Introduction to Bayesian linear regression with brms

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Installation

- Safe method:
  - Install Rstan first: https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started (see Installation of Rstan, Checking the C++ Toolchain, and Configuration of the C++ Toolchain).
    - Note that in Checking the C++ Toolchain details differ depending on OS.
  - Install brms: https://github.com/paul-buerkner/brms#how-do-i-install-brms.

Random variables

- We have a question about the world, so we collect data (sample from a population).
  - y $= (y_1, y_2, y_3, y_4, ..., y_n)$
- We want to know how the data (the sample) was generated.
- In probability theory, data is generated by a random variable Y.

Random variables

- Y is uncertain.
  - We can describe Y as a probability distribution, expressed by a set of parameters $\Theta = (\theta_1, ..., \theta_n)$.
- Probability distributions:
  - $Normal(\mu, \sigma)$,
  - $Binomial(n, p)$,
  - ...

Random variables

$$vot_i \sim Normal(\mu, \sigma)$$

$$voiced_i \sim Bernoulli(p)$$

$$DoubleDative_i \sim Poisson(\lambda)$$

Frequentist vs Bayesian view

- Parameters: μ, σ, p, λ, ...
- Frequentist view:
  - The parameters are **fixed** (they are unknown but certain).
  - They take on a specific value.
- Bayesian view:
  - The parameters are **random variables** (they are unkown and uncertain).
  - We describe each parameter as a probability distribution, expressed by a set of **hyperparameters**.

Continous random variable

$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(\mu_1, \sigma_1)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Bayes' Theorem

$$P(\theta \mid d) = \frac{P(d \mid \theta)\, P(\theta)}{P(d)}$$

Bayes' Theorem

$$posterior\ probability = \frac{likelihood \times prior}{marginal\ likelihood}$$

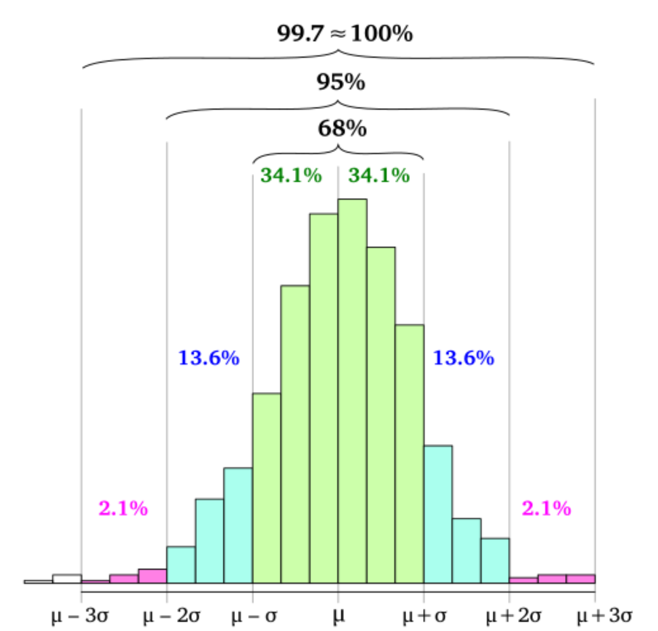
Priors

- We can incorporate previous knowledge about the hyperparameters as **priors** (prior distributions).
- Priors are chosen based on expert knowledge, previous studies, pilot data...
  - Priors must **not** be chosen based on the data to be analysed.

Priors

- Informative and weakly informative priors.
- Uninformative or diffuse priors.
    - Uniform distribution.
- Regularising priors.

