

Compairing (ap_v) and (η) -rules.

$$\frac{\Gamma \vdash f : A \rightarrow B}{\Gamma, x : A \vdash f(x) : B} (\eta), \text{ where } x \notin \text{FV}(\Gamma). \quad \frac{\Gamma, x : A \vdash f : A \rightarrow B}{\Gamma, x : A \vdash f(x) : B} (\text{ap}_v).$$

If A is N , trace chasing would be as follows.

$$\frac{\Gamma \vdash f : \mathbf{N} \rightarrow B}{\Gamma, x : \mathbf{N} \vdash f(x) : B} (\eta), \text{ where } x \notin \text{FV}(\Gamma). \quad \frac{\Gamma, x : N \vdash f : \mathbf{N} \rightarrow B}{\Gamma, x : \mathbf{N} \vdash f(x) : B} (\text{ap}_v).$$

Define terms **sum** and **sumd** by:

$$\begin{aligned} \text{sum} &= \lambda mn. \text{cond}(n, \lambda m'. \text{sumd}(m', \mathbf{S}n))(m) \\ \text{sumd} &= \lambda mn. (\lambda k. \text{sum}(m)(n))(m) \end{aligned}$$

Remark There are two occurrences of m in **sumd**. The inner one works as a conter, and the outer one is meaningless that is just consumed by the dummy variable k .

We sometimes write $f(x, y)$ instead of $f(x)(y)$, for readability.

Example

$$\text{sumd}(0, 2) \mapsto (\lambda k. \text{sum}(0, 2))(0) \mapsto \text{sum}(0, 2) \mapsto \text{cond}(2, \lambda m'. \text{sumd}(m', \mathbf{S}2))(0) \mapsto 2$$

With (η) , we need to use the rule (ap) to handle the subterm $(\lambda k. \text{sum}(m)(n))(m)$ of **sumd** because of the side-condition of (η) .

Lemma 0.1 $\text{sumd} : N \rightarrow N \rightarrow N$ is well-typed with (η) .

Proof

$$\begin{aligned} & \frac{\frac{\frac{\vdash \text{sumd} : \mathbf{N} \rightarrow N \rightarrow N}{m' : \mathbf{N} \vdash \text{sumd}(m') : N \rightarrow N} (\eta) \quad \frac{\overline{n : N \vdash n : N}}{n : N \vdash \mathbf{S}n : N}}{n : N, m' : \mathbf{N} \vdash \text{sumd}(m')(\mathbf{S}n) : N}}{n : N \vdash n : N \quad n : N \vdash \lambda m'. \text{sumd}(m')(\mathbf{S}n) : \mathbf{N} \rightarrow N} (\text{cond}) \\ & \frac{n : N \vdash \text{cond}(n, \lambda m'. \text{sumd}(m')(\mathbf{S}n)) : \mathbf{N} \rightarrow N}{m : \mathbf{N}, n : N \vdash \text{cond}(n, \lambda m'. \text{sumd}(m')(\mathbf{S}n))(m) : N} (\eta) \\ & \frac{\frac{\frac{\vdash \text{sum} : \mathbf{N} \rightarrow N \rightarrow N}{k : N \vdash \text{sum} : \mathbf{N} \rightarrow N \rightarrow N} (\eta) \quad \frac{m : \mathbf{N}, k : N \vdash \text{sum}(m) : N \rightarrow N}{m : \mathbf{N}, n : N, k : N \vdash \text{sum}(m)(n) : N} (\eta)}{m : \mathbf{N}, n : N \vdash \lambda k. \text{sum}(m)(n) : N \rightarrow N} \quad \overline{m : N \vdash m : N}}{m : \mathbf{N}, n : N \vdash (\lambda k. \text{sum}(m)(n))(m) : N} (\text{ap}) \\ & \frac{}{\vdash \text{sumd} : \mathbf{N} \rightarrow N \rightarrow N} \end{aligned}$$

This satisfies GTC since the unique infinite path contains a progressing trace (the sequence of \mathbf{N}). \square

However, $\mathbf{sumd} : N \rightarrow N \rightarrow N$ cannot be typed with (\mathbf{ap}_v) , since we need to use the rule (\mathbf{ap}_v) to handle the subterm $(\lambda k.\mathbf{sum}(m)(n))(m)$ of \mathbf{sumd} . This causes a situation of missing trace.

$$\begin{array}{c}
\frac{\frac{\vdash \mathbf{sumd} : N \rightarrow N \rightarrow N}{m' : N \vdash \mathbf{sumd}(m') : N \rightarrow N} (\eta) \quad \frac{\overline{n : N \vdash n : N}}{n : N \vdash \mathbf{Sn} : N}}{\frac{n : \mathbf{N}, m' : N \vdash \mathbf{sumd}(m')(\mathbf{Sn}) : N}{n : \mathbf{N} \vdash \lambda m'.\mathbf{sumd}(m')(\mathbf{Sn}) : N \rightarrow N}} \\
\frac{\overline{n : N \vdash n : N} \quad \frac{n : \mathbf{N} \vdash \lambda m'.\mathbf{sumd}(m')(\mathbf{Sn}) : N \rightarrow N}{m : N, n : \mathbf{N} \vdash \mathbf{cond}(n, \lambda m'.\mathbf{sumd}(m')(\mathbf{Sn})) : N \rightarrow N} (\mathbf{cond})}{m : N, n : \mathbf{N} \vdash \mathbf{cond}(n, \lambda m'.\mathbf{sumd}(m')(\mathbf{Sn}))(m) : N} (\mathbf{ap}_v) \\
\frac{\vdash \mathbf{sum} : N \rightarrow \mathbf{N} \rightarrow N}{m : N, n : N, k : \mathbf{N} \vdash \mathbf{sum} : N \rightarrow \mathbf{N} \rightarrow N} (\mathbf{ap}_v) \\
\frac{m : N, n : N, k : \mathbf{N} \vdash \mathbf{sum}(m) : \mathbf{N} \rightarrow N}{m : N, n : \mathbf{N}, k : \mathbf{N} \vdash \mathbf{sum}(m)(n) : N} (\mathbf{ap}_v) \\
\frac{m : N, n : \mathbf{N}, k : \mathbf{N} \vdash \mathbf{sum}(m)(n) : N}{m : N, n : \mathbf{N} \vdash \lambda k.\mathbf{sum}(m)(n) : \mathbf{N} \rightarrow N} (\mathbf{ap}_v) \\
\frac{m : \mathbf{N}, n : \mathbf{N} \vdash \lambda k.\mathbf{sum}(m)(n) : \mathbf{N} \rightarrow N}{\vdash \mathbf{sumd} : \mathbf{N} \rightarrow \mathbf{N} \rightarrow N} (\mathbf{ap}_v)
\end{array}$$

This does not satisfies GTC, since the unique infinite path contains no progressing trace.