^*\}$ be given. Define*

$$\eta \stackrel{\mathbb{N}\to\tau_0}{=} \mathrm{EPS}^\omega_{\langle\rangle}(\varepsilon)(q);$$

$$p_s(x) \stackrel{\tau_1}{=} \overline{\mathrm{EPS}^\omega_{s*x}(\varepsilon)}(q_{s*x}).$$

Then, for any $n \leq \omega(\eta)$ we have

$$\begin{aligned} \eta(n) &\stackrel{\tau_0}{=} \varepsilon_{[\eta](n)}(p_{[\eta](n)}) \\ q(\eta) &\stackrel{\tau_1}{=} \overline{\varepsilon_{[\eta](n)}}(p_{[\eta](n)}). \end{aligned}$$

For the proof of this theorem and for a complete analysis of the main properties of the product of selection functions we refer to [32, 69].

4.3 The Case of Two Colors

The goal of this section is to present a *recursive* construction $\Phi(\sigma)$ for Theorem 4.1.2 when $n = 2$. In the sub-sequent section we then show how to make Φ *sub-recursive* by defining it in T extended with Spector's bar recursion.

In [14], Blackwell provided a proof of the following corollary of the Finite Ramsey Theorem.

Theorem 4.3.1 (Erdős-Szerkeres Theorem)**.** *Let n and m be natural numbers. Given a sequence with more than $n \cdot m$ distinct natural numbers, if any decreasing subsequence has at most m elements, then there exists an increasing subsequence with more than n elements.*

We first generalize Blackwell's proof of Erdős-Szerkeres Theorem for the Weak Ramsey Theorem.

Theorem 4.3.2. *Let $c : [\mathbb{N}]^2 \to 2$, then there exists either an infinite sequence in color 0 or an infinite homogeneous set in color 1.*

Proof. Given a well ordered set X we say that a non-empty finite or infinite sequence s is a *leftmost* sequence of X if and only if all $s_i \in X$ and

- $s_0 = \min X$;
- $s_i < s_{i+1}$;
- $c(\{s_i, s_{i+1}\}) = 0$;
- $\forall x^X (s_i < x < s_{i+1} \implies c(\{s_i, x\}) \neq 0)$.

We construct a sequence of leftmost sequences, as Blackwell does, as follows.

- r_0 is the maximal *leftmost* sequence of $\mathbb{N}$.
- r_i is the maximal *leftmost* sequence of $\mathbb{N} \setminus \bigcup\{r_j \mid j < i\}$.

We stop in the first infinite leftmost sequence, otherwise we produce an infinite sequence of sequences. Since $\mathbb{N}$ is infinite we have either an infinite sequence r_i or infinitely many finite non-empty sequences r_i. In the first case we will have an infinite sequence in color 0. In the second case let $\{l_i \mid i \in \mathbb{N}\}$ be the set of last elements of each sequence r_i. Since l is an infinite sequence of pairwise distinct natural numbers, it should contain an infinite increasing subsequence. This is an infinite homogeneous set in color 1 by construction, since we only ever stop a sequence r_i if there are no more edges in color 0 from the last element of r_i. In fact the last element l_i of each sequence r_i is by construction related by an edge in color 1 to each element of $\{x \in \mathbb{N} \mid x > l_i \wedge x \notin \bigcup\{r_j \mid j \leq i\}\}$. In particular, for all $j > i$ if $l_j > l_i$, then l_i, l_j are related by an edge in color 1. $\square$

Note that Theorem 4.3.2 implies Theorem 4.1.3 in the case of two colors, since a homogeneous set is in particular a homogeneous sequence. The existence of a homogeneous set in color 1 is important to generalize the proof of Theorem 4.3.2 for n-many colors (see Section 4.5)

A coloring $c\colon [\mathbb{N}]^2 \to n$ is said to be transitive if $c(\{x,y\}) = c(\{y,z\}) = k$ implies $c(\{x,z\}) = k$. The Transitive Ramsey Theorem for pairs says that any transitive coloring $c\colon [\mathbb{N}]^2 \to n$ has an infinite homogeneous set. If the coloring is transitive and $n = 2$ then Theorem 4.3.2 gives us a homogeneous set in one of the two colors. Note, however, that if the coloring is not transitive, then we have either an infinite homogeneous sequence in color 0 or an infinite homogeneous set in color 1 (instead of infinite homogeneous set in both cases as in Ramsey's Theorem).

The proof of Theorem 4.3.2 is classical, but we can extract the constructive idea behind the argument to prove Theorem 4.1.2 for two colors. Let $\mathcal{R}$ be a program with its transition relation R. Assume that σ is such that $\sigma_0 \in \mathrm{I}$, where I is the set of the initial states of $\mathcal{R}$ and that there exist R_0, R_1, ω_0, ω_1 as in the hypothesis of Theorem 4.1.2. We have to construct a $\Phi(R,\omega_0,\omega_1,R_0,R_1,\sigma)$ and show that

$$\exists m < \Phi(R,\omega_0,\omega_1,R_0,R_1,\sigma)\ \neg(\sigma_{m+1} R \sigma_m).$$

Definition 4.3.3. Assume $R_0 \cup R_1$ is a disjunctively well-founded transition invariant for $\mathcal{R}$. Define the coloring $c : [\mathbb{N}]^2 \to 2$ on pairs as $c(\{i,j\}) = d$, where $d \in 2$ is the least such that $\sigma_j R_d \sigma_i$.

The idea is to consider finite approximations of the sequence of sequences given by the leftmost sequences of the Blackwell proof for the coloring c, approximations large enough to compute the value of $\Phi(R,\omega_0,\omega_1,R_0,R_1,\sigma)$.

In order to do that we need to define the following functions. The first one which we call $\beta(x,s)$ is the "partial" function which gives us the successor (in the sense of the leftmost sequences) of a node x with respect to a finite sequence s, representing some finite well-founded set X. If a successor of x does not exist in s the function $\beta(x,s)$ returns `err`. The return value `err` is a special symbol denoting the fact that a suitable successor of x was not found in the given finite sequence s.

Notation 4.3.4. Define $x <_i y$ if and only if $x < y$ and $c(\{x,y\}) = i$. Given a non-empty sequence s, we define $\mathrm{hd}(s)$ as the first element of s, $\mathrm{tl}(s)$ as the tail of s, and $\mathrm{last}(s)$ as the last element of s.

Definition 4.3.5. Let $x \in \mathbb{N}$ and let $x_0 < \cdots < x_{n-1} \in \mathbb{N}$. Define

$$\beta(x,\langle x_0,\ldots,x_{n-1}\rangle) = \begin{cases} x_i & \text{for the least } i < n \text{ such that } x <_0 x_i \\ \texttt{err} & \text{if no such } i < n \text{ exists.} \end{cases}$$

Observe that β is primitive recursive in the coloring c. Given an increasing sequence of natural numbers, let r_i be the i-th leftmost sequence as defined in the proof of Theorem 4.3.2. Now we define a function $\varphi(s)$ which given a sequence $s = \langle x_0,\ldots,x_{n-1}\rangle$, with

$x_0 < \cdots < x_{n-1} \in \mathbb{N}$, constructs a finite sequence of finite sequences $\{r_{i,j} \mid i \leq k, j < |r_i|\}$ consisting only of elements of s, i.e. for each i, j, we have $r_{i,j} = x_h$ for some $h < n$. We want this sequence of sequences to have the same properties as the infinite Blackwell sequence of sequences, namely

- for all $i \leq k$, r_i is a finite, non-empty $<_0$-sequence.
- for all $i < k$ and $\mathrm{last}(r_i) < y \in s$ we have $c(\{r_i, y\}) = 1$.

In particular it follows that for all $i < j \leq k$, if $\mathrm{last}(r_i) < \mathrm{last}(r_j)$ then $\mathrm{last}(r_i) <_1 \mathrm{last}(r_j)$.

We introduce now a procedure φ which can be described informally as follows. Start with the empty list of lists $r = \langle\rangle$. The variable x, which starts holding the first element of s, will hold the value of the last element considered. The assignment $(x,s) = (\mathrm{hd}(s), \mathrm{tl}(s))$ means that $x = \mathrm{hd}(s)$ and that the variable s holds the tail of the original s. At each iteration of the loop we check whether a link in color 0 exists from x into the remaining elements in s. If so, $\beta(x,s)$ will return the first such v. If no such v exists then we start a new row in the sequence of sequences r with the first element not yet considered.

Definition 4.3.6. Define the function $\varphi\colon \mathbb{N}^* \to (\mathbb{N}^*)^*$ as

$$
\begin{array}{ll}
\varphi(s)\ \{ & \\
\quad (r,i,j) = (\langle\langle\rangle\rangle, 0, 0) & \\
\quad (x,s) = (\mathrm{hd}(s), \mathrm{tl}(s)) & \\
\quad \text{while } (s \neq \langle\rangle) & \\
\quad\quad v = \beta(x,s) & \\
\quad\quad \text{if } v \neq \texttt{err} \text{ then} & (\textit{continue } <_0\textit{-sequence}) \\
\quad\quad\quad (x,s) = (v, s\backslash\{v\}) & \\
\quad\quad\quad (r_{i,j+1}, j) = (x, j+1) & \\
\quad\quad \text{else} & (\textit{start new } <_0\textit{-sequence}) \\
\quad\quad\quad (x,s) = (\mathrm{hd}(s), \mathrm{tl}(s)) & \\
\quad\quad\quad (r_{i+1,0}, i, j) = (x, i+1, 0) & \\
\quad \text{return } w & \\
\} &
\end{array}
$$

Observe that for conciseness we have identified sequences and arrays and use destructive array update.

Finally, we define the function $\mathrm{Inc}(b)$ which calculates an increasing subsequence of a given finite sequence of natural numbers b, starting with b_0.

Definition 4.3.7. Let $\langle x_0, \ldots, x_{n-1}\rangle$ be a sequence of natural numbers. Define

$$
\mathrm{Inc}(\langle x_0, \ldots, x_{n-1}\rangle) = \begin{cases} \langle\rangle & \text{if } n = 0 \\ \langle x_0 \rangle & \text{if } \forall k < n (x_0 \geq x_k) \\ \langle x_0 \rangle * \mathrm{Inc}(\langle x_j, \ldots, x_{n-1}\rangle) & \text{if } j = \mu k < n (x_0 < x_k). \end{cases}
$$

Given a finite sequence of non-empty finite sequences of natural number $r = \langle r_i \mid i \leq k \rangle$, let us denote by b the sequence of the last elements of these sequences, i.e.

$$b = \langle \mathrm{last}(r_i) \mid i \leq k \rangle.$$

Recall that under our assumption we have "moduli of termination" ω_i for each of the relations R_i, i.e.

$$\forall \sigma \exists j < \omega_i(\sigma) \neg(\sigma_{j+1} R_i \sigma_j).$$

For any given infinite sequence the modulus of termination gives us an upper bound on the point where termination is guaranteed to have happened. We must consider a large enough approximation of the Blackwell sequence of sequences, so as to make sure that we obtain a counter-example on one of the two colors.

Assuming we can perform *unbounded search*, we may define a functional ξ that computes a finite approximation to the Blackwell sequence of sequences which is big enough to obtain the desired counter-example, namely

$$\xi(\sigma) = \langle r_i \mid i \leq k \rangle = \varphi(\langle 0, \dots, n \rangle)$$

where n is the least such that, for b defined as above, $\exists i \leq k \, (\omega_0(\widehat{r_i}) < |r_i|) \vee \omega_1(\widehat{\mathrm{Inc}(b)}) < |\mathrm{Inc}(b)|$. Observe that $\xi(\sigma)$ depends not just on ω_0, ω_1 but also on R_0, R_1 and σ via the dependence on coloring c (see Definition 4.3.3).

Finally, we can define the bound $\Phi(R, \omega_0, \omega_1, R_0, R_1, \sigma)$ as the maximal element of the union of the finite sequence of sequences $\xi(\sigma)$. This Φ is a witness for Theorem 4.1.2.

We show that Φ is sub-recursive by giving a definition of ξ in system T plus Spector's bar recursion.

4.4 Bar Recursive Definition of $\xi(\sigma)$

Observe that $\xi(\sigma)$ builds a finite sequence of sequences which is an approximation of Blackwell's sequence of sequences. Firstly we prove that given $\gamma : \mathbb{N} \to \mathbb{N}^*$ satisfying certain conditions, we can compute $\xi(\sigma)$ by primitive recursion. Then we will approximate $\gamma(k)$ via bar recursion.

4.4.1 Construction of $\xi(\sigma)$ given oracle γ

Let $i, \ j, \ k : \mathbb{N}$ be variables. Define $A(i,j,k)$ as

$$A(i,j,k) \ \equiv \ j > k \wedge c(\{i,j\}) = 0.$$

First, let us assume that we have an infinite sequence γ of finite sequences such that $\gamma(k)$ finds for any $i \leq k$ the least witness for $\exists j A(i,j,k)$ whenever such a witness exists, and returning $k+1$, some dummy value greater than k, otherwise. Obviously γ is non-computable and by definition we have $(\gamma(k))_i > k$ for any $i \leq k$.

$$\forall k \forall i \leq k(\exists j A(i,j,k) \Leftrightarrow A(i,(\gamma(k))_i,k) \wedge \forall j < (\gamma(k))_i \neg A(i,j,k)). \tag{4.2}$$

Let us first show that given such oracle γ the construction of the finite sequence of finite sequences $\xi(\sigma)$ is primitive recursive. We will then find a bar-recursive approximation to γ which will be good enough for our purposes.

Let us denote with $X : \mathbb{N}^*$ any finite sequence over $\mathbb{N}$ – we use X to represent a finite subset of $\mathbb{N}$. Given γ which satisfies condition (4.2) and $X : \mathbb{N}^*$ we define $\gamma'(X) : \mathbb{N} \to \mathbb{N}$ as

$$\gamma'(X)(i) = (\gamma(k_i^X))_i, \quad \text{where } k_i^X = \mu k \leq \max(X)(i \leq k \wedge (\gamma(k))_i \notin X).$$

Observe that k_i^X exists for any i and X, since $(\gamma(\max(X)))_i > \max(X)$.

The sequence $\gamma'(X)$ is primitive recursively definable given γ and X. Define $A'(i,j,X)$ as

$$A'(i,j,X) \equiv j > i \wedge c(\{i,j\}) = 0 \wedge j \notin X.$$

It is straightforward to show directly that:

$$\exists j A'(i,j,X) \Leftrightarrow A'(i,\gamma'(X)(i),X) \wedge \forall j < \gamma'(X)(i) \neg A'(i,j,X) \tag{4.3}$$

Lemma 4.4.1. *Let $X : \mathbb{N}^*$ and $i \in \mathbb{N}$. If $\neg A'(i,\gamma'(X)(i),X)$ then*

$$\forall j \notin X(j > i \implies c(\{i,j\}) = 1).$$

Proof. By condition (4.3) from $\neg A'(i,\gamma'(X)(i),X)$ we obtain $\neg \exists j A'(i,j,X)$ and we are done. □

Let us denote by $\mu_{\mathrm{Sp}}(\omega)(\eta)$ the primitive recursive functional (cf. [68, Lemma 1.2]) that finds the first point n such that $\omega(\widehat{[\eta](n)}) < n$, for any ω continuous.[4]

Let $s, X : \mathbb{N}^*$. Define $B(s,X)$ as

$$B(s,X) \equiv \forall j < |s| - 1(A'(s(j),s(j+1),X)).$$

The formula $B(s,X)$ holds if s is an increasing sequence in color 0 whose elements s_i do not belong to X, for $0 < i < |s|$.

Define as usual $f^n(i)$ for any $f : \mathbb{N} \to \mathbb{N}$ and any $i,n \in \mathbb{N}$:

$$\begin{aligned} f^0(i) &= i \\ f^{n+1}(i) &= f(f^n(i)). \end{aligned}$$

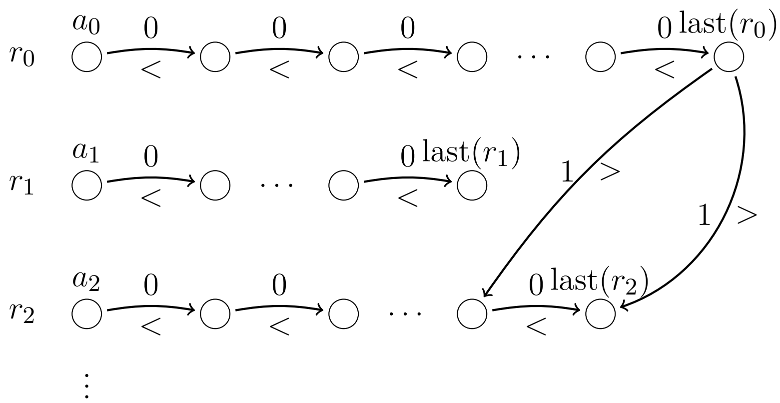
The informal idea behind our construction is as follows. We first consider the sequence $\alpha_0 = \lambda j.(\gamma'(\emptyset))^j(0)$. By definition of the coloring c and since R_0 is well-founded, any

[4]Notice that μ_{Sp} cannot by defined if ω is not continuous. For instance let $\omega(\eta) = \mu n(\eta(n) < n)$ if there is such an element, $\omega(\eta) = 0$ otherwise. Then for any n, $\omega(\widehat{[\mathrm{id}](n)}) = \omega(\widehat{\langle 0,\ldots,n-1\rangle}) \geq n$.

increasing sequence in color 0 is finite. Therefore, in particular, there exists a maximal finite initial segment r_0 of α_0 which satisfies $B(r_0, \emptyset)$. By the properties of $\gamma(\emptyset)$, r_0 is the first leftmost sequence of Blackwell's sequence of sequences. In fact, since it satisfies $B(r_0, \emptyset)$, r_0 is an increasing sequence in color 0 and moreover by maximality of r_0 and by Lemma 4.4.1 any element x which is bigger than $\mathrm{last}(r_0)$ is such that $c(\{\mathrm{last}(r_0), x\}) = 1$. Let X_0 be the set whose elements are the ones appearing in r_0 and let a_0 be the minimum element which is not in X_0. We now repeat the same argument by considering $\alpha_1 = \lambda j.(\gamma'(X_0))^j(a_0)$ in order to produce the second leftmost sequence, and so on. Formally:

Definition 4.4.2. Let ω_0 be the given modulus of well-foundedness for color 0. Let $a_{-1} = 0$ and $X_{-1} = \emptyset$. Define simultaneously α_i, r_i, X_i, and a_i, by primitive recursion over i, as:

$$
\begin{aligned}
\alpha_i &= \lambda j.(\gamma'(X_{i-1}))^j(a_{i-1});\\
r_i &= \text{maximal finite prefix of } \alpha_i \text{ such that } B(r_i, X_{i-1}) \text{ holds;}\\
X_i &= X_{i-1} \cup \{r_i(j) \mid j < |r_i|\};\\
a_i &= \mu j \le \max(X_i) + 1 (j \notin X_i).
\end{aligned}
$$



It is not self-evident that the sequences r_i are finite. This can be justified as follows:

- α_i is the sequence generated by $\gamma'(X_{i-1})$ starting with the smallest number not yet used, i.e. a_{i-1}, while avoiding all numbers already used, i.e. X_{i-1}.

