

Introduction to Bayesian linear regression with brms

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18/01/2020

Road map

1. Basic concepts.

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2. Choosing priors.

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3. Model fitting with brms.

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 - 3.1 Normal.

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 - 3.3 Binomial (Bernoulli).

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3. Model fitting with brms.
 - 3.1 Normal.
 - 3.2 Log-normal.
 - 3.3 Binomial (Bernoulli).
4. Bayesian inference.

BASIC CONCEPTS

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 - $y = (y_1, y_2, y_3, y_4, \dots, y_n)$
- We want to know how the data (the sample y) was generated.
- In probability theory, data is generated by a random variable Y .

- Y is a variable whose value is generated by a random event.

Random variables

- Y is a variable whose value is generated by a random event.
- Y is uncertain.

Random variables

- Y is a variable whose value is generated by a random event.
- Y is uncertain.
 - We can describe Y as a probability distribution.

Random variables

- Probability distribution.

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Random variables

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 - A list of the values a random variable could take on along with their corresponding probability.
- Probability distributions can be expressed by a set of parameters $\Theta = (\theta_1, \dots, \theta_n)$.
- Some probability distributions:
 - *Normal*(μ, σ),
 - *Binomial*(n, p),
 - *Poisson*(λ)

$$y_i \sim \textit{Normal}(\mu, \sigma)$$

Frequentist vs Bayesian view

- Parameters: $\mu, \sigma, p, \lambda, \dots$

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- Parameters: $\mu, \sigma, p, \lambda, \dots$
- Frequentist view:
 - The parameters are **fixed** (they are unknown but certain).
 - They take on a specific value.
- Bayesian view:
 - The parameters are **random variables** (they are unknown and uncertain).
 - We describe each parameter as a probability distribution, expressed by a set of **hyperparameters**.

- We want to estimate the parameters μ and σ from the data.

Priors

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- We can incorporate previous knowledge (belief) about the parameters using **priors** (*prior probability distributions*).

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Priors

- We want to estimate the parameters μ and σ from the data.
- We can incorporate previous knowledge (belief) about the parameters using **priors** (*prior probability distributions*).
 - We specify the hyperparameters of the prior probability distributions.
- Priors are chosen based on expert knowledge, previous studies, pilot data...
 - Priors must **not** be chosen based on inspection of the data to be analysed.

$$\textit{observed data} \times \textit{prior belief} = \textit{posterior belief}$$

Bayesian belief update

<https://nanx.shinyapps.io/conjugate-normal-umkv/>

CHOOSING PRIORS

- Toy example with F1.

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 - C2 = /t, d, k, g/.

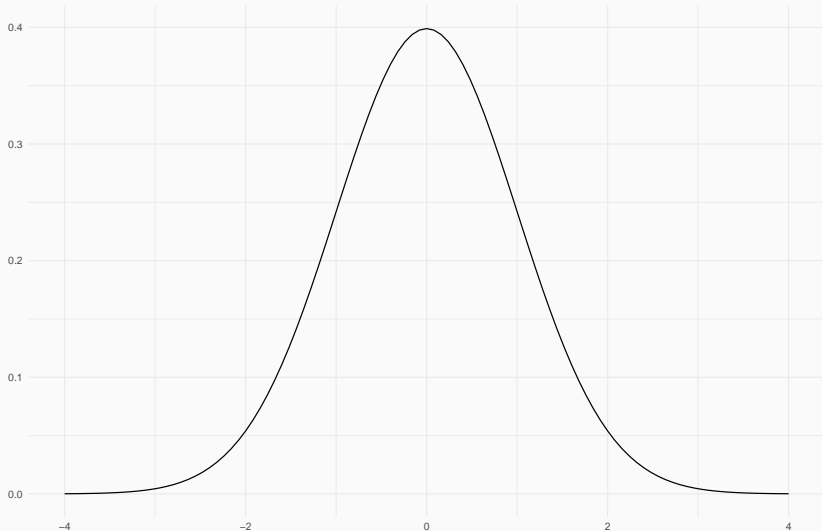
Formant values are roughly distributed according to a normal (Gaussian) distribution.

$$F1_i \sim \text{Normal}(\mu, \sigma)$$

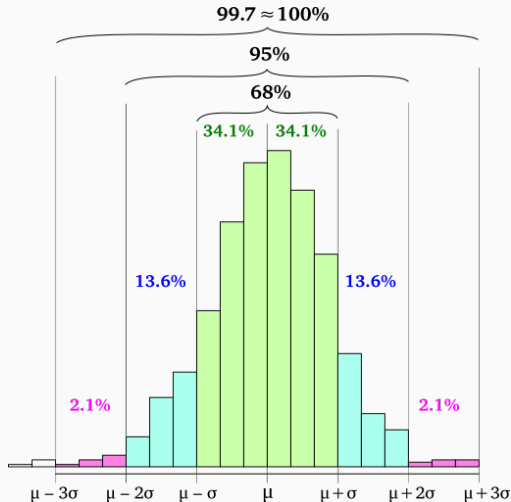
Normal (Gaussian) distribution

Standard normal (Gaussian) distribution

mean = 0, SD = 1



Normal (Gaussian) distribution



Melikamp, https://commons.wikimedia.org/wiki/File:Empirical_rule_histogram.svg (CC BY-SA 4.0)

- $Normal(\mu, \sigma)$ has two parameters, μ and σ .

