CTλ

$$A ::= \alpha \mid A \to A \mid N$$

$$t ::= x^A \mid \lambda x^A.t \mid tt \mid \mathrm{cond}_x(t,t) \mid 0 \mid St$$

$$\frac{}{\Gamma, x:A \vdash x:A}(Ax) \qquad \frac{\Gamma \vdash t:A\to B \quad \Gamma \vdash u:A}{\Gamma \vdash tu:B}(App) \qquad \frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x^A.t : A\to B}(Abs)$$

$$\frac{\Gamma \vdash f:A \quad \Gamma, x:N \vdash g:A}{\Gamma \vdash \mathrm{cond}_x(f,g) : N\to A}(cond) \qquad \frac{}{\Gamma \vdash 0:N}(zero) \qquad \frac{\Gamma \vdash t:N}{\Gamma \vdash St:N}(Succ)$$

$$\frac{\Gamma \vdash t:A\to B}{\Gamma, x:A \vdash tx:B}(\eta)$$

$\vdash$ is the gratest fixed pt that satisfies $(D_1) \sim (D_7)$

(D_1) $((\Gamma, x:A), x, A) \in\ \vdash$

(D_2) $(\Gamma, t, A\to B), (\Gamma, u, A) \in\ \vdash \ \Rightarrow\ (\Gamma, tu, B) \in\ \vdash$

(D_3) $((\Gamma, x:A), t, B) \in\ \vdash \ \Rightarrow\ (\Gamma, \lambda x.t, A\to B) \in\ \vdash$

(D_4) $(\Gamma, 0, N) \in\ \vdash$

(D_5) $(\Gamma, t, N) \in\ \vdash \ \Rightarrow\ (\Gamma, St, N) \in\ \vdash$

(D_6) $(\Gamma, t, A), ((\Gamma, x:N), u, A) \in\ \vdash \ \Rightarrow\ (\Gamma, \mathrm{cond}_x(t,u), N\to A) \in\ \vdash$

(D_7) $(\Gamma, t, A\to B) \in\ \vdash \ \Rightarrow\ ((\Gamma, x:A), tx, B) \in\ \vdash$

$$\Gamma \vdash t:A \ \overset{def}{\Longleftrightarrow}\ (\Gamma, t, A) \in\ \vdash$$

$$\frac{\dfrac{\dfrac{}{x:N \vdash x:N} \qquad \dfrac{\dfrac{\dfrac{\vdash \mathrm{sum} : N\to N\to N}{x:N \vdash \mathrm{sum}(x) : N\to N}}{x:N,\ y:N \vdash \mathrm{sum}(x)(y) : N}}{x:N,\ y:N \vdash S(\mathrm{sum}(x)(y)) : N}}{x:N \vdash \mathrm{cond}_y(x, S(\mathrm{sum}(x)(y))) : N\to N}}{\dfrac{\vdash \lambda x.\mathrm{cond}_y(x, S(\mathrm{sum}(x)(y))) : N\to N\to N}{\vdash \mathrm{sum} : N\to N\to N}}$$

$$\mathrm{sum} = \lambda x.\mathrm{cond}_y(x, S(\mathrm{sum}(x)(y)))$$

global trace condition : Any infinite path contains a progressing trace

Reduction

(1-step full reduction)

$$(\lambda x. t)u \xrightarrow{1} t[u/x]$$

$$\mathrm{cond}_x(t, u)(0) \xrightarrow{1} t$$

$$\mathrm{cond}_x(t,u)(Sv) \xrightarrow{1} u[v/x]$$

& closed under (finite?) substitution

(1-step safe reduction) $t \xrightarrow{1} u$ with "safe" condition (not inside cond)

Possibly infintary many times reduction $\xrightarrow{\omega}$ (full reduction)

$\xrightarrow{\omega}$ (safe reduction)

Finitely many times redution $\xrightarrow{*}$, $\xrightarrow{*}$

Lem (Substitution) $\Gamma, x : A \vdash t : B$ & $\Gamma \vdash u : A \implies \Gamma \vdash t[u/x] : B$

Prop Subject reduction

Def closed t^{N} is total $\overset{def}{\iff}$ $t \overset{*}{\longrightarrow} n \in N$

closed t^{N} is a value $\overset{def}{\iff}$ $t \in N$

closed $t^{A \to B}$ is total $\overset{def}{\iff}$ $\forall a^{A}$: closed value . $(ta)^{B}$: total

closed $t^{A \to B}$ is a value $\overset{def}{\iff}$ t is total

$\underset{\vec{x} = FV(t)}{\underset{\vec{x} : \vec{A}}{t[\vec{x}]}}$ is total $\overset{def}{\iff}$ $\forall \vec{a} : \vec{A}$: closed value . $t[\vec{a}]$: total