- Define
$$n_i = \mu_{\mathrm{Sp}}(\omega_0)(\alpha_i).$$
n_i is the first natural number such that $\omega_0(\widehat{[\alpha_i](n_i)}) < n_i$. By the assumption on ω_0, we know that before point n_i we will find a link in color 1 in the sequence α_i, i.e. a point j such that $c(\{\alpha_i(j), \alpha_i(j+1)\}) = 1$. For such j we have $\neg B([\alpha_i](j+2), X_{i-1})$.

Hence there exists a maximal increasing sequence in color 0 r_i, with n_i being a bound on the length of r_i. As discussed before such r_i form the rows of Blackwell's sequence of sequences.

Definition 4.4.3. Let the sequences r_i be as above and $b_i = \mathrm{last}(r_i)$. Define the sequence of indexes η inductively as

$$\eta_i = \begin{cases} 0 & \text{if } i = 0 \\ \eta_{i-1} + \mu j < b_{\eta_{i-1}}(b_{\eta_{i-1}+j} > b_{\eta_i}) & \text{if } i > 0. \end{cases}$$

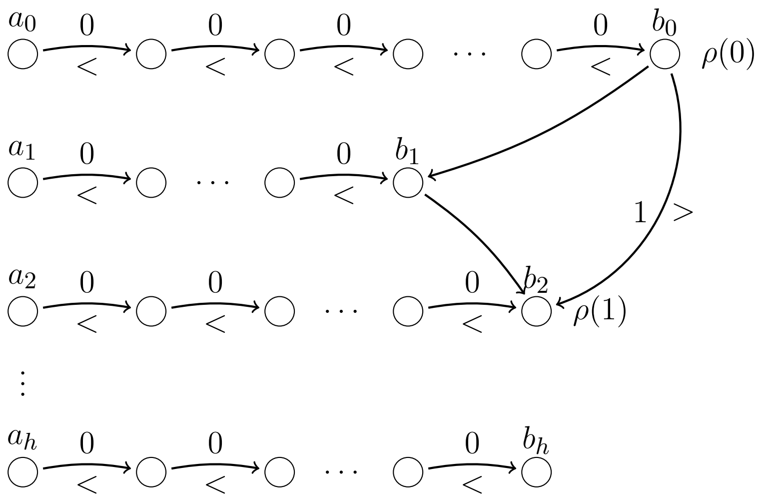
Finally, let ρ be the sub-sequence of b given by the indexes ρ, i.e. $\rho_i = b_{\eta_i}$, and let t be the maximal finite 1-prefix[5] of ρ.

Observe that the sequences $\{r_i \mid i \in \mathbb{N}\}$ are pairwise disjoint, therefore b is a sequence of pairwise distinct integers. This means for any i the point after ρ_i comes after at most ρ_i-many steps in b. Moreover t is finite since $|t| \le \mu_{\mathrm{Sp}}(\omega_1)(\rho)$. Therefore the above are well-defined.

Given the sequence of sequences r and the sequence t as defined above, define the finite Blackwell's sequence of sequences $\tilde{\xi}(\sigma)(\gamma)$ (parametrized by γ) as

$$\tilde{\xi}(\sigma)(\gamma)(i) = r_i$$

for any $i \le h$, where h is such that $\mathrm{last}(r_h) = \mathrm{last}(t)$. Note that $h \le \mathrm{last}(t)$. Moreover, since γ returns the minimal successor in color 0, we have $\tilde{\xi}(\sigma)(\gamma) = \xi(\sigma)$.



Summary of the definition of $\tilde{\xi}(\sigma)(\gamma)$ in T. In this subsection we have produced an approximation to the Blackwell sequence of leftmost sequences given an oracle γ which satisfies (4.2), namely

$$\tilde{\xi}(\sigma)(\gamma) = \langle r_i \mid i \le h \rangle,$$

[5]We say that s is a finite 1-prefix of ρ if it is a finite prefix of ρ such that $\forall i < |s| - 1(c(\{s_i, s_{i+1}\}) = 1)$.

where we recall that

$$
\begin{aligned}
r_i &= \text{maximal finite prefix of } \alpha_i \text{ such that } B(r_i, X_{i-1}) \text{ holds;}\\
X_i &= X_{i-1} \cup \{r_i(j) \mid j < |r_i|\};\\
\alpha_i &= \lambda j.(\gamma'(X_{i-1}))^j(a_{i-1});\\
a_i &= \mu j \leq \max(X_i) + 1(j \notin X_i);\\
h &= \mu j \leq \text{last}(t)(\text{last}(r_j) = \text{last}(t));\\
b &= \langle \text{last}(r_i) \mid i \in \mathbb{N}\rangle;\\
\eta_i &= \begin{cases} 0 & \text{if } i = 0\\ \eta_{i-1} + \mu j < b_{\eta_{i-1}}(b_{\eta_{i-1}+j} > b_{\eta_i}) & \text{if } i > 0; \end{cases}\\
\rho_i &= b_{\eta_i};\\
t &= \text{maximal finite 1-prefix of } \rho.
\end{aligned}
$$

Let $q(\gamma)$ keep track of the largest index used in building $\tilde{\xi}(\sigma)(\gamma)$; i.e.

$$q(\gamma) = \max \tilde{\xi}(\sigma)(\gamma) = \max(X_k) = \text{last}(t).$$

By construction, $\Phi(R, \omega_0, \omega_1, R_0, R_1, \sigma) = q(\gamma) + 1$ is a bound for the number of steps of the computation σ. In fact, by construction $c(\{\sigma_{q(\gamma)}, \sigma_{q(\gamma)+1}\}) \neq 0$ and $c(\{\sigma_{q(\gamma)}, \sigma_{q(\gamma)+1}\}) \neq 1$. Therefore $\neg(\sigma_{q(\gamma)} R \sigma_{q(\gamma)+1})$ and so the computation is over before $q(\gamma) + 1$ many steps. Since the construction of $\tilde{\xi}(\sigma)(\gamma)$ is primitive recursive in γ the last step we need in order to obtain a sub-recursive bound is to provide a sub-recursive approximation of γ.

4.4.2 Approximating γ via bar recursion

Now we want to define a sub-recursive approximation $\tilde{\gamma}$ for γ using the product of selection functions, up to the maximal point needed in the construction of $\tilde{\xi}(\sigma)(\gamma)$. We will construct the approximation $\tilde{\gamma}$ for γ so that $\tilde{\xi}(\sigma)(\gamma) = \tilde{\xi}(\sigma)(\tilde{\gamma})$.

Let $q, \omega : (\mathbb{N} \to \mathbb{N}^*) \to \mathbb{N}$ be such that $q\gamma = \omega\gamma = \max(\tilde{\xi}(\sigma)(\gamma)) = \text{last}(t)$: it is the greatest element in Blackwell's sequence of sequences.

Recall that $A(i,j,k) \equiv j > k \wedge c(\{i,j\}) = 0$. Our goal is to build a finite approximation to γ, in the sense that it will only satisfy

$$
\begin{aligned}
&\forall i \leq k (\exists j \leq q\,\tilde{\gamma}\, A(i,j,k)\\
&\qquad \iff A(i, (\tilde{\gamma}(k))_i, k) \wedge \forall j < (\tilde{\gamma}(k))_i \neg A(i,j,k) \wedge (\tilde{\gamma}(k))_i \leq q\,\tilde{\gamma})
\end{aligned}
\tag{4.4}
$$

for all $k \leq \omega(\tilde{\gamma})$. If for some $i \leq k$, $\neg(\exists j \leq q\,\tilde{\gamma}\, A(i,j,k))$, then $(\tilde{\gamma}(k))_i = k+1$, some dummy value greater than k. But this approximation is clearly enough to give us a proper bound, since we only ever use γ up to the point $\omega(\tilde{\gamma})$ when building $\tilde{\xi}(\sigma)(\gamma)$.

Definition 4.4.4. Define selection functions $\varepsilon_k, \varepsilon_{\mathbf{s}} \colon (\mathbb{N}^* \to \mathbb{N}) \to \mathbb{N}^*$ as:

- For any $k \in \mathbb{N}$.

$$
\begin{array}{ll}
\varepsilon_k(p) = \{ & \\
& \text{changed} = \mathsf{true} \\
& t = \langle k+1 \mid i \leq k \rangle \\
& v = \langle \mathsf{true} \mid i \leq k \rangle \\
& \text{while (changed)} \\
& \quad \text{changed} = \mathsf{false} \\
& \quad \text{for } (i = 0, \ldots, k) \\
& \quad\quad \text{if } v_i \wedge \exists j \in (k, p(t)](c(\{i,j\}) = 0) \text{ then} \\
& \quad\quad\quad t_i = \mu j \in (k, p(t)](c(\{i,j\}) = 0) \\
& \quad\quad\quad v_i = \mathsf{false} \\
& \quad\quad\quad \text{changed} = \mathsf{true} \\
& \text{return } t \\
\} &
\end{array}
$$

- For any $\mathbf{s} : (\mathbb{N}^*)^*$, let
$$\varepsilon_{\mathbf{s}}(p) = \varepsilon_{|\mathbf{s}|}(p).$$

Observe that:

- ε_k (and hence $\varepsilon_{\mathbf{s}}$) are well-defined, since after at most k-many iterations t will not change. By definition, any execution of the for-loop in ε_k may change t_i only if $v_i = \mathsf{true}$, and in this case v_i is set to false. This means that any t_i may change at most once, and whenever the for-loop changes no t_i, then the variable "changed" is false and the while-loop ends. Therefore, after k-many iterations either there is no i such that $v_i = \mathsf{true}$ or not. In the first case we stop, otherwise, if there exists i such that $v_i = \mathsf{true}$, then it means that t changed at most $k-1$ many times in k many loops. Therefore the computation has already ended.

- For $i \leq k$, if $\exists j \in (k, p(\varepsilon_k(p))](c(\{i,j\}) = 0)$ then at the end of the computation of $\varepsilon_k(p)$ we must have had that $v_i = \mathsf{false}$.

Lemma 4.4.5. *For any $k \in \mathbb{N}$, ε_k satisfies, for all $p : \mathbb{N}^* \to \mathbb{N}$,*

$$\forall i \leq k (\exists j < p(t) A(i,j,k) \implies A(i,t_i,k) \wedge \forall j < t_i \neg A(i,j,k) \wedge t_i \leq p(t)).$$

where $t = \varepsilon_k(p)$.

Proof. Fix p and $i \leq k$, and assume $\exists j \in (k, p(t)](c(\{i,j\}) = 0)$. Then $v_i = \mathsf{false}$ at the end of the computation of $\varepsilon_k(p)$, which means that $(\varepsilon_k(p))_i = j$ where j is the least number greater than k such that $c(\{i,j\}) = 0$. $\square$

Define

$$\tilde{\gamma} \stackrel{\mathbb{N}\to\mathbb{N}^*}{=} \mathrm{EPS}^{\omega}_{\langle\rangle}(\varepsilon)(q);$$
$$p_s(x) \stackrel{\mathbb{N}}{=} \overline{\mathrm{EPS}^{\omega}_{s*x}(\varepsilon)}(q_{s*x}).$$

By Theorem 4.2.3 for the case $\tau_0 = \mathbb{N}^*$ and $\tau_1 = \mathbb{N}$, for any $k \leq \omega\tilde{\gamma}$,

$$\tilde{\gamma}(k) = \varepsilon_{[\tilde{\gamma}](k)}(p_{[\tilde{\gamma}](k)}) \tag{4.5}$$
$$q(\tilde{\gamma}) = \overline{\varepsilon_{[\tilde{\gamma}](k)}}(p_{[\tilde{\gamma}](k)}) = p_{[\tilde{\gamma}](k)}(\varepsilon_{[\tilde{\gamma}](k)}(p_{[\tilde{\gamma}](k)})). \tag{4.6}$$

Since $\varepsilon_s = \varepsilon_{|s|}$ and $|[\tilde{\gamma}](k)| = k$, we have that for all $k \in \mathbb{N}$:

$$\varepsilon_{[\tilde{\gamma}](k)} = \varepsilon_k.$$

Therefore $\tilde{\gamma}$ satisfies for any $k \leq \omega\tilde{\gamma}$ and for any $\forall i \leq k$

$$\exists j \in (k, q\tilde{\gamma}]\ (c(\{i,j\}) = 0) \iff ((\tilde{\gamma}(k))_i > k\ \wedge\ c(\{i, (\tilde{\gamma}(k))_i\}) = 0 \wedge (\tilde{\gamma}(k))_i \leq q\tilde{\gamma}).$$

In fact let $k \leq \omega\tilde{\gamma}$; if $\exists j \in (k, q\tilde{\gamma}](c(\{i,j\}) = 0)$ holds, then by (4.6) above we have

$$\exists j \in (k, p_{[\tilde{\gamma}](k)}(\varepsilon_{[\tilde{\gamma}](k)}(p_{[\tilde{\gamma}](k)}))](c(i,j) = 0).$$

By definition of the selection functions and Lemma 4.4.5, it implies:

$$(\varepsilon_{[\tilde{\gamma}](k)}(p_{[\tilde{\gamma}](k)}))_i > k\ \wedge\ c(\left\{i, (\varepsilon_{[\tilde{\gamma}](k)}(p_{[\tilde{\gamma}](k)}))_i\right\}) = 0$$
$$\wedge\ (\varepsilon_{[\tilde{\gamma}](k)}(p_{[\tilde{\gamma}](k)}))_i \leq p_{[\tilde{\gamma}](k)}(\varepsilon_{[\tilde{\gamma}](k)}(p_{[\tilde{\gamma}](k)})).$$

Hence, by (4.5) and (4.6) above, we obtain:

$$(\tilde{\gamma}(k))_i > k\ \wedge\ c(\{k, (\tilde{\gamma}(k))_i\}) = 0\ \wedge\ (\tilde{\gamma}(k))_i \leq q\tilde{\gamma}.$$

To conclude observe that $\mathrm{EPS}^{\omega}_{\langle\rangle}(\varepsilon)(q)$ is defined by bar Recursion of type $\mathbb{N}$ then, thanks to Schwichtenberg's result [82] and by assuming that ω_0 and ω_1 are in T, we can conclude that $\tilde{\gamma}$ is in T.

Summary of the definition of $\xi(\sigma)$ in $\mathsf{T}+$ bar recursion. In this subsection we produced a bar-recursive approximation $\tilde{\gamma}$ of γ up to the point we need in the construction of $\tilde{\xi}(\sigma)(\gamma)$

$$\tilde{\gamma} \stackrel{\mathbb{N}\to\mathbb{N}^*}{=} \mathrm{EPS}^{\omega}_{\langle\rangle}(\varepsilon)(q)$$

where $q, \omega : (\mathbb{N} \to \mathbb{N}^*) \to \mathbb{N}$ are such that $q\gamma = \omega\gamma = \max(\tilde{\xi}(\sigma)(\gamma))$ and ε is defined by Definition 4.4.4.

Since $\tilde{\gamma}$ is such that $\tilde{\xi}(\sigma)(\gamma) = \tilde{\xi}(\sigma)(\tilde{\gamma})$ and $\xi(\sigma) = \tilde{\xi}(\sigma)(\gamma)$, we get:

$$\xi(\sigma) = \tilde{\xi}(\sigma)(\tilde{\gamma}).$$

Since $\tilde{\gamma}$ is definable in $\mathsf{T}+$ bar recursion, $\Phi(R,\omega_0,\omega_1,R_0,R_1,\sigma)$ and $\tilde{\xi}(\sigma)(\tilde{\gamma})$ also are.

4.5 The General Case

In this section we prove Theorem 4.1.2 in the general case of n colors. This will be done by induction on n making use of the same construction for two colors presented in Section 4.3 and Section 4.4. Firstly we want to generalize Blackwell's proof of Theorem 4.3.2 in order to prove Theorem 4.1.3.

Proof of 4.1.3. Proof by induction on n. The base case, $n=2$, follows from Theorem 4.3.2. Assume that the result holds for n, and let a coloring $c:[\mathbb{N}]^2 \to n+1$ be given. Analogously to the case of two colors, given a well ordered set X we define the leftmost sequences of X (see the proof of Theorem 4.3.2).

- r_0 is the maximal *leftmost* sequence of $\mathbb{N}$.
- r_i is the maximal *leftmost* sequence of $\mathbb{N} \setminus \bigcup \{r_j \mid j<i\}$.

Since $\mathbb{N}$ is infinite we have either an infinite sequence or, if there is none, infinitely many finite sequences. In the first case we will have an infinite sequence in color 0 and we are done. In the second case let $\{l_i \mid i \in \mathbb{N}\}$ be the set of the last element of each such sequence. Since it is an infinite sequence of natural numbers, it should contain an infinite increasing subsequence. This is an infinite set in n colors, by construction. Then by induction hypothesis there exists an infinite homogeneous sequence. $\square$

We now prove Theorem 4.1.2 by induction on n. Define $\Phi_1(R,\omega_0,R_0,\sigma)=\omega_0(\sigma)$. It is straightforward to show directly that if $R^+ \cap (\mathrm{Acc}\times\mathrm{Acc}) \subseteq R_0$ and R_0 has modulo of well-foundedness ω_0, then $\Phi_1(R,\omega_0,R_0,\sigma)$ is a bound on the length of the computation σ. In the case of two relations we define

$$\Phi_2(R,\omega_0,\omega_1,R_0,R_1,\sigma)=\Phi(R,\omega_0,\omega_1,R_0,R_1,\sigma),$$

for Φ defined in Section 4.3. As shown in Section 4.3, if the input satisfies the hypothesis of Theorem 4.1.2 then $\Phi(R,\omega_0,\omega_1,R_0,R_1,\sigma)$ is a bound for σ. Assume that Theorem 4.1.2 holds for n with witness Φ_n, we will prove it for $n+1$.

Let $\mathcal{R}$ be a program with its transition relation R. Assume that σ is such that $\sigma_0 \in \mathrm{I}$, where I is the set of the initial states of $\mathcal{R}$ and that there exist $R_0, \ldots, R_n$, $\omega_0, \ldots, \omega_n$ as in the hypothesis of Theorem 4.1.2. We have to prove that there exists Φ_{n+1} such that

$$\exists m < \Phi_{n+1}(R,\omega_0,\ldots,\omega_n,R_0,\ldots,R_n,\sigma)\ \neg(\sigma_{m+1} R \sigma_m).$$

Let the functions Inc, β and φ be as defined in Section 4.3. Similarly to Section 4.3, define

$$\xi_{n+1}(\sigma)=\varphi(\langle 0,\ldots,m\rangle)$$

where m is the least such that

$$\exists i \leq m(\omega_0(\widehat{r_i}) < |r_i|) \vee \Phi_n(R, \omega_1, \ldots, \omega_n, R_1, \ldots, R_n, \widehat{\mathrm{Inc}(b)}) < |\mathrm{Inc}(b)|$$

where $r = \varphi(\langle 0, \ldots, m\rangle)$ and $b_i = \mathrm{last}(r_i)$. We then let $\Phi_{n+1}(R, \omega_0, \ldots, \omega_n, R_0, \ldots, R_n, \sigma)$ be the maximal element of the finite sequence of sequences $\xi_{n+1}(\sigma)$. In order to justify this definition, observe that in this construction the sequence $\mathrm{Inc}(b)$ is a homogeneous set in n colors, thus we are in the following case:

$$(R \upharpoonright \mathrm{Inc}(b))^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \subseteq R_1 \cup \cdots \cup R_n \text{ and } R_1, \ldots, R_n \text{ well-founded.}$$

Hence $R \upharpoonright \mathrm{Inc}(b) = \mathrm{Inc}(b)$ is well-founded (with modulus Φ_n). Note that, as in Section 4.3, the definition of $\Phi_{n+1}(R, \omega_0, \ldots, \omega_n, R_0, \ldots, R_n, \sigma)$ is recursive, but not primitive recursive. As in Section 4.4, we can give a definition in $\mathsf{T}+$ bar recursion, by using the same argument we used in the case with two colors. When the input R_i and ω_i are T-definable, by Swichtenberg's result, $\Phi_{n+1}(R, \omega_0, \ldots, \omega_n, R_0, \ldots, R_n, \sigma)$ is also T-definable.

