Introduction to Bayesian linear regression with brms

Stefano Coretta

18/01/2020

Installation

Safe method:

- **Install Rstan first**: https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started (see *Installation of Rstan*, *Checking the C++ Toolchain*, and *Configuration of the C++ Toolchain*).
    - Note that details in *Checking the C++ Toolchain* differ depending on your OS.
- **Install brms**: https://github.com/paul-buerkner/brms#how-do-i-install-brms.

Road map

- Basic concepts
- Model fitting with brms
- Bayesian inference
- BYOD clinic

Learning outcomes

- Understand the basic concepts behind Bayesian statistics and how this differs from frequentist statistics.
- Be able to set up and fit a Bayesian linear regression model using brms.
- Understand Bayesian inference methods.

BASIC CONCEPTS

Random variables

- We have a question about the world, so we collect data (sample from a population).
    - y $= (y_1, y_2, y_3, y_4, ..., y_n)$
- We want to know how the data (the sample y) was generated.
- In probability theory, data is generated by a random variable Y.

Random variables

- Y is a variable whose value is generated by a random event.
- Y is uncertain.
    - We can describe Y as a probability distribution.

Random variables

- Probability distribution.
    - A list of the values a random variable could take on along with their corresponding probability.
- Probability distributions can be expressed by a set of parameters $\Theta = (\theta_1, ..., \theta_n)$.
- Some probability distributions:
    - $Normal(\mu, \sigma)$,
    - $Binomial(n, p)$,
    - $Poisson(\lambda)$

Frequentist vs Bayesian view

- Parameters: μ, σ, p, λ, ...
- Frequentist view:
  - The parameters are **fixed** (they are unknown but certain).
  - They take on a specific value.
- Bayesian view:
  - The parameters are **random variables** (they are unkown and uncertain).
  - We describe each parameter as a probability distribution, expressed by a set of **hyperparameters**.

Continuous random variables

$$vot_i \sim Normal(\mu, \sigma)$$

Continous random variable

- We want to estimate the parameters μ and σ from the data.
    - We describe the parameters using probability distributions (defined by hyperparameters).

Continous random variable

$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(\mu_1, \sigma_1)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Priors

- We can incorporate previous knowledge (belief) about the parameters using **priors** (*prior probability distributions*).
  - We specify the hyperparameters μ_1, σ_1, x_0, γ of the prior probability distributions.
- Priors are chosen based on expert knowledge, previous studies, pilot data…
  - Priors must **not** be chosen based on the data to be analysed.

Bayesian belief update

$$prior\ probability \times evidence\ (da$$

Bayesian belief update

https://nanx.shinyapps.io/conjugate-normal-umkv/

MODEL FITTING

Italian VOT

- Voice Onset Time (VOT): Time difference (ms) between the release of a stop and the onset of voicing (vocal fold vibration).
- Toy study of Italian VOT in stops.

