

Introduction to Bayesian linear regression with brms

Stefano Coretta

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Installation

Safe method:

- **Install Rstan first:** <https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started> (see *Installation of Rstan*, *Checking the C++ Toolchain*, and *Configuration of the C++ Toolchain*).
 - Note that details in *Checking the C++ Toolchain* differ depending on your OS.
- **Install brms:** <https://github.com/paul-buerkner/brms#how-do-i-install-brms>.

Road map

- Basic concepts
- Model fitting with brms
- Bayesian inference
- BYOD clinic

Learning outcomes

- Understand the basic concepts behind Bayesian statistics and how this differs from frequentist statistics.
- Be able to set up and fit a Bayesian linear regression model using brms.
- Understand Bayesian inference methods.

BASIC CONCEPTS

Random variables

- We have a question about the world, so we collect data (sample from a population).
 - $y = (y_1, y_2, y_3, y_4, \dots, y_n)$
- We want to know how the data (the sample y) was generated.
- In probability theory, data is generated by a random variable Y .

Random variables

- Y is a variable whose value is generated by a random event.
- Y is uncertain.
 - We can describe Y as a probability distribution.

Random variables

- Probability distribution.
 - A list of the values a random variable could take on along with their corresponding probability.
- Probability distributions can be expressed by a set of parameters $\Theta = (\theta_1, \dots, \theta_n)$.
- Some probability distributions:
 - *Normal*(μ, σ),
 - *Binomial*(n, p),
 - *Poisson*(λ)

Frequentist vs Bayesian view

- Parameters: $\mu, \sigma, p, \lambda, \dots$
- Frequentist view:
 - The parameters are **fixed** (they are unknown but certain).
 - They take on a specific value.
- Bayesian view:
 - The parameters are **random variables** (they are unknown and uncertain).
 - We describe each parameter as a probability distribution, expressed by a set of **hyperparameters**.

$$vot_i \sim Normal(\mu, \sigma)$$

Continuous random variable

- We want to estimate the parameters μ and σ from the data.
 - We describe the parameters using probability distributions (defined by hyperparameters).

Continuous random variable

$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(\mu_1, \sigma_1)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Priors

- We can incorporate previous knowledge (belief) about the parameters using **priors** (*prior probability distributions*).
 - We specify the hyperparameters μ_1 , σ_1 , x_0 , γ of the prior probability distributions.
- Priors are chosen based on expert knowledge, previous studies, pilot data...
 - Priors must **not** be chosen based on the data to be analysed.

Bayesian belief update

prior probability \times *evidence* (data)

Bayesian belief update

<https://nanx.shinyapps.io/conjugate-normal-umkv/>

MODEL FITTING

- Voice Onset Time (VOT): Time difference (ms) between the release of a stop and the onset of voicing (vocal fold vibration).
- Toy study of Italian VOT in stops.

$$vot_i \sim Normal(\mu, \sigma)$$

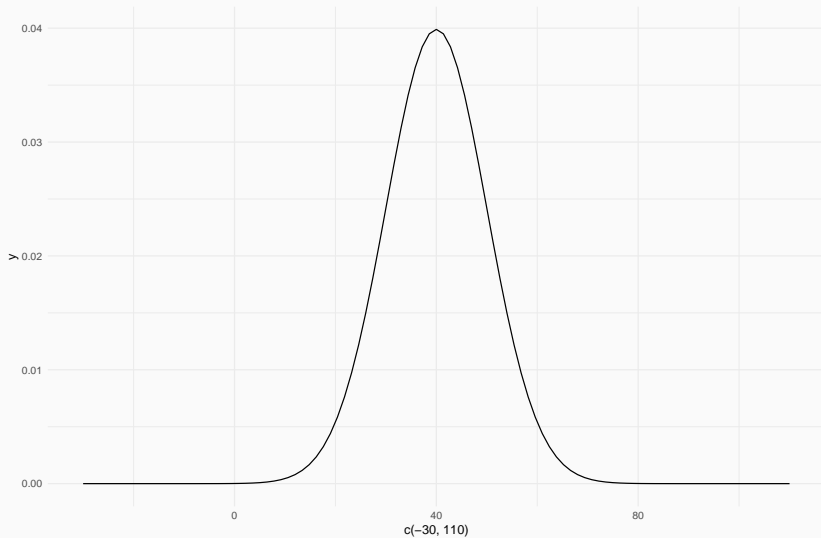
- Previous literature on VOT in Italian (Esposito, 2002; Stevens & Hajek, 2010) report VOT values for voiceless stops in the range of 20–60 ms.
- We can include our knowledge with a prior for μ .
 - *Normal*(40, 10).
 - This is a somewhat strongly informative prior.

```
ggplot() +  
  aes(x = c(-30, 110)) +  
  stat_function(fun = dnorm, n = 101, args = list(40,  
  labs(title = "Normal (Gaussian) distribution", subti-
```