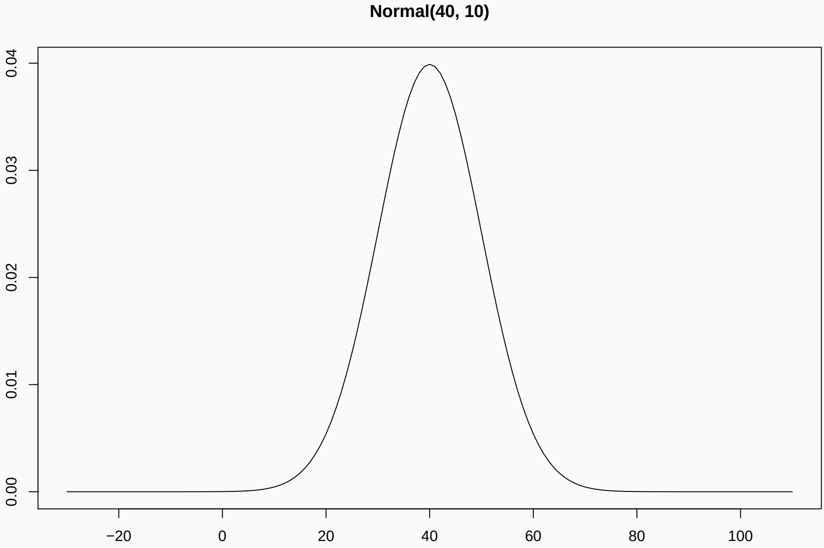
Normal prior



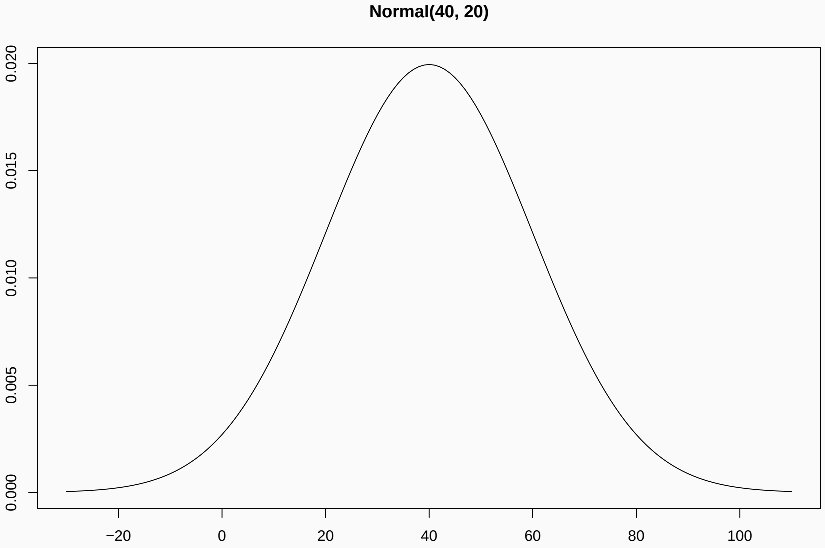
Italian VOT

- Previous literature on VOT in Italian (Esposito, 2002; Stevens & Hajek, 2010) report VOT values for voiceless stops in the range of 20–60 ms.
  - We can express this knowledge with the prior $Normal(40, 10)$.
  - This is a somewhat strongly informative prior.