$$vot_i \sim Normal(\mu, \sigma)$$

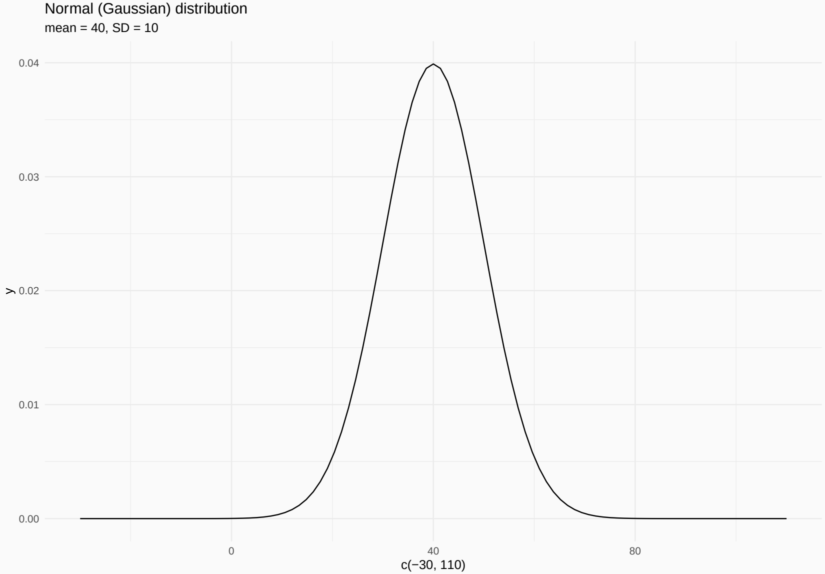
Italian VOT

- Previous literature on VOT in Italian (Esposito, 2002; Stevens & Hajek, 2010) report VOT values for voiceless stops in the range of 20–60 ms.
- We can include our knowledge with a prior for μ.
  - $Normal(40, 10)$.
  - This is a somewhat strongly informative prior.

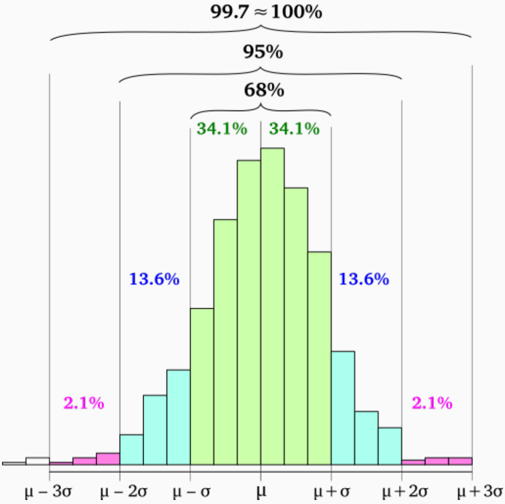
Italian VOT

```r
ggplot() +
  aes(x = c(-30, 110)) +
  stat_function(fun = dnorm, n = 101, args = list(40,
  labs(title = "Normal (Gaussian) distribution", subti
```

Italian VOT


Normal (Gaussian) distribution
mean = 40, SD = 10
0.04
0.03
0.02
0.01
0.00
y
0
40
80
c(−30, 110)

Normal prior

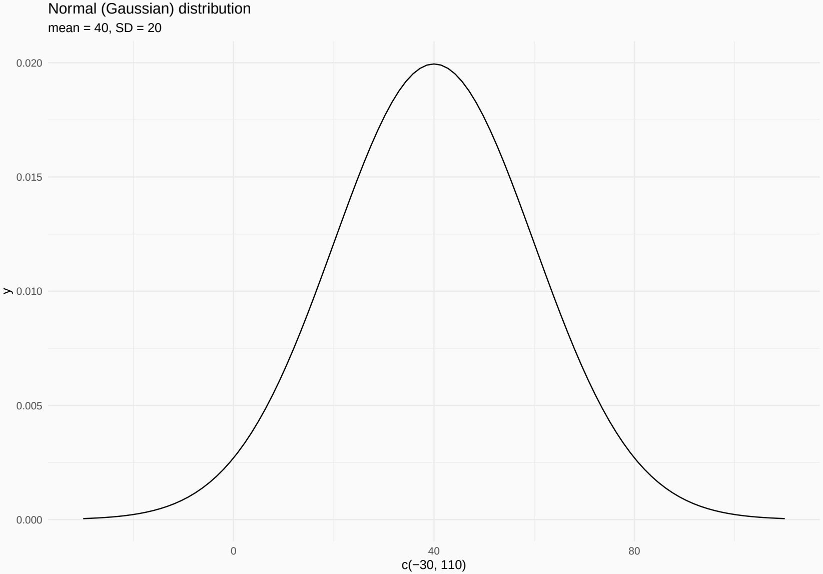


Melikamp, https://commons.wikimedia.org/wiki/File:Empirical_rule_histogram.svg (CC BY-SA 4.0)

Italian VOT

```r
ggplot() +
  aes(x = c(-30, 110)) +
  stat_function(fun = dnorm, n = 101, args = list(40, 20)) +
  labs(title = "Normal (Gaussian) distribution", subtitle = "mean = 40, SD = 20")
```

Italian VOT


Normal (Gaussian) distribution
mean = 40, SD = 20
0.020
0.015
0.010
0.005
0.000
y
0
40
80
c(−30, 110)

Italian VOT

$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(40, 10)$$

What about σ?