Proposition 4.5.1. *For $n \geq 2$ the functional $\Phi_n(\sigma)$ is in $\mathsf{T}+$ bar recursion.*

Proof. Again by induction. For $n = 2$ we proved it in the previous section. Assume that for any σ, $\Phi_n(\sigma)$ is in $\mathsf{T}+$ bar recursion. Then, by applying the same procedure we used for the case in two colors with $\omega_1 = \Phi_n$, we start building the sequences r_i in color 0. Let ρ be the increasing subsequence of the sequence composed of the last elements of the finite sequences r_i, as defined in Section 4.4.1. Therefore we consider $\mu_{\mathrm{SP}}(\Phi_n)(\rho)$, as in Section 4.4.1, to find a bound for the number of sequences in $\xi(\sigma)$. By induction hypothesis $\Phi_n(\rho)$, and hence also $\Phi_{n+1}(\sigma)$, are in $\mathsf{T}+$ bar recursion. □

4.6 Further work

The main goal of this chapter was to obtain a *sub-recursive*[6] bound for the Podelski-Rybalchenko Termination Theorem. In here we proved, using Schwichtenberg's result [82], that we can extract a bound from the proof of Theorem 4.1.2 which is in system T when the given ω_i are also in T. Schwichtenberg's proof, however, is indirect, going via α-recursion and an infinitary version of system T. As pointed out by Alexander Kreuzer, Howard provided an ordinal analysis of the constant of bar recursion of type 0 [48]. Kreuzer refined Howard's ordinal analysis of bar recursion to term in Grzegorczyk's hierarchy [55]. This approach should provide the computational level of Φ in system T. We are currently working on a direct proof of Schwichtenberg's construction in order to obtain an explicit construction of Φ in T when we are given concrete $\omega_i \in \mathsf{T}$.

System T has primitive recursion in all finite types. If we assume that the ω_i are primitive recursive (i.e. in T_0), the bound provided is in system T (apparently there are no trivial arguments which guarantee that it is in T_0). Hence in this specific case it is larger than the bounds obtained in Chapter 2 and in Chapter 5.

[6]We stress that obtaining a *recursive* bound is trivial, since, once we know that the program terminates, we can simply search for the point of termination.

Chapter 5

Reverse Mathematical bounds for the Termination Theorem

The goal of this chapter is to study the H-closure Theorem and the Termination Theorem from the viewpoint of Reverse Mathematics. Results from Reverse Mathematics may be applied to several uses, in this thesis we apply them to the extraction of bounds. The first question is whether the H-closure Theorem and the Termination Theorem are equivalent over RCA_0 to Ramsey's Theorem for pairs. Due to our analysis we answer to [38, Open Problem 2] posed by Gasarch: finding a natural example showing that the Termination Theorem requires the full Ramsey Theorem for pairs. In this chapter we prove that such program cannot exist. We also answer negatively to [38, Open Problem 3] posed by Gasarch: is the Termination Theorem equivalent to Ramsey's Theorem for pairs?

In [34] Figueira et al. show that the Termination Theorem is a consequence of Dickson's Lemma[1] by observing that any relation is well-founded if and only if it is embedded into a well-quasi-ordering. However this property of well-quasi-orderings is equivalent to ACA_0 over RCA_0 and therefore in order to analyse the strength of the Termination Theorem we need a different point of view. In Section 5.2 we prove that the Termination Theorem is equivalent over RCA_0 to a weak version of Ramsey's Theorem for pairs. As a corollary of this result we have that for any natural number k, CAC (the Chain-AntiChain principle) is stronger than the Termination Theorem for k many relations. The latter is the statement: given a binary relation R, if there exist k-many well-founded relations whose union contains the transitive closure of R, then R is well-founded.

From our results and by using [20, 33, 72] we obtain a different proof for the characterization of transition-based programs proved to be terminating by the Termination Theorem. In order to provide more precise termination bounds, in Section 5.5 we study the reverse mathematical strength of some bounded versions of both the H-closure Theorem and the Termination Theorem. These two results in the full case are not

[1]Dickson's Lemma states that $(\mathbb{N}^k, \leq)$ (where $\leq$ is the componentwise order) is a well-quasi order [57, 64]; i.e. every infinite sequence σ of elements of $\mathbb{N}^k$ is such that there exists $n < m$ with $\sigma(n) \leq \sigma(m)$.

equivalent, in the restricted ones they turn out to be equivalent. Moreover we prove they are equivalent to a weaker version of the Paris Harrington Theorem [71]. Due to our analysis and by using the relationship between the Paris Harrington Theorem and the Fast-Growing Hierarchy, in Section 5.6 we investigate Question 0.1.2, how to characterize the programs which are proved to be terminating using a fixed number of well-founded relations. As a side results we provide a proof in RCA_0^*[2] of the implication from $\mathrm{Tot}(\mathcal{F}_{k+1})$ to the Paris Harrington Theorem for pairs and k colors.

In this chapter we work in subsystems of second order arithmetic. Results in this chapter (except those of Subsection 5.6.2) are a joint work with Keita Yokoyama [87].

5.1 Ramsey's Theorem in reverse mathematics

Ramsey's Theorem, and in particular Ramsey's Theorem for pairs in two colors, is a central argument of study in Reverse Mathematics. In this section we summarize some main facts about the strength of Ramsey's Theorem and of some of its corollaries we use in this chapter.

Notation 5.1.1. For given $k, n \in \mathbb{N}$, $k \geq 2$ we define the following statements.

RT^n_k**.** For any $c : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|c''[H]^n| = 1$.

RT^n**.** $\forall k \in \mathbb{N}\ \mathrm{RT}^n_k$.

For any natural numbers n and $k \geq 2$ it is straightforward to show directly both RT^n_{k+1} from RT^n_k, and RT^n_2 from RT^{n+1}_2 in RCA_0. Moreover, due to [19, 50, 84], it is well-known that in RCA_0 the situation is the following:

$$\mathrm{RT}^1_2 < \mathrm{RT}^1 < \mathrm{RT}^2_2 < \mathrm{RT}^2 < \mathrm{RT}^3_2 = \mathrm{ACA}_0 = \mathrm{RT}^3 = \cdots = \mathrm{RT}^n < \forall n\, \mathrm{RT}^n .$$

Ramsey's Theorem for pairs in two colors has been proved to be not equivalent to any of the "Big Five" principles of reverse mathematics [19, 60].[3] In reverse mathematics there exists a zoo of consequences of Ramsey's Theorem for pairs. Throughout this chapter we mainly focus on WRT (Weak Ramsey's Theorem) [66] which states that any coloring over the edges of a complete graph with countably many nodes admits an infinite homogeneous sequence, as introduced in Chapter 4 (Theorem 4.1.3). In order to analyse it we need to recall some more famous principles of reverse mathematics: CAC and ADS. Here a chain is a totally ordered subset, and an antichain is a subset of pair-wise unrelated elements.

Definition 5.1.2. Let $k \in \mathbb{N}$.

[2]The system RCA_0^* is a subsystem of RCA_0, defined for the language of second order arithmetic enriched with an exponential operator. RCA_0^* consists of axioms of arithmetic, exponentiation axioms, Δ^0_0-induction and Δ^0_1-comprehension.

[3]As remarked by Alexander Kreuzer there are other theorems know to be not equivalent to the Big Five before Ramsey's Theorem for pairs. For instance the Infinite Pigeonhole Principle [46] and the Weak Weak König Lemma [94].

WRT^2_k. For any $c:[\mathbb{N}]^2 \to k$, there exist $h \in k$ and an infinite set $H = \{x_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for any $i \in \mathbb{N}$ $c(\{x_i, x_{i+1}\}) = h$.

CAC. Every infinite poset has an infinite chain or antichain.

ADS. Every infinite linear ordering has an infinite ascending or descending sequence.

As shown in [45], ADS is equivalent to the Transitive Ramsey Theorem for pairs (TRT^2_2), stated below. A coloring $c:[\mathbb{N}]^2 \to k$ is said to be transitive if for any $x, y, z \in \mathbb{N}$,

$$c(\{x,y\}) = c(\{y,z\}) \implies c(\{x,z\}) = c(\{x,y\}).$$

The Transitive Ramsey Theorem for pairs states that every transitive coloring $c:[\mathbb{N}]^2 \to k$ admits an infinite homogeneous set. It is straightforward to prove directly that CAC implies ADS in RCA_0. Moreover for any $k \in \mathbb{N}$, WRT^2_k lies between them.

Proposition 5.1.3 (RCA_0). *Let k be a natural number. Then*

1. $\mathsf{WRT}^2_2 = \mathsf{ADS}$;
2. $\mathsf{CAC} \implies \mathsf{WRT}^2_k$.

Proof. 1. "$\mathsf{WRT}^2_2 \implies \mathsf{ADS}$": Let $(\mathbb{N}, \prec)$ be an infinite linear ordering. Let $c:[\mathbb{N}]^2 \to 2$ be such that $c(\{x,y\}) = 0$ if and only if $x \prec y$. Then, by WRT^2_2 there exists an infinite homogeneous sequence. If the sequence is in color 0 we have an infinite increasing sequence; otherwise since the order is total we have an infinite decreasing sequence.

"$\mathsf{ADS} \implies \mathsf{WRT}^2_2$": Given $c:[\mathbb{N}]^2 \to 2$, let $c^* : [\mathbb{N}]^2 \to 2$ defined by induction over $|x-y|$ as follows

$$c^*(\{x,x+1\}) = c(\{x,x+1\});$$
$$c^*(\{x,y\}) = \begin{cases} c^*(\{x,z\}) & \text{if } \exists z < y(x < z \wedge c^*(\{x,z\}) = c^*(\{z,y\})) \\ c(\{x,y\}) & \text{otherwise.} \end{cases}$$

A priori, the definition $c^*(\{x,y\}) = c^*(\{x,z\})$ could assign more than one value to $c^*(\{x,y\})$. However, c^* is well-defined since if there exist $x < z < z' < y$ such that

$$c^*(\{x,z\}) = c^*(\{z,y\}) \neq c^*\left(\left\{x,z'\right\}\right) = c^*\left(\left\{z',y\right\}\right)$$

then either $c^*(\{z,z'\}) = c^*(\{x,z\})$ or $c^*(\{z,z'\}) = c^*(\{z',y\})$. If $c^*(\{z,z'\}) = c^*(\{x,z\})$ by definition we get $c^*(\{x,z'\}) = c^*(\{x,z\})$, and this is a contradiction. Otherwise, if $c^*(\{z,z'\}) = c^*(\{z',y\})$, then $c^*(\{z,y\}) = c^*(\{z',y\})$ and this is once again a contradiction. It is straightforward to prove that c^* is transitive. Since we assumed ADS, by [45] we can use TRT^2_2 to derive the existence of an infinite homogeneous set H^* for c^* in color i. From H^* we obtain the infinite homogeneous sequence H for c. In fact let $\{x_n \mid n \in \omega\}$ be an increasing enumeration of H^*. Define H by: $x \in H$ if and only if $\exists n, m < x$ such that x belongs to the minimum, with respect to the lexicographical order, of the shortest paths in color i from x_n to x_m.

2. By induction over k. First of all we prove that $\mathrm{RCA}_0 \vdash \mathrm{CAC} \implies \mathrm{WRT}^2_2$. Given $c : [\mathbb{N}]^2 \to 2$, let c^* as above and define the poset $(\mathbb{N}, \prec)$, where

$$n \prec m \iff n < m \wedge c^*(\{n, m\}) = 0.$$

By CAC we have that there exists either an infinite chain or an infinite antichain. In the first case and since $\mathbb{N}$ is well-founded we obtain an infinite sequence in color 0. In the second one we have an infinite homogeneous set in color 1. Anyway we are done.

Assume now that $\mathrm{RCA}_0 \vdash \mathrm{CAC} \implies \mathrm{WRT}^2_k$ in order to prove that $\mathrm{RCA}_0 \vdash \mathrm{CAC} \implies \mathrm{WRT}^2_{k+1}$. Assume that $c : [\mathbb{N}]^2 \to k+1$ is given and define the poset $(\mathbb{N}, \prec)$ as above. Again by CAC we have that there exists either an infinite chain or an infinite antichain. In the former case we obtain an infinite sequence in color 0 as well. In the latter one we have an infinite set X whose nodes cannot be connected in color 0. Since X is infinite, there exists a bijection between X and $\mathbb{N}$. Therefore by applying (CAC and) the inductive hypothesis to $c \restriction [X]^2$, we obtain an infinite homogeneous sequence. □

Hence we have the following situation

$$\mathrm{RCA}_0 < \mathrm{ADS} = \mathrm{TRT}^2_2 = \mathrm{WRT}^2_2 \leq \mathrm{WRT}^2_3 \leq \cdots \leq \mathrm{WRT}^2_k \leq \mathrm{CAC} < \mathrm{RT}^2_2 = \cdots = \mathrm{RT}^2_k.$$

We heard that Patey pointed out the equivalence between ADS and WRT^2_2 independently. Since Lerman, Solomon and Towsner in [59] proved that ADS < CAC we get that at least one of the previous inequalities is strict. Apparently Patey [74] separates CAC from WRT^2_k. As far as we know the equivalence between WRT^2_k and WRT^2_{k+1} is still open. The separation between CAC and RT^2_2 was firstly proved by Hirschfeldt and Shore in [45].

Observe that, since the proof of Proposition 5.1.3.2 is by induction over k, we cannot conclude from this proof that $\mathrm{CAC} \geq \forall k\, \mathrm{WRT}^2_k$ in RCA_0.

5.2 Termination Theorem and H-closure Theorem in RM's zoo

The first question about the Termination Theorem and the H-closure Theorem is whether they are equivalent to Ramsey's Theorem for pairs over RCA_0. In this section we prove that actually the H-closure Theorem is equivalent to Ramsey's Theorem for pairs. On the other hand the Termination Theorem is equivalent to Weak Ramsey's Theorem.

5.2.1 Remark on inductive well-foundedness

As shown in Section 1.2, inductive well-foundedness is intuitionistically stronger than the standard notion of weakly well-foundedness (no infinite decreasing sequences exist, Definition 1.2.1.1). Moreover even if classically they are equivalent, they are not the same in RCA_0. Indeed in this system we can prove that weakly well-foundedness is stronger than inductively well-foundedness, but the other implication is equivalent to ACA_0, as noticed by Marcone [62]. For short, throughout this chapter we write "well-founded" instead of "weakly well-founded".

Lemma 5.2.1 (RCA_0)**.** *R well-founded implies R inductively well-founded.*

Proof. Assume by contradiction that R is not inductively well-founded. Let X be an inductive set such that $\mathbb{N}\setminus X$ is not empty and let $x \in \mathbb{N}\setminus X$. Then we can define by primitive recursion and minimization[4] the following infinite decreasing R-sequence.

$$f(n) = \begin{cases} \mu y\ (yRf(n-1)\ \wedge\ y \notin X) & \text{if } n > 0 \\ x & \text{if } n = 0. \end{cases} \qquad \square$$

Proposition 5.2.2 (RCA_0, [62])**.** *The following are equivalent:*

- ACA_0 *;*
- *for any binary relation R, R inductively well-founded implies R well-founded.*

Proof. We use the fact that ACA_0 is equivalent to "every map $f : \mathbb{N} \to \mathbb{N}$ has a range" and to "every 1-1 map $f : \mathbb{N} \to \mathbb{N}$ has a range" [83].

"$\Downarrow$": Assume that ACA_0 holds and assume that R is not well-founded. Then there exists an infinite decreasing R-sequence f. By ACA_0 we can define X to be the range of f. Then $\overline{X} = \mathbb{N}\setminus X$ is an inductive set which witnesses that R is not inductively well-founded.

"$\Uparrow$": Assume that for any binary relation R, R inductively well-founded implies R well-founded. Given a 1-1 function f we have to prove that f has a range. Define the binary relation R on the set $\{a_n \mid n \in \mathbb{N}\} \cup \{b_n \mid n \in \mathbb{N}\}$, with $a_n \neq b_m$ for any n, m, as follows:

$$\begin{aligned} b_n R a_k &\iff f(n) = k \\ a_k R b_n &\iff f(n+1) = k \end{aligned}$$

R is not well-founded as witnessed by the descending sequence $a_{f(0)}, b_0, a_{f(1)}, b_1, \ldots$. Indeed for any $i \in \mathbb{N}$, $a_{f(i+1)} R b_i$ and $b_{i+1} R a_{f(i+1)}$. Then by hypothesis it is not inductively well-founded. Hence there exists $X \subseteq \mathbb{N}$ inductive such that $\overline{X} = \mathbb{N}\setminus X$ is not empty. Therefore $\overline{X}$ contains no a_k such that k is not in the range of f, since all these elements

[4] Recall that, within RCA_0, the universe of total number-theoretic functions is closed under composition, primitive recursion, and minimization [83, Section II.3].

have no R-predecessors. Moreover there exists $m \in \mathbb{N}$ such that $\overline{X}$ contains every $a_{f(n)}$ with $n \geq m$. In fact since $\overline{X}$ is non-empty we have two possibilities: either $b_i \in \overline{X}$ for some i and then for any $n \geq i+1$ we have $a_{f(n)} \in \overline{X}$, or $a_k \in \overline{X}$ for some k, but since it does not belong to X also its predecessor belongs to $\overline{X}$ and it is b_i for some i, therefore we are done again. Hence we can define the range of f as follows:

$$\{n \in \mathbb{N} \mid a_n \in X\} \cup \{f(n) \mid n < m\} \qquad \square$$

However as witnessed by Lemma 5.2.1, the standard definition of well-foundedness implies the inductive definition and moreover if the binary relation is transitive then also the other implication turns out to be true.

Lemma 5.2.3 (RCA$_0$)**.** *If R is transitive and inductively well-founded, then it is well-founded.*

Proof. Assume by contradiction that R is not well-founded, then there exists an infinite decreasing transitive R-sequence: $f : \mathbb{N} \to \mathbb{N}$.