Italian VOT



Italian VOT



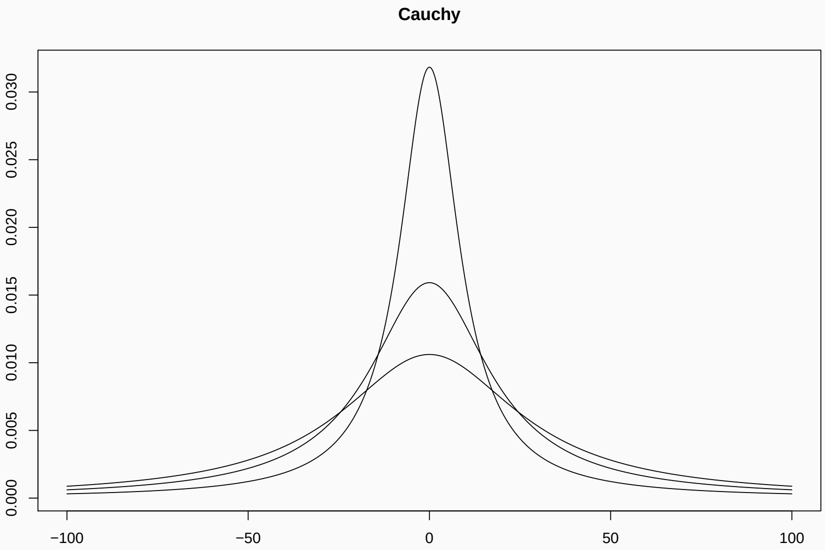
Italian VOT

$$vot_i \sim Normal(\mu, \sigma)$$

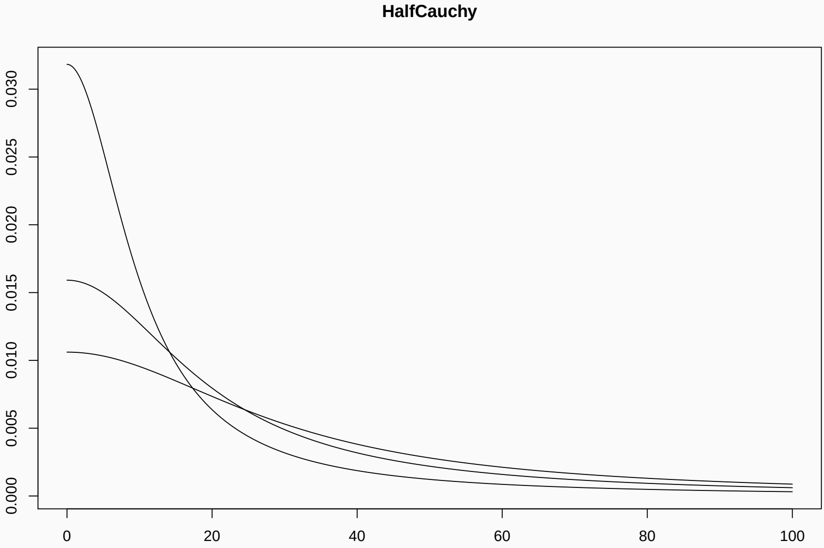
$$\mu \sim Normal(40, 10)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Cauchy prior



Cauchy prior



Italian VOT

$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(40, 10)$$

$$\sigma \sim HalfCauchy(0, 10)$$

Italian VOT

- We have a model which incorporates (some of) our knowledge about VOT (through the priors for μ and σ).
- Now we want to obtain the **posterior distributions** of μ and σ.
    - The posterior distribution is the prior distribution *conditioned* on the data.
- **brms** R package: Bayesian Regression Models using Stan (Bürkner, 2018).

brms

- Stan (Stan Development Team, 2017).
  - Statistical programming language written in C++ for fitting Bayesian models (calculate posterior distributions).
  - Calculation can be complex and/or impossible, so we take many samples from the data and from the possible parameter values to find the posterior distributions of the hyperparameters.
  - Markov Chain Monte Carlo (MCMC) sampling using the No-U-Turn sampler (NUTS).
- brms is an interface between R and Stan.
- `brm()` function from brms.
  - lme4 syntax (`y ~ x + (1|w)`).
  - Creates a Stan model, which is compiled and run.

brms

```
library(brms)

vot1 <- brm(
  <model_formula>,
  <family>,
  <prior>,
  <data>,
  chains = 4,
  iter = 2000
)
```

brms

```
library(brms)

vot1 <- brm(
  vot ~ 1,
  family = gaussian(),
  <prior>,
  data = ita_egg,
  chains = 4,
  iter = 2000
)
```

Get prior

```
get_prior(
  vot ~ 1,
  family = gaussian(),
  data = ita_egg
)
```

```
##                    prior     class coef group resp dpar nl
## 1 student_t(3, 19, 14) Intercept
## 2  student_t(3, 0, 14)     sigma
```