Italian VOT

$$vot_i \sim Normal(\mu, \sigma)$$

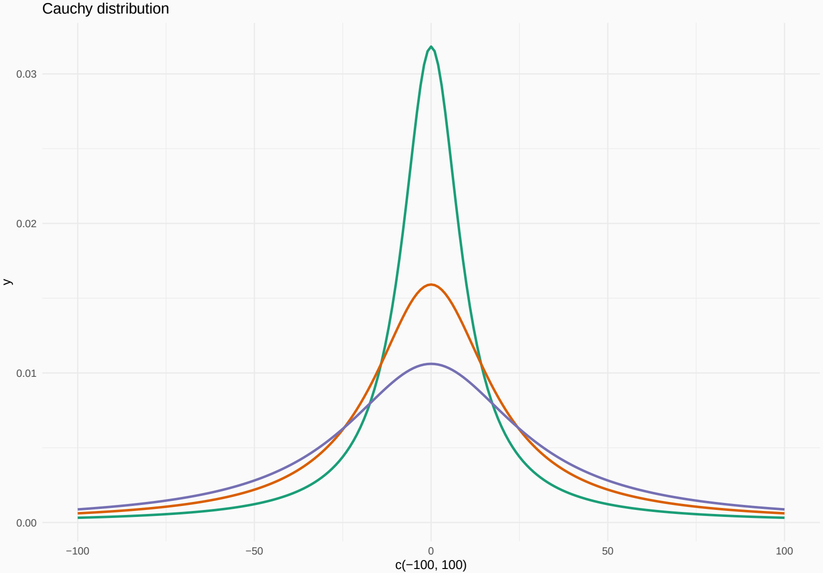
$$\mu \sim Normal(40, 10)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Cauchy prior

```
ggplot() +
  aes(x = c(-100, 100)) +
  stat_function(fun = dcauchy, n = 201, args = list(0, 10), colour = "#1b9e77", size = 1) +
  stat_function(fun = dcauchy, n = 201, args = list(0, 20), colour = "#d95f02", size = 1) +
  stat_function(fun = dcauchy, n = 201, args = list(0, 30), colour = "#7570b3", size = 1) +
  labs(title = "Cauchy distribution")
```