We have two possibilities:

1. f is not injective. Let n, m such that $n < m$ and $f(n) = f(m)$. Then define

$$\begin{aligned} g : \mathbb{N} &\longrightarrow \mathbb{N} \\ k &\longmapsto f(n + (k \mod (m-n))). \end{aligned}$$

2. f is injective. Then define $h : \mathbb{N} \to \mathbb{N}$ as follows

$$h(k) = \begin{cases} f(0) & \text{if } k = 0; \\ \mu m(m > h(k-1) \wedge \forall i < m(f(m) > f(i))) & \text{if } k > 0. \end{cases}$$

Let $g : \mathbb{N} \to \mathbb{N}$ be such that $g(k) = f(h(k))$.

The function g is a R-sequence. In the case (1), we have

$$g(0) = f(n), g(1) = f(n+1), \ldots, g(m-n-1) = f(m-1), g(m-n) = f(n) = f(m) = g(0),$$

therefore

$$g(0)Rg(1)R\ldots Rg(m-n-1)Rg(m-n) = g(0).$$

In the case (2), g is an infinite subsequence of a decreasing transitive R-sequence, therefore g is a decreasing R-sequence.

There exists X which is the range of g. In the case (1) the set X exists because it is finite. In the case (2), g is increasing, therefore $\exists m(g(m) = n)$ is equivalent to

$$\exists m \leq n(g(m) = n),$$

which is Δ^0_1. Let $\overline{X} = \mathbb{N} \setminus X$. $\overline{X}$ is different from $\mathbb{N}$: if we prove that X is R-inductive we conclude that R is not inductive. In fact

$$\forall y(\forall z(z \succ y \implies z \in \overline{X}) \implies y \in \overline{X})$$

holds. We prove the contrapositive. If $y \notin \overline{X}$, then $y \in X$ and since g is an infinite decreasing R-sequence, this implies there exists z in X such that $z \succ y$. $\square$

5.2.2 H-closure Theorem and Ramsey's Theorem for pairs

The main problem we have in order to prove this equivalence is that the H-closure Theorem uses the inductive definition of well-foundedness which is not equivalent to the standard one as shown by Proposition 5.2.2. Fortunately we can show that if R is inductively H-well-founded then it is H-well-founded as well. And by using this result we obtain the desired equivalence.

Let Seq be the set of code for finite sequences. First of all we can observe that $H(R)$ is defined by the following formula

$$s \in H(R) \iff s \in \mathrm{Seq} \wedge \forall i, j < |s|(i < j \implies s(j)Rs(i)).$$

Lemma 5.2.4 (RCA_0)**.** *Let R be a binary relation. If R is inductively H-well-founded then R is H-well-founded.*

Proof. Assume by contradiction that R is not H-well-founded. Then there exists an infinite decreasing $\succ$-sequence $f : \mathbb{N} \to H(R)$. Let $X = \mathrm{ran}(f)$, we may prove in RCA_0 that X exists since $s \in \mathrm{ran}(f)$ if and only if $\exists x \leq |s|(f(x) = s)$. Hence put $\overline{X} = H(R) \setminus X$. This is a counterexample to the inductiveness of $\succ$ in $H(R)$. In fact

$$\forall y(\forall z(z \succ y \implies z \in \overline{X}) \implies y \in \overline{X})$$

holds (where $z \succ y$ if and only if $s(z) \succ s(y)$). We prove the contrapositive. If $y \notin \overline{X}$, then $y \in X$ and since f is an infinite decreasing $\succ$-sequence, this implies there exists z in X such that $z \succ y$. $\square$

Due to the previous lemma we can prove that the H-closure Theorem is equivalent to Ramsey's Theorem for pairs by considering the standard definition of well-foundedness instead of the intuitionistic one.

Theorem 5.2.5 (RCA_0)**.** *Let k be a natural number, then the following are equivalent*

1. *the H-closure Theorem for k-many relation, i.e. the union of k-many inductively H-well-founded relations is inductively H-well-founded;*

2. *the union of k-many H-well-founded relations is H-well-founded;*

3. RT^2_k.

Proof. "$1 \Leftrightarrow 2$": It follows from Lemma 5.2.1 and Lemma 5.2.4.

"$2 \Rightarrow 3$": Let R'_i be symmetric relations defined by a k-coloring on $[\mathbb{N}]^2$ such that

$$R'_0 \cup \cdots \cup R'_{k-1} = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x \neq y\}.$$

We need to prove that there exists an infinite homogeneous set $X \subseteq \mathbb{N}$. For any $i < k$, put

$$R_i = \left\{(x,y) \mid xR'_i y \wedge x > y\right\}$$

R_i is defined by Δ^0_1-comprehension. Then

$$R_0 \cup \cdots \cup R_{k-1} = \{(x,y) \mid x > y\},$$

and $\{n \mid n \in \mathbb{N}\}$ is an infinite transitive decreasing $(R_0 \cup \cdots \cup R_{k-1})$-sequence. By applying the H-closure Theorem we obtain there exists an infinite transitive decreasing sequence $f_i : \mathbb{N} \to H(R_i)$ for some $i < k$.

Define $\tilde{f}_i : \mathbb{N} \to \mathbb{N}$ such that $\tilde{f}_i(n)$ is the last element of $f_i(n)$, i.e. $\tilde{f}_i(n) = f_i(n)(n-1)$. Let X be the range of $\tilde{f}_i$. We may prove in RCA_0 that X exists since $\tilde{f}_i$ is increasing. Then X is an infinite homogeneous subset of $\mathbb{N}$.

"$3 \Rightarrow 2$": Suppose that there exists an infinite transitive decreasing $(R_0 \cup \cdots \cup R_{k-1})$-sequence: $f : \mathbb{N} \to \mathbb{N}$. For any $i < k$, put

$$R'_i = \{(m,n) \mid (m < n \wedge f(n)R_i f(m)) \vee (n < m \wedge f(m)R_i f(n))\},$$

Since f is $(R_0 \cup \cdots \cup R_{k-1})$-transitive for any $m,\ n \in \mathbb{N}$ we have

$$m < n \implies f(n)(R_0 \cup \cdots \cup R_{k-1})f(m),$$

then $(R'_0 \cup \cdots \cup R'_{k-1}) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \neq n\}$. Thanks to RT^2_k there exists an infinite homogeneous set $X \subseteq \mathbb{N}$ for some R'_i, for some $i < k$. Then

$$\forall m,n \in X (m < n \implies nR'_i m).$$

Then define $h : \mathbb{N} \to \mathbb{N}$ as follows

$$h(k) = \begin{cases} \mu m(m \in X) & \text{if } k = 0; \\ \mu m(m \in X \wedge m > h(k-1)) & \text{if } k > 0. \end{cases}$$

Hence for any $m < n$, we have $h(m) < h(n)$ and since $h(m), h(n) \in X$ we get $h(n)R'_i h(m)$. Therefore h is an infinite transitive decreasing R'_i-sequence. By definition of R'_i, h is also an infinite decreasing R_i-sequence and therefore $H(R_i)$ is ill-founded. $\square$

5.2.3 Termination Theorem and Weak Ramsey's Theorem

In this subsection we are going to prove that the Termination Theorem is equivalent in RCA_0 to Weak Ramsey's Theorem. Moreover we can observe that these two results are equivalent to a weak version of H-closure, which we call Weak H-closure.

Theorem 5.2.6 (Weak H-closure Theorem.)**.** *Given any relations $R_0,\dots,R_{k-1}$ with $k\in\mathbb{N}$, if R_i is well-founded for every $i<k$ then $\bigcup\{R_i \mid i<k\}$ is H-well-founded.*

This result follows from H-closure Theorem by applying Proposition 2.2.3.1. Observe that thanks to Lemma 5.2.4 the theorem above is equivalent to the statement "the union of well-founded relations is inductively H-well-founded". While the statement "the union of inductively-well-founded relations is (inductively) H-well-founded" is stronger by Proposition 5.2.2.

Theorem 5.2.7 (RCA_0)**.** *Let k be a natural number. Then the following are equivalent:*

1. *the Termination Theorem for transition invariants composed by k-many relations: given a binary relation R, if there exist k-many well-founded relations whose union contains the transitive closure of R, then R is well-founded;*
2. WRT^2_k*;*
3. *Weak H-closure Theorem for k-many relations: the union of k-many well-founded relations is H-well-founded.*

Proof. "$1\Rightarrow 2$": Let $c:[\mathbb{N}]^2\to k$ be a coloring. For any $i<k$ define a binary relation R_i as follows:

$$xR_iy \iff (x>y)\wedge c(\{x,y\})=i.$$

Assume by contradiction that there are no infinite sequences for any R_i. Put $R=\{(x+1,x)\mid x\in\mathbb{N}\}$ then $\bigcup\{R_i\mid i<k\}=R^+=\{(x,y)\mid x>y\}$. Then by applying the Termination Theorem R is well-founded. Contradiction.

"$2\Rightarrow 3$": Assume that $R_0,\dots,R_{k-1}$ are well-founded and suppose by contradiction that there exists $f:\mathbb{N}\to\mathbb{N}$ such that it is a transitive decreasing $R_0\cup\dots\cup R_{k-1}$-sequence. For any $i<k$ define R'_i as in the proof of Theorem 5.2.5:

$$R'_i=\{(m,n)\mid (m<n\wedge f(n)R_if(m))\vee(n<m\wedge f(m)R_if(n))\}.$$

By using WRT^2_k instead of RT^2_k we obtain that h (defined as in the proof of Theorem 5.2.5) is an infinite decreasing sequence for R'_i for some $i<k$. This is a contradiction with the fact that R_i is well-founded.

"$3\Rightarrow 1$": Assume that $R_0,\dots,R_{k-1}$ are well-founded and that

$$\bigcup\{R_i\mid i<k\}\supseteq R^+,$$

in order to prove that R is well-founded. Assume by contradiction that R is non well-founded. Then R^+ is not well-founded, hence since it is transitive, it is not H-well-founded by Proposition 2.2.3. Then also $\bigcup\{R_i \mid i < k\}$ is not H-well-founded as well. Then by Weak H-closure there exists $i < k$ such that R_i is not well-founded. Contradiction. □

Remark 5.2.8. By using the same argument we can prove that for any natural number k the following are equivalent over RCA_0.

- For any binary relation R, if there exist k-many inductively well-founded relations whose union contains the transitive closure of R then R is well-founded.
- The union of k-many inductively well-founded relations is H-well-founded.

Observe that by Proposition 5.1.3.2 and since $\mathrm{RT}^2_2 > \mathrm{CAC}$ [45], we have that the Termination Theorem for transition invariant composed of k-many relations is strictly weaker that Ramsey's Theorem for pairs and k-many colors. Hence we can provide an answer to [38, Open Problem 2] posed by Gasarch. There is no program proved to be terminating by the Termination Theorem, such that the proof of this fact requires the full Ramsey Theorem for pairs.

Moreover notice that $\forall k\, \mathrm{WRT}^2_k$ is provable from CAC plus the full induction. On the other hand CAC plus the full induction does not imply RT^2_2 (and with more reason $\forall k\, \mathrm{RT}^2_k$), since the separation between CAC and RT^2_2 provided in [45] is done over ω-models [5], which always enjoy the full induction. We conclude that $\forall k\, \mathrm{WRT}^2_k$ does not imply $\forall k\, \mathrm{RT}^2_k$ (even RT^2_2). Thanks to Theorem 5.2.7, we get a negative answer to [38, Open Problem 3] posed by Gasarch: is the Termination Theorem equivalent to full Ramsey Theorem for pairs? In fact the full Termination Theorem is equivalent to the full Weak Ramsey Theorem which is strictly weaker than the full Ramsey Theorem over RCA_0.

5.3 Weight functions, bounds, and H-bounds

In the study of the termination analysis, it is important to investigate a bound for the number of steps required by a program to terminate by analysing the structure of the program. For this purpose, we need a formal notion of bounds.

Definition 5.3.1. Let R be a binary relation on S.

- A *bound* for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing sequence $\langle a_0, \dots, a_{l-1}\rangle$, $l \le f(a_0)$, i.e., any decreasing R-sequence starting from a is shorter than $f(a)$.
- A *H-bound* for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing transitive sequence $\langle a_0, \dots, a_{l-1}\rangle$, $l \le f(a_0)$, i.e., any decreasing transitive R-sequence starting from a is shorter than $f(a)$.

[5] Models whose first order part is standard (e.g. [83]).

It is easy to see that in ACA_0, R has a bound if and only if R has a weight function (as defined in Section Definition 1.2.26). However one of the implications cannot be proved in RCA_0.

Proposition 5.3.2 (ACA_0)**.** *Given a binary relation R. R has a bound if and only if R has a weight function.*

Proof. "$\Rightarrow$": If R has a bound $f : S \to \mathbb{N}$ then we can define $f^* : S \to \mathbb{N}$ as follows

$$f^*(x) = \max\{l \mid l \text{ is the length of a decreasing } R\text{-sequence from } x\}.$$

For any $x \in S$, $f^*(x) \in \mathbb{N}$ since $f^*(x) \le f(x)$ and f^* is a weight function by definition. "$\Leftarrow$": If R has a weight function $f : S \to \mathbb{N}$, then if $\langle a_i \mid i \in l\rangle$ is a decreasing R-sequence, $f(a_i) \ge l$. □

The second implication is in RCA_0, while the first one requires Π^0_1-comprehension. If we assume that R is finitely branching, i.e. there exists $\delta : S \to \mathcal{P}_{<\omega}(S) = \{X \subseteq S \mid |X| < \omega\}$ such that xRy if and only if $x \in \delta(y)$, then also the first implication turns out to be provable in RCA_0. In fact

$$f^*(x) = \max\{l \mid \delta^l(x) \ne \emptyset\},$$

where $\delta^{l+1}(x) = \bigcup\{\delta(y) : y \in \delta^l(x)\}$. This set exists for any l since it is a finite set. As above $f^*(x)$ is bounded by $f(x)$, therefore $f^*(x) \in \mathbb{N}$. However in the general case, we have the following.

Theorem 5.3.3. *The following are equivalent over* RCA_0.

1. WKL_0.

2. *For any relation $R \subseteq S^2$, R has a bound if and only if R has a weight function.*

Proof. We reason within RCA_0. Without loss of generality, we may assume that $S = \mathbb{N}$. Let $S_n = n = \{0, \ldots, n-1\}$ and $R_n = R \cap {S_n}^2$ then the pair S_n, R_n associates two finite subsets of S, R respectively. One can easily check the following over RCA_0:

- $f : S \to \mathbb{N}$ is a weight function on S if and only if $f \restriction S_n$ is a weight function on S_n for all $n \in \mathbb{N}$.

- $f : S \to \mathbb{N}$ is a bound on S if and only if $f \restriction S_n$ is a bound on S_n for all $n \in \mathbb{N}$.

- If R_n has a bound $h : S_n \to \mathbb{N}$, then R_n has a weight function $f : S_n \to \mathbb{N}$ such that $f \le h$ (as the above proposition).

"$\Downarrow$": Assume that $R \subseteq \mathbb{N}^2$ has a bound $h : \mathbb{N} \to \mathbb{N}$. Define a tree $\mathrm{Tr} \subseteq {}^{<\mathbb{N}}\mathbb{N}$ as follows:

$$\sigma \in \mathrm{Tr} \iff n = |\sigma| \wedge \sigma : S_n \to \mathbb{N} \text{ is a weight function on } S_n \wedge \forall k\, \sigma(k) \le h(k).$$

Then, by the last point above, this Tr is infinite. Thus, by bounded König's Lemma[6], Tr has an infinite path $f : \mathbb{N} \to \mathbb{N}$ such that $f \leq h$. This f is a weight function for R.

"$\Uparrow$": We show (the restricted version of) Σ^0_1-separation.[7] Let $p, q : \mathbb{N} \to \mathbb{N}$ be one-to-one functions such that $\mathrm{ran}(p) \cap \mathrm{ran}(q) = \emptyset$. We want to find a set X such that $\mathrm{ran}(p) \subseteq X \subseteq \mathbb{N} \setminus \mathrm{ran}(q)$. Let $S = \mathbb{N} \times 4$. We claim there is some relation $R \subseteq S^2$ such that $(p(n),3)R(n,1)R(p(n),0)$ and $(q(n),0)R(n,2)R(q(n),3)$, and there is no other relation. We may prove in RCA_0 that R exists because R has a Δ^0_1 definition:

$$\begin{aligned}(n,i)R(m,j) \iff & (p(m) = n \wedge (i,j) = (3,1)) \vee (p(n) = m \wedge (i,j) = (1,0)) \\ & \vee (q(n) = m \wedge (i,j) = (2,3)) \vee (q(m) = n \wedge (i,j) = (0,2)),\end{aligned}$$

i.e. if $p(n) = m$ then $(m,3)R(n,1)R(m,0)$, if $q(n) = m$ then $(m,0)R(n,2)R(m,3)$, and there is no other relation. All R-sequences have at most three elements, because no element of S has both the form $p(n)$ and the form $q(n')$, for any $n, n' \in \mathbb{N}$. Put $h : S \to \mathbb{N}$ as $h((n,i)) = 2$, then h is a bound for R. By the assumption (2),there is some weight function $f : S \to \mathbb{N}$. If m is in $\mathrm{ran}(p)$ then $m = p(n)$ for some n and there is a R-sequence $(m,3)R(n,1)R(m,0)$, hence $f((m,3)) < f((m,0))$. If m is in $\mathrm{ran}(q)$ then $m = q(n)$ for some n and there is a R-sequence $(m,0)R(n,2)R(m,3)$, hence $f((m,0)) < f((m,3))$. Thus, $X = \{m \mid f((m,3)) < f((m,0))\}$ is a set separating $\mathrm{ran}(p)$ and $\mathrm{ran}(q)$. □

5.4 Proof-theoretic strength of termination

In this section we apply the result we obtained about the reverse mathematical strength of the Termination Theorem to get bounds. As standard we denote Σ^0_1-induction as $\mathrm{I}\Sigma^0_1$ and with $\mathrm{B}\Sigma^0_2$ the bounding principle for Σ^0_2-formulas [44], i.e. for any $\varphi(x,y)$ Σ^0_2 for which n does not occur freely,

$$\forall m[(\forall i < m \exists j \varphi(i,j)) \implies (\exists n \forall i < m \exists j < n \varphi(i,j))].$$

In order to do that we need to recall some classical results. The first one is by Parsons in 1970 (see e.g., [33]):

Theorem 5.4.1. *The class of provable recursive functions of* $\mathrm{I}\Sigma^0_1$ *is exactly the same as the class of primitive recursive functions.*

The second result we need is by Paris and Kirby [72]:

Theorem 5.4.2. $\mathrm{B}\Sigma^0_2$ *is a* Π^0_3*-conservative extension of* $\mathrm{I}\Sigma^0_1$*.*

Moreover, as shown by Chong, Slaman and Yang [20]:

Theorem 5.4.3. $\mathrm{WKL}_0 + \mathrm{CAC}$ *is a* Π^1_1*-conservative extension of* $\mathrm{B}\Sigma^0_2$*.*

Thus, we can conclude the following.

[6] Bounded König's Lemma is equivalent to WKL_0 [83, Lemma IV.1.4].
[7] Σ^0_1-separation is equivalent to WKL_0 [83, Lemma IV.4.4].