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- μ and σ are random variables (unknown and uncertain).

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- μ and σ are random variables (unknown and uncertain).
- We express these parameters as priors (probability distributions with hyperparameters).

$$F1_i \sim \text{Normal}(\mu, \sigma)$$

What prior for μ ?

$$F1_i \sim \text{Normal}(\mu, \sigma)$$

$$\mu \sim \text{Normal}(\mu_1, \sigma_1)$$

Prior for the mean

Normal(0, 500) as the prior for μ .

- This means that we believe μ to be between -1000 and +1000 at 95% confidence.

Prior for the mean

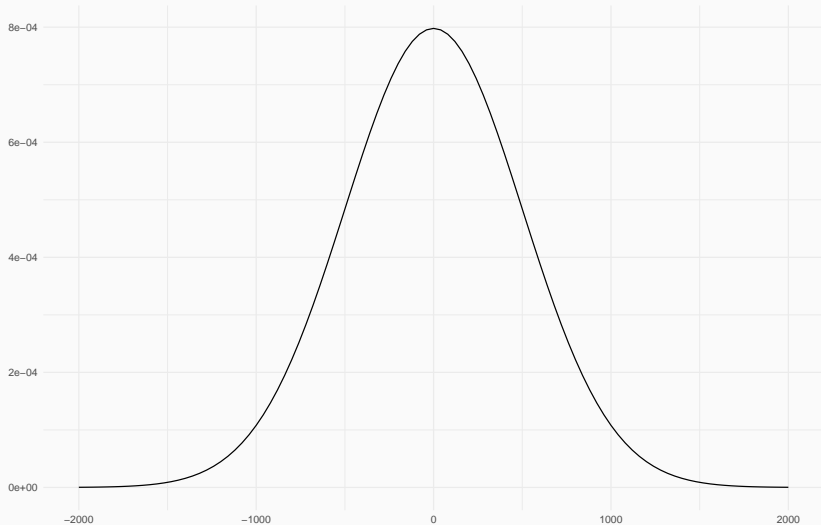
Normal(0, 500) as the prior for μ .

- This means that we believe μ to be between -1000 and +1000 at 95% confidence.
 - 95% credible interval (CI) = $[\mu_1 - 2\sigma_1, \mu_1 + 2\sigma_1]$.

Prior for the mean

Prior for μ

$\mu = 0, \sigma = 500$



Type of priors

- Informative and weakly informative priors.

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Type of priors

- Informative and weakly informative priors.
- Uninformative or diffuse priors.
 - Uniform distribution.
- Regularising priors.

$$F1_i \sim \text{Normal}(\mu, \sigma)$$

$$\mu \sim \text{Normal}(0, 500)$$

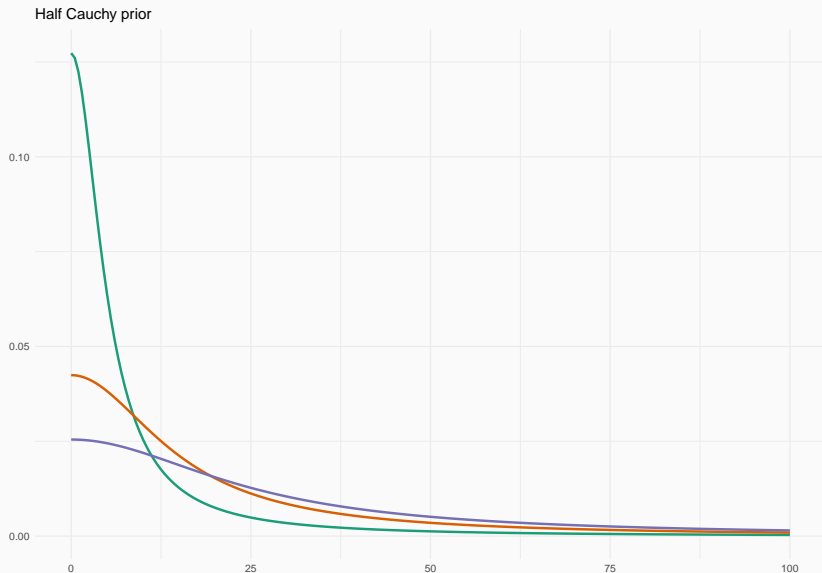
What about σ ?

$$Fl_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(0, 500)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Prior for the standard deviation



Prior for the standard deviation

```
library(HDInterval)
inverseCDF(c(0.025, 0.975), phcauchy, 5)

## [1] 0.1964676 127.2584987

inverseCDF(c(0.025, 0.975), phcauchy, 15)

## [1] 0.5893621 381.7754937

inverseCDF(c(0.025, 0.975), phcauchy, 25)

## [1] 0.9822792 636.2924897
```

$$F1_i \sim \text{Normal}(\mu, \sigma)$$

$$\mu \sim \text{Normal}(0, 500)$$

$$\sigma \sim \text{HalfCauchy}(0, 25)$$

MODEL FITTING

- We have a model which incorporates our (vague) knowledge about F1 (through the priors for μ and σ).