Lem ① t^A : closed & $t \to u$: total $\Longrightarrow$ t : total

② $\vec{x} = FV(t)$ & $\vec{x} : \vec{B}$ & $\left.\begin{array}{l}\forall \vec{u} : \vec{B} \\ \forall \vec{a} : \vec{A}\end{array}\right\}$ closed value, $t^{\vec{A} \to N}[\vec{u}]\,\vec{a}$: total $\Longrightarrow$ t : total

∵) ① Ind on A

- $A = N$ $\quad \underbrace{t^N}_{\text{closed.}} \xrightarrow{SR} \underbrace{u^N}_{\text{closed total}} \xrightarrow{*} n \qquad \therefore\ t^N$: total

- $A = A_1 \to A_2$ Take $a : A_1$: closed total

$$\underbrace{(t\,a)^{A_2}}_{\text{closed}} \xrightarrow{SR} \underbrace{(u^{A_1 \to A_2}\,a)^{A_2}}_{\text{total by } u : \text{total}}$$

$\therefore\ t\,a$: total by I.H.

Then t : total.

② Assume that $\vec{x} = FV(t)$ & $\vec{x} : \vec{B}$ & $\left.\begin{array}{l}\forall \vec{u} : \vec{B} \\ \forall \vec{a} : \vec{A}\end{array}\right\}$ closed value, $t^{\vec{A} \to N}[\vec{u}]\,\vec{a}$: total

Ind on $|\vec{A}|$

- $|\vec{A}| = 0$: By def
- $\vec{A} = A_0 \vec{A}'$: Take $\vec{u}^{\vec{B}}$: closed values, $\vec{a}'^{\vec{A}'}$: closed values, $a_0^{A_0}$: closed value

By asmp. $(t[\vec{u}]\,a_0\,\vec{a}')^N$: total

By IH. $(t[\vec{u}]\,a_0)^{\vec{A}' \to N}$: total

$\therefore\ t[\vec{u}]^{A_0 \to \vec{A}' \to A}$: total by def

By def $t[\vec{x}]$: total

Thm t: not total $\Rightarrow$ $\Gamma \nvdash t : A$ in $CT\lambda$

$\because$) Assume that t: not total & $\vec{x}:\vec{D} \vdash t : \vec{A} \to N$ has a $CT\lambda$-proof Π

By Lem②, $\exists \vec{a}^{\vec{A}}, \vec{d}^{\vec{D}}$: closed values s.t. $t[\vec{d}]\vec{a}$: not total

We inductively construct $(e_i, \vec{d}_i, \vec{a}_i)$ for each $i \in \mathbb{N}$

s.t. $\left\{\begin{array}{l} e_i : \text{node of } \Pi \text{ where } \Pi e_i = (\vec{x}_i : \vec{D}_i \vdash t_i : \vec{A}_i \to N) \ \& \ e_{i+1} \text{ is a child of } e_i \\ t_i : \text{not total} \\ \vec{d}_i^{\vec{D}_i}, \vec{a}_i^{\vec{A}_i} : \text{closed values s.t. } (t_i[\vec{d}_i]\vec{a}_i)^N : \text{not total} \ \& \end{array}\right.$ $\begin{pmatrix}\vec{d}_i \\ \vec{a}_i\end{pmatrix} \longrightarrow \begin{pmatrix}\vec{d}_{i+1} \\ \vec{a}_{i+1}\end{pmatrix}$

connection from i-th to $(i+1)$-th

- $(e_0, \vec{d}_0, \vec{a}_0) \stackrel{\text{def}}{=} (\varepsilon, \vec{d}, \vec{a})$ where $\Pi\varepsilon = (\vec{x}:\vec{D} \vdash t : \vec{A} \to N)$

- Assume that we already have $(e_i, \vec{d}_i, \vec{a}_i)$

Case (Ax): $\Pi e_i = \Gamma, x:A \vdash \underbrace{x}_{t_i}:A$ not the case (x: total)

Case (Zero): $\Pi e_i = \Gamma \vdash \underbrace{0}_{t_i}:N$ not the case (0: total)

Case (Succ): $\dfrac{\Pi e_{i+1} \stackrel{\text{def}}{=} \Gamma \vdash t_{i+1} : N}{\Pi e_i = \Gamma \vdash \underbrace{S t_{i+1}}_{t_i} : N}$ By IH $(S t_{i+1})[\vec{d}_i]$: not total & $\vec{a}_i = \emptyset$

