Benders decomposition for choice-based optimization problems with discrete upper-level variables

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- Many classes of discrete choice models cannot be easily integrated in mixed integer optimization models $\rightarrow$ **choice-based optimization**
- The majority of the works in the literature sacrifice complexity either at **demand level** or at supply level for the sake of tractability.
- Alternative approach: trying to circumvent issues related to non-linearity and non-convexity of the demand function using **simulation**.

Simulation-based linearization of choice probabilities[1]

- Let I be the universal choice set and N be the set of heterogeneous customers.
- Random utility models:

$$U_{in} = V_{in} + \epsilon_{in} \qquad \forall i \in I, \forall n \in N.$$

- Choice probabilities:

$$P_{in} = \Pr[V_{in} + \epsilon_{in} = \max_{j \in I}(V_{jn} + \varepsilon_{jn})].$$

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- Linearization:

$$U_{inr} = V_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R,$$

$$x_{inr} = \begin{cases} 1 \text{ if } U_{inr} = \max_{j \in I} U_{jnr}, \\ 0 \text{ otherwise} \end{cases} \qquad \forall i \in I, \forall n \in N, \forall r \in R,$$

$$P_{in} = \frac{1}{|R|} \sum_{r \in R} x_{inr} \qquad \forall i \in I, \forall n \in N.$$

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Previous research

Applications:

- Optimizing prices for uncapacitated and capacitated services.[2]
- Computing approximate equilibrium solutions for competitive markets.[3]
- Determining optimal price-based regulation of transport markets.[4]

Open questions:

- Scalability.
- Extension to variables other than prices.

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[2] Pacheco Paneque et al., "Integrating advanced discrete choice models in mixed integer linear optimization" (2021).

[3] Bortolomiol, Lurkin, and Bierlaire, "A simulation-based heuristic to find approximate equilibria with disaggregate demand models" (2021).

[4] Bortolomiol, Lurkin, and Bierlaire, "Price-based regulation of oligopolistic markets under discrete choice models of demand" (2021).

Continuous Pricing Problem (CPP)

- A supplier wants to maximize profits obtained by controlling alternatives $I_k \subset I$.
- The utilities of the customers are price-dependent variables:

$$U_{inr} = \beta_{p,inr} p_i + \hat{q}_{inr} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R.$$

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$$\max_{p} \quad \pi = \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} \frac{1}{|R|} \theta_n p_i x_{inr}, \tag{1}$$

$$s.t. \quad \sum_{i \in I} x_{inr} = 1 \qquad \forall n \in N, \forall r \in R, \tag{2}$$

$$\sum_{j \in I} U_{jnr} x_{jnr} \geq U_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{3}$$

$$0 \leq p_i \leq M_i^p \qquad \forall i \in I, \tag{4}$$

$$x_{inr} \in \{0, 1\} \qquad \forall i \in I, \forall n \in N, \forall r \in R. \tag{5}$$

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- The linearization of the product $p_i \cdot x_{inr}$ (continuous and binary) can be done using big-M constraints.

Discrete Pricing Problem (DPP)

- For each alternative $i \in I_k$ we constrain prices p_i to the set $Q_i = \{p_i^1, p_i^2, ..., p_i^{|Q|}\}$.
- Utilities become parameters of the optimization model: $\hat{U}_{inr} = \beta_{p,inr}\hat{p}_i + \hat{q}_{inr} + \xi_{inr}$.