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 - The posterior distribution is the prior distribution *conditioned* on the data.
- **brms** R package: Bayesian Regression Models using Stan (Bürkner, 2018).

Installation

Safe method:

- **Install Rstan first:** <https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started> (see *Installation of Rstan, Checking the C++ Toolchain, and Configuration of the C++ Toolchain*).

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 - Note that details in *Checking the C++ Toolchain* differ depending on your OS.
- **Install brms:** <https://github.com/paul-buerkner/brms#how-do-i-install-brms>.

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 - Statistical programming language written in C++.
 - Fit Bayesian models (calculate posterior distributions).
- Calculation can be complex and/or impossible, so we take many samples from the data and from the possible parameter values to find the posterior distributions of the hyperparameters.
 - Markov Chain Monte Carlo (MCMC) sampling using the No-U-Turn sampler (NUTS).

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- Stan code is run in Stan via Rstan, an R interface to Stan.
- **brm()** function from brms.
 - lme4 syntax ($y \sim x + (1|w)$).
 - Creates a Stan model, which is compiled (in C++) and run.

```
library(brms)

f1 <- brm(
  <model_formula>,
  <family>,
  <prior>,
  <data>,
  chains = 4,
  iter = 2000
)
```

```
library(brms)

f1 <- brm(
  f1 ~ 1,
  family = gaussian(),
  <prior>,
  data = f_end,
  chains = 4,
  iter = 2000
)
```

Get prior

```
get_prior(  
  f1 ~ 1,  
  family = gaussian(),  
  data = f_end  
)
```

```
##               prior      class coef group resp dpar nlpar bound  
## 1 student_t(3, 495, 167) Intercept  
## 2  student_t(3, 0, 167)    sigma
```


Prior predictive checks

```
nsim <- 1000
nobs <- 1000

y <- matrix(rep(NA, nsim * nobs), ncol = nobs)

mu <- rnorm(nsim, 0, 500)
sigma <- rhcauchy(nsim, 25)

for (i in 1:nsim) {
  y[i,] <- rnorm(nobs, mean = mu[i], sd = sigma[i])
}
```

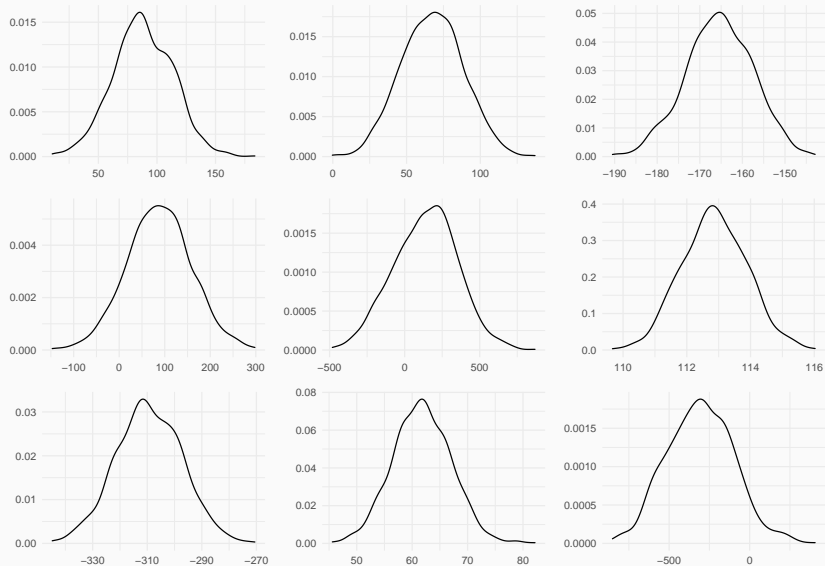
Prior predictive checks

```
rand_sample <- sample(1:nsim, 9, replace = FALSE)
plots <- list()

j = 1

for (i in rand_sample) {
  my_data <- enframe(y[i,], name = NULL)
  plots[[j]] <- ggplot(data = my_data) +
    aes(x = value) +
    geom_density() +
    labs(x = element_blank(), y = element_blank())
  j = j + 1
}
```