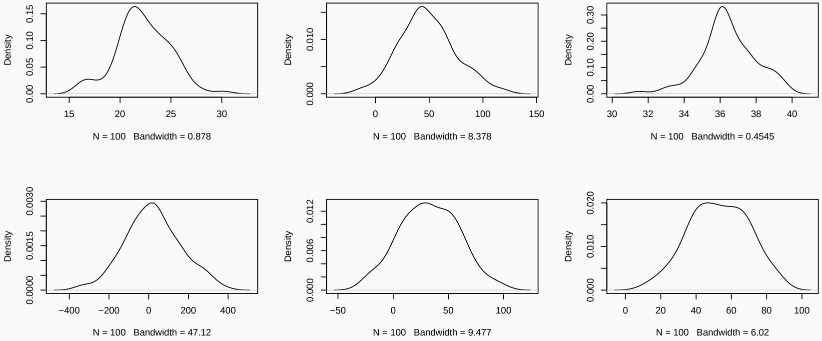
Set prior

Prior predictive checks

```r
nsim <- 1000
nobs <- 100

y <- matrix(rep(NA, nsim * nobs), ncol = nobs)

mu <- rnorm(nsim, 40, 10)
sigma <- rhcauchy(nsim, 10)

for (i in 1:nsim) {
  y[i,] <- rnorm(nobs, mean = mu[i], sd = sigma[i])
}
```

Prior predictive checks

```
op <- par(mfrow = c(3, 3))
rand_sample <- sample(1:nsim, 9)

for (i in rand_sample) {
  plot(density(y[i,]), main = "")
}
```



Run the model

```r
vot1 <- brm(
  vot ~ 1,
  family = gaussian(),
  prior = priors,
  data = ita_egg,
  chains = 4,
  iter = 2000,
  file = "./cache/vot1"
)
```

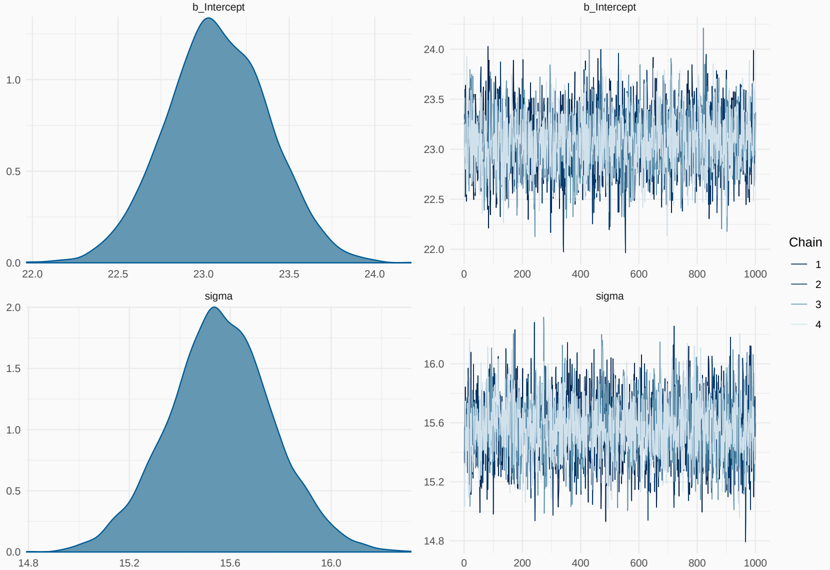
Model summary

```
vot1
```

```
##  Family: gaussian
##   Links: mu = identity; sigma = identity
## Formula: vot ~ 1
##    Data: ita_egg (Number of observations: 2624)
## Samples: 4 chains, each with iter = 2000; warmup = 1000;
##          total post-warmup samples = 4000
##
## Population-Level Effects:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk
## Intercept    23.08      0.30    22.49    23.67 1.00
##
## Family Specific Parameters:
##       Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS
## sigma    15.56      0.21    15.15    15.98 1.00      3391
```