$\therefore$ $t_{i+1}[\vec{d}_i]$: not total

Take $\vec{d}_{i+1} \stackrel{\text{def}}{=} \vec{d}_i$ & $\vec{a}_{i+1} \stackrel{\text{def}}{=} \emptyset$ & $\begin{pmatrix}\vec{d}_i \\ \emptyset\end{pmatrix} \xrightarrow{id} \begin{pmatrix}\vec{d}_{i+1} \\ \emptyset\end{pmatrix}$

Case (Abs): $\dfrac{\Pi e_{i+1} \stackrel{\text{def}}{=} \Gamma, z:A \vdash t_{i+1} : \vec{A} \to N}{\Pi e_i = \Gamma \vdash \underbrace{\lambda z. t}_{t_i} : A \to \vec{A} \to N}$ By IH $(\lambda z. t)[\vec{d}_i]\underbrace{a\vec{a}}_{\vec{a}_i}$: not total

$(\lambda z. t)[\vec{d}_i] a \vec{a} = (\lambda z. t[\vec{d}_i, z]) a \vec{a} \longrightarrow t[\vec{d}_i, a]\vec{a}$

not total by & Lem①

Take $\vec{d}_{i+1} \stackrel{\text{def}}{=} \vec{d}_i, a$ & $\vec{a}_{i+1} \stackrel{\text{def}}{=} \vec{a}$ & $\begin{pmatrix}\vec{d}_i \\ a \\ \vec{a}_i\end{pmatrix} \xrightarrow{id} \begin{pmatrix}\vec{d}_i \\ a \\ \vec{a}_i\end{pmatrix}$

Case (App): $\dfrac{\Gamma \vdash t : C \to \vec{A} \to N \quad \Gamma \vdash u : C}{\Pi e_i = \Gamma \vdash \underbrace{tu}_{t_i} : \vec{A} \to N}$ By IH. $(tu)[\vec{d}_i]\vec{a}_i$: not total

Subcase: $u[\vec{d}_i]$: not total (let $C = \vec{C} \to N$. There exist $\vec{c}^{\vec{C}}$: closed total s.t. $u[\vec{d}_i]\vec{c}$: not total)

Take $\Pi e_{i+1} \stackrel{\text{def}}{=} (\Gamma \vdash u : C)$, $t_{i+1} \stackrel{\text{def}}{=} u$, $\vec{d}_{i+1} \stackrel{\text{def}}{=} \vec{d}_i$, $\vec{a}_{i+1} \stackrel{\text{def}}{=} \vec{c}$ & $\begin{pmatrix}\vec{d}_i \\ \vec{a}_i\end{pmatrix} \xrightarrow{id} \begin{pmatrix}\vec{d}_i \\ \vec{c}\end{pmatrix}$

Subcase: $u[\vec{d}_i]$: total.

Take $\Pi e_{i+1} \stackrel{\text{def}}{=} (\Gamma \vdash t : A \to \vec{A} \to N)$, $t_{i+1} \stackrel{\text{def}}{=} t$, $\vec{d}_{i+1} \stackrel{\text{def}}{=} \vec{d}_i$, $\vec{a}_{i+1} \stackrel{\text{def}}{=} u[\vec{a}_i], \vec{a}_i$ & $\begin{pmatrix}\vec{d}_i \\ \vec{a}_i\end{pmatrix} \xrightarrow{id} \begin{pmatrix}\vec{d}_i \\ u[\vec{d}_i] \\ \vec{a}_i\end{pmatrix}$

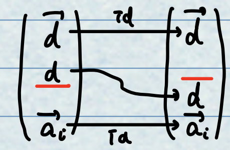
Case (η) : $\Pi e_{i+1} \overset{\text{def}}{=} \dfrac{\Gamma \vdash t : N \to A}{\Gamma, x:N \vdash \underbrace{tx}_{t_i} : A}$ (x:fresh) $\quad$ By IH. $\underbrace{(tx)[\vec{d}d]}_{\vec{d_i}}\vec{a_i}$: not total

$\Pi e_i =$

$\Vert$

$t[\vec{d}]d\vec{a_i}$

Take $t_{i+1} \overset{\text{def}}{=} t$ & $\vec{d_{i+1}} \overset{\text{def}}{=} \vec{d}$ & $\vec{a_{i+1}} \overset{\text{def}}{=} d\vec{a_i}$ &



Case (cond):

$\Pi e_i = \dfrac{\Gamma \vdash t : A \qquad \Gamma, z:N \vdash u : A}{\Gamma \vdash \mathrm{cond}_{z.}(t,u) : N \to A}$ $\quad$ By IH. $\mathrm{cond}_{z.}(t,u)[\vec{d_i}]n\vec{a}$: not total