Prior predictive checks



Set prior

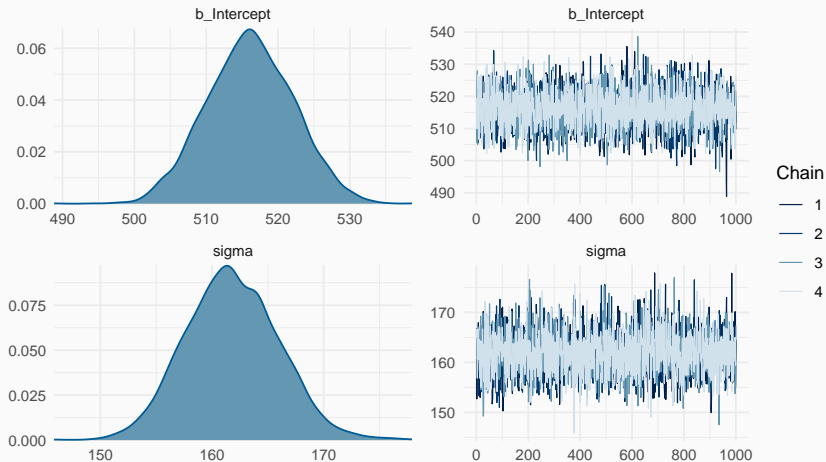
```
priors <- c(  
  prior(normal(0, 500), class = Intercept),  
  prior(cauchy(0, 25), class = sigma)  
)
```

Run the model

```
f1_bm <- brm(  
  f1 ~ 1,  
  family = gaussian(),  
  prior = priors,  
  data = f_end,  
  chains = 1,  
  iter = 2000,  
  file = "./cache/f1"  
)
```

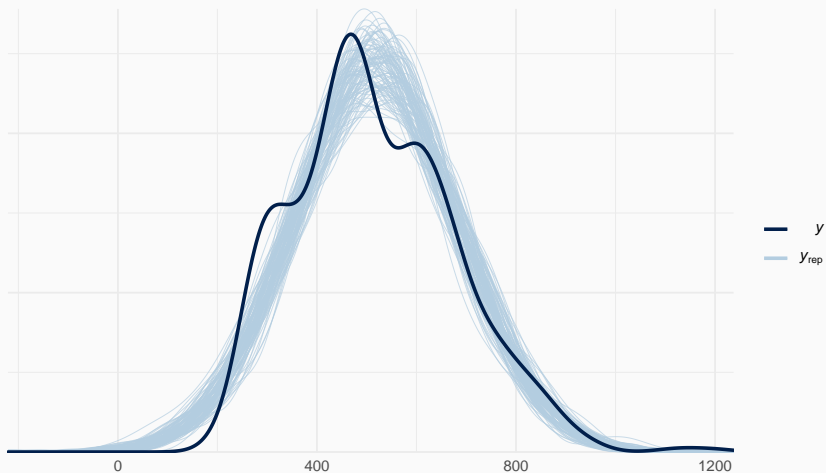
Plot model

```
plot(f1_bm, ask = FALSE)
```



Posterior predictive check

```
pp_check(f1_bm, nsamples = 100)
```



Model summary

f1_bm

```
## Family: gaussian
## Links: mu = identity; sigma = identity
## Formula: f1 ~ 1
## Data: f_end (Number of observations: 748)
## Samples: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
##           total post-warmup samples = 4000
##
## Population-Level Effects:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## Intercept   516.14      6.01   504.35   527.81 1.00     3358     2498
##
## Family Specific Parameters:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## sigma    161.81      4.18   153.95   170.03 1.00     3071     2682
##
## Samples were drawn using sampling(NUTS). For each parameter, Bulk_ESS
## and Tail_ESS are effective sample size measures, and Rhat is the potential
## scale reduction factor on split chains (at convergence, Rhat = 1).
```


$$F1_i \sim \text{Normal}(\mu, \sigma)$$

$$\mu \sim \text{Normal}(0, 500)$$

$$\sigma \sim \text{HalfCauchy}(0, 25)$$

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Let's add the predictor `vowel`.

$$F1_i \sim \text{Normal}(\mu_i, \sigma)$$

Adding predictors

$$F1_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times O_i + \beta_2 \times U_i$$

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$$\mu_i = \alpha + \beta_1 \times O_i + \beta_2 \times U_i$$

$$\alpha \sim \text{Normal}(\mu_1, \sigma_1) \text{ [vowel = /a/]}$$

Adding predictors

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$$\alpha \sim \text{Normal}(\mu_1, \sigma_1) \text{ [vowel = /a/]}$$

$$\beta_1 \sim \text{Normal}(\mu_2, \sigma_2) \text{ [vowel = /o/]}$$

$$\beta_2 \sim \text{Normal}(\mu_3, \sigma_3) \text{ [vowel = /u/]}$$

Adding predictors

$$F1_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times O_i + \beta_2 \times U_i$$