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$$\max_{y} \quad \pi = \sum_{i \in I_k^{exp}} \sum_{n \in N} \sum_{r \in R} \frac{1}{|R|} \theta_n \hat{p}_i x_{inr}, \tag{6}$$

$$s.t. \quad \sum_{j \in I_i^{exp}} y_j = 1 \qquad \forall i \in I, \tag{7}$$

$$\sum_{i \in I^{exp}} x_{inr} = 1 \qquad \forall n \in N, \forall r \in R, \tag{8}$$

$$x_{inr} \leq y_i \qquad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \tag{9}$$

$$\sum_{j \in I^{exp}} \hat{U}_{jnr} x_{jnr} \geq \hat{U}_{inr} y_i \qquad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \tag{10}$$

$$x_{inr} \in \{0,1\} \qquad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \tag{11}$$

$$y_i \in \{0,1\} \qquad \forall i \in I^{exp}. \tag{12}$$

Numerical experiments

$\|R\|$	CPP		DPP			Gap
	Time	Opt	$\|I_i^{exp}\|$	Time	Opt	
20	0.45	71774.95	21	1.42	70390.50	1.93%
			51	7.18	71316.20	0.64%
			101	8.89	71379.90	0.55%
50	10.46	72423.71	21	14.59	71889.00	0.74%
			51	31.51	72106.36	0.44%
			101	89.91	72185.30	0.33%
100	101.64	66452.18	21	34.48	66118.40	0.50%
			51	161.03	66255.90	0.30%
			101	395.86	66341.32	0.17%
200	288.89	70788.17	21	139.17	69859.60	1.31%
			51	415.90	70489.95	0.42%
			101	1829.24	70571.67	0.31%

Table: High-speed rail pricing: solving CPP and DPP to optimality with CPLEX.

Assortment and Continuous Pricing Problem (ACPP)

- We include the decision about whether or not to offer any given product $i \in I_k$ to the customers.
- The actual utility for the customer is $U^a_{inr} = U_{inr} \cdot y_i$.

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- The actual utility for the customer is $U^a_{inr} = U_{inr} \cdot y_i$.
- Customers must choose the alternative with the highest utility among those that are made available by the supplier:

$$
\begin{aligned}
& U_{inr} = \beta_{p,inr} p_i + q_{inr} + \xi_{inr} && \forall i \in I, \forall n \in N, \forall r \in R, && (13)\\
& U^a_{inr} \leq U_{inr} && \forall i \in I, \forall n \in N, \forall r \in R, && (14)\\
& U_{inr} \leq U^a_{inr} + M^U_{inr}(1 - y_i) && \forall i \in I, \forall n \in N, \forall r \in R, && (15)\\
& U^a_{inr} \leq M^U_{inr} y_i && \forall i \in I, \forall n \in N, \forall r \in R, && (16)
\end{aligned}
$$

Assortment and Discrete Pricing Problem (ADPP)

- The formulation of the DPP **still applies**, with a small change:

$$\max_{y} \quad \pi = \sum_{i\in I_k^{exp}} \sum_{n\in N} \sum_{r\in R} \frac{1}{|R|} \theta_n \hat{p}_i x_{inr}, \tag{17}$$

$$s.t. \quad \cancel{\sum_{j\in I_i^{exp}} y_j = 1} \qquad \cancel{\forall i \in I}, \tag{18}$$

$$\sum_{i\in I^{exp}} x_{inr} = 1 \qquad \forall n \in N, \forall r \in R, \tag{19}$$

$$x_{inr} \leq y_i \qquad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \tag{20}$$

$$\sum_{j\in I^{exp}} \hat{U}_{jnr} x_{jnr} \geq \hat{U}_{inr} y_i \qquad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \tag{21}$$

$$x_{inr} \in \{0, 1\} \qquad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \tag{22}$$

$$y_i \in \{0, 1\} \qquad \forall i \in I^{exp}. \tag{23}$$

Numerical experiments

$\lvert R \rvert$	ACPP		ADPP			Gap
	Time	Opt	$\lvert I_i^{exp} \rvert$	Time	Opt	
10	11706	907.8	16	132	864.0	4.82%
			31	800	876.0	3.50%
20	129600*	877.0*	16	429	842.0	3.99%
			31	2778	862.5	1.65%
50	129600*	842.8*	16	837	816.4	3.13%
			31	12191	830.4	1.47%
100	129600*	844.0*	16	3419	828.2	1.87%
			31	39425	831.8	1.45%

Table: Parking assortment and pricing: solving ACPP and ADPP to optimality with CPLEX.