$\Vert$

$\mathrm{cond}_{z.}(t[\vec{d_i}], u[\vec{d_i}, z])n\vec{a}$

Subcase ($n=0$) :

$\mathrm{cond}_{z.}(t[\vec{d_i}], u[\vec{d_i}, z])0\vec{a} \longrightarrow \underline{t[\vec{d_i}]\vec{a}}$

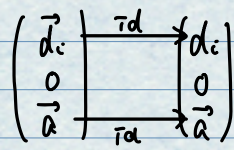
not total by & Lem ①

Take $\Pi e_{i+1} \overset{\text{def}}{=} (\Gamma \vdash t : A)$

$t_{i+1} \overset{\text{def}}{=} t$

$\vec{d_{i+1}} \overset{\text{def}}{=} \vec{d_i}$ &

$\vec{a_{i+1}} \overset{\text{def}}{=} \vec{a}$



Subcase ($n = Sn'$) :

$\mathrm{cond}_{z.}(t[\vec{d_i}], u[\vec{d_i}, z])(Sn')\vec{a} \longrightarrow \underline{u[\vec{d_i}, n']\vec{a}}$

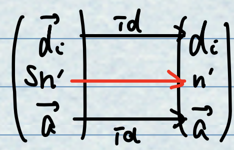
not total by & Lem ①

Take $\Pi e_{i+1} \overset{\text{def}}{=} (\Gamma, z:N \vdash u : A)$

$t_{i+1} \overset{\text{def}}{=} u$

$\vec{d_{i+1}} \overset{\text{def}}{=} \vec{d_i}, n'$ &

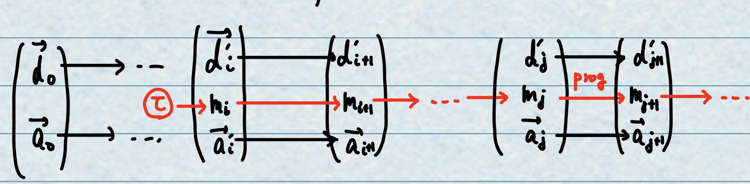
$\vec{a_{i+1}} \overset{\text{def}}{=} \vec{a}$



☆ $(e_i)_{i\in\omega}$ is an inf path in Π

By regularity, $(e_i)_{i\in\omega}$ contains progressing trace $(\tau_i)_{i\in\omega}$

We have



$m_{j+1} = m_j$ if j : not prog. pt.

$m_{j+1} = m_j - 1$ if j : not prog. pt. $\quad\therefore$ $(m_j)_{j\in\omega}$ is infinitely decreasing seq. of ≥ 0 contradiction! //

Cor $\quad \Gamma \vdash t : A$ in CT-λ $\Longrightarrow$ t : total

Cor $\quad \vdash t : N \Longrightarrow \exists n \in \mathbb{N}\,(t \overset{*}{\longrightarrow} n)$

Unique N.F

Def t, u: closed terms of type A. We define $t \sim_A u$

$$t \sim_N u \overset{def}{\iff} \forall t', u' \in NF\,(t \xrightarrow{*} t' \ \&\ u \xrightarrow{*} u' \Rightarrow t' = u')$$

$$t \sim_{A\to B} u \overset{def}{\iff} \forall a_1^A, a_2^A : \text{closed}\,(a_1 \sim_A a_2 \Rightarrow t a_1 \sim_B u a_2)$$

$$t[\vec{x}^{\vec{A}}] \sim_A u[\vec{x}^{\vec{A}}] \overset{def}{\iff} \forall \vec{a}_1, \vec{a}_2 : \text{closed}\,(\vec{a}_1 \sim_{\vec{A}} \vec{a}_2 \Rightarrow t[\vec{a}_1] \sim_A u[\vec{a}_2])$$

Lem $t \sim_N t \Rightarrow t$ has a unique n.f.

$\because)$ trivial //

Lem $t \sim_N u \Rightarrow \exists n \in N\,\left(\begin{array}{l} t \xrightarrow{*} n \ \&\ u \xrightarrow{*} n \ \& \\ \&\ \forall t' \in NF\,(t \xrightarrow{*} t' \Rightarrow t' = n) \\ \&\ \forall u' \in NF\,(u \xrightarrow{*} u' \Rightarrow u' = n) \end{array}\right)$

$\because)$ By the previous Cor, $t \xrightarrow{*} m$ & $u \xrightarrow{*} n$. By $t \sim_N u$, $m = n$