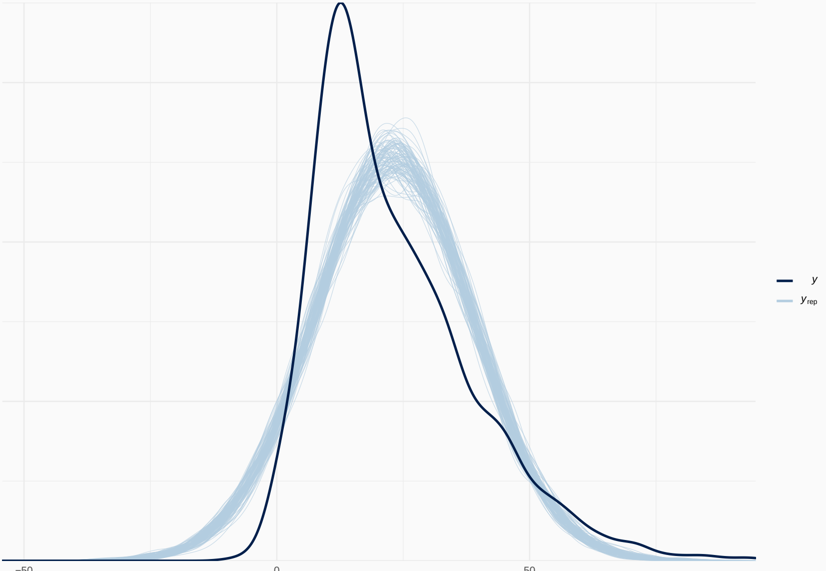
Plot model

```r
plot(vot1)
```



Posterior predictive check

```r
pp_check(vot1, nsamples = 100)
```



Adding predictors

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times coronal_i + \beta_2 \times velar_i$$

$$\alpha \sim Normal(\mu_1, \sigma_1)$$

$$\beta_1 \sim Normal(\mu_2, \sigma_2)$$

$$\beta_2 \sim Normal(\mu_3, \sigma_3)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Adding predictors

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times coronal_i + \beta_2 \times velar_i$$

$$\alpha \sim Normal(25, 10)$$

$$\beta_1 \sim Normal(10, 10)$$

$$\beta_2 \sim Normal(20, 10)$$

$$\sigma \sim HalfCauchy(0, 10)$$

Adding predictors

```
get_prior(
  vot ~ 1 + c1_place,
  family = gaussian(),
  data = ita_egg
)
```

```
##                   prior     class               coef group r
## 1                               b
## 2                               b c1_placecoronal
## 3                               b   c1_placevelar
## 4 student_t(3, 19, 14) Intercept
## 5  student_t(3, 0, 14)     sigma
```

Adding predictors

```
priors <- c(
  prior(normal(25, 10), class = Intercept),
  prior(cauchy(0, 10), class = sigma),
  prior(normal(10, 10), class = b, coef = "c1_placecoronal"
  prior(normal(20, 10), class = b, coef = "c1_placevelar")
)
```

Adding predictors

```
vot2 <- brm(
  vot ~ 1 + c1_place,
  family = gaussian(),
  prior = priors,
  data = ita_egg,
  chains = 4,
  iter = 2000,
  file = "./cache/vot2"
)
```

Random effects

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i =$$
$$\alpha + \alpha_{speaker[i]} + (\beta_1 + \beta_{1speaker[i]}) \times coronal_i + (\beta_2 + \beta_{2speaker[i]}) \times velar_i$$

$$\begin{bmatrix} \alpha_{speaker} \\ \beta_{1speaker[i]} \\ \beta_{2speaker[i]} \end{bmatrix} \sim MVNormal(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, S)$$

$$\alpha \sim Normal(25, 10)$$

$$\alpha_{speaker} \sim Normal(0, \sigma_{speaker})$$

$$\beta_1 \sim Normal(10, 10)$$

$$\beta_2 \sim Normal(20, 10)$$

$$\sigma_{\alpha speaker} \sim Normal(0, \sigma_{speaker})$$

$$\sigma_{\beta 1speaker} \sim HalfCauchy(0, 10)$$

Random effects

```
get_prior(
  vot ~ 1 + c1_place + (1 + c1_place | speaker),
  family = gaussian(),
  data = ita_egg
)
```

```
##                          prior     class               coef    grou
## 1                                      b
## 2                                      b c1_placecoronal
## 3                                      b   c1_placevelar
## 4                       lkj(1)       cor
## 5                                    cor                     speake
## 6  student_t(3, 19, 14) Intercept
## 7   student_t(3, 0, 14)        sd
## 8                                     sd                     speake
## 9                                     sd c1_placecoronal speake
```