$$\alpha \sim \text{Normal}(\mu_1, \sigma_1) \text{ [vowel = /a/]}$$

$$\beta_1 \sim \text{Normal}(\mu_2, \sigma_2) \text{ [vowel = /o/]}$$

$$\beta_2 \sim \text{Normal}(\mu_3, \sigma_3) \text{ [vowel = /u/]}$$

$$\sigma \sim \text{HalfCauchy}(x_0, \gamma)$$

Adding predictors

$$F1_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times O_i + \beta_2 \times U_i$$

$$\alpha \sim \text{Normal}(0, 500)$$

$$\beta_1 \sim \text{Normal}(0, 750)$$

$$\beta_2 \sim \text{Normal}(0, 750)$$

$$\sigma \sim \text{HalfCauchy}(0, 25)$$

Adding predictors

```
get_prior(  
  f1 ~ 1 + vowel,  
  family = gaussian(),  
  data = f_end  
)
```

```
##           prior      class  coef group resp dpar nlpar bound  
## 1                    b  
## 2                    b vowelo  
## 3                    b vowelu  
## 4 student_t(3, 495, 167) Intercept  
## 5  student_t(3, 0, 167)    sigma
```

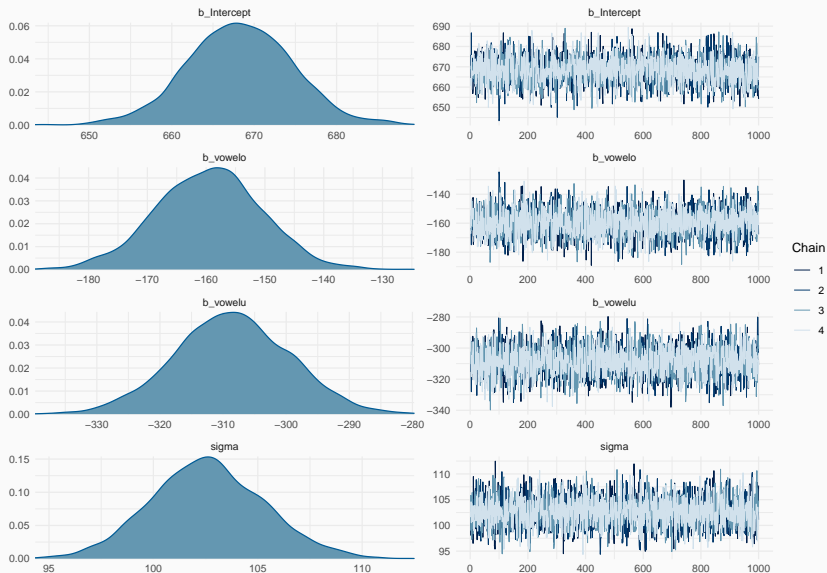
Adding predictors

```
priors <- c(  
  prior(normal(0, 500), class = Intercept),  
  prior(cauchy(0, 25), class = sigma),  
  prior(normal(0, 750), class = b, coef = "vowel_o"),  
  prior(normal(0, 750), class = b, coef = "vowel_u")  
)
```

Adding predictors

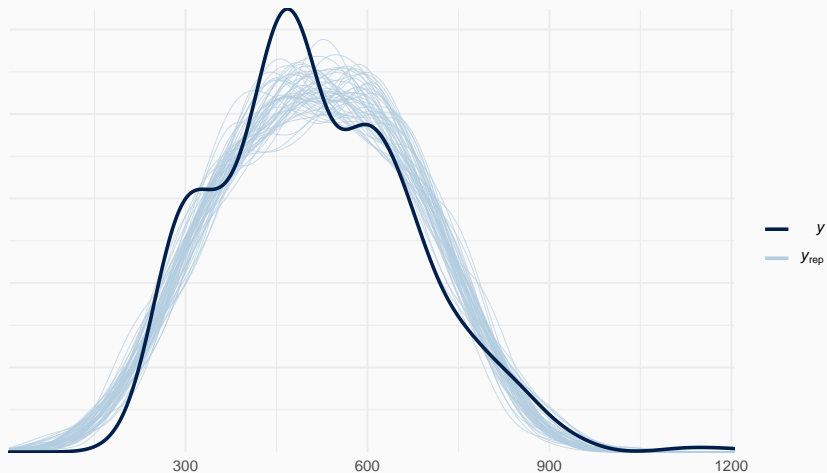
```
f1_2 <- brm(  
  f1 ~ 1 + vowel,  
  family = gaussian(),  
  prior = priors,  
  data = f_end,  
  chains = 4,  
  iter = 2000,  
  file = "./cache/f1_2"  
)
```

