About equality on infinite terms

Remark We assume the following (not formally shown yet):

Any $t \in \mathtt{GTC}$ has a unique normal form $\mathtt{Lim}(t)$ (limit of safe-reductions).

Purpose of this article We want to characterlize the limit equivalence of terms in $\mathtt{GTC}$, which means two terms have the same limit: $\mathtt{Lim}(t) = \mathtt{Lim}(u)$. More formally, we try to coinductively define a binary relation $\approx$ that satisfies:

$$t \approx u \quad \Longleftrightarrow \quad \mathtt{Lim}(t) = \mathtt{Lim}(u).$$

A *1-context* is a finite context with several number of distinguished holes $[-]$.

Definition 0.1 (0-context) $C ::= x \mid \lambda x.C \mid CC \mid 0 \mid \mathtt{S}(C) \mid \mathtt{cond}(C, [-])$

We write $C[-,-,-]$ if C has three holes, and write $C[t,u,v]$ for the resulting term obtained by filling terms t,u,v for the holes.

A term can be divided into the 0-context part and the terms part filling the holes of the 0-context, namely the following lemma holds.

Lemma 0.2 *For any $t \in \mathbb{A}$, there uniquely exists $(C, \vec{u})$ such that $t = C[\vec{u}]$.*

For a binary relation R on terms, we write $(t_1, \ldots, t_n)R(u_1, \ldots, u_n)$ if $t_i R u_i$ holds for all i.

Let $\mathtt{NF}_0$ be the set of 0-safe normal forms. Note that if both $C_1[\vec{t_1}]$ and $C_2[\vec{t_2}]$ are 0-safe normal forms of t, then $C_1 = C_2$.

Definition 0.3 (0-bisimulation) A binary relation R on $\mathbb{A}$ is said to be a *0-bisimulation* if it satisfies the following:

$$\begin{gathered} tRu \text{ and } t \mapsto^*_{0,\mathtt{safe}} C_t[\vec{t}] \in \mathtt{NF}_0 \text{ and } u \mapsto^*_{0,\mathtt{safe}} C_u[\vec{u}] \in \mathtt{NF}_0 \\ \text{implies} \quad C_t = C_u \text{ and } \vec{t} R \vec{u}. \end{gathered}$$

We define $\approx$ by the largest 0-bisimulation.

Lemma 0.4 *$\approx$ is an equivalence relation.*

Proof Reflexivity for $\approx$ holds by $= \subseteq \approx$, since $=$ is a 0-bisimulation. Symmetricty holds by $\approx^{-1} \subseteq \approx$, since $\approx^{-1}$ is a 0-bisimulation. Transitivity holds by $(\approx \circ \approx) \subseteq \approx$, since $\approx \circ \approx$ is a 0-bisimulation. $\square$

Finally we show the relation $\approx$ characterizes the limit equivalence.

Proposition 0.5 *$t \approx u$ and $\mathtt{Lim}(t) = \mathtt{Lim}(u)$ are equivalent.*

Proof ($\Leftarrow$): Let R be the limit equivalence relation, namely $tRu \Leftrightarrow \mathtt{Lim}(t) = \mathtt{Lim}(u)$. Then R is a 0-bisimulation. Hence $\mathtt{Lim}(t) = \mathtt{Lim}(u)$ implies $t \approx u$ by $R \subseteq \approx$.

($\Rightarrow$): We show this direction by proof by contradiction. Define

$$X = \{(t,u) \mid t \approx u \text{ and } \mathtt{Lim}(t) \neq \mathtt{Lim}(u)\}.$$

Assume that $X \neq \emptyset$ (there is a counter example of the ($\Rightarrow$)-direction). For $(t,u) \in X$, we define the difference depth $\sharp(t,u)$ by $\min\{\,|\pi| \mid (\mathtt{Lim}(t))_\pi \neq (\mathtt{Lim}(u))_\pi\,\}$. Fix a $(t_0,u_0) \in X$ that has the least difference depth. Take 0-safe normal forms $C_t[\vec{t'}]$ and $C_u[\vec{u'}]$ of t_0 and u_0, respectively. Then, by $t_0 \approx u_0$, we have $C_t = C_u$ and $\vec{t'} \approx \vec{u'}$. Also we have $C_t[\overrightarrow{\mathtt{Lim}(t')}] = \mathtt{Lim}(t_0)$ and $C_u[\overrightarrow{\mathtt{Lim}(u')}] = \mathtt{Lim}(u_0)$. By $\mathtt{Lim}(t_0) \neq \mathtt{Lim}(u_0)$ and $C_t = C_u$, there is j such that $\mathtt{Lim}(t'_j) \neq \mathtt{Lim}(u'_j)$. Hence we have $(t'_j, u'_j) \in X$. However we also have $\sharp(t'_j, u'_j) < \sharp(t_0,u_0)$, since, when we consider the path to t'_j from the root of t_0, there must be at least one `cond` on the path. This contradicts the leastness of (t_0,u_0). Therefore we have $X = \emptyset$, namely the ($\Rightarrow$)-direction holds. $\square$