If $t \xrightarrow{*} t' \in NF$. By $t \sim_N u$, $t' = n$

If $u \xrightarrow{*} u' \in NF$. By $t \sim_N u$, $u' = n$ //

Lem $\forall \vec{u}^{\vec{B}}, \vec{u}'^{\vec{B}}, \vec{a}^{\vec{A}}, \vec{a}'^{\vec{A}}$: closed

$(\vec{u} \sim \vec{u}' \ \&\ \vec{a} \sim \vec{a}' \Rightarrow t[\vec{u}]\vec{a} \sim_N t'[\vec{u}']\vec{a}') \Rightarrow t[\vec{x}] \sim_{\vec{A}\to N} t'[\vec{x}]$

$\because)$ Assume $\forall \vec{u}^{\vec{B}}, \vec{u}'^{\vec{B}}, \vec{a}^{\vec{A}}, \vec{a}'^{\vec{A}}$: closed $(\vec{u} \sim \vec{u}' \ \&\ \vec{a} \sim \vec{a}' \Rightarrow t[\vec{u}]\vec{a} \sim_N t'[\vec{u}']\vec{a}')$.

Take $\vec{u}, \vec{u}', \vec{a}, \vec{a}'$ s.t. $\vec{u} \sim \vec{u}'$ & $\vec{a} \sim \vec{a}'$. Then $t[\vec{u}]\vec{a} \sim_N t'[\vec{u}']\vec{a}'$.

$\therefore\ t[\vec{u}] \sim_{\vec{A}\to N} t'[\vec{u}']$

$\therefore\ t[\vec{x}] \sim_{\vec{A}\to N} t'[\vec{x}]$ //

Thm $t \not\sim t \implies \Gamma \nvdash t : A$ in $CT\lambda$

$\because$) Assume that $t \not\sim t$ & $\vec{x}:\vec{D} \vdash t : \vec{A} \to N$ has a $CT\lambda$-proof Π (Show contradiction)

Then, by Lem, $t[\vec{d}\,]\vec{a} \not\sim_N t'[\vec{d'}]\vec{a'}$ for some $\vec{a} \sim_{\vec{A}} \vec{a'}$ & $\vec{d} \sim_{\vec{D}} \vec{d'}$

We inductively construct $(e_i, \vec{d_i}, \vec{a_i}, \vec{d'_i}, \vec{a'_i})$ for each $i \in \omega$

where $\Pi e_i = \overrightarrow{x_i : D_i} \vdash t_i : \vec{A_i} \to N$

& $\vec{d_i} \sim_{\vec{D_i}} \vec{d'_i}$ & $\vec{a_i} \sim_{\vec{A_i}} \vec{a'_i}$

& $t_i[\vec{d_i}]\vec{a_i} \not\sim t_i[\vec{d'_i}]\vec{a'_i}$

& $\begin{pmatrix}\vec{d_i}\\ \vec{a_i}\end{pmatrix} \Longrightarrow \begin{pmatrix}\vec{d_{i+1}}\\ \vec{a_{i+1}}\end{pmatrix}$, $\begin{pmatrix}\vec{d'_i}\\ \vec{a'_i}\end{pmatrix} \Longrightarrow \begin{pmatrix}\vec{d'_{i+1}}\\ \vec{a'_{i+1}}\end{pmatrix}$ are trace compatible

- $(e_0, \vec{d_0}, \vec{a_0}, \vec{d'_0}, \vec{a'_0}) \stackrel{\text{def}}{=} (\varepsilon, \vec{d}, \vec{a}, \vec{d'}, \vec{a'})$ where $\Pi\varepsilon = (\vec{x}:\vec{D} \vdash t : \vec{A} \to N)$

- Assume that we already have $(e_i, \vec{d_i}, \vec{a_i}, \vec{d'_i}, \vec{a'_i})$

Case (Ax): $\Pi e_i = \Gamma, x : A \vdash \underbrace{x}_{t_i} : A$ not the case (by $x[\vec{d_i}] \sim x[\vec{d'_i}]$)

Case (Zero): $\Pi e_i = \Gamma \vdash 0 : N$ not the case (by $0[\vec{d_i}] = 0 \sim 0 = 0[\vec{d'_i}]$)

Case (Succ): $\Pi e_{i+1} \stackrel{\text{def}}{=} \Gamma \vdash t_{i+1} : N$

$$\Pi e_i = \frac{\Gamma \vdash t_{i+1} : N}{\Gamma \vdash \underbrace{S t_{i+1}}_{t_i} : N}$$

By IH $(S t_{i+1})[\vec{d_i}] \not\sim_N (S t_{i+1})[\vec{d'_i}]$ & $\vec{a_i} = \vec{a'_i} = \phi$