Adding predictors



Adding predictors

```
pp_check(f1_2, nsamples = 50)
```

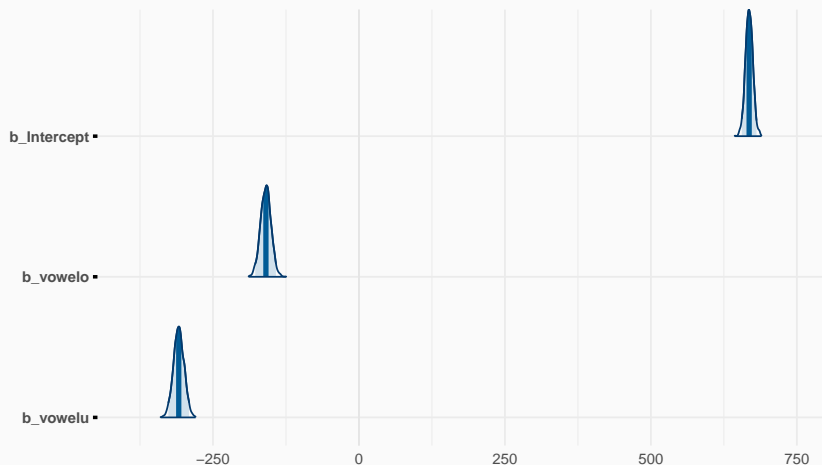


Adding predictors

```
## Family: gaussian
## Links: mu = identity; sigma = identity
## Formula: f1 ~ 1 + vowel
## Data: f_end (Number of observations: 748)
## Samples: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
##           total post-warmup samples = 4000
##
## Population-Level Effects:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## Intercept    668.30      6.38   655.69   680.85 1.00     3249     2620
## vowel0       -159.54      8.94  -177.58  -142.24 1.00     3355     2903
## vowel1       -308.57      9.06  -326.33  -291.04 1.00     3281     2734
##
## Family Specific Parameters:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## sigma    102.60      2.70    97.38   108.10 1.00     3470     2818
##
## Samples were drawn using sampling(NUTS). For each parameter, Bulk_ESS
## and Tail_ESS are effective sample size measures, and Rhat is the potential
## scale reduction factor on split chains (at convergence, Rhat = 1).
```

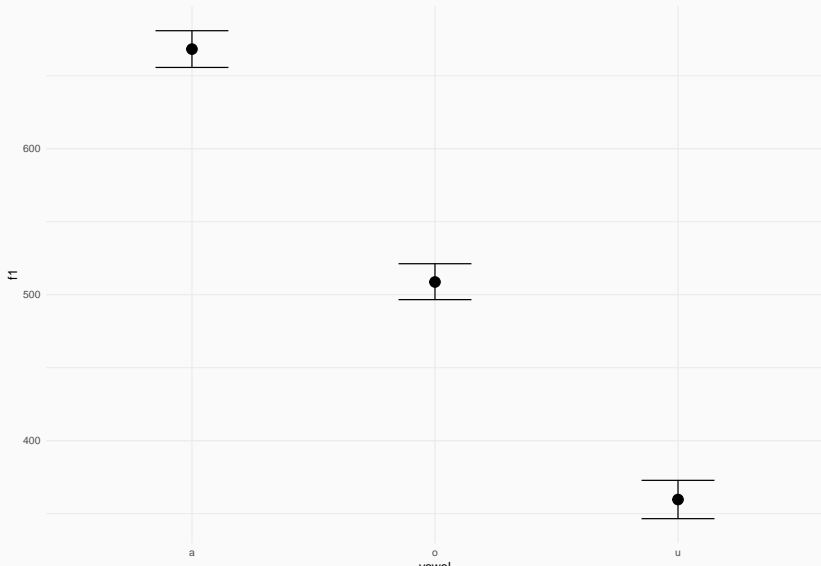
Adding predictors

```
mcmc_areas(f1_2, regex_pars = "b_", prob = 0.95)
```



Adding predictors

```
conditional_effects(f1_2)
```



Random effects

$$F1_i \sim \text{Normal}(\mu_i, \sigma)$$

$$\mu_i = \alpha + \alpha_{\text{speaker}[i]} + (\beta_1 + \beta_{1\text{speaker}[i]}) \times O_i + (\beta_2 + \beta_{2\text{speaker}[i]}) \times U_i$$

$$\begin{bmatrix} \alpha_{\text{speaker}[i]} \\ \beta_{1\text{speaker}[i]} \\ \beta_{2\text{speaker}[i]} \end{bmatrix} \sim \text{MVNormal}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, S\right)$$