Corollary 5.4.4. *The class of provable recursive functions of* $\mathrm{WKL}_0 + \mathrm{CAC}$ *is exactly the same as the class of primitive recursive functions.*

On the other hand, by Theorem 5.2.7 and Proposition 5.1.3, we have the following.

Proposition 5.4.5. *The following is provable within* $\mathrm{WKL}_0 + \mathrm{CAC}$.

$(*)$ *any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many relations $R_1 \cup \cdots \cup R_{k-1} \supseteq R^+$ with bounds is well-founded.*

Consider now the special case of $(*)$ in the real world: if R is a primitive recursive relation generated by a primitive recursive transition function (in particular it is deterministic), and each R_i has a primitive recursive bounds. Note that any primitive recursive function is strongly represented within RCA_0. Then, $(*)$ (together with WKL_0) means that

$(**)$ for any state a, there exists a bound $b \in \mathbb{N}$ of R-sequences from a.

Since $(**)$ is a Π^0_2-statement provable in $\mathrm{WKL}_0 + \mathrm{CAC}$, the function $a \mapsto b$ must be bounded by a primitive recursive function. Thus, we have the following.

Corollary 5.4.6. *Any relation generated by a primitive recursive transition function for which there exists a disjunctively well-founded transition invariant composed of k-many relations with primitive recursive bounds has a primitive recursive bound.*

Observe that since we worked in WKL_0 and thanks to Theorem 5.3.3 this is another version of the result obtained in Section 2.3 by using the constructive proof of the Termination Theorem. In the next sections we will study the reverse mathematical strength of the H-closure Theorem and the Termination Theorem for relations of height ω and for relations with bounds and H-bounds, to strengthen investigations about bounds.

5.5 Bounded versions of Termination Theorem

The goal of this section is to study the strength of some bounded versions of the Termination Theorem. They turn out to be equivalent to suitable versions of Paris and Harrington's Theorem. Paris and Harrington's Theorem is a strengthened version of finite Ramsey's Theorem which is unprovable in Peano Arithmetic, since it implies the consistency of Peano Arithmetic [71]. For all bounded versions proposed in this section the "bounded Termination Theorem" and the "bounded H-closure Theorem" turn out to be equivalent.

Since Paris Harrington's Theorem for pairs and k many colors is provable within RCA_0 for any k, throughout this section we work in the subsystem RCA_0^*, defined for the language of second order arithmetic enriched with an exponential operation (e.g. [83]). RCA_0^* consists of the basic axioms together with the exponentiation axioms (elementary function arithmetic), Δ^0_0 induction and Δ^0_1-comprehension. Since in RCA_0 there exists the exponential function, $\mathrm{RCA}_0 \equiv \mathrm{RCA}_0^* + \Sigma^0_1$-induction [83, Section X.4].

Fast Growing Hierarchy. We denote with $\mathcal{F}_k$ the usual k-class of the Fast Growing Hierarchy [61] as briefly introduced in Chapter 3. Define

$$\begin{cases} F_0(x) = x+1, \\ F_{n+1}(x) = F_n{}^{(x+1)}(x). \end{cases}$$

Then $\mathcal{F}_k$ is the closure under limited recursion and substitution of the set of functions defined by constant, projections, sum and F_h for any $h \leq k$. We need to recall also some results by Löb and Wainer [61].

Proposition 5.5.1. *1. For any $k, k', n, m, x, y \in \mathbb{N}$:*

- $x < y \implies F_k^n(x) < F_k^n(y)$;
- $m < n \implies F_k^m(x) < F_k^n(x)$;
- $k < k' \implies F_k^n(x) \leq F_{k'}^n(x)$.

2. For any $k \in \mathbb{N}$ and for each $f \in \mathcal{F}_k$, there exists $n \in \mathbb{N}$ such that for any x $f(x) \leq F_k^n(x)$.

Due to these results we get the following.

Corollary 5.5.2. *If $f \in \mathcal{F}_k$ and $g \in \mathcal{F}_{k'}$, for some $k, k' \in \mathbb{N}$, then the function $h(x) = f^{g(x)}(x)$ is in $\mathcal{F}_{\max\{k+1,k'\}}$.*

Proof. Let $m, n \in \mathbb{N}$ such that $f(x) \leq F_k^m(x)$ and $g(x) \leq F_{k'}^n(x)$. Therefore

$$h(x) \leq F_k^{mF_{k'}^n(x)}(x) \leq F_k^{mF_{k'}^n(x)}(mF_{k'}^n(x)) \leq F_{k+1}(mF_{k'}^n(x)). \qquad \square$$

5.5.1 Termination Theorem for relations of height ω

In here we present the equivalence in RCA_0^* between the Termination Theorem for relations of height ω, H-closure for relations of height ω and the principle we call Weak Paris Harrington Theorem. This equivalence holds level by level: i.e. we prove that H-closure for k-many relations of height ω is equivalent to the Termination Theorem for k-many relations of height ω and to the Weak Paris Harrington Theorem for k-many colors.

First of all we state the theorems we deal with. We say that a set X is 1-*large* if $\min X < |X|$. Given a coloring $c : [X]^2 \to k$, a set $Y \subseteq X$ is *weakly homogeneous* if its increasing enumeration is a homogeneous sequence for c. Therefore if $Y = \{y_0 < y_1 < \cdots < y_n < \ldots\}$, there exists $i \in k$ such that $c(\{y_n, y_{n+1}\}) = i$ for all n. Paris and Harrington's Theorem is a strengthened version of finite Ramsey's Theorem. It has been the first natural example of a theorem not provable in Peano arithmetic, indeed it implies the consistency of Peano arithmetic [71]. Paris and Harrington's Theorem guarantees that for any infinite $X \subseteq \mathbb{N}$ and for any coloring over the edges of the complete graph on X with finitely many colors there exists a finite homogeneous 1-large set. Weak Paris Harrington Theorem is the infinite version of the path-homogeneous variant studied by Erdős and Mills in [28]. Weak Paris Harrington Theorem states the existence of a finite weakly homogeneous 1-large set. To perform our analysis we need to slice these results.

Definition 5.5.3. For given $h,k \in \mathbb{N}$, we define the following statements.

1. $\mathrm{PH}^{h,2}_k$: Given $f:\mathbb{N}\to\mathbb{N}$ such that for all $n\in\mathbb{N}$ $f(n+1)<F_h(f(n))$, for all coloring $c:[\mathrm{ran}(f)]^2\to k$, there exists a homogeneous set for c which is 1-large.

2. $\mathrm{WPH}^{h,2}_k$: Given $f:\mathbb{N}\to\mathbb{N}$ such that for all $n\in\mathbb{N}$ $f(n+1)<F_h(f(n))$, for all coloring $c:[\mathrm{ran}(f)]^2\to k$, there exists a weakly homogeneous set for c which is 1-large.

3. k-HCT^h: If, for all $i<k$, R_i is a binary relation of height ω whose weight function f_i is such that $f_i(n)<F_h(n)$ for all n, then any transitive $R_0\cup\cdots\cup R_{k-1}$-decreasing sequence $f:\mathbb{N}\to\mathbb{N}$ such that for all $n\in\mathbb{N}$ $f(n+1)<F_h(f(n))$ is finite.

4. k-TT^h: Let R be a deterministic binary relation, whose transition function $f:\mathbb{N}\to\mathbb{N}$ is such that for all $n\in\mathbb{N}$ $f(n+1)<F_h(f(n))$. If there exists a disjunctively well-founded transition invariant for R composed of k-many relations of height ω whose weight functions f_i are such that $f_i(n)<F_h(n)$ for all n, R is well-founded.

By using these definitions we can prove the following.

Theorem 5.5.4 (RCA_0^*+$\mathrm{Tot}(F_h)$)**.** *For every natural number k,*

$$\mathrm{WPH}^{h,2}_k = k\text{-}\mathrm{HCT}^h = k\text{-}\mathrm{TT}^h.$$

Proof. "$\mathrm{WPH}^{h,2}_k \Rightarrow k\text{-}\mathrm{HCT}^h$": Let binary relations $\left\{R_i\subseteq S^2 \mid i<k\right\}$ and weight functions $\{f_i:S\to\mathbb{N}\mid i<k\}$ be given as in the hypotheses of k-HCT^h. For the sake of contradiction, assume $R=\bigcup\{R_i\mid i<k\}$ and let $\langle a_j\mid j\in\mathbb{N}\rangle$ be an infinite transitive sequence for R such that $a_{i+1}<F_h(a_i)$. Define $f:\mathbb{N}\to\mathbb{N}$ as

$$f(n)=\max\{F_h(a_j)+(n-j+1)\mid j\le n\}.$$

Observe that for any n, $f(n)\ge F_h(a_n)+1$, hence:

$$\begin{aligned} f(n+1) &= \max\{f(n)+1, F_h(a_{n+1})+1\} < \max\{f(n)+1, F_h(F_h(a_n))+1\} \\ &\le \max\{f(n)+1, F_h(F_h(a_n)+1)\} \le F_h(f(n)). \end{aligned}$$

Since f is increasing, we can define within RCA_0 $X=\{f(j)\mid j\in\mathbb{N}\}$. Define $c:[X]^2\to k$ as $c(\{f(n),f(m)\})=\min\{i<k\mid a_m R_i a_n\}$. By $\mathrm{WPH}^{h,2}_k$, take a weakly homogeneous set H for color $i<k$ with $|H|>\min H$. Write $H=\{f(h_0),f(h_1),\ldots,f(h_{l-1})\}$. Then, $l>f(x_{h_0})$. By definition, we have $a_{h_{n+1}}R_i a_{h_n}$ for any $n<n+1<l$. Thus, we have

$$f_i(a_{h_{l-1}})<\cdots<f_i(a_{h_1})<f_i(a_{h_0})<F_h(a_{h_0})<f(h_0)<l,$$

which is a contradiction.

"k-HCT$^h \Rightarrow k$-TT^h": Assume there exists a disjunctively well-founded transition invariant

$$T = R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$$

where each relation R_i is well-founded and has height ω with weight function as in the hypothesis. By applying k-HCT^h their union is such that each transitive decreasing sequence f' such that for any n $f'(n+1) < F_h(f'(n))$ is finite. So it holds also for R^+, since it is preserved between subsets. Therefore, since R is the graph of f as in the hypothesis of k-TT^h, there are no infinite R-decreasing sequences. Hence R is well-founded.

"k-TT$^h \Rightarrow$ WPH$^{h,2}_k$": Assume by contradiction that there exist X and $c : [X]^2 \to k$ such that, there is no weakly homogeneous 1-large set. For any $i \in k$, define R_i as follows:

$$xR_iy \iff y < x \wedge c(\{y,x\}) = i.$$

We claim that R_i has height ω for any $i \in k$. In fact, if $X = \{x_i \mid i \in \mathbb{N}\}$, we can define a weight function $f_i : X \to \mathbb{N}$, by bounded recursion:

$$f_i(x_n) = \begin{cases} x_0 & \text{if } n = 0; \\ \min\{\{f_i(x_m) - 1 \mid m < n \ \wedge \ c(\{x_m, x_n\}) = i\} \cup \{x_n\}\} & \text{otherwise;} \end{cases}$$
$$f_i(x_n) \leq x_n.$$

f_i is a weight function, since if xR_iy then $c(\{y,x\}) = i$ and $y < x$ and so $f_i(x) < f_i(y)$. Moreover for any $x \in X$ we have $f_i(x) \geq 0$. Otherwise there should exist $y_0 > \cdots > y_l$ such that

$$y_0R_iy_1 \ldots R_iy_l = x,$$

where $l > y_0$, due to the definition of f_i and since $X \subseteq \mathbb{N}$. This is a contradiction since we assumed that there is no weakly homogeneous sets for c. Then each R_i has height ω.

Therefore, by applying k-TT^h, $R = \bigcup\{R_i \mid i \in k\}$ should be well-founded, but this is a contradiction since $R = [X]^2$. □

We can consider also the relativized versions of the statements in Definition 5.5.3. Formally:

Definition 5.5.5. For given $k \in \mathbb{N}$ and for given $X \subseteq \mathbb{N}$, we define the following statements.

1. PH$^{*2}_k$: for any infinite set Y and any coloring function $c : [Y]^2 \to k$, there exists a homogeneous set for c which is 1-large.

2. WPH$^{*2}_k$: for any infinite set Y and any coloring function $c : [Y]^2 \to k$, there exists a weakly homogeneous set for c which is 1-large.

3. k-HCT$^*_\omega$: if R_i is a binary relation on S of height ω for any $i < k$, then $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded.

4. k-TT^*_ω: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many relations in of height ω is well-founded.

It is easy to prove in RCA_0^* that, by using the previous definitions, we have:

$$\mathrm{WPH}^{*2}_k = k\text{-}\mathrm{HCT}^*_\omega = k\text{-}\mathrm{TT}^*_\omega.$$

For any fixed k, each of these statements proves that for any infinite set there exists a 1-large subset, which is just another form of $\mathrm{I}\Sigma^0_1$. Thus they are all equivalent to RCA_0 over RCA_0^*. Nonetheless the full versions are not provable over RCA_0 and our results imply that within RCA_0^*

$$\forall k\ \mathrm{WPH}^{*2}_k = \forall k\ k\text{-}\mathrm{HCT}^*_\omega = \forall k\ k\text{-}\mathrm{TT}^*_\omega.$$

5.5.2 Termination Theorem for bounded relations

Here we consider the formulations obtained by using bounds instead of weight functions. For the application to real programs, finding a bound seems to be as difficult as finding a weight function. So, this relaxed notion would be nonsense. On the other hand, H-bound could be found more easily, e.g., if R has no loop and $a \in S$ has only m-many predecessors, then putting $f(a) = m$ is good enough to obtain a H-bound (Definition 5.3.1).

Then we can define the version of H-closure and of the Termination Theorem with bounds and with H-bounds.

Definition 5.5.6. For given $h, k \in \mathbb{N}$, we define the following statements.

1. k-HCT^h_b: If, for any $i < k$, R_i is a binary relation with bound f_i such that $f_i(n) < F_h(n)$ for any n, then any transitive $R_0 \cup \cdots \cup R_{k-1}$-decreasing sequence $f : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$ $f(n+1) < F_h(f(n))$ is finite.

2. k-HCT^h_H: If, for any $i < k$, R_i is a binary relation with H-bound f_i such that $f_i(n) < F_h(n)$ for any n, then any transitive $R_0 \cup \cdots \cup R_{k-1}$-decreasing sequence $f : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$ $f(n+1) < F_h(f(n))$ is finite.

3. k-TT^h_b: Let R be a deterministic binary relation, whose transition function $f : \mathbb{N} \to \mathbb{N}$ is such that for any $n \in \mathbb{N}$ $f(n+1) < F_h(f(n))$. If there exists a disjunctively well-founded transition invariant for R composed of k-many relations with bounds f_i such that $f_i(n) < F_h(n)$ for any n, R is well-founded.

4. k-TT^h_H: Let R be a deterministic binary relation, whose transition function $f : \mathbb{N} \to \mathbb{N}$ is such that for any $n \in \mathbb{N}$ $f(n+1) < F_h(f(n))$. If there exists a disjunctively well-founded transition invariant for R composed of k-many relations with H-bounds f_i such that $f_i(n) < F_h(n)$ for any n, R is well-founded.

Then, we have the following.

Theorem 5.5.7 (RCA_0^*+Tot(F_h))**.** *For any natural number k,*

$$\mathrm{WPH}_k^{h,2} = k\text{-}\mathrm{HCT}_b^h = k\text{-}\mathrm{TT}_b^h.$$

Proof. "$\mathrm{WPH}_k^{h,2} \Rightarrow k\text{-}\mathrm{HCT}_b^h$": The argument of Theorem 5.5.4 provides a decreasing R_i-sequence from a_{h_0} of length greater than $f_i(a_{h_0})$. Hence the thesis.

"$\mathrm{HCT}_b^h \Rightarrow k\text{-}\mathrm{TT}_b^h$": This proof is the same of the one in Theorem 5.5.4.

"$k\text{-}\mathrm{TT}_b^h \Rightarrow \mathrm{WPH}_k^{h,2}$": Thanks to Theorem 5.5.4 we have that $k\text{-}\mathrm{TT}^h \Rightarrow \mathrm{WPH}_k^{h,2}$. Moreover $k\text{-}\mathrm{TT}_b^h$ implies $k\text{-}\mathrm{TT}^h$ since if the relation R has a weight function then it has also a bound. Therefore we are done. □

This implies that $k\text{-}\mathrm{HCT}_b^h = k\text{-}\mathrm{HCT}^h$ and $k\text{-}\mathrm{TT}_b^h = k\text{-}\mathrm{TT}^h$. In the case of H-bounds, the bounded versions are stronger. In fact, they are equivalent to the Paris Harrington Theorem for k-many colors.

Theorem 5.5.8 (RCA_0^*+Tot(F_h))**.** *For any natural number k,*

$$\mathrm{PH}_k^{h,2} = k\text{-}\mathrm{HCT}_H^h = k\text{-}\mathrm{TT}_H^h.$$

Proof. "$\mathrm{PH}_k^{h,2} \Rightarrow k\text{-}\mathrm{HCT}_H^h$". As we did in the proof of Theorem 5.5.4, let binary relations $\left\{R_i \subseteq S^2 \mid i < k\right\}$ and H-bounds $\{f_i : S \to \mathbb{N} \mid i < k\}$ be given as in the hypothesis of $k\text{-}\mathrm{HCT}_H^h$. For the sake of contradiction, assume $R = \bigcup\{R_i \mid i < k\}$ and let $\langle a_j \mid j \in \mathbb{N}\rangle$ be an infinite transitive sequence for R such that $a_{i+1} < F_h(a_i)$. Define $f : \mathbb{N} \to \mathbb{N}$ as

$$f(n) = \max\{f_i(a_{n+1}) + (n-j+1) \mid i < k, j \le n\}.$$

Let $X = \{f(j) : j \in \mathbb{N}\}$, and define $c : [X]^2 \to k$ as $c(\{f(n), f(m)\}) = \min\{i < k \mid a_m R_i a_n\}$. By $\mathrm{PH}_k^{h,2}$, take a homogeneous set H for color $i < k$ with $|H| > \min H$. Write $H = \{x_{h_0} < x_{h_1} < \cdots < x_{h_{l-1}}\}$. Then, $l > x_{h_0}$. By definition, we have $a_{h_n} R_i a_{h_m}$ for any $m < n < l$. Thus there exists a transitive decreasing R_i-sequence from a_{h_0} of length l, but $f_i(a_{h_0}) < x_{h_0} < l$ which is a contradiction.

"$\mathrm{HCT}_H^h \Rightarrow k\text{-}\mathrm{TT}_H^h$". As in Theorem 5.5.4.