$\therefore \exists u, u' \in NF\ (u \neq u'$ & $S(t_{i+1}[\vec{d_i}]) \xrightarrow{*} u$ & $S(t_{i+1}[\vec{d'_i}]) \xrightarrow{*} u')$

$\therefore$ $u = Sv$ & $u' = Sv'$ & $v \neq v'$ & $t_{i+1}[\vec{d_i}] \xrightarrow{*} v$ & $t_{i+1}[\vec{d'_i}] \xrightarrow{*} v'$ for some $\overset{N}{v}, \overset{N}{v'}$

$\therefore$ $t_{i+1}[\vec{d_i}] \not\sim_N t_{i+1}[\vec{d'_i}]$

Take $\vec{d_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}$ & $\overrightarrow{d'_{i+1}} \stackrel{\text{def}}{=} \vec{d'_i}$ & $\vec{a_{i+1}} = \vec{a'_{i+1}} \stackrel{\text{def}}{=} \phi$ & $\begin{pmatrix}\vec{d_i}\\ \phi\end{pmatrix} \xrightarrow{id} \begin{pmatrix}\vec{d_{i+1}}\\ \phi\end{pmatrix}$ & $\begin{pmatrix}\vec{d'_i}\\ \phi\end{pmatrix} \xrightarrow{id} \begin{pmatrix}\vec{d'_{i+1}}\\ \phi\end{pmatrix}$

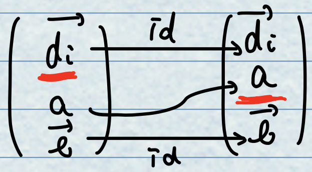
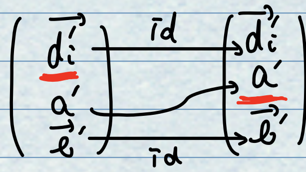
Case (Abs): $$\begin{array}{rl} \Pi e_{i+1} = & \dfrac{\Gamma, x : A \vdash t_{i+1} : \vec{A} \to N}{\Gamma \vdash \underbrace{\lambda x. t_{i+1}}_{t_i} : A \to \vec{A} \to N} \\ \Pi e_i = & \end{array}$$

By IH, $(\lambda x. t_i[\vec{d_i}, x]) \underbrace{a \vec{b}}_{\vec{a_i}} \not\sim_N (\lambda x. t_i[\vec{d'_i}, x]) \underbrace{a' \vec{b'}}_{\vec{a'_i}}$

$\therefore$ $t_i[\vec{d_i}, a] \vec{b} \not\sim_N t_i[\vec{d'_i}, a'] \vec{b'}$ $\left(\text{since } \begin{array}{ccc} (\lambda x \ldots) a \vec{b} & & (\lambda x \ldots) a' \vec{b'} \\ \downarrow & & \downarrow \\ t_i[\ ] \vec{b} & & t_i[\ ] \vec{b'} \\ \downarrow^* & & \downarrow \\ \cdot & \neq & \cdot \end{array} \right)$

Take $\vec{d_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}, a$ & $\vec{d'_{i+1}} \stackrel{\text{def}}{=} \vec{d'_i}, a'$

$\vec{a_{i+1}} \stackrel{\text{def}}{=} \vec{b}$ & $\vec{a'_{i+1}} \stackrel{\text{def}}{=} \vec{b'}$

Then  & 

Case (App): $$\dfrac{\Gamma \vdash t : A \to \vec{A} \to N \qquad \Gamma \vdash u : A}{\Gamma \vdash \underbrace{tu}_{t_i} : \vec{A} \to N}$$

$\Pi e_i =$ (the derivation above)

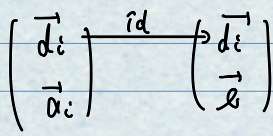
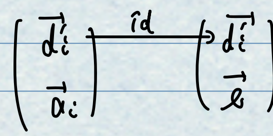
By IH, $(t[\vec{d_i}])(u[\vec{d_i}]) \vec{a_i} \not\sim_N (t[\vec{d'_i}])(u[\vec{d'_i}]) \vec{a'_i}$

Subcase 1: $u[\vec{d_i}] \not\sim_{\vec{B} \to N} u[\vec{d'_i}]$.