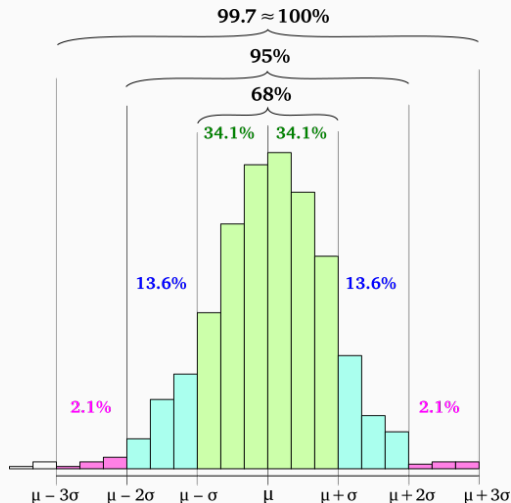
Italian VOT

Normal (Gaussian) distribution

mean = 40, SD = 10



Normal prior



Melikamp,

https://commons.wikimedia.org/wiki/File:Empirical_rule_histogram.svg

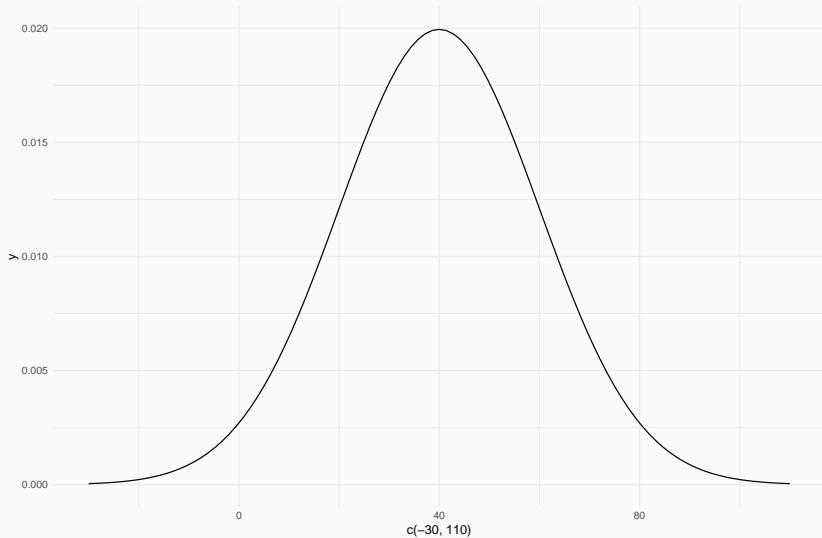
(CC BY-SA 4.0)

```
ggplot() +  
  aes(x = c(-30, 110)) +  
  stat_function(fun = dnorm, n = 101, args = list(40,  
  labs(title = "Normal (Gaussian) distribution", subti-
```

Italian VOT

Normal (Gaussian) distribution

mean = 40, SD = 20



$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(40, 10)$$

What about σ ?