"$k\text{-}\mathrm{TT}_H^h \Rightarrow \mathrm{PH}_k^{h,2}$." Assume that $c : [X]^2 \to k$ has no homogeneous 1-large set. Then we define, as we did in Theorem 5.5.4,

$$x R_i y \iff y < x \wedge c(\{y,x\}) = i.$$

Then let f be the identity function on X. We claim it is a H-bound for any R_i. In fact, let $\langle x_j \mid j \in l\rangle$ be a decreasing transitive R_i-sequence: by definition unfolding we have $x_0 < x_1 < \cdots < x_{l-1}$ and $x_b R_i x_a$ for any $0 \le a < b < l$. If $x_0 = f(x_0) < l$ then we obtain a homogeneous 1-large set and this is a contradiction. Then $x_0 = f(x_0) \ge l$: this means that the identity function is a H-bound. So thanks to $k\text{-}\mathrm{TT}_H^h$, $[X]^2 = \bigcup\{R_i \mid i \in k\}$ should be well-founded, and this is a contradiction. □

Also in this case, we can consider also the relativized versions of the statements above.

Definition 5.5.9. For given $k \in \mathbb{N}$ and for given $X \subseteq \mathbb{N}$, we define the following statements.

1. k-HCT^*_b: if R_i is a binary relation with bound for any $i < k$, then $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded.

2. k-TT^*_b: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many relations with bounds is well-founded.

3. k-HCT^*_H: if R_i is a binary relation with H-bound for any $i < k$, then $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded.

4. k-TT^*_H: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many relations with H-bounds is well-founded.

As a corollary we get within RCA_0^*:

$$\forall k\ \mathrm{WPH}^{*2}_k = \forall k\ k\text{-}\mathrm{HCT}^*_b = \forall k\ k\text{-}\mathrm{TT}^*_b.$$

$$\forall k\ \mathrm{PH}^{*2}_k = \forall k\ k\text{-}\mathrm{HCT}^*_H = \forall k\ k\text{-}\mathrm{TT}^*_H.$$

5.6 Bounding via Fast Growing Hierarchy

Is there a correspondence between the complexity of a primitive recursive transition relation and the number of relations which compose the transition invariant? Here we prove that actually there is, by applying the results obtained in Section 5.5. As a corollary we get the following proposition.

Proposition 5.6.1. *For a transition relation $R \subseteq \mathbb{N}^2$, the following are equivalent.*

(1) R is primitive recursively bounded.

(2) R is k-disjunctively linearly H-bounded for some $k \in \omega$.

Moreover, if R is a deterministic transition relation, the following is also equivalent to the above.

(3) R is k-disjunctively linearly bounded for some $k \in \omega$.

5.6.1 First comparisons

The main result we are going to prove here is the equivalence between $\forall k\ k\text{-}\mathrm{TT}^*_\omega$ and the relativized version of the totality of the fast growing hierarchy.

Given an increasing function $f : \mathbb{N} \to \mathbb{N}$ and a natural number k, we define $F_{k,f}$ as follows:

$$\begin{cases} F_{0,f}(x) = f(x) + 1, \\ F_{n+1,f}(x) = F^{(x+1)}_{n,f}(x). \end{cases}$$

Then $\mathcal{F}_k^f$ is the closure under limited recursion and substitution of the set of functions defined by constant, projections, sum and $F_{h,f}$ for any $h \leq k$. For any natural number k, let $\mathrm{Tot}^*(\mathcal{F}_k)$ be the relativized version of $\mathrm{Tot}(\mathcal{F}_k)$, namely: for any function f, any function in $\mathcal{F}_k^f$ is total.

Summing up the results presented in this subsection we have the following:

$$\forall k \ \mathrm{WPH}^{*2}_k = \forall k \ k\text{-}\mathrm{TT}^*_\omega = \forall k \ k\text{-}\mathrm{HCT}^*_\omega = \forall k \ \mathrm{Tot}^*(\mathcal{F}_k) = \forall k \ \mathrm{PH}^{*2}_k.$$

Recall that, although for any natural number k PH^{*2}_k and $\mathrm{Tot}^*(\mathcal{F}_k)$ hold within RCA_0, $\forall k \ \mathrm{PH}^{*2}_k$ and $\forall k \ \mathrm{Tot}^*(\mathcal{F}_k)$ do not.

From termination to totality

In order to prove that $\forall k \ k\text{-}\mathrm{TT}^*_\omega \implies \forall k \ \mathrm{Tot}^*(\mathcal{F}_k)$, we recall the following intermediate statement introduced by Ketonen and Solovay in [51]. We say that a finite set X is 0-*large* if it is not empty, X is $k+1$-*large* if

$$X \setminus \{\min X\} = \bigsqcup \{X_i \mid i \in n\},$$

where $n \geq \min X$ and X_i are disjoint k-*large* sets.

Notice that, for any set X such that $X = \bigsqcup\{X_i \mid i \in n\}$ for some $n > \min X$, if X_i is k-large for all $i \in n$, then X is $k+1$-large.

Definition 5.6.2. For given $h, k \in \mathbb{N}$ and for given $X \subseteq \mathbb{N}$, we define the following statements.

k-LRG [8]: any infinite set $X \subseteq \mathbb{N}$ contains a k-large set.

k-LRG^h: Given any function $f : \mathbb{N} \to \mathbb{N}$ such that for any n $f(n+1) < F_h(f(n))$, $\mathrm{ran}(f)$ contains a k-large set.

Proposition 5.6.3 (RCA_0). *For any $k \in \mathbb{N}$, we have*

$$\mathrm{WPH}^{*2}_k \Rightarrow k\text{-LRG}.$$

Proof. Given any infinite $X \subseteq \mathbb{N}$, we want to find $L \subseteq X$ such that L is k-large. For any $a, b \in \mathbb{N}$, let $[a, b) = \{x \in X \mid a \leq x < b\}$. Define k-many sequences as follows:

$$\begin{aligned} x_0^i &= \min X; \\ L_i &= \left\{ y \mid \exists n \leq y (y = x_n^i) \right\}; \\ x_{n+1}^i &= \min \left\{ y \mid y \in L_i \wedge [x_n^i, y) \text{ is } i\text{-large} \right\} \end{aligned}$$

[8] Following the notation of [47] being k-large is equivalent to being ω^k-large (or ω_0^k-large). In this section we prove that in RCA_0^* k-LRG^h is equivalent to $\mathrm{Tot}(\mathcal{F}_k^{F_h})$ for any natural numbers h, k, as for ω^k-large in [47].

Observe that x^0_n is the n-th element of X, and that x^i_m may be undefined. However, given any $x \in X$, and $i, m \in \mathbb{N}$, we may decide whether x is of the form x^i_m or not. Any $x \in X$ is x^0_n for some n. Moreover the following properties hold.

Claim. *Let i be a natural number.*

1. *If $a > \min X$ and $[a,b)$ is i-large then there exists $x^i_j \in (a,b]$ for some j.*
2. *If $n_0 < \cdots < n_{l-1}$ and $x^i_{n_0} < l$, then $[x^i_{n_0}, x^i_{n_{l-1}})$ is $(i+1)$-large.*
3. *If $n_0 < \cdots < n_{l-1}$ and $x^i_{n_0} < l$, then there exists $j \in \mathbb{N}$ such that $x^{i+i}_j \in (x^i_{n_0}, x^i_{n_{l-1}}]$.*

Proof. 1. Let j' be the maximum such that $x^i_{j'} \leq a$. Such element exists since $\min X = x^i_0 \leq a$. Then $[x^i_{j'}, b)$ is i-large, therefore $x^i_{j'+1} \leq b$ and by the choice of j' $x^i_{j'+1} > a$.

2. Observe that
$$[x^i_{n_0}, x^i_{n_{l-1}}) = [x^i_{n_0}, x^i_{n_1}) \cup \cdots \cup [x^i_{n_{l-2}}, x^i_{n_{l-1}}).$$
Since $n_{m+1} > n_m + 1$ and $[x^i_{n_m}, x^i_{n_{m+1}})$ is i-large, we are done.

3. The thesis follows by points (1) and (2). □

Then consider the coloring $c : [X]^2 \to k$, such that for any $x < y$ we have

$$c(\{x,y\}) = \max\left\{ i < k \mid \exists x^i_j \in [x,y) \right\}$$

By applying WPH^{*2}_k there exists $H = \{h_0 < \cdots < h_{l-1}\}$ which is weakly homogeneous and 1-large. We claim that the color of this sequence is $k-1$, i.e. $c(\{h_0,h_1\}) = k-1$. There are two cases. If $h_0 = \min X$ then $h_0 = x^{k-1}_0$ and therefore $c(\{h_0,h_1\}) = k-1$ by definition. Otherwise assume by contradiction that $c(\{h_0,h_1\}) = i < k-1$ then for any $m < l-1$ the interval $[h_m, h_{m+1})$ contains an element of the form $x^i_{j_m}$. Since $h_0 \neq \min X$, let j_0 be the maximum such that $x^i_{j_0} < h_0$. Therefore $x^i_{j_0} < x^i_{j_1} < \cdots < x^i_{j_{l-2}}$ and, since $h_0 < l$, we have $x^i_{j_0} < l-1$. By applying point (3) we get that there exists $x^{i+1}_j \in (x^i_{j_0}, x^i_{j_{l-2}}]$, for some j. Since $x^{i+1}_j = x^i_{j'}$ for some j' and by the definition of j_0 we have $x^{i+1}_j \geq h_0$. Hence $x^{i+1}_j \in [h_0, h_{l-1})$ and this implies $c(\{h_0, h_{l-1}\}) \geq i+1$. Contradiction.

By applying the same argument we have that $[h_0, h_{l-1})$ contains $(l-1)$-many x^{k-1}_j, hence $[h_0, h_{l-1})$ is k-large. □

Due to the equivalence proved in the previous section we can easily obtain that for any natural number k:

$$\forall k\ k\text{-}\mathrm{TT}^*_\omega = \forall k\ k\text{-}\mathrm{HCT}^*_\omega = \forall k\ \mathrm{WPH}^{*2}_k \geq \forall k\ k\text{-LRG}.$$

Moreover we have that $\forall k\ k$-LRG $\Rightarrow \forall k\ \mathrm{Tot}^*(\mathcal{F}_k)$, and so the first goal is proved.

Proposition 5.6.4 (RCA_0)**.** $\forall k\ k$-LRG $\Rightarrow \forall k\ \mathrm{Tot}^*(\mathcal{F}_k)$

Proof. Given an increasing function $f:\mathbb{N}\to\mathbb{N}$, we define for any $k\in\mathbb{N}$ the function f_k as follows:

$$f_k(a)=\min\Big\{|X|\mid X \text{ is } k\text{-large on } f''(a,\infty)\Big\}+1.$$

Observe that $f_0(a)>f(a)+1$. For any k, if k-LRG holds, then we can define f_k. We claim that for any $a\in\mathbb{N}$

$$f_k(a)>f_{k-1}^{a+1}(a).$$

Let X be k-large on $f''(a,\infty)$. By definition of k-large there exists $n>a$ and there exists X_i $(k-1)$-large for any $i\in n$ such that $|X\setminus\{\min X\}|=\bigsqcup\{|X_i|\mid i\in n\}$. Hence since $n\geq \min X\geq f(a)+1\geq a+1$:

$$f_k(a)\geq |X|>f_{k-1}^{n}(a)\geq f_{k-1}^{a+1}(a).$$

We claim that

$$\forall k\forall a\ \exists b\ <f_k(a)(F_{k,f}(a)=b).$$

We prove it by induction on k (this sentence is Π^0_1). If $k=0$, $F_{0,f}(a)=f(a)+1<f_0(a)$. Assume that it holds for $k-1$, then the it is true also for k since $f_k(a)>f_{k-1}^{a+1}(a)$ and, by induction hypothesis, $f_{k-1}(a)$ is bigger than $F_{k-1,f}(a)$. This proves $\forall k\ \mathrm{Tot}^*(\mathcal{F}_k)$. □

Note that we can prove by external induction on k that $\mathrm{RCA}_0^*\vdash k\text{-LRG}^h\implies \mathrm{Tot}(\mathcal{F}_k^h)$, by slightly modifying the argument of Proposition 5.6.4.

From totality to termination

Here we apply a result by Solovay and Ketonen to prove that if we have $\forall k\ \mathrm{Tot}^*(\mathcal{F}_k)$, then we have $\forall k\ k\text{-TT}^*_\omega$.

Proposition 5.6.5 (RCA_0)**.** *$\forall k\ \mathrm{Tot}^*(\mathcal{F}_k)\implies\forall k\ k$-LRG.*

Proof. Let X be a set and let $f:\mathbb{N}\to X$ the increasing enumeration of X. We claim that

$$\forall k\forall n[n,F_{k,f}(n))\cap X \text{ is } k\text{-large}.$$

We prove it by induction (since this formula is Π^0_1). If $k=0$ then

$$f(n)\in[n,f(n)+1)\cap X.$$

Suppose the statement is true for k and let us prove it for $k+1$. Fix n, by definition $F_{k+1}(n)=F_{k,f}^{n+1}(n)$. Hence we have

$$[n,F_{k+1,f}(n))\cap X=([n,F_{k,f}(n))\cap X)\cup\cdots\cup([F_{k,f}^{n}(n),F_{k,f}^{n+1}(n))\cap X).$$

Then it is $k+1$-large, since for each $i\in n$,

$$[F_{k,f}^{i}(n),F_{k,f}^{i+1}(n))\cap X$$

is k-large. □

Observe that almost the same argument as above shows that for any natural number k, $\mathrm{RCA}_0^* \vdash \mathrm{Tot}(\mathcal{F}_k^h) \implies k\text{-LRG}^h$.

By Proposition 5.6.4 and Proposition 5.6.5 we get:

Corollary 5.6.6 (RCA_0)**.** $\forall k\ \mathrm{Tot}^*(\mathcal{F}_k) = \forall k\ k\text{-LRG}$.

In [51] Solovay and Ketonen proved the following result in RCA_0.

Theorem 5.6.7 (RCA_0^*, Solovay/Ketonen [51])**.** *For any natural number k,*

$$(k+5)\text{-LRG}^h \implies \mathrm{PH}_k^{h,2}$$

Thus, by composing Theorem 5.6.7 and Corollary 5.6.6 we obtain

Corollary 5.6.8 (RCA_0)**.** $\forall k\ \mathrm{Tot}^*(\mathcal{F}_k) \implies \forall k\ \mathrm{PH}^{*2}_k$.

Since for any h, $\mathrm{PH}_k^{h,2} \geq \mathrm{WPH}_k^{h,2} = k\text{-TT}_b^h$, $\mathrm{Tot}(F_{k+h}) \geq k\text{-LRG}^h$ and thanks to Theorem 5.6.7 we get the following bound.

Corollary 5.6.9. *For any $k, h \in \mathbb{N}$ and for any $R \subseteq \mathbb{N}^2$, R is bounded by F_{k+h+5} if there exist $R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ and each R_i is bounded by F_h.*

Observe that the proofs of Proposition 5.6.3, Proposition 5.6.4 and of Proposition 5.6.5 cannot be carried out within RCA_0^*. In fact Σ_1^0-induction is required to apply unbounded primitive recursive definitions, unbounded minimalization, and Π_1^0-induction.

5.6.2 An optimal bound

Due to the equivalence between the Termination Theorem for k-many relations with bounds and the Weak Paris Harrington Theorem and by using some results about Paris Harrington Theorem, namely Theorem 5.6.7, we conclude that if we have $(k+5)$-many relations whose weight functions are in $\mathcal{F}_h$ then R is bounded by a function in $\mathcal{F}_{k+5+h}$, as shown in Corollary 5.6.9. In this section we improve this result, providing an optimal bound for the Termination Theorem, which makes use of a combinatorial argument. Thanks to Theorem 5.5.8 this induces an interesting consequence about the Paris Harrington Theorem.

In [34], Figueira et al. provided an optimal bound for programs which admit a disjunctively well-founded transition invariant of height ω and which have an exponential control on the ranks by using the Dickson Lemma. More precisely the Dickson Lemma states that for any natural number k, every infinite sequence σ of elements in $\mathbb{N}^k$ is *good*; i.e. for any infinite sequence σ of elements in $\mathbb{N}^k$ there exist natural numbers $n < m$ such that $\sigma(n) \leq \sigma(m)$ [57, 64] (where $\leq$ is the componentwise order). In [34] it is shown that given a control function $f : \mathbb{N} \to \mathbb{N}$ in $\mathcal{F}_h$, there exists a function $L_{k,f}$ in $\mathcal{F}_{k+h-1}$, such that the length of the *bad* (not good) sequences for which there exists a natural number

t such that $\forall n \in \mathbb{N}\ \forall i < k\ (\sigma(n)_i < f(n+t))$ is bounded by $L_{k,f}(t)$. If $h = 1$ then the bound is in $\mathcal{F}_k$.

Now assume given a program $\mathcal{R}$ with control function in $\mathcal{F}_h$ (where $h \geq 2$) for which there exists a transition invariant composed by k-many primitive recursive relations $R_0, \ldots, R_{k-1}$ with weight functions $f_0, \ldots, f_{k-1} \in \mathcal{F}_h$. By mapping each state s is the k-tuple $\sigma(s) = \langle f_0(s), \ldots, f_{k-1}(s)\rangle$, any computation σ' of $\mathcal{R}$ is mapped in a bad sequence. In fact, by definition of weight function and since for $m < n$ there exists $i \in k$ such that $\sigma'(n) R_i \sigma'(m)$, we have that for any $n < m$ there exists $i \in k$ such that $f_i(\sigma'(n)) < f_i(\sigma'(m))$. Hence $\sigma(\sigma'(m)) \nleq \sigma(\sigma'(n))$. Therefore Figueira et al. provided a bound in $\mathcal{F}_{h+k-1}$ for $\mathcal{R}$.

We conjecture that these results may be formalized in RCA_0^*+$\mathrm{Tot}(\mathcal{F}_h)$. Since we proved that the Termination Theorem for deterministic programs which admit a k-disjunctively bounded transition invariant is equivalent to the Weak Paris Harrington Theorem, we would extract a proof of $\mathrm{Tot}(\mathcal{F}_{k+\max\{1,h-1\}}) \implies \mathrm{WPH}_k^{h,2}$.

Notice that the argument above does not apply if the function f_i are H-bounds instead of weight functions. In fact in this case it is not true that the map σ applied to computations of $\mathcal{R}$ produces bad sequences.

Here we study an alternative argument for the H-bounded version of the Termination Theorem in order to produce an analogous result for the Paris Harrington Theorem. As a consequence we get that for any natural number k, $\mathrm{Tot}(\mathcal{F}_{k+\max\{1,h-1\}}) \implies \mathrm{PH}_k^{h,2}$ in RCA_0^*.

Theorem 5.6.10 (RCA_0^*+$\mathrm{Tot}(\mathcal{F}_h)$)**.** *Let $R, R_0, \ldots, R_{k-1}$, be binary relations on $\mathbb{N}$. Assume that:*

- *R is a graph of a function in $\mathcal{F}_h$;*
- *$R_0 \cup \cdots \cup R_{k-1} \subseteq R^+$;*
- *each R_i admits a H-bound in $\mathcal{F}_h$.*

Then $\mathrm{Tot}(\mathcal{F}_{k+\max\{1,h-1\}})$ *implies that R is well-founded.*

Proof. Let R be the graph of a function $t \in \mathcal{F}_h$. The proof is developed in two steps: first of all we provide a bound by induction over k (with basic case $k = 2$), then we analyse the complexity of such bound in order to prove that it belongs to $\mathcal{F}_{k+\max\{1,h-1\}}$.