$u[\vec{d_i}] \vec{b} \not\sim_N u[\vec{d'_i}] \vec{b'}$ & $\vec{b} \sim_{\vec{B}} \vec{b'}$ for some $\vec{b}, \vec{b'}$

Take $\Pi e_{i+1} \stackrel{\text{def}}{=} \Gamma \vdash u : A$

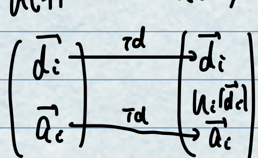
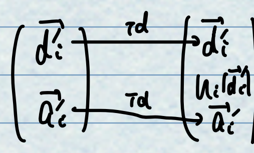
$\vec{d_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}$ & $\vec{d'_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}$ & $\vec{a_{i+1}} \stackrel{\text{def}}{=} \vec{b}$ & $\vec{a'_{i+1}} \stackrel{\text{def}}{=} \vec{b'}$

Then  & 

Subcase 2: $u[\vec{d_i}] \sim_A u[\vec{d'_i}]$.

Take $\Pi e_{i+1} \stackrel{\text{def}}{=} \Gamma \vdash t : A \to \vec{A} \to N$

$\vec{d_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}$, $\vec{d'_{i+1}} \stackrel{\text{def}}{=} \vec{d'_i}$ & $\vec{a_{i+1}} \stackrel{\text{def}}{=} u[\vec{d_i}], \vec{a_i}$ & $\vec{a'_{i+1}} \stackrel{\text{def}}{=} u[\vec{d'_i}], \vec{a'_i}$

Then  & 

Case (η) :

$$\begin{array}{ll} \Pi e_{i+1} \overset{\text{def}}{=} & \Gamma \vdash t_{i+1} : A \to B \\ \hline \Pi e_i \overset{\text{def}}{=} & \Gamma, x : A \vdash t_{i+1}\, x : B \end{array}$$

By IH, $\vec{d_i} = \vec{c}, c$ & $\vec{d_i'} = \vec{c'}, c'$ & $t_{i+1}[\vec{c}]\, c\, \vec{a_i} \not\sim_N t_{i+1}[\vec{c'}]\, c'\, \vec{a_i'}$

Take $\vec{d_{i+1}} \overset{\text{def}}{=} \vec{c}$ & $\vec{d_{i+1}'} \overset{\text{def}}{=} \vec{c'}$ & $\vec{a_{i+1}} \overset{\text{def}}{=} c, \vec{a_i}$ & $\vec{a_{i+1}'} = c', \vec{a_i'}$

Then

$$\begin{tikzcd} \vec{c} \arrow[r, "\mathrm{id}"] & \vec{c} \\ c \arrow[r] & c \\ \vec{a_i} \arrow[r, "\mathrm{id}"'] & \vec{a_i} \end{tikzcd} \quad \& \quad \begin{tikzcd} \vec{c'} \arrow[r, "\mathrm{id}"] & \vec{c'} \\ c' \arrow[r] & c' \\ \vec{a_i'} \arrow[r, "\mathrm{id}"'] & \vec{a_i'} \end{tikzcd}$$

Case (cond) :

$$\Pi e_i = \frac{\Gamma \vdash t : A \qquad \Gamma, z : N \vdash u : A}{\Gamma \vdash \mathrm{cond}_z(t, u) : N \to A}$$

By IH. $\mathrm{cond}_z(t[\vec{d_i}], u[\vec{d_i}, x])\, \underbrace{b\, \vec{b}}_{\vec{a_i}} \not\sim_N \mathrm{cond}_z(t[\vec{d_i'}], u[\vec{d_i'}, x])\, \underbrace{b'\, \vec{b'}}_{\vec{a_i'}}$

By $b \sim_N b'$, $\exists n \in N\ (b \xrightarrow{*} n \xleftarrow{*} b')$

Case $n = 0$:

$$\begin{tikzcd} \mathrm{cond}_z(t[\vec{d_i}], u[\vec{d_i}, x])\, b\, \vec{b} \arrow[d, "*"'] \arrow[r, phantom, "\not\sim_N"] & \mathrm{cond}_z(t[\vec{d_i'}], u[\vec{d_i'}, x])\, b'\, \vec{b'} \arrow[d, "+"] \\ \mathrm{cond}_z(t[\vec{d_i}], u[\vec{d_i}, x])\, 0\, \vec{b} \arrow[d] & \mathrm{cond}_z(t[\vec{d_i'}], u[\vec{d_i'}, x])\, 0\, \vec{b'} \arrow[d] \\ t[\vec{d_i}]\, \vec{b} \arrow[r, phantom, "\not\sim"] & t[\vec{d_i'}]\, \vec{b'} \end{tikzcd}$$

