Compairing ($\mathrm{ap_v}$) and (η)-rules.

$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash f(x) : B}\ (\eta),\ \text{where } x \notin \mathrm{FV}(\Gamma). \qquad \frac{\Gamma, x : A \vdash f : A \to B}{\Gamma, x : A \vdash f(x) : B}\ (\mathrm{ap_v}).$$

If A is N, trace chasing would be as follows.

$$\frac{\Gamma \vdash f : \mathbf{N} \to B}{\Gamma, x : \mathbf{N} \vdash f(x) : B}\ (\eta),\ \text{where } x \notin \mathrm{FV}(\Gamma). \qquad \frac{\Gamma, x : \mathbf{N} \vdash f : \mathbf{N} \to B}{\Gamma, x : \mathbf{N} \vdash f(x) : B}\ (\mathrm{ap_v}).$$

Define terms $\mathtt{sum}$ and $\mathtt{sumd}$ by:

$$\begin{aligned}\mathtt{sum} &= \lambda mn.\mathrm{cond}(n, \lambda m'.\mathtt{sumd}(m', \mathtt{S}n))(m)\\ \mathtt{sumd} &= \lambda mn.(\lambda k.\mathtt{sum}(m)(n))(m)\end{aligned}$$

Remark There are two occurences of m in $\mathtt{sumd}$. The inner one works as a conter, and the outer one is meaningless that is just consumed by the dummy variable k.

We sometimes write $f(x, y)$ instead of $f(x)(y)$, for readability.

Example

$\mathtt{sumd}(0, 2) \mapsto (\lambda k.\mathtt{sum}(0, 2))(0) \mapsto \mathtt{sum}(0, 2) \mapsto \mathrm{cond}(2, \lambda m'.\mathtt{sumd}(m', \mathtt{S}2))(0) \mapsto 2$

With (η), we need to use the rule (ap) to handle the subterm $(\lambda k.\mathtt{sum}(m)(n))(m)$ of $\mathtt{sumd}$ because of the side-condition of (η).

Lemma 0.1 $\mathtt{sumd} : N \to N \to N$ *is well-typed with* (η).

Proof

$$\dfrac{\dfrac{}{n : N \vdash n : N} \qquad \dfrac{\dfrac{\dfrac{\vdash \mathtt{sumd} : \mathbf{N} \to N \to N}{m' : \mathbf{N} \vdash \mathtt{sumd}(m') : N \to N}\ (\eta) \qquad \dfrac{\dfrac{}{n : N \vdash n : N}}{n : N \vdash \mathtt{S}n : N}}{n : N, m' : \mathbf{N} \vdash \mathtt{sumd}(m')(\mathtt{S}n) : N}}{n : N \vdash \lambda m'.\mathtt{sumd}(m')(\mathtt{S}n) : \mathbf{N} \to N}}{\dfrac{n : N \vdash \mathrm{cond}(n, \lambda m'.\mathtt{sumd}(m')(\mathtt{S}n)) : \mathbf{N} \to N}{m : \mathbf{N}, n : N \vdash \mathrm{cond}(n, \lambda m'.\mathtt{sumd}(m')(\mathtt{S}n))(m) : N}\ (\eta)}\ (\mathrm{cond})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathtt{sum} : \mathbf{N} \to N \to N}{k : N \vdash \mathtt{sum} : \mathbf{N} \to N \to N}}{m : \mathbf{N}, k : N \vdash \mathtt{sum}(m) : N \to N}\ (\eta)}{m : \mathbf{N}, n : N, k : N \vdash \mathtt{sum}(m)(n) : N}\ (\eta)}{m : \mathbf{N}, n : N \vdash \lambda k.\mathtt{sum}(m)(n) : N \to N} \qquad \dfrac{}{m : N \vdash m : N}}{m : \mathbf{N}, n : N \vdash (\lambda k.\mathtt{sum}(m)(n))(m) : N}\ (\mathrm{ap})}{\vdash \mathtt{sumd} : \mathbf{N} \to N \to N}$$

This satisfies GTC since the unique infinite path contains a progressing trace (the sequence of $\mathbf{N}$). $\square$

However, $\texttt{sumd} : N \to N \to N$ cannot be typed with $(\mathrm{ap_v})$, since we need to use the rule $(\mathrm{ap_v})$ to handle the subterm $(\lambda k.\texttt{sum}(m)(n))(m)$ of $\texttt{sumd}$. This causes a situation of missing trace.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n : N \vdash n : N} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \texttt{sumd} : {\color{red}\mathbf{N}} \to N \to N}{m' : N \vdash \texttt{sumd} : {\color{red}\mathbf{N}} \to N \to N}}{m' : {\color{red}\mathbf{N}} \vdash \texttt{sumd}(m') : N \to N}\,(\mathrm{ap_v}) \quad \dfrac{\dfrac{}{n : N \vdash n : N}}{n : N \vdash \mathtt{S}n : N}}{n : N, m' : {\color{red}\mathbf{N}} \vdash \texttt{sumd}(m')(\mathtt{S}n) : N}}{n : N \vdash \lambda m'.\texttt{sumd}(m')(\mathtt{S}n) : {\color{red}\mathbf{N}} \to N}}{n : N \vdash \mathrm{cond}(n, \lambda m'.\texttt{sumd}(m')(\mathtt{S}n)) : {\color{red}\mathbf{N}} \to N}\,(\mathrm{cond})}{m : N, n : N \vdash \mathrm{cond}(n, \lambda m'.\texttt{sumd}(m')(\mathtt{S}n)) : {\color{red}\mathbf{N}} \to N}}{m : {\color{red}\mathbf{N}}, n : N \vdash \mathrm{cond}(n, \lambda m'.\texttt{sumd}(m')(\mathtt{S}n))(m) : N}\,(\mathrm{ap_v})}{\vdash \texttt{sum} : {\color{red}\mathbf{N}} \to N \to N}}{m : N, n : N, k : {\color{blue}\mathbf{N}} \vdash \texttt{sum} : {\color{red}\mathbf{N}} \to N \to N}}{m : {\color{red}\mathbf{N}}, n : N, k : {\color{blue}\mathbf{N}} \vdash \texttt{sum}(m) : N \to N}\,(\mathrm{ap_v})}{m : {\color{red}\mathbf{N}}, n : N, k : {\color{blue}\mathbf{N}} \vdash \texttt{sum}(m)(n) : N}\,(\mathrm{ap_v})}{m : {\color{red}\mathbf{N}}, n : N \vdash \lambda k.\texttt{sum}(m)(n) : {\color{blue}\mathbf{N}} \to N}}{m : \mathbf{N}, n : N \vdash (\lambda k.\texttt{sum}(m)(n))(m) : N}\,(\mathrm{ap_v})}{\vdash \texttt{sumd} : \mathbf{N} \to N \to N}
$$

This proof satisfies GTC, since the only infinite path contains an infinitely progressing trace $(\mathbf{N}, \mathbf{N}, {\color{red}\mathbf{N}}, \ldots)$