Proof of the two-relations case. Assume that we have two relations R_0 and R_1 such that $R_0 \cup R_1 \supseteq R^+$ and that f_0 and f_1 are their H-bounds. Given a state s we want to find a bound on the number of steps we can do from s. Define a coloring $c : [\mathbb{N}]^2 \to 2$ such that $c(\{t^i(s), t^j(s)\}) = 0$ if and only if $i < j$ and $t^j(s) R_0 t^i(s)$.

First of all we want to find a bound on the number of steps we can do before finding an element which is connected with s in color 0. We build over the proof of this result by using Erdős' tree, as defined in Subsection 1.1.4. We use also the following property of binary trees by observing that if a binary tree has at least 2^n nodes, then it has some branch with n edges (with $n+1$ nodes).

Lemma 5.6.11. *Let* Tr *be the Erdős' tree which corresponds to the computation which starts in some state.*

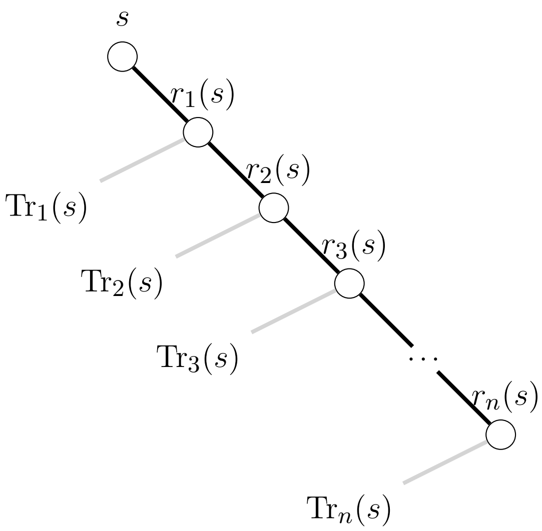
1. *Assume that $x \in \mathrm{Tr}$ has ancestors x_0, x_1 in* Tr *such that there is an edge in color* 0 *between x_0 and x and there is an edge in color* 1 *between x_1 and x. Then $|\mathrm{Tr}(x)| < 2^{f_0(x_0)+f_1(x_1)}$.*

2. *As before, assume that $x, x_0, x_1 \in \mathrm{Tr}$ are such that x_0 is an ancestor in color* 0 *and x_1 is an ancestor in color* 1 *of x. If we have two subsets I_0 and I_1 of nodes such that $x_0 \in I_0$ and $x_1 \in I_1$, then $|\mathrm{Tr}(x)| < 2^{\max\{f_0(z) \mid z \in I_0\} + \max\{f_1(z) \mid z \in I_1\}}$.*

Proof. 1. Assume $x_0, x_1, x \in \mathrm{Tr}$ are such that x_0 and x_1 are both ancestors of x, the former in color 0 and the latter in color 1. Assume by contradiction that $|\mathrm{Tr}| = 2^{f_0(x_0)+f_1(x_1)}$, then since Tr is a binary tree there exists a branch with $(f_0(x_0) + f_1(x_1))$-many edges. This means that in this branch there are either $f_0(x_0)$-many edges in color 0 or $f_1(x_1)$-many edges in color 1. Without loss of generality assume that we have $f_0(x_0)$-many edges in color 0. Hence if we consider the first nodes in each of these edges and the last node of the last edge we obtain a homogeneous sequence in color 0 with $f_0(x_0) + 1$-many nodes. Thus, this is a transitive R_0-sequence such whose length is greater than $f_0(x_0)$. This is a contradiction.

2. It follows by the hypotheses and by point (1). □

Let $\mathrm{Tr}(s)$ be the Erdős' tree whose root is s. For any state s, let $r(s)$ be the branch of $\mathrm{Tr}(s)$ whose elements are all connected in color 1 with the root and between them. For any n let $r_n(s)$ be the n-th node of $r(s)$, and $root_n(s)$ the child in color 0 of $r_n(s)$, if they exist. Let $\mathrm{Tr}_n(s) = \mathrm{Tr}(root_n(s))$ be the subtree of Tr having root $root_n(s)$. The father of $\mathrm{Tr}_n(s)$ is the n-th element of $r(s)$. Then s is the 0-th element of $r(s)$.



We want to define a bound for the size of any $\mathrm{Tr}_n(s)$. In order to do that we shall apply Lemma 5.6.11.1. So for any $\mathrm{Tr}_n(s)$ we will find two nodes x_0 and x_1 in $\mathrm{Tr}(s)$

such that they are an ancestor in color 0 and an ancestor in color 1 for any element of $\mathrm{Tr}_n(s)$. This will provide the nodes x_0 and x_1 we need in order to apply Lemma 5.6.11.1. If we assume that any element of $\mathrm{Tr}(s)$ is connected with s in color 1 we will choose $x_1 = s$, connected to any element of $\mathrm{Tr}_n(\mathrm{s})$ with color 1. The natural choice for a node x_0 connected to any element of $\mathrm{Tr}_n(s)$ with color 0 is $r_n(s)$. Therefore we can define a map $b^0_n(s)$ returning an upper bound for $|\mathrm{Tr}_n(s)|$. To verify this is a bound we apply Lemma 5.6.11.2.

Definition 5.6.12. For any state s, define $b^0_n(s)$ by induction on n:

$$b^0_0(s) = 1$$
$$b^0_{n+1}(s) = 2^{\max\left\{f_0(t^i(s))\ \mid\ i \le \sum_{j=0}^{n} b^0_j(s)\right\} + f_1(s)}.$$

Proposition 5.6.13. *Assume that any element of the Erdős' tree is connected in color* 1 *with* s. *Then*

1. *for any* $n \ge 1$, *and for any* $x \in \mathrm{Tr}_n(s)$, xR_1s;
2. *for any* $n \ge 1$, $b^0_n(s)$ *is a bound for the size of* $\mathrm{Tr}_n(s)$;
3. *for any* $n \ge 1$, $r_{n+1}(s) \in \left\{t^i(s) \mid i \le \sum_{j=0}^{n} b^0_j(s)\right\}$; *i.e. after* $\sum_{j=0}^{n} b^0_j(s)$*-many steps we find* $r_{n+1}(s)$, *the* $(n+1)$*-th element of* $r(s)$.

Proof. By induction over n. Assume that $n = 1$.

1. It follows since any x is connected to s in color 1.
2. Observe that
$$r_1(s) = t(s) \in \left\{t^i(s) \mid i \le 1\right\}.$$
Therefore due to point (1) above we can apply Lemma 5.6.11.2 for $I_0 = \left\{t^i(s) \mid i \le 1\right\}$ and $I_1 = \{s\}$. Hence we get $|\mathrm{Tr}_1(s)| < b^0_1(s)$.
3. Due to point (2) above, in $b^0_0(s) + b^0_1(s)$ many steps we find the second element of $r(s)$.

Assume now that the thesis holds for n and we prove it for $n+1$. So assume that $b^0_n(s)$ is a bound for $\mathrm{Tr}_n(s)$ and that after $(\sum_{j=0}^{n} b^0_j(s))$-many steps we find the $(n+1)$-th element of $r(s)$.

1. Again it follows since any x is connected to s in color 1.
2. Due to the inductive hypothesis we have
$$r_n(s) \in \left\{t^i(s) \mid i \le \sum_{j=0}^{n-1} b^0_j(s)\right\}$$

Again, by using point (1) and this remark, we can apply Lemma 5.6.11.2 by putting $I_0 = \left\{t^i(s) \mid i \leq \sum_{j=0}^{n-1} b_j^0(s)\right\}$ and $I_1 = \{s\}$. Hence we get the thesis.

3. Thanks to point (2), in less than $\sum_{j=0}^{n+1} b_j^0(s)$-many steps we complete each Tr_i for any $i < n+2$ and so, since by hypothesis we are assuming that any element is connected to s in color 1, we are forced to add a new element $r_{n+2}(s)$ of $r(s)$. $\square$

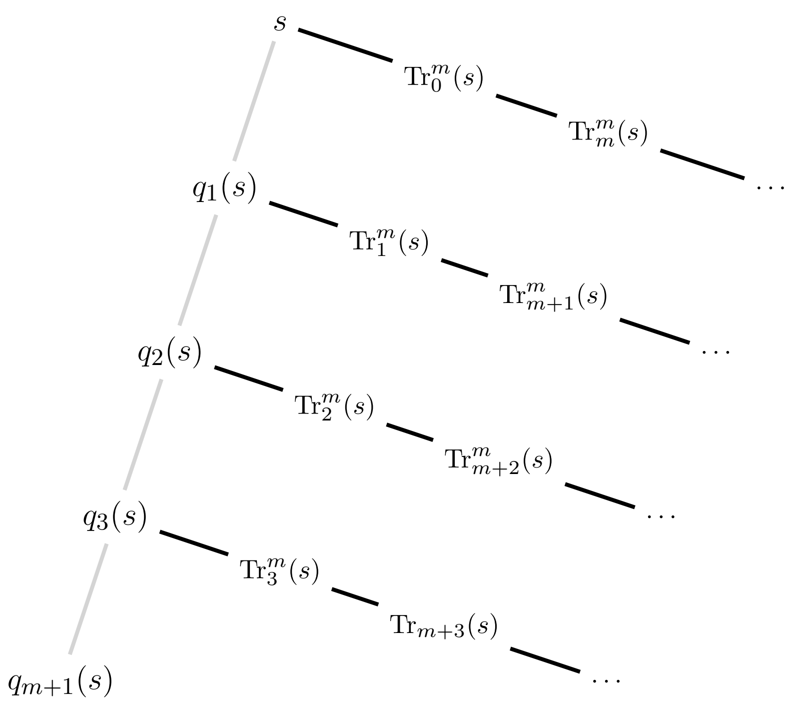
Now we study a bound for the size of the whole Erdős' tree.

Definition 5.6.14. Put

$$
\begin{aligned}
b_0^{m+1}(s) &= \sum_{i=0}^{m \cdot \max\left\{f_1(t^i(s)) \,\mid\, i < b_0^m(s)\right\}} b_i^m(s) \\
b_{n+1}^{m+1}(s) &= 2^{\max\left\{f_0(t^i(s)) \,\mid\, i \leq \sum_{j=0}^{n} b_j^{m+1}(s)\right\} + \max\left\{f_1(t^i(s)) \,\mid\, i < b_0^m(s)\right\}} \\
f_2(s) &= b_0^{f_0(s)}
\end{aligned}
$$

Observe that $b_0^1(s)$ is a bound for the number of steps required to find an element which is connected to s in color 0. While we will prove $f_2(s)$ is a bound for the whole computation as guaranteed by the following results.

Let $q(s)$ be the branch of $\mathrm{Tr}(s)$ whose elements are all connected in color 0 with the root and between them. Let $q_m(s)$ be the m-th node of $q(s)$.



Let $r_n^m(s)$ be the ancestor in color 0 of the root of $\mathrm{Tr}_n^m(s)$ and let $\phi : \mathbb{N} \to \{r_n^m \mid n \in \mathbb{N}\}$ be such that $\phi(l)$ is the l-th element of $\{r_n^m \mid n \in \mathbb{N}\}$ which appears in the computation. As a corollary of the following result we get a bound for $\phi(n)$.

Proposition 5.6.15. *For any $m \in \mathbb{N}$.*

1. $q_m(s) \in \left\{t^i(s) \mid i < b_0^m(s)\right\}$; *i.e. after $b_0^m(s)$-many steps we find $q_m(s)$, the $m+1$-th element of $q(s)$.*

2. *For any n, $b_{n+1}^m(s)$ is a bound for $\mathrm{Tr}_{\phi(n)}^m(s)$.*

Proof. By induction on m.

- If $m = 0$. It follows by Proposition 5.6.13.
- Assume that the thesis holds for m. We prove it for $m+1$.
  1. Observe that by induction hypothesis for any $j \in m+1$,
  $$f_1(q_j(s)) < \max\left\{f_1(t^i(s)) \mid i < b_0^m(s)\right\}.$$
  Therefore $\max\left\{f_1(t^i(s)) \mid i < b_0^m\right\}$ is a bound for the homogeneous 1-branch which starts in $q_j(s)$ for any $j < m+1$. Thus
  $$(m+1)\max\left\{f_1(t^i(s)) \mid i < b_0^m(s)\right\}$$
  is a bound for the number of trees of the form Tr_l^m. Since by induction hypothesis on point (2) every $\mathrm{Tr}_{\phi(n)}^m$ is bounded by $b_{n+1}^m(s)$, the thesis follows.
  2. By induction on n. If $n = 0$, then by point (1), there exists i such that $t^i(s) = q_{m+1}(s)$ and $b_0^{m+1}(s) \geq i+1$. Then the root of $\mathrm{Tr}_{\phi(0)}^{m+1}(s)$ appears in $b_0^{m+1}(s)$-many steps. Therefore, following the proof of Proposition 5.6.13, $b_1^{m+1}(s)$ is a bound for $\mathrm{Tr}_{\phi(0)}^{m+1}$.

  Assume that the thesis holds for n. Then, by induction hypothesis in $b_n^{m+1}(s)$-many steps we find a bound for $\left|\bigcup\left\{\mathrm{Tr}_{\phi(i)}^{m+1} \mid i < n\right\}\right|$. Therefore the root of $\mathrm{Tr}_{\phi(n)}^{m+1}$ appears in $b_n^{m+1}(s)$-many steps. Again as shown in Proposition 5.6.13, $b_{n+1}^{m+1}(s)$ is a bound for $\mathrm{Tr}_{\phi(n)}^{m+1}$. □

Corollary 5.6.16. *$f_2(s)$ is a bound for the number of nodes of the Erdős' tree associated to the computation which starts in s.*

Proof. Since $f_0(s)$ is a bound for the length of $q(s)$, by Proposition 5.6.15 we get that $f_2(s)$ is a bound for the size of $\mathrm{Tr}(s)$. □

Proof of the $(k+1)$-relations case. Assume we proved that the bound for k-many colors and weight functions in $\mathcal{F}_h$ is given by $f_k \in \mathcal{F}_{h+k-1}$, we want to prove that the bound for $k+1$-many colors and weight functions in $\mathcal{F}_h$ is in $\mathcal{F}_{h+k}$. We define $b_n^m(s)$ as before by putting $f_1 = f_k$ and then a possible bound we obtain is $f_{k+1}(s) = b_0^{f_0(s)}(s)$.

Complexity. Here we analyze the complexity of the bounds we found. Assume that $f_0 \in \mathcal{F}_h$, $f_k \in \mathcal{F}_{h+k-1}$, $k \geq 2$ and $h \geq 2$. Hence $h+k-1 \geq h+1$.

Let us prove that $b_n^{m+1}(s) \in \mathcal{F}_{h+k-1}$. By applying Proposition 5.5.1.2, let $l_0, l_1, l_2 \in \mathbb{N}$ be such that $t(x) \leq F_h^{l_0}(x)$, $f_k(x) \leq F_{h+k-1}^{l_1}(x)$, and $f_0(x) \leq F_h^{l_2}(x)$. Put

$$u(x) = 2F_3(2F_{k+h-1}^{l_1}(F_{h+1}(l_0(x)))).$$

Then we can define the following function:

$$\begin{aligned} f(0,s) &= b_0^m(s) \\ f(n+1,s) &= f(n,s) + 2^{\max\left\{f_0(t^i(s)) \mid i \leq f(n,s)\right\} + \max\left\{f_k(t^i(s)) \mid i < b_0^m(s)\right\}} \\ f(n,s) &\leq u^n(\max\{b_0^m(s), s\}). \end{aligned}$$

Observe that the third inequality holds. In fact assume by induction that $f(n,s) \leq u^n(\max\{b_0^m(s), s\})$. Then

$$\begin{aligned} & f(n+1,s) \leq f(n,s) + 2^{\max\left\{f_0(t^i(s)) \mid i \leq f(n,s)\right\} + \max\left\{f_k(t^i(s)) \mid i < f(n,s)\right\}} \\ & \leq f(n,s) + 2^{\max\left\{F_h^{l_2}({F_h^{l_0}}^i(s)) \mid i \leq f(n,s)\right\} + \max\left\{F_{h+k-1}^{l_1}({F_h^{l_0}}^i(s)) \mid i \leq f(n,s)\right\}} && \text{(by Prop. 5.5.1.1)} \\ & \leq f(n,s) + 2^{F_h^{l_2}({F_h^{l_0}}^{f(n,s)}(s)) + F_{h+k-1}^{l_1}({F_h^{l_0}}^{f(n,s)}(s))} \\ & \leq f(n,s) + 2^{2F_{h+k-1}^{l_1}({F_h^{l_0}}^{f(n,s)}(s))} && \text{(by Prop. 5.5.1.1)} \\ & \leq f(n,s) + 2^{2F_{h+k-1}^{l_1}({F_h^{l_0}}^{u^n(\max\{b_0^m(s),s\})}(s))} && \text{(by Ind. Hyp.)} \\ & \leq f(n,s) + 2^{2F_{h+k-1}^{l_1}(F_{h+1}(l_0 u^n(\max\{b_0^m(s),s\})))} && \text{(by Cor. 5.5.2)} \\ & \leq f(n,s) + F_2(2F_{h+k-1}^{l_1}(F_{h+1} l_0 u^n(\max\{b_0^m(s), s\}))) && \text{(by Prop. 5.5.1.1)} \\ & \leq 2F_2(2F_{h+k-1}^{l_1}(F_{h+1} l_0 u^n(\max\{b_0^m(s), s\}))). && (F_j \text{ increasing }) \\ & = u(u^n(\max\{b_0^m(s), s\})) \end{aligned}$$

Hence $b_n^m(s) = f(n,s) \leq u^n(\max(\{b_0^m(s), s\}))$. Since $u \in \mathcal{F}_{h+k-1}$, let l_3 be such that $u(x) \leq F_{h+k-1}^{l_3}(x)$.

$$\begin{aligned} b_0^{m+1}(s) &\leq m f_k(t^{b_0^m(s)}(s)) b^m_{m f_k(t^{b_0^m(s)}(s))} \\ &\leq 2u^{m f_k(t^{b_0^m(s)}(s))}(\max\{b_0^m(s), s\}) \\ &\leq 2F_{h+k-1}(l_3 \max\left\{m f_k(t^{b_0^m(s)}(s)), s\right\}) \\ &\leq F_{h+k-1}(l_3 m f_k(F_{h+1}(\max\{b_0^m(s), s\}))) \\ &\leq F_{h+k-1}(l_3 m F_{h+k-1}^{l_1}(F_{h+1}(\max\{b_0^m(s), s\}))) \end{aligned}$$

Define $v(m,x) = F_{h+k-1}(l_3 m F_{h+k-1}^{l_1}(F_{h+1}(x)))$. Since $b_0^0(s) = 1$, by applying the argument above we get $b_0^1(s) \leq v(1,s)$. So, since $v(1,s) \geq s$ and $v(2,s) \geq v(1,s)$, $b_1^2(s) \leq v^2(2,s)$. Hence:

$$b_0^m(s) \leq v^m(m,s).$$

Therefore $f_{k+1}(s) = b_0^{f_0(s)}(s) \leq v^{f_0(s)}(f_0(s),s)$ is in $\mathcal{F}_{h+k}$.