Take $\Pi e_{i+1} \overset{\text{def}}{=} \Gamma \vdash t : A$

$\vec{d_{i+1}} \overset{\text{def}}{=} \vec{d_i}$ & $\vec{d_{i+1}'} \overset{\text{def}}{=} \vec{d_i'}$ & $\vec{a_{i+1}} \overset{\text{def}}{=} \vec{b}$ & $\vec{a_{i+1}'} = \vec{b'}$

$$\begin{tikzcd} \vec{d_i} \arrow[r, "\mathrm{id}"] & \vec{d_i} \\ b & \\ \vec{b} \arrow[r, "\mathrm{id}"'] & \vec{b} \end{tikzcd} \quad \& \quad \begin{tikzcd} \vec{d_i'} \arrow[r, "\mathrm{id}"] & \vec{d_i'} \\ b' & \\ \vec{b'} \arrow[r, "\mathrm{id}"'] & \vec{b'} \end{tikzcd}$$

Case $n = Sn'$:

$$\begin{tikzcd} \mathrm{cond}_z(t[\vec{d_i}], u[\vec{d_i}, x])\, b\, \vec{b} \arrow[d, "*"'] \arrow[r, phantom, "\not\sim_N"] & \mathrm{cond}_z(t[\vec{d_i'}], u[\vec{d_i'}, x])\, b'\, \vec{b'} \arrow[d, "+"] \\ \mathrm{cond}_z(t[\vec{d_i}], u[\vec{d_i}, x])\, (Sn')\, \vec{b} \arrow[d] & \mathrm{cond}_z(t[\vec{d_i'}], u[\vec{d_i'}, x])\, (Sn')\, \vec{b'} \arrow[d] \\ u[\vec{d_i}, n']\, \vec{b} \arrow[r, phantom, "\not\sim"] & u[\vec{d_i'}, n']\, \vec{b'} \end{tikzcd}$$

Take $\quad \Pi_{e_{i+1}} \stackrel{\text{def}}{=} \Gamma, z : N \vdash u : A$

$\overrightarrow{d_{i+1}} \stackrel{\text{def}}{=} \vec{d_i}, n'$ & $\overrightarrow{d'_{i+1}} \stackrel{\text{def}}{=} \vec{d'_i}, n'$ & $\overrightarrow{a_{i+1}} \stackrel{\text{def}}{=} \vec{b}$ & $\overrightarrow{a'_{i+1}} \stackrel{\text{def}}{=} \vec{b'}$

Then
$$\begin{pmatrix} \vec{d_i} \\ b \\ \vec{b} \end{pmatrix} \begin{matrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{matrix} \begin{pmatrix} \vec{d_i} \\ n' \\ \vec{b} \end{pmatrix} \quad \& \quad \begin{pmatrix} \vec{d'_i} \\ b' \\ \vec{b'} \end{pmatrix} \begin{matrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{matrix} \begin{pmatrix} \vec{d'_i} \\ n' \\ \vec{b'} \end{pmatrix}$$

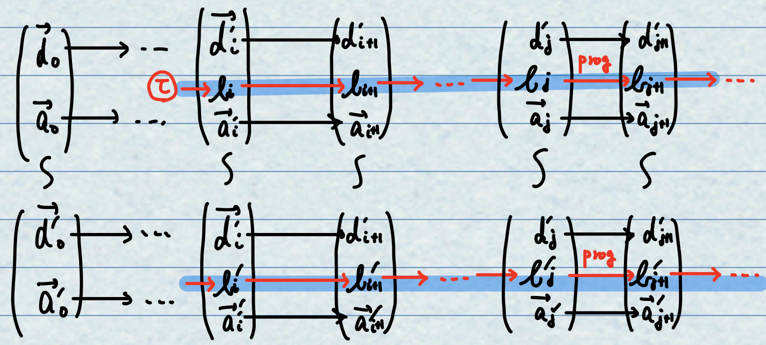
⊛ For lines of type N, $b \sim_N b'$ has a unique n.f. $\in \mathbb{N}$

$$\begin{tikzcd} b \arrow[r, squiggly] \arrow[d, phantom, "\wr_N"] & n' \arrow[d, phantom, "\|"] \\ b' \arrow[r, squiggly, "\text{prog pt}"'] \arrow[d, "*"] & n' \\ Sn' & \end{tikzcd}$$

⊛ $(e_i)_{i\in\omega}$ is an inf path in Π

By regularity, $(e_i)_{i\in\omega}$ contains progressing trace $(\tau_i)_{i\in\omega}$ of type N

We have



Define $\quad m_j \stackrel{\text{def}}{=}$ n.f. of $b_j : N$

$m_{j+1} = m_j \quad$ if j : not prog. pt.

$m_{j+1} = m_j - 1 \quad$ if j : not prog. pt.

$\therefore \ (m_j)_{j\in\omega}$ is infinitely decreasing seq. of ≥ 0 contradiction! //