What about Benders and discrete supply variables?

- Let's fix the discrete supply variables of the supplier to $\mathbf{y}^*$.
- The lower-level utility maximization problem for a single customer n and scenario r is as follows:

$$\max_{x} \quad U = \sum_{i \in I} \hat{U}_i x_i, \tag{24}$$

$$s.t. \quad \sum_{i \in I} x_i = 1, \tag{25}$$

$$x_i \leq y_i^* \qquad \forall i \in I, \tag{26}$$

$$x_i \geq 0 \qquad \forall i \in I. \tag{27}$$

- This is a continuous knapsack problem, where the knapsack's capacity is equal to 1 and each item (alternative) i has a weight of 1 and a value of $\hat{U}_{inr}$.

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3. Solve current RMP. Save the solution y^*, z^*. Let $f(y^*, z^*)$ be the optimal objective value. Update $LB = f(y^*, z^*)$.

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4. Given y^*, compute $f(y^*)^{ADPP}$ for the original problem by deriving the choices for all customers and scenarios. Update $UB = \min\{UB, f(y^*)^{ADPP}\}$.

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4. Given y^*, compute $f(y^*)^{ADPP}$ for the original problem by deriving the choices for all customers and scenarios. Update $UB = \min\{UB, f(y^*)^{ADPP}\}$.
5. If $UB - LB \leq \epsilon$, then stop.
   Else, solve the dual worker problem for $y = y^*$. Using the optimal dual variables, add to the master problem an optimality cut of the following form:

$$z \geq \sum_{n \in N} \sum_{r \in R} (\sum_{i \in I} m_i y_i + q) \tag{31}$$

   and go to step 3.

Branch-and-Benders-cut

- Solving the master problem at each iteration is inefficient.
- Benders cuts can be inserted while processing the branch-and-bound tree of the master problem.[5]

[5] Fischetti, Ljubić, and Sinnl, "Redesigning Benders decomposition for large-scale facility location" (2017).

Preliminary results

$\lvert R \rvert$	$\lvert I_i^{exp} \rvert$	Opt	CPLEX	BBC
5	3	2399.20	13.20	42.81
5	6	2526.20	165.68	230.17
5	12	2641.60	3685.49	1793.00
10	3	2330.20	87.79	106.40
10	6	2727.20	703.47	587.74
10	12	2795.10	10931.09	7627.22
20	3	2333.90	363.22	256.94
20	6	2585.15	1066.06	1669.50
20	12	2638.08	54336.94	27043.61

Table: Parking assortment and pricing, $\lvert N \rvert = 80$, $\lvert I_k \rvert = 12$: solving ADPP to optimality with CPLEX and BBC algorithm (single-thread).

Enhancements and future work

- Classical Benders cuts provide slow convergence $\rightarrow$ efficient cuts are key to the success of this approach:
  - Pareto-optimal cuts;[6]
  - minimal infeasible subset cuts;
  - partial Benders decomposition.[7]

[6] Magnanti and Wong, "Accelerating Benders decomposition: Algorithmic enhancement and model selection criteria" (1981).

[7] Crainic et al., "Partial benders decomposition: general methodology and application to stochastic network design" (2021).

Summary

- Supply problems with **advanced discrete choice models** of demand can be written as stochastic optimization problems by relying on **simulation**.
- Choice-based optimization problems with discrete upper-level variables exhibit a **block-diagonal structure** which make them particularly suitable to the use of decomposition techniques such as Benders.
- We are working on **efficient enhancements** for our Benders approach to generate tighter cuts and reduce computational times.
- The **trade-off** between the increased realism of the demand model and the computational complexity of the resulting optimization/equilibrium problem must be evaluated on a case-by-case basis.