By using the argument above we get the following.

- Assume that $h < 2$. Hence:
  - if $k = 2$, we get $f_2 \in \mathcal{F}_3 = \mathcal{F}_{k+1}$.
  - if $k > 2$, we get $f_k \in \mathcal{F}_{k+1}$.
- Assume that $h \geq 2$. Hence:
  - if $k = 2$, we get $f_2 \in \mathcal{F}_{h+1} = \mathcal{F}_{h+k-1}$;
  - if $k > 2$, we get $f_k \in \mathcal{F}_{h+k-1}$. □

Figueira et al. in [34] proved that the bound provided for the miniaturization of the Dickson Lemma is optimal; i.e. there are examples of programs with control functions in $\mathcal{F}_h$ and a transition invariant composed of k-many relations with weight functions in $\mathcal{F}_h$ for which the computations cannot be bounded by a function in $\mathcal{F}_{k+h-2}$. Since any weight function is a bound and any bound is a H-bound, the example provided in [34] shows that also our bound is optimal.

Example 5.6.17. Consider the following program.

```
while  ( x > 0 and y > 0 )
     if(y > 1)
          (x, y, z) = (x, y - 1, 2 * z)
     else (x, y, z) = (x - 1, 2 * z, 2 * z)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$$R_1 = \Big\{(\langle x,y,z\rangle, \langle x',y',z'\rangle) \mid y > 0 \wedge y' < y\Big\} \qquad \text{Bounded by } F_0$$
$$R_2 = \Big\{(\langle x,y,z\rangle, \langle x',y',z'\rangle) \mid x > 0 \wedge x' < x\Big\} \qquad \text{Bounded by } F_0$$

By Theorem 5.6.10 and since R is the graph of function in $\mathcal{F}_2$, R is bounded by F_{2+1}. It is straightforward to prove direct that such bound is optimal since for any $x > y > 0$, the computation which starts in $(x,y,1)$ has length greater than $F_2^x(y)$.

The previous result guarantees that $\mathrm{RCA}_0^* + \mathrm{Tot}(\mathcal{F}_{k+\max\{1,h-1\}}) \vdash k\text{-}\mathrm{TT}_H^{h+1}$. By using Theorem 5.5.8 we get the following for any natural numbers h and k.

Corollary 5.6.18 (RCA_0^*)**.** $\mathrm{Tot}(\mathcal{F}_{k+\max\{1,h-1\}}) \implies \mathrm{PH}_k^{h+1,2}$.

In particular for any natural number k, $\mathrm{Tot}(\mathcal{F}_{k+1}) \implies \mathrm{PH}^{2,2}_k$, which implies the usual lightface version of Paris Harrington Theorem (e.g. [47]). Hence over RCA_0^* we have:

$$\mathrm{Tot}(\mathcal{F}_3) \geq \mathrm{PH}^{2,2}_2 \geq \mathrm{WPH}^{2,2}_k \not\leq \mathrm{Tot}(\mathcal{F}_2).$$

In fact, thanks to Example 5.6.17, we know that we cannot have $\mathrm{Tot}(\mathcal{F}_2) \implies \mathrm{WPH}^{2,2}_2$ in RCA_0^*.

5.6.3 From bounds to transition invariants

Here we study a kind of vice versa of the results obtained in the previous subsections. Let k be a natural number. Assume that we have a deterministic relation R which is bounded by F_k, how many linearly bounded relations do we need to obtain a transition invariant? In here we prove that if R is bounded by F_k (the usual k-th fast growing function defined in the previous section) then it has a $k+2$-disjunctively well-founded transition invariant bounded by F_0 and that if R is deterministic then there exists a $k+2$-disjunctively linearly bounded one. Up to now we do not know if such number is the minimum possible.

Theorem 5.6.19 (RCA_0^*+$\mathrm{Tot}(F_k)$)**.** *For any deterministic transition relation $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $T_0, \dots, T_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq T_0 \cup \cdots \cup T_{k+1}$ and each T_i is bounded by F_0.*

Proof. Let $R \subseteq \mathbb{N}^2$ be a deterministic transition relation which is bounded by F_k. Note that for a deterministic transition relation R generated by a transition function, R^+ is Δ^0_1-definable, thus it exists as a set within RCA_0^*. Define $T_<$ and $T_>$ as

$$xT_<y \iff xR^+y \wedge x < y,$$
$$xT_>y \iff xR^+y \wedge x > y.$$

Trivially, $T_<$ is bounded by F_k because every bound for R is a bound for R^+ and for $T_<$, and $R^+ = T_< \cup T_>$. Now, define $d_{T_<} : \mathbb{N}^2 \to \mathbb{N} \cup \{\infty\}$ as

$$d_{T_<}(x,y) = \begin{cases} \max\{m \mid \exists \langle x_i \mid i \leq m\rangle (x = x_0 T_< x_1 T_< \cdots T_< x_m = y)\} & \text{if } xT_<y, \\ \infty & \text{otherwise.} \end{cases}$$

$T_>$ is decreasing, therefore it is bounded by $\mathrm{id}(x) = x$ and with more reason by $F_0(x) = x+1$. Hence we only need to decompose $T_<$ into $k+1$-many F_0-bounded relations. For each x, by F_k-boundedness, we may effectively compute the list of states $x = x_0, \dots, x_m$ reachable from x by R and with more reason by $T_<$. Thus, for each $i \leq k$ we define by bounded induction in RCA_0^*+$\mathrm{Tot}(F_k)$ $\mathrm{rank}_i(x)$ for each state x as follows:

- for any $x \in \mathbb{N}$, $\mathrm{rank}_i(x) \geq 0$,

- $\mathrm{rank}_i(x) \geq n+1$ if there exists $y \in \{x_0, \ldots, x_m\}$ such that $d_{T_<}(x,y) \geq F_i(x)$ and $\mathrm{rank}_i(y) \geq n$,
- $\mathrm{rank}_i(x) = n$ if $\mathrm{rank}_i(x) \geq n$ and $\mathrm{rank}_i(x) \ngeq n+1$,
- $\mathrm{rank}_i(x) \leq m$.

Now, we put T_i for $i \leq k$ as follows:

$$xT_iy \iff xT_<y \wedge i = \min\{j \leq k \mid \mathrm{rank}_j(x) = \mathrm{rank}_j(y)\}.$$

Note that by definition, T_i is transitive, and if $xT_<y$ then $\mathrm{rank}_i(x) \geq \mathrm{rank}_i(y)$ for any $i \leq k$. Since R is deterministic, if $xT_<y$, $xT_<z$, and $\mathrm{rank}_i(y) > \mathrm{rank}_i(z)$, then $yT_<z$.

Now, for the sake of contradiction, we assume that $\langle x_n \mid n \leq m \rangle$ is a T_i-sequence such that $m > F_0(x_0) = x_0+1$. Then, $\mathrm{rank}_i(x_0) = \cdots = \mathrm{rank}_i(x_m)$. If $i = 0$, this is impossible since $d_{T_<}(x_0, x_m) \geq m > F_0(x_0)$, which means $\mathrm{rank}_0(x_0) > \mathrm{rank}_0(x_m)$. If $i > 0$, then, $\mathrm{rank}_{i-1}(x_0) > \mathrm{rank}_{i-1}(x_1) > \cdots > \mathrm{rank}_{i-1}(x_m)$, and hence $\mathrm{rank}_{i-1}(x_0) - x_0 - 1 > \mathrm{rank}_{i-1}(x_0) - m \geq \mathrm{rank}_{i-1}(x_m)$. By the definition of rank_{i-1}, there exists $\langle y_n : n \leq x_0+1 \rangle$ such that $x_0 = y_0$, $d_{T_<}(y_n, y_{n+1}) \geq F_{i-1}(y_n)$ and $\mathrm{rank}_{i-1}(y_n) = \mathrm{rank}_{i-1}(x_0) - n$. Recall that $x_0 = y_0 < y_1 < \cdots < y_m$ since $T_<$ is an increasing relation. Moreover $d_{T_<}(y_n, y_{n+1}) \geq F_{i-1}(y_n)$ implies that $y_{n+1} \geq F_{i-1}(y_n)$, since there exists a decreasing sequence composed of $F_{i-1}(y_n)$ from y_{n+1}. Hence we have $d_{T_<}(y_0, y_{x_0+1}) \geq F_{i-1}(y_{x_0}) \geq F_{i-1}(F_{i-1}(y_{x_0-1})) \geq \cdots \geq F_{i-1}^{(x_0+1)}(y_0) = F_{i-1}^{(x_0+1)}(x_0)$. Moreover, since $\mathrm{rank}_{i-1}(y_{x_0+1}) = \mathrm{rank}_{i-1}(x_0) - x_0 - 1 > \mathrm{rank}_{i-1}(x_m)$, we have $y_{x_0+1} \neq x_m$. Thus $y_{x_0+1}T_<x_m$. Thus $d_{T_<}(x_0, x_m) > d_{T_<}(x_0, y_{x_0+1}) \geq F_{i-1}^{(x_0+1)}(x_0) = F_i(x_0)$. This means $\mathrm{rank}_i(x_0) > \mathrm{rank}_i(x_m)$, which is a contradiction. □

The relations T_i provided by the previous proof are not computable. We wondered how many relations would we need in order to have computable witnesses, but by now we do not know.

Another question we may ask is: can we generalize this result for non-deterministic transition relations? A partial answer to this question is the following: in the general case we need to use H-bounds instead of bounds.

Theorem 5.6.20 ($\mathrm{RCA}_0^*+\mathrm{Tot}(F_k)$)**.** *For any transition relation $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $T_0, \ldots, T_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq T_0 \cup \cdots \cup T_{k+1}$ and each T_i is H-bounded by F_0.*

Proof. Let $R \subseteq \mathbb{N}^2$ be a transition relation which is bounded by F_k. Define $T_<$ and $T_>$ and $d_{T_<}$ as in the proof of Theorem 5.6.19. Trivially, $T_<$ is H-bounded by F_k, $T_>$ is bounded by F_0, and $R^+ = T_< \cup T_>$. Now we define T_i for each $i \leq k$ as

$$xT_iy \iff xT_<y \wedge i = \min\{j \mid d_{T_<}(x,y) \leq F_j(x)\}.$$

Then, we have $T_0 \cup \cdots \cup T_k = T_<$. Thus, we only need to check that each T_i is F_0-H-bounded. Assume that $\langle x_n \mid n \leq m \rangle$ is a T_i-homogeneous sequence such that $m >$

$F_0(x_0) = x_0 + 1$. If $i = 0$, this is impossible since $d_{T_<}(x_0, x_m) \geq m > F_0(x_0)$. If $i > 0$, then, $d_{T_<}(x_n, x_{n+1}) > F_{i-1}(x_n)$, and hence $x_{n+1} > F_{i-1}(x_n)$ because any T_i-homogeneous decreasing sequence has decreasing values, and there is some T_i-homogeneous decreasing sequence from x_{n+1} of length $F_{i-1}(x_n)$. Thus,

$$d_{T_<}(x_0, x_m) > d_{T_<}(x_{m-1}, x_m) > F_{i-1}(x_{m-1}) > \cdots > F_{i-1}^{(m)}(x_0) > F_i(x_0),$$

which is a contradiction because the definition of T_i-homogeneous sequence implies $x_0 T_i x_m$, therefore $d_{T_<}(x_0, x_m) \leq F_i(x_0)$. $\square$

5.7 Related works

The relation between bounds and provability in a certain theory has been explored. As pointed out by Sylvain Schmitz, in [16] Buchholz provided bounds for the lengths of reduction sequences in term rewriting systems, which are proven to terminate thanks to recursive (resp. lexicographic) path ordering. The technique is to show that if a finite rewrite system is reducing under such orderings then the termination can be proven within the fragment Σ^0_1-IA (resp. Π^0_2-IA) of Peano Arithmetic.

Subsection 5.6.2 highlights the connection between Dickson Lemma and Paris Harrington Theorem. In [70] Omata and Pelupessy perform a precise analysis of the relation of a miniaturized version of Dickson's Lemma as in [34] and Weak Paris Harrington Theorem over RCA_0. They also present a different proof of the bound provided in [34] for the length of the bad sequence, which corresponds to the one provided in Theorem 5.6.10.

5.8 Open questions

In Section 5.2 we proved that the full Termination Theorem is equivalent to the full Weak Ramsey Theorem. Moreover, since the full Weak Ramsey Theorem is provable within CAC plus full induction, which is strictly weaker than the full Ramsey Theorem for pairs, we answered negatively to [38, Open Problem 3]. However we wonder if these two results have the same verification power, namely if the first order statements which are provable within $\forall k\, \mathrm{WRT}^2_k$ and within $\forall k\, \mathrm{RT}^2_k$ are the same.

Question 5.8.1. Do $\forall k\, \mathrm{WRT}^2_k$ and $\forall k\, \mathrm{RT}^2_k$ have the same first order part?

The proofs presented in Subsection 5.6.1 cannot be carried out within RCA^*_0. We conjecture that such results hold over RCA^*_0. As shown in Subsection 5.6.2 we have the following situation within RCA^*_0

$$\mathrm{Tot}(\mathcal{F}_{k+\max\{1,h-1\}}) \geq \mathrm{PH}^{h,2}_k \geq \mathrm{WPH}^{h,2}_k.$$

Hence a question we wonder is:

Question 5.8.2. Does $\mathrm{WPH}_k^{h,2}$ imply $\mathrm{Tot}(\mathcal{F}_{\max\{k,h\}})$ over RCA_0^*?

Hájek and Pudlák in [47, Problem 3.37] proposed the following problem: find a reasonably simple proof of $\mathrm{I}\Sigma_1 \vdash (W)_n \implies (\mathrm{PH})_n$, where $(W)_n$ is a principle equivalent to the statement[9] $\forall k \forall \alpha < \omega_{n-1}^k \mathrm{Tot}(F_\alpha)$ and $(\mathrm{PH})_n$ is the statement[10] $\forall k \mathrm{PH}_k^{n+1}$. Corollary 5.6.18 guarantees that $\mathrm{RCA}_0^* \vdash \mathrm{Tot}(\mathcal{F}_{k+1}) \implies \mathrm{PH}_k^2$. However, since the proof is by induction over k, we cannot conclude directly that $\mathrm{RCA}_0^* \vdash \forall k \mathrm{Tot}(\mathcal{F}_{k+1}) \implies \forall k \mathrm{PH}_k^2$. Hence a third question we are currently working on is:

Question 5.8.3. Can we use a similar argument to prove $(W)_1 \implies (\mathrm{PH})_1$ in $\mathrm{I}\Sigma_1$?

Moreover, given a natural number $n > 2$:

Question 5.8.4. Can we generalize our argument to prove PH_k^n from $\mathrm{Tot}(\mathcal{F}_{\omega_{n-2}^{k+1}})$?

Eventually, in Subsection 5.6.3, we proved that given a deterministic relation bounded by F_k we can find a transition invariant composed of $(k+2)$-many linearly bounded relations. Moreover given a relation bounded in F_k we provided a transition invariant composed of $(k+2)$-many linearly H-bounded relations. Thus we wonder whether such results are improvable:

Question 5.8.5. Given a relation bounded by F_k, what is the minimal number of linearly bounded relations whose union contains the transitive closure of R?

[9] For any natural numbers n and k $\omega_0^k = \omega^k$ and $\omega_{n+1}^k = \omega^{\omega_n^k}$. By adding $F_\alpha(x) = F_{\{\alpha\}(x)}(x+1)$ to the definition of Fast Growing Hierarchy we obtain the definition of Schwichtenberg-Wainer used in [47]. We refer to [47] for details.

[10] PH_k^{n+1} is the lightface version of $\mathrm{PH^*}_k^{n+1}$ for sets X which are intervals.

Conclusion

The aim of this thesis has been to investigate bounds for termination by using proof theoretical tools. The H-closure Theorem provides an intuitionistic proof for both the Termination Theorem and the SCT Theorem. Bar recursion yields a semi-constructive interpretation for the Termination Theorem. A reverse mathematical's analysis shows the strength of the Termination Theorem. Several questions arises from this work, beyond the ones introduced in each chapter.

Bounds for the Termination Theorem. In Chapter 2 we proved that any program with deterministic transition relation R (with primitive recursive transition function) for which there exists a disjunctively well-founded transition invariant with height ω, has a primitive recursive bound. In Chapter 4 we relaxed the hypotheses, by asking that R has some modulus of well-foundedness in system T (which contains primitive recursion in all finite types). Under these hypothesis we obtained a sub-recursive bound in system T. A strengthened analysis is presented in Chapter 5, where we proved that any program with deterministic transition relation R (with transition function in $\mathcal{F}_h$) for which there exists a k-disjunctively well-founded transition invariant with H-bounds in $\mathcal{F}_h$ has a bound in $\mathcal{F}_{k+\max 1, h-1}$.

We conjecture that we could extract more information than the one presented in this work from the construction of bounds in Chapter 2 and in Chapter 4. In particular we conjecture that the proof of Theorem 2.3.10 provides a bound in $\mathcal{F}_{k+1}$ anyhow the weight functions are in $\mathcal{F}_1$. By using Howard and Kreuzer's ordinal analysis we could extract more information also from the result in Chapter 4. Hence it could be possible to compare the results more precisely.

Bounds for the SCT **Theorem.** The SCT Theorem is analysed in Chapter 3. Bounds for this result are investigated following the approach of Chapter 2. We are currently working on a reverse mathematical analysis of the SCT Theorem. The importance in studying the SCT Theorem lies in the fact that the functions proved to be terminating by the SCT Theorem are the multiple-recursive functions [6]. For instance the Ackermann function is proved to be terminating by the SCT Theorem. Classifying the SCT Theorem in the hierarchy of corollaries of Ramsey's Theorem for pairs, would help to understand the relation between the two different notions of termination in order to make the situation clear.

Connections with rewriting theory. As highlighted in Subsection 1.2.3, there is a strong connection between termination (as well-foundedness property) and rewriting theory. In particular, state transitions can be related to the notion of dependency pair in rewriting theory (e.g. see [88]). A possible future direction is to investigate the relation between results presented this thesis and the complexity results based on dependency pairs (e.g. see [65]).

Formalization in proof assistants. The constructive versions of termination results presented in [93] are also formalized in mechanized proof verification systems, which justifies them. Similarly, a formalization of results in Chapter 2 and Chapter 3 in Coq, Agda would be possible since all the proofs are carried out in HAS. For instance, the second reviewer of [86] formalized some results of Chapter 3 in Agda.

Applications. There are other applications of Ramsey's Theorem in Computer Science, beyond termination, for which our approach could be significant, for instance in Automata Theory. Namely the original proof by Büchi of the complementation of non-deterministic Büchi automata uses Ramsey's Theorem for pairs [17]. Since there are constructive proofs of this results by using Safra's automata [79], it should be interesting to isolate the fragment of Ramsey's Theorem for pairs used in Büchi's proof. The proof by Safra guarantees that this fragment (and its consequences) is constructive. It would be worthwhile to also study some of the several applications of Ramsey's Theorem for pairs in mathematics (e.g. see [4, 49]), in order to see whether constructive proofs can be provided.

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