Random effects

$$\alpha \sim \text{Normal}(0, 500)$$

$$\beta_1 \sim \text{Normal}(0, 750)$$

$$\beta_2 \sim \text{Normal}(0, 750)$$

$$\alpha_{\text{speaker}} \sim \text{Normal}(0, \sigma_{\alpha_{\text{speaker}}})$$

$$\sigma_{\alpha_{\text{speaker}}} \sim \text{HalfCauchy}(0, 15)$$

$$\beta_{1\text{speaker}} \sim \text{Normal}(0, \sigma_{\beta_{1\text{speaker}}})$$

$$\sigma_{\beta_{1\text{speaker}}} \sim \text{HalfCauchy}(0, 15)$$

$$\beta_{2\text{speaker}} \sim \text{Normal}(0, \sigma_{\beta_{2\text{speaker}}})$$

$$\sigma_{\beta_{2\text{speaker}}} \sim \text{HalfCauchy}(0, 15)$$

$$\sigma \sim \text{HalfCauchy}(0, 15)$$

Random effects

```
get_prior(  
  f1 ~ 1 + vowel + (1 + vowel | speaker),  
  family = gaussian(),  
  data = f_end  
)
```

```
##              prior      class      coef  group resp dpar nlpar bound  
## 1              b  
## 2              b  vowel0  
## 3              b  vowel1  
## 4      lkj(1)      cor  
## 5              cor      speaker  
## 6 student_t(3, 495, 167) Intercept  
## 7      student_t(3, 0, 167)      sd  
## 8              sd      speaker  
## 9              sd Intercept speaker  
## 10             sd  vowel0 speaker  
## 11             sd  vowel1 speaker  
## 12 student_t(3, 0, 167)      sigma
```

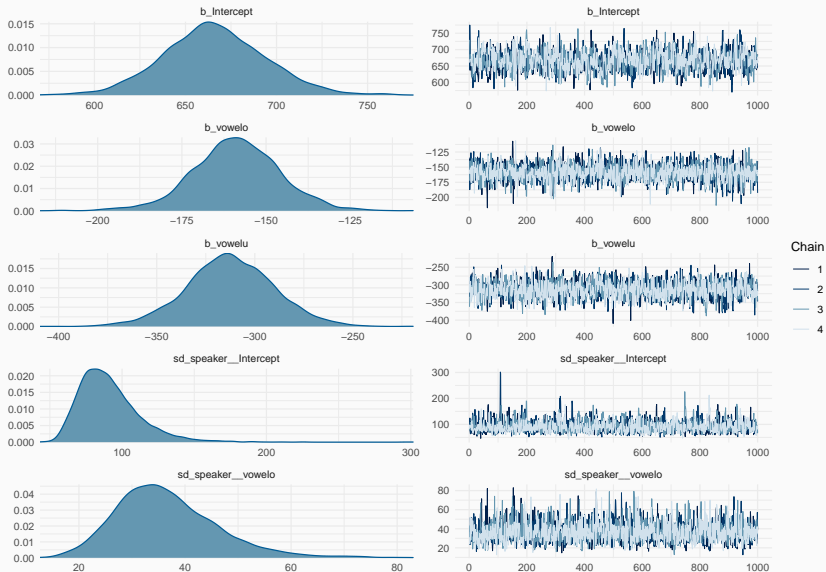
Random effects

```
priors <- c(  
  prior(normal(0, 500), class = Intercept),  
  prior(cauchy(0, 15), class = sigma),  
  prior(normal(0, 750), class = b, coef = "vowel_o"),  
  prior(normal(0, 750), class = b, coef = "vowel_u"),  
  prior(cauchy(0, 15), class = sd),  
  prior(lkj(2), class = cor)  
)
```

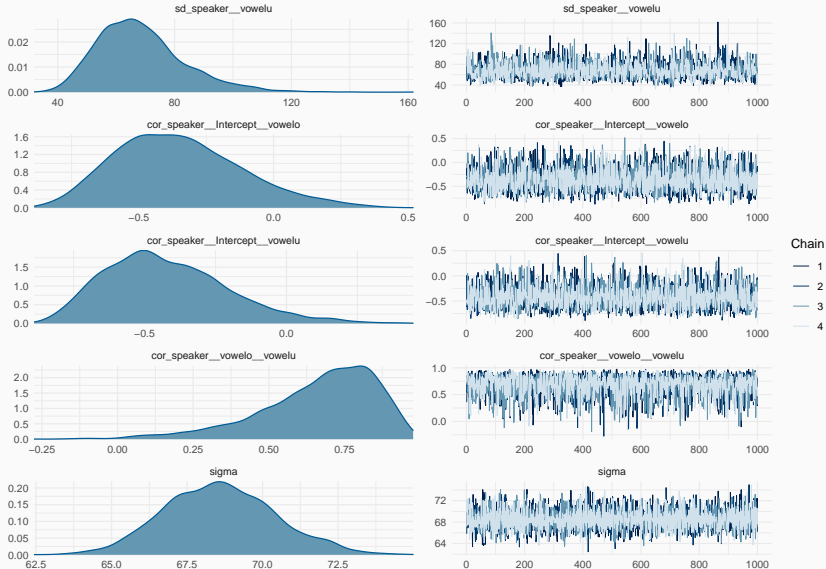
Random effects

```
f1_3 <- brm(  
  f1 ~ 1 + vowel + (1 + vowel | speaker),  
  family = gaussian(),  
  prior = priors,  
  data = f_end,  
  chains = 4,  
  iter = 2000,  
  file = "./cache/f1_3"  
)
```