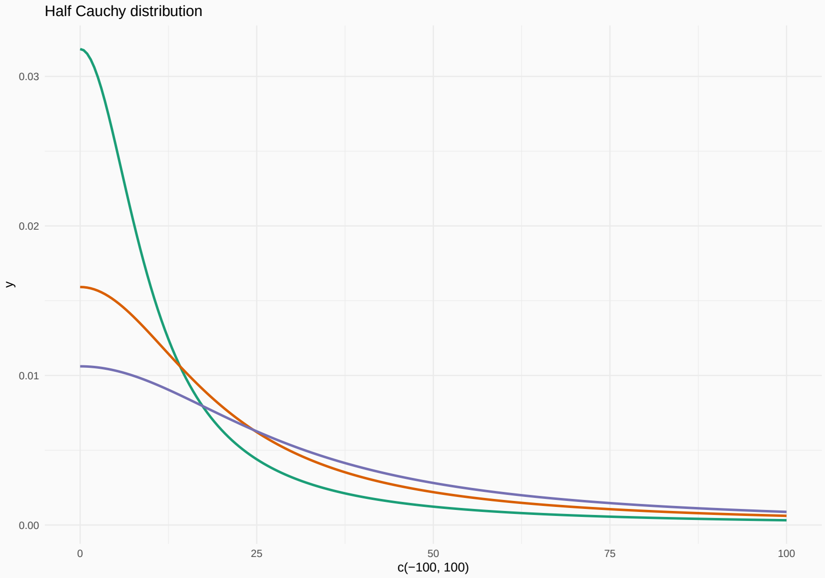
Cauchy prior



Cauchy prior

```r
ggplot() +
  aes(x = c(-100, 100)) +
  stat_function(fun = dcauchy, n = 201, args = list(0, 10), colour = "#1b9e77", size = 1) +
  stat_function(fun = dcauchy, n = 201, args = list(0, 20), colour = "#d95f02", size = 1) +
  stat_function(fun = dcauchy, n = 201, args = list(0, 30), colour = "#7570b3", size = 1) +
  labs(title = "Half Cauchy distribution") +
  xlim(0, 100)
```

Cauchy prior



Italian VOT

$$vot_i \sim Normal(\mu, \sigma)$$

$$\mu \sim Normal(40, 10)$$

$$\sigma \sim HalfCauchy(0, 10)$$

Italian VOT

- We have a model which incorporates (some of) our knowledge about VOT (through the priors for μ and σ).
- Now we want to obtain the **posterior distributions** of μ and σ.
    - The posterior distribution is the prior distribution *conditioned* on the data.
- **brms** R package: Bayesian Regression Models using Stan (Bürkner, 2018).

brms

- Stan (Stan Development Team, 2017).
  - Statistical programming language written in C++.
  - Fit Bayesian models (calculate posterior distributions).
- Calculation can be complex and/or impossible, so we take many samples from the data and from the possible parameter values to find the posterior distributions of the hyperparameters.
  - Markov Chain Monte Carlo (MCMC) sampling using the No-U-Turn sampler (NUTS).

brms

- brms translates R code into Stan code.
- Stan code is run in Stan via Rstan, an R interface to Stan.
- `brm()` function from brms.
    - lme4 syntax (`y ~ x + (1|w)`).
    - Creates a Stan model, which is compiled (in C++) and run.

brms

```r
library(brms)

vot1 <- brm(
  <model_formula>,
  <family>,
  <prior>,
  <data>,
  chains = 4,
  iter = 2000
)
```

brms

```
library(brms)

vot1 <- brm(
  vot ~ 1,
  family = gaussian(),
  <prior>,
  data = ita_egg,
  chains = 4,
  iter = 2000
)
```

Get prior

```r
get_prior(
  vot ~ 1,
  family = gaussian(),
  data = ita_egg
)
```

```
##                   prior     class coef group resp dpar nlpar bound
## 1 student_t(3, 19, 14) Intercept
## 2  student_t(3, 0, 14)     sigma
```

Prior predictive checks

```r
nsim <- 1000
nobs <- 1000

y <- matrix(rep(NA, nsim * nobs), ncol = nobs)

mu <- rnorm(nsim, 40, 10)
sigma <- rhcauchy(nsim, 10)

for (i in 1:nsim) {
  y[i,] <- rnorm(nobs, mean = mu[i], sd = sigma[i])
}
```