$$vot_i \sim Normal(\mu, \sigma)$$

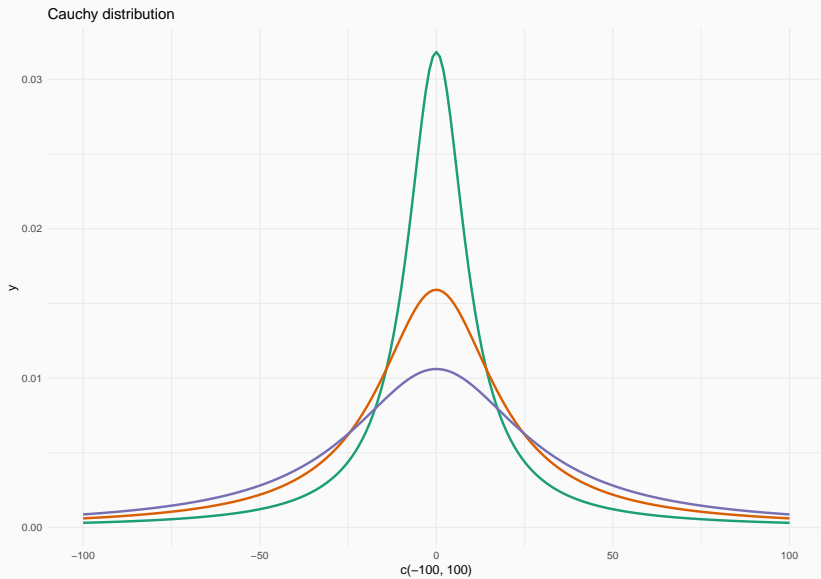
$$\mu \sim Normal(40, 10)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Cauchy prior

```
ggplot() +  
  aes(x = c(-100, 100)) +  
  stat_function(fun = dcauchy, n = 201, args = list(0,  
  stat_function(fun = dcauchy, n = 201, args = list(0,  
  stat_function(fun = dcauchy, n = 201, args = list(0,  
  labs(title = "Cauchy distribution")
```

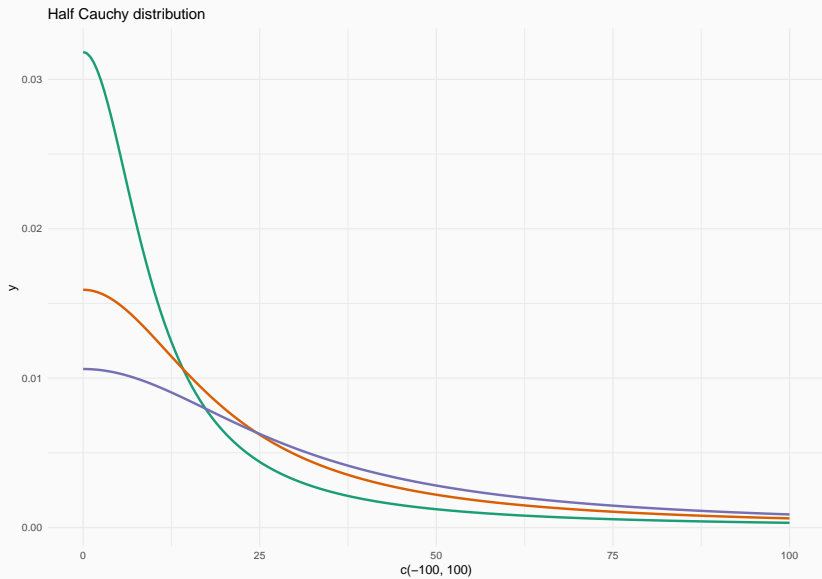
Cauchy prior



Cauchy prior

```
ggplot() +  
  aes(x = c(-100, 100)) +  
  stat_function(fun = dcauchy, n = 201, args = list(0,  
  stat_function(fun = dcauchy, n = 201, args = list(0,  
  stat_function(fun = dcauchy, n = 201, args = list(0,  
  labs(title = "Half Cauchy distribution") +  
  xlim(0, 100)
```

Cauchy prior



$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(40, 10)$$

$$\sigma \sim HalfCauchy(0, 10)$$

- We have a model which incorporates (some of) our knowledge about VOT (through the priors for μ and σ).
- Now we want to obtain the **posterior distributions** of μ and σ .
 - The posterior distribution is the prior distribution *conditioned* on the data.
- **brms** R package: Bayesian Regression Models using Stan (Bürkner, 2018).

- Stan (Stan Development Team, 2017).
 - Statistical programming language written in C++.
 - Fit Bayesian models (calculate posterior distributions).
- Calculation can be complex and/or impossible, so we take many samples from the data and from the possible parameter values to find the posterior distributions of the hyperparameters.
 - Markov Chain Monte Carlo (MCMC) sampling using the No-U-Turn sampler (NUTS).