Random effects



Random effects



Random effects

```
## Family: gaussian
## Links: mu = identity; sigma = identity
## Formula: f1 ~ 1 + vowel + (1 + vowel | speaker)
## Data: f_end (Number of observations: 748)
## Samples: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
##           total post-warmup samples = 4000
##
## Group-Level Effects:
## ~speaker (Number of levels: 11)
##
```

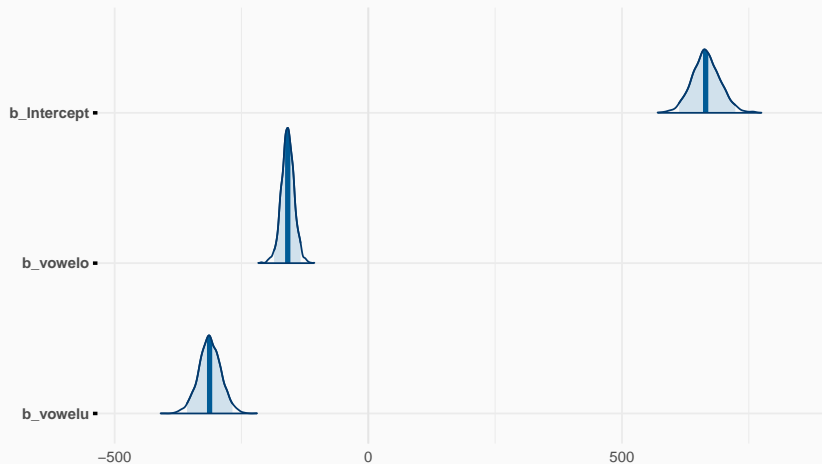
	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS
## sd(Intercept)	91.16	21.51	60.80	142.19	1.00	1539
## sd(vowelo)	36.58	9.68	21.10	59.01	1.00	1892
## sd(vowelu)	68.40	15.01	44.88	103.57	1.00	1908
## cor(Intercept,vowelo)	-0.35	0.23	-0.74	0.17	1.00	3159
## cor(Intercept,vowelu)	-0.42	0.22	-0.77	0.07	1.00	2846
## cor(vowelo,vowelu)	0.66	0.19	0.18	0.93	1.00	1935

Random effects

```
## Population-Level Effects:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## Intercept   665.93    28.22   612.57   723.44 1.01     1155     1495
## vowel0     -158.86    12.97  -185.75  -133.49 1.00     2084     2298
## vowel1     -312.55    22.39  -357.34  -267.79 1.00     1977     2518
##
## Family Specific Parameters:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## sigma      68.59      1.84    65.20    72.34 1.00     6014     3028
##
## Samples were drawn using sampling(NUTS). For each parameter, Bulk_ESS
## and Tail_ESS are effective sample size measures, and Rhat is the potential
## scale reduction factor on split chains (at convergence, Rhat = 1).
## NA
## NA
## NA
```

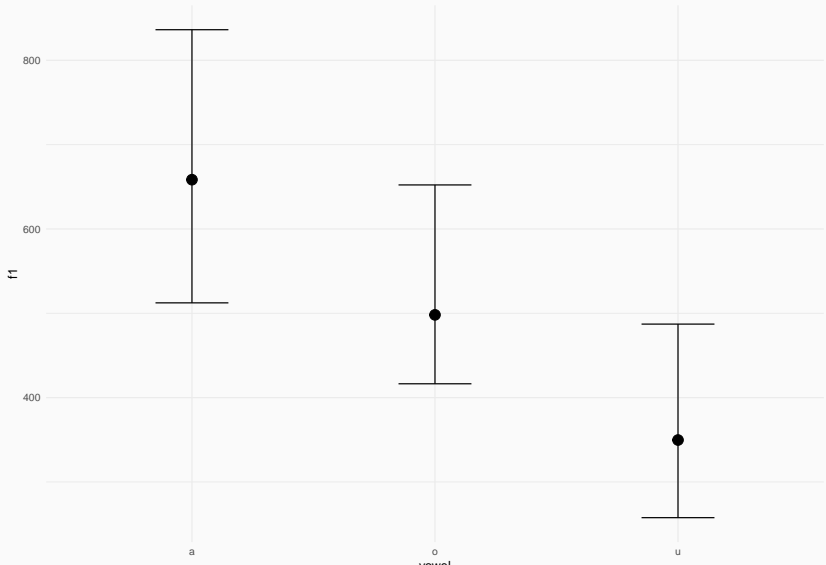
Random effects

```
mcmc_areas(f1_3, regex_pars = "b_", prob = 0.95)
```



Random effects

```
conditional_effects(f1_3, re_formula = NULL)
```



References

- Bürkner, Paul-Christian. 2018. Advanced Bayesian multilevel modeling with the R package brms. *The R Journal* 10(1). 395–411.
- Stan Development Team. 2017. Stan: A C++ library for probability and sampling, version 2.14.0. <http://mc-stan.org/>.