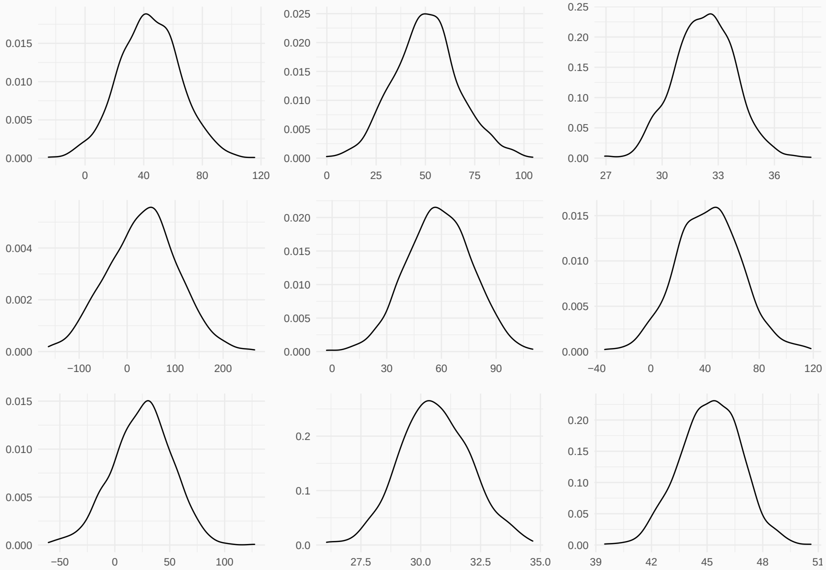
Prior predictive checks

```r
rand_sample <- sample(1:nsim, 9, replace = FALSE)
plots <- list()

j = 1

for (i in rand_sample) {
  my_data <- enframe(y[i,], name = NULL)
  plots[[j]] <- ggplot(data = my_data) +
    aes(x = value) +
    geom_density() +
    labs(x = element_blank(), y = element_blank())
  j = j + 1
}
```

Prior predictive checks



Set prior

```r
priors <- c(
  prior(normal(40, 10), class = Intercept),
  prior(cauchy(0, 10), class = sigma)
)
```

Run the model

```r
vot1 <- brm(
  vot ~ 1,
  family = gaussian(),
  prior = priors,
  data = ita_egg,
  chains = 4,
  iter = 2000,
  file = "./cache/vot1"
)
```

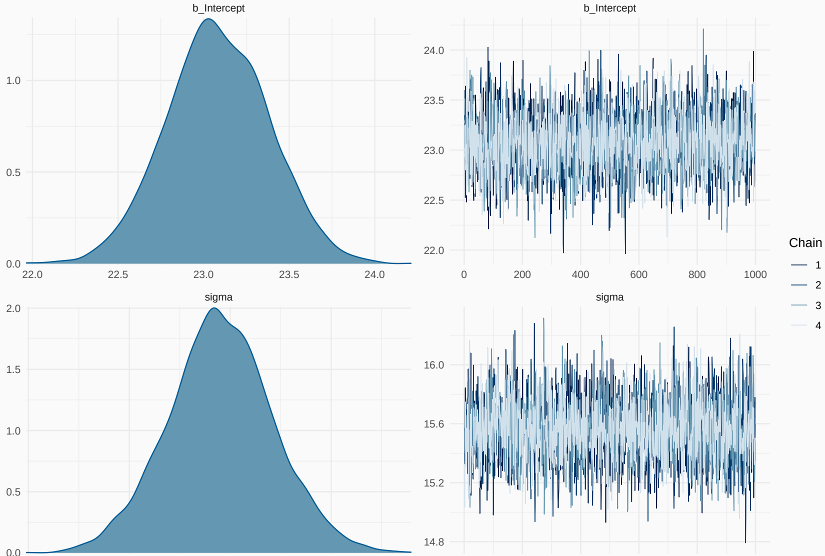
Model summary

```
vot1
```

```
##  Family: gaussian
##   Links: mu = identity; sigma = identity
## Formula: vot ~ 1
##    Data: ita_egg (Number of observations: 2624)
## Samples: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
##          total post-warmup samples = 4000
##
## Population-Level Effects:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## Intercept    23.08      0.30    22.49    23.67 1.00     3184     2635
##
## Family Specific Parameters:
##       Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## sigma    15.56      0.21    15.15    15.98 1.00     3391     2325
##
## Samples were drawn using sampling(NUTS). For each parameter, Bulk_ESS
## and Tail_ESS are effective sample size measures, and Rhat is the potential
## scale reduction factor on split chains (at convergence, Rhat = 1).
```