- brms translates R code into Stan code.
- Stan code is run in Stan via Rstan, an R interface to Stan.
- `brm()` function from brms.
 - lme4 syntax ($y \sim x + (1|w)$).
 - Creates a Stan model, which is compiled (in C++) and run.

```
library(brms)

vot1 <- brm(
  <model_formula>,
  <family>,
  <prior>,
  <data>,
  chains = 4,
  iter = 2000
)
```

```
library(brms)

vot1 <- brm(
  vot ~ 1,
  family = gaussian(),
  <prior>,
  data = ita_egg,
  chains = 4,
  iter = 2000
)
```

Get prior

```
get_prior(  
  vot ~ 1,  
  family = gaussian(),  
  data = ita_egg  
)
```

```
##               prior      class coef group resp dpa  
## 1 student_t(3, 19, 14) Intercept  
## 2  student_t(3, 0, 14)      sigma
```

Prior predictive checks

```
nsim <- 1000
nobs <- 100

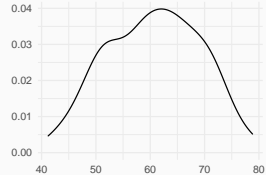
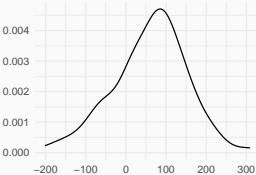
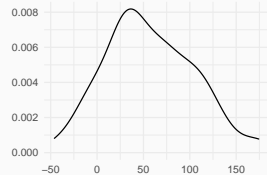
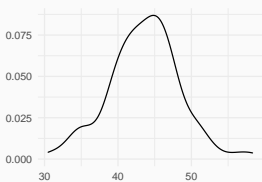
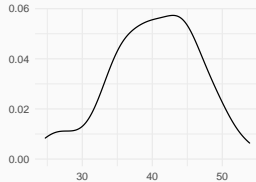
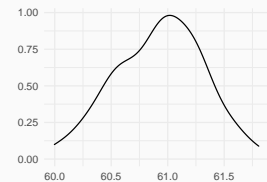
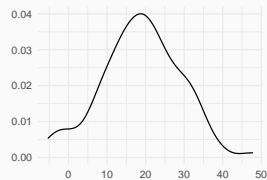
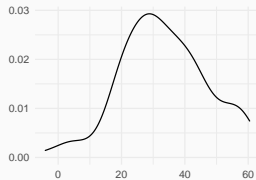
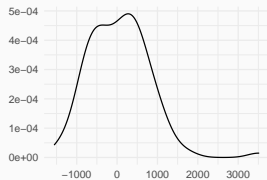
y <- matrix(rep(NA, nsim * nobs), ncol = nobs)

mu <- rnorm(nsim, 40, 10)
sigma <- rhcauchy(nsim, 10)

for (i in 1:nsim) {
  y[i,] <- rnorm(nobs, mean = mu[i], sd = sigma[i])
}
```

Prior predictive checks

Prior predictive checks



Set prior

```
priors <- c(  
  prior(normal(40, 10), class = Intercept),  
  prior(cauchy(0, 10), class = sigma)  
)
```


Run the model

```
vot1 <- brm(  
  vot ~ 1,  
  family = gaussian(),  
  prior = priors,  
  data = ita_egg,  
  chains = 4,  
  iter = 2000,  
  file = "./cache/vot1"  
)
```

Model summary

```
vot1
```

```
## Family: gaussian
```

```
## Links: mu = identity; sigma = identity
```

```
## Formula: vot ~ 1
```

```
## Data: ita_egg (Number of observations: 2624)
```

```
## Samples: 4 chains, each with iter = 2000; warmup = 1000
```

```
## total post-warmup samples = 4000
```

```
##
```

```
## Population-Level Effects:
```

```
##           Estimate Est.Error 1-95% CI u-95% CI Rhat
```

```
## Intercept      23.08       0.30   22.49   23.67 1.00
```

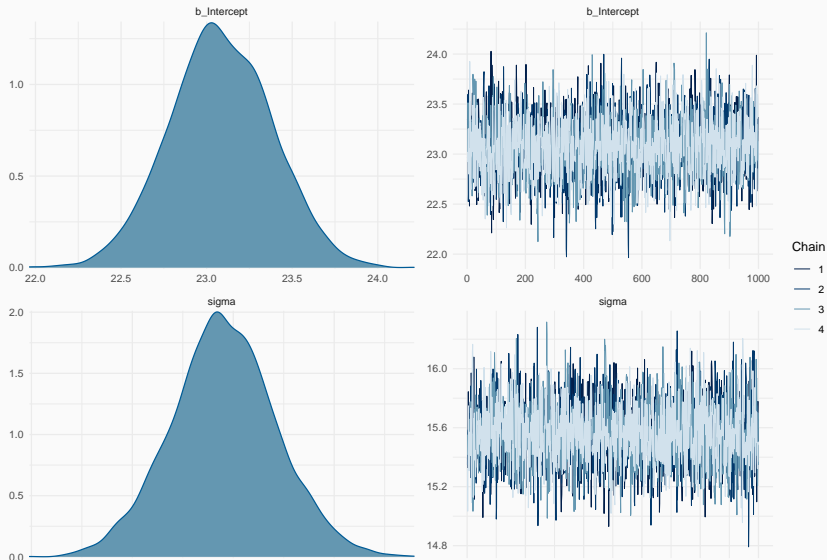
```
##
```

```
## Family Specific Parameters:
```

```
##           Estimate Est. Error 1-95% CI u-95% CI Rhat Bulk
```