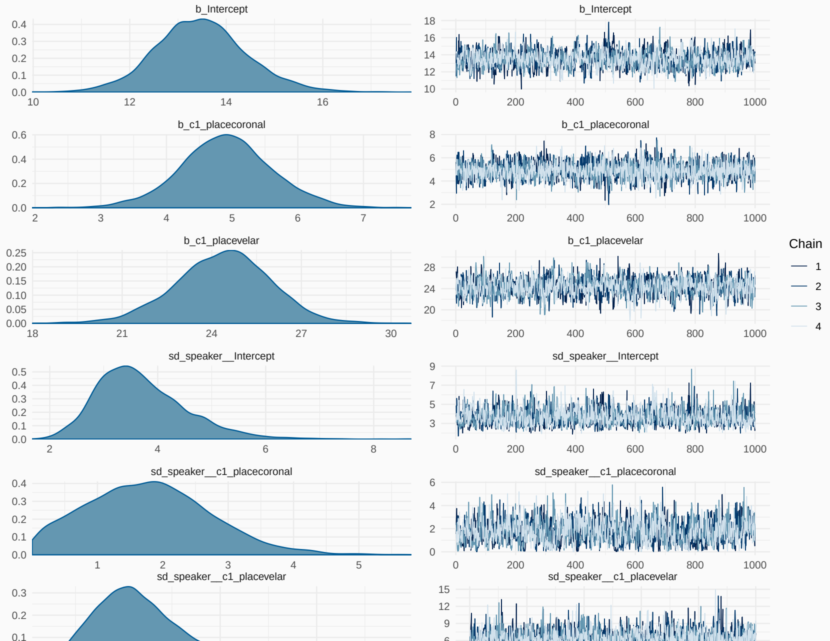
Random effects

```
priors <- c(
  prior(normal(40, 10), class = Intercept),
  prior(cauchy(0, 10), class = sigma),
  prior(normal(10, 10), class = b, coef = "c1_placecoronal'
  prior(normal(20, 10), class = b, coef = "c1_placevelar"),
  prior(normal(0, 25), class = sd),
  prior(lkj(2), class = cor)
)
```

Random effects

```r
vot3 <- brm(
  vot ~ 1 + c1_place + (1 + c1_place | speaker),
  family = gaussian(),
  prior = priors,
  data = ita_egg,
  chains = 4,
  iter = 2000,
  file = "./cache/vot3"
)
```

Random effects



Random effects

```r
vot3
```

```
##  Family: gaussian
##   Links: mu = identity; sigma = identity
## Formula: vot ~ 1 + c1_place + (1 + c1_place | speaker)
##    Data: ita_egg (Number of observations: 2624)
## Samples: 4 chains, each with iter = 2000; warmup = 1000;
##          total post-warmup samples = 4000
##
## Group-Level Effects:
## ~speaker (Number of levels: 18)
##                                           Estimate Est.Error l-
## sd(Intercept)                                 3.70      0.83
## sd(c1_placecoronal)                           1.78      0.94
## sd(c1_placevelar)                             6.46      1.38
## cor(Intercept,c1_placecoronal)               -0.24      0.32
```