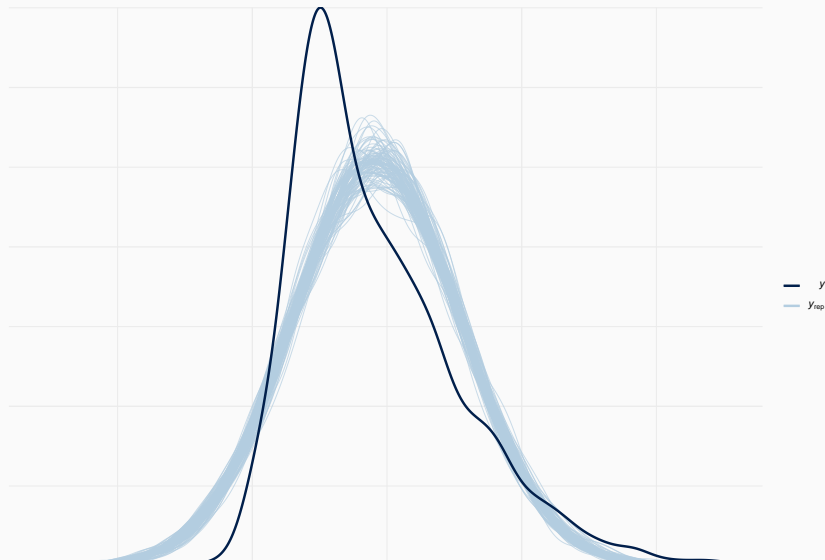
Plot model

```
plot(vot1)
```



Posterior predictive check

```
pp_check(vot1, nsamples = 100)
```



Adding predictors

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times coronal_i + \beta_2 \times velar_i$$

$$\alpha \sim Normal(\mu_1, \sigma_1)$$

$$\beta_1 \sim Normal(\mu_2, \sigma_2)$$

$$\beta_2 \sim Normal(\mu_3, \sigma_3)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Adding predictors

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times coronal_i + \beta_2 \times velar_i$$

$$\alpha \sim Normal(25, 10)$$

$$\beta_1 \sim Normal(10, 10)$$

$$\beta_2 \sim Normal(20, 10)$$

$$\sigma \sim HalfCauchy(0, 10)$$

Adding predictors

```
get_prior(  
  vot ~ 1 + c1_place,  
  family = gaussian(),  
  data = ita_egg  
)
```

##		prior	class	coef	gro
## 1			b		
## 2			b	c1_placecoronal	
## 3			b	c1_placevelar	
## 4	student_t(3, 19, 14)	Intercept			
## 5	student_t(3, 0, 14)		sigma		

Adding predictors

```
priors <- c(  
  prior(normal(25, 10), class = Intercept),  
  prior(cauchy(0, 10), class = sigma),  
  prior(normal(10, 10), class = b, coef = "c1_placecor"),  
  prior(normal(20, 10), class = b, coef = "c1_placevel")  
)
```


Adding predictors

```
vot2 <- brm(  
  vot ~ 1 + c1_place,  
  family = gaussian(),  
  prior = priors,  
  data = ita_egg,  
  chains = 4,  
  iter = 2000,  
  file = "./cache/vot2"  
)
```

Random effects

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \alpha_{speaker[i]} + (\beta_1 + \beta_{1speaker[i]}) \times coronal_i + (\beta_2 + \beta_{2speaker[i]}) \times velar_i$$

$$\begin{bmatrix} \alpha_{speaker} \\ \beta_{1speaker[i]} \\ \beta_{2speaker[i]} \end{bmatrix} \sim MVNormal\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, S\right)$$

$$\alpha \sim Normal(25, 10)$$

$$\alpha_{speaker} \sim Normal(0, \sigma_{speaker})$$

$$\beta_1 \sim Normal(10, 10)$$

$$\beta_2 \sim Normal(20, 10)$$

$$\sigma_{\alpha speaker} \sim Normal(0, \sigma_{speaker})$$

Random effects

```
get_prior(  
  vot ~ 1 + c1_place + (1 + c1_place | speaker),  
  family = gaussian(),  
  data = ita_egg  
)
```

##		prior	class	coef
## 1			b	
## 2			b c1_placecoronal	
## 3			b c1_placevelar	
## 4		lkj(1)	cor	
## 5			cor	sp
## 6	student_t(3, 19, 14)		Intercept	
## 7	student_t(3, 0, 14)		sd	51
## 8				

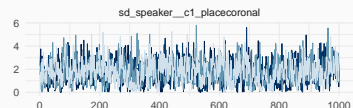
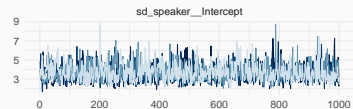
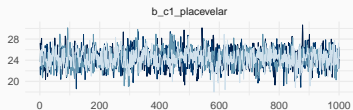
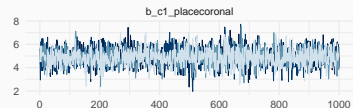
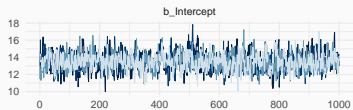
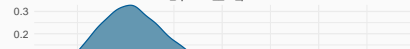
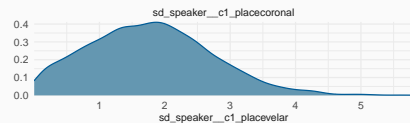
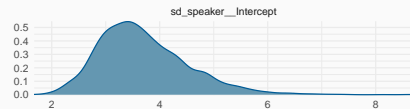
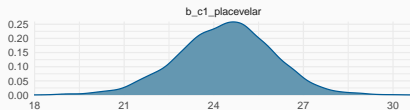
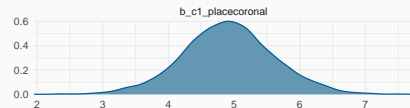
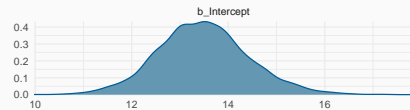
Random effects

```
priors <- c(  
  prior(normal(40, 10), class = Intercept),  
  prior(cauchy(0, 10), class = sigma),  
  prior(normal(10, 10), class = b, coef = "c1_placecor"),  
  prior(normal(20, 10), class = b, coef = "c1_placevel"),  
  prior(normal(0, 25), class = sd),  
  prior(lkj(2), class = cor)  
)
```

Random effects

```
vot3 <- brm(  
  vot ~ 1 + c1_place + (1 + c1_place | speaker),  
  family = gaussian(),  
  prior = priors,  
  data = ita_egg,  
  chains = 4,  
  iter = 2000,  
  file = "./cache/vot3"  
)
```

Random effects



Chain



Random effects

vot3

```
## Family: gaussian
```

```
## Links: mu = identity; sigma = identity
```

```
## Formula: vot ~ 1 + c1_place + (1 + c1_place | speaker)
```

```
## Data: ita_egg (Number of observations: 2624)
```

```
## Samples: 4 chains, each with iter = 2000; warmup = 1000
```

```
## total post-warmup samples = 4000
```

```
##
```

```
## Group-Level Effects:
```

```
## ~speaker (Number of levels: 18)
```

```
##
```

Estimate Est.Error

```
## sd(Intercept) 3.70 0.8
```

```
## sd(c1_placecoronal) 1.78 0.9
```

```
## sd(c1_placevelar) 6.46 1.1
```

Binomial logistic regression

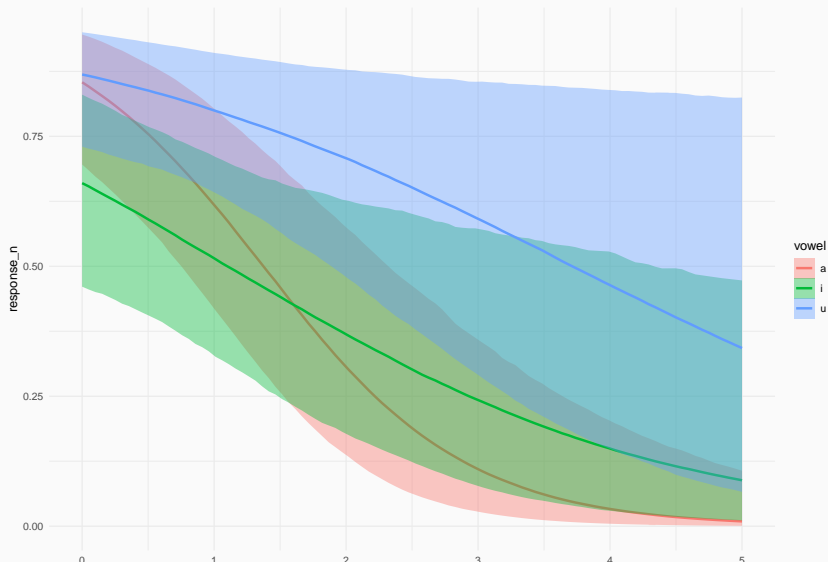
```
priors <- c(  
  prior(student_t(3, 0, 2), class = Intercept),  
  prior(student_t(3, 0, 2), class = b),  
  prior(cauchy(0, 1), class = sd),  
  prior(lkj(2), class = cor)  
)
```


Binomial logistic regression

```
burst1 <- brm(  
  response_n ~  
    burst *  
    vowel +  
    (1+burst|participant),  
  data = burst,  
  prior = priors,  
  family = bernoulli,  
  file = "./cache/burst1",  
  control = list(adapt_delta = 0.999)  
)
```

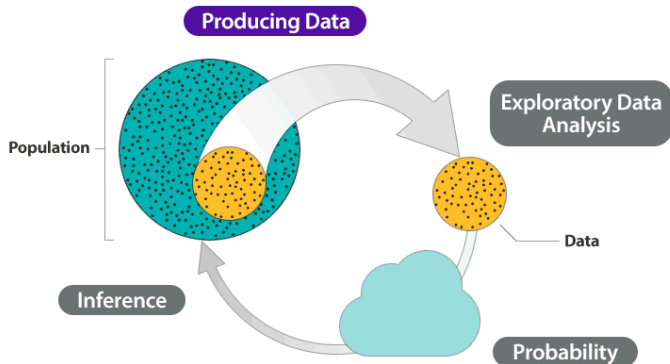
Binomial logistic regression

```
conditional_effects(burst1, effects = "burst:vowel")
```



BAYESIAN INFERENCE

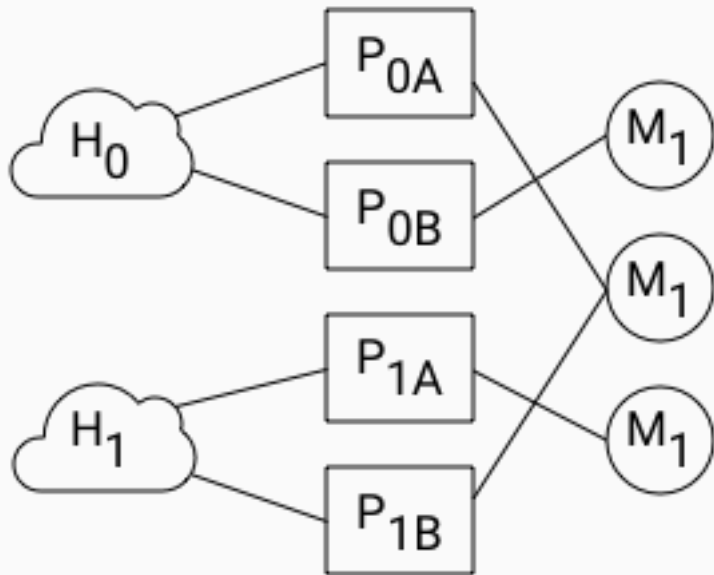
Inference



Open Learning Initiative,

<https://oli.cmu.edu/courses/concepts-of-statistics/> (CC BY-NC-SA 4.0)

Inference



Inference

- We want to know two things:
 - If there is **evidence for our hypothesis H** (or for the value of the parameter θ), and
 - What the **strength** of the evidence is.

- **Inferential statistics.**
- We test H against empirical data (hypothesis testing).
 - It is important to decide in advance the details of the analysis (model and prior specification among other things).
- Inference is ultimately a long-term endeavour (via accumulation of knowledge).

Inference

- Three ways of doing inference (hypothesis testing) with Bayesian statistics:
 - Inference from the **posterior**.
 - Inference using a **Region Of Practical Equivalence (ROPE)**.
 - Inference using the **Bayes factor**.

Inference from the posterior

- **H:** Condition B decreases reaction times relative to Condition A.
 - You have chosen a prior which appropriately conveys the content of this H.
- **Posterior:** Condition B 95% CI = [-80, -15] ms.
- **Inference:** The posterior suggests that Condition B decreases reaction times by 15 to 80 ms at 95% confidence.

Inference from the posterior

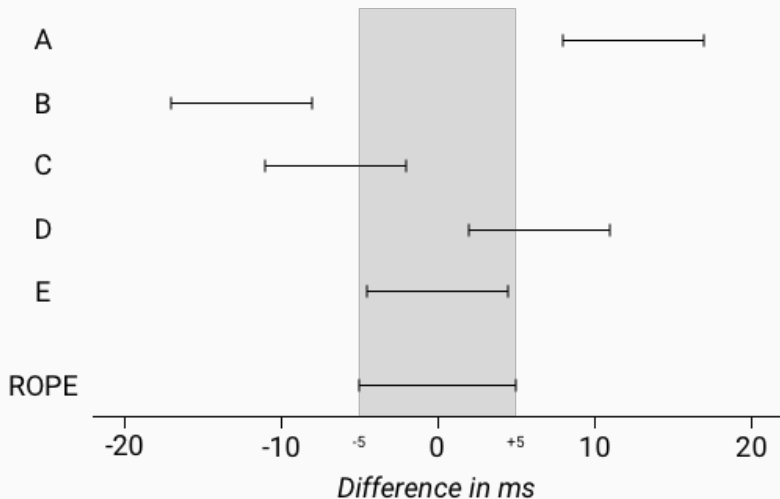
- **H:** Condition B decreases reaction times relative to Condition A *by 100 ms*.
 - You have chosen a prior which appropriately conveys the content of this H.
- **Posterior:** Condition B 95% CI = [-80, -15] ms.
- **Inference:** The posterior suggests that Condition B decreases reaction times by a smaller amount than expected from H (15 to 80 ms at 95% confidence).

Inference with a ROPE

H0 vs H1

- H1 states that Condition B increases segment duration (alternative hypothesis), while H0 states that Condition B does not increase segment duration (null hypothesis, null effect).
 - $H_1 : \beta > 0$
 - $H_0 : \beta = 0$
- Region of Practical Equivalence (ROPE):
 - Define a region around $\beta = 0$ that practically corresponds to a null effect.
 - For example: $[-5, +5]$ ms ($-5 \geq \beta \leq +5 = \text{null effect}$).
 - This ROPE has a width of 10 ms.
 - Choose a minimal sample size (ideally based on prospective power analyses).

Inference with a ROPE



Bayes Factor

The Bayes factor is the ratio of the likelihood of H_1 to the likelihood of H_2 .

$$BF_{12} = \mathcal{L}(H_1)/\mathcal{L}(H_2)$$

Bayes Factor

BF	$p(M1 D)$	evidence
1–3	0.5–0.75	weak
3–20	0.75–0.95	positive
20–150	0.95–0.99	strong
> 150	> 0.99	very strong

Bayes Factor

```
vot3_bf <- brm(  
  vot ~ 1 + c1_place + (1 + c1_place | speaker),  
  family = gaussian(),  
  prior = priors,  
  data = ita_egg,  
  chains = 4,  
  iter = 20000,  
  file = "./cache/vot3_bf",  
  save_all_pars = TRUE  
)
```

```
vot3_bf_null <- brm(  
  vot ~ 1 + c1_place + (1 | speaker),  
  family = gaussian(),
```

Bayes Factor

```
bayes_factor(vot3_bf, vot3_bf_null)
```


Sensitivity analysis

References

- Bürkner, Paul-Christian. 2018. Advanced Bayesian multilevel modeling with the R package brms. *The R Journal* 10(1). 395–411. doi:10.32614/RJ-2018-017.
- Esposito, Anna. 2002. On vowel height and consonantal voicing effects: Data from Italian. *Phonetica* 59(4). 197–231. doi:10.1159/000068347.
- Stan Development Team. 2017. Stan: A C++ library for probability and sampling, version 2.14.0. <http://mc-stan.org/>.

Stevens, Mary & John Hajek. 2010. Post-aspiration in standard Italian: some first cross-regional acoustic evidence. Paper presented at Interspeech, 26-30 September 2010, Makuhari, Chiba, Japan.