Binomial logistic regression

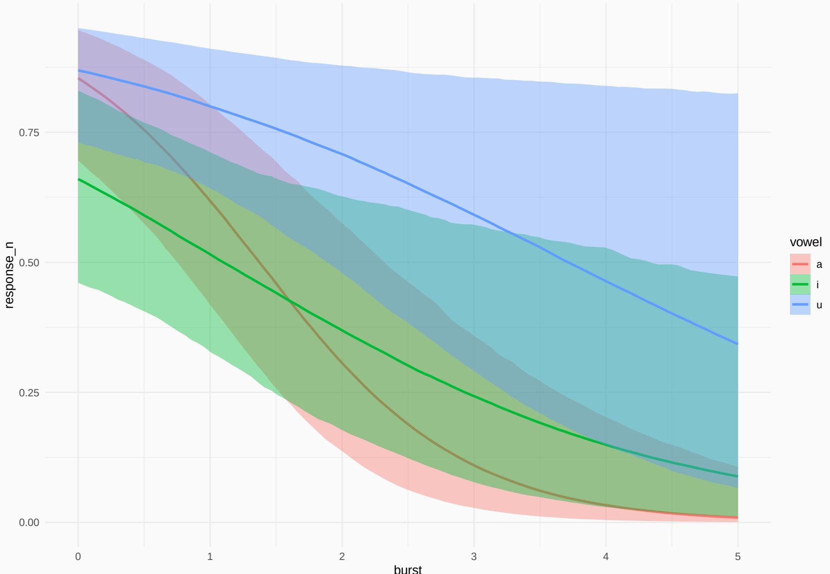
```
priors <- c(
  prior(student_t(3, 0, 2), class = Intercept),
  prior(student_t(3, 0, 2), class = b),
  prior(cauchy(0, 1), class = sd),
  prior(lkj(2), class = cor)
)
```

Binomial logistic regression

```
burst1 <- brm(
  response_n ~
    burst *
    vowel +
    (1+burst|participant),
  data = burst,
  prior = priors,
  family = bernoulli,
  file = "./cache/burst1",
  control = list(adapt_delta = 0.999)
)
```

Binomial logistic regression

```
conditional_effects(burst1, effects = "burst:vowel")
```



Inference

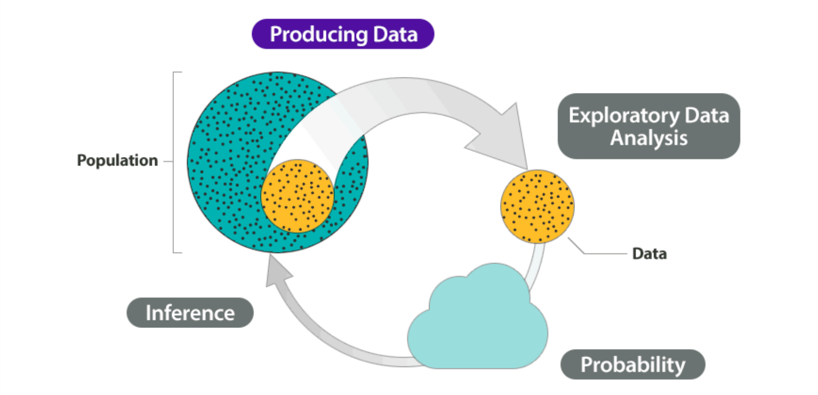
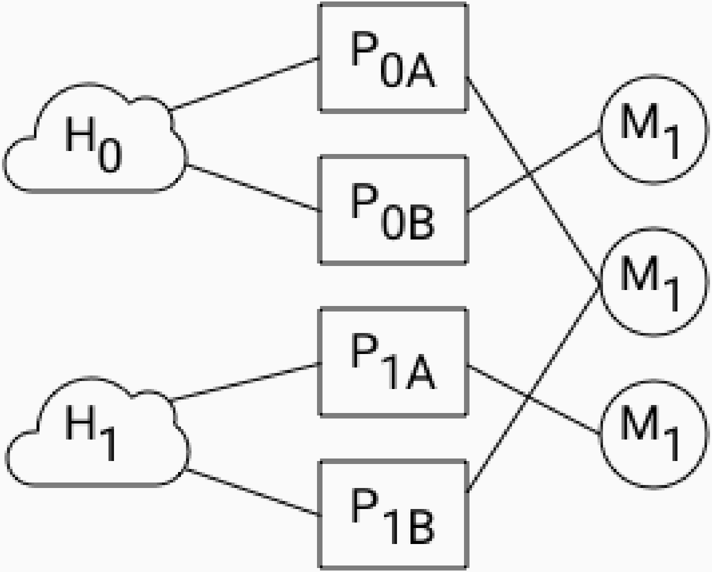


Figure 2: statistical inference

Open Learning Initiative, https://oli.cmu.edu/courses/concepts-of-statistics/ (CC BY-NC-SA 4.0)

Inference



Inference

- We want to know two things:
  - If there is **evidence for our hypothesis H** (or for the value of the parameter θ), and
  - What the **strength** of the evidence is.

Inference

- **Inferential statistics**.
- We test H against empirical data (hypothesis testing).
    - It is important to decide in advance the details of the analysis (model and prior specification among other things).
- Inference is ultimately a long-term endaveour (via accumulation of knowledge).

Inference

- Three ways of doing inference (hypothesis testing) with Bayesian statistics:
  - Inference from the **posterior**.
  - Inference using a **Region Of Practical Equivalence** (ROPE).
  - Inference using the **Bayes factor**.

Inference from the posterior

- **H**: Condition B decreases reaction times relative to Condition A.
  - You have chosen a prior which appropriately conveys the content of this H.
- **Posterior**: Condition B 95% CI = [-80, -15] ms.
- **Inference**: The posterior suggests that Condition B decreases reaction times by 15 to 80 ms at 95% confidence.

Inference from the posterior

- **H**: Condition B decreases reaction times relative to Condition A *by 100 ms*.
  - You have chosen a prior which appropriately conveys the content of this H.
- **Posterior**: Condition B 95% CI = [-80, -15] ms.
- **Inference**: The posterior suggests that Condition B decreases reaction times by a smaller amount than expected from H (15 to 80 ms at 95% confidence).

Inference with a ROPE

H0 vs H1

- H1 states that Condition B increases segment duration (alternative hypothesis), while H0 states that Condition B does not increase segment duration (null hypthesis, null effect).
  - $H_1 : \beta > 0$
  - $H_0 : \beta = 0$
- Region of Practical Equivalence (ROPE):
  - Define a region around $\beta = 0$ that practically correponds to a null effect.
    - For example: [-5, +5] ms ($-5 \geq \beta \leq +5 =$ null effect).
    - This ROPE has a width of 10 ms.
  - Choose a minimal sample size (ideally based on prospective power analyses).
  - Collect data until the 95% CI of β has width equal to or smaller than the width of the ROPE.

Inference with a ROPE

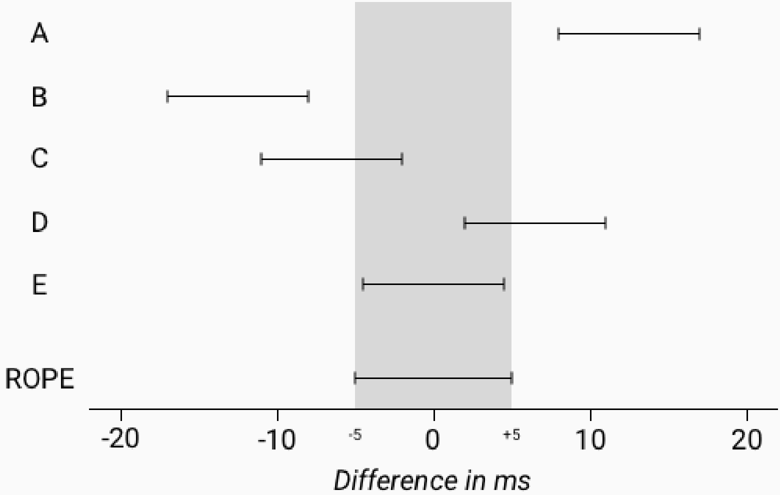


Figure 4: Possible scenarios

Bayes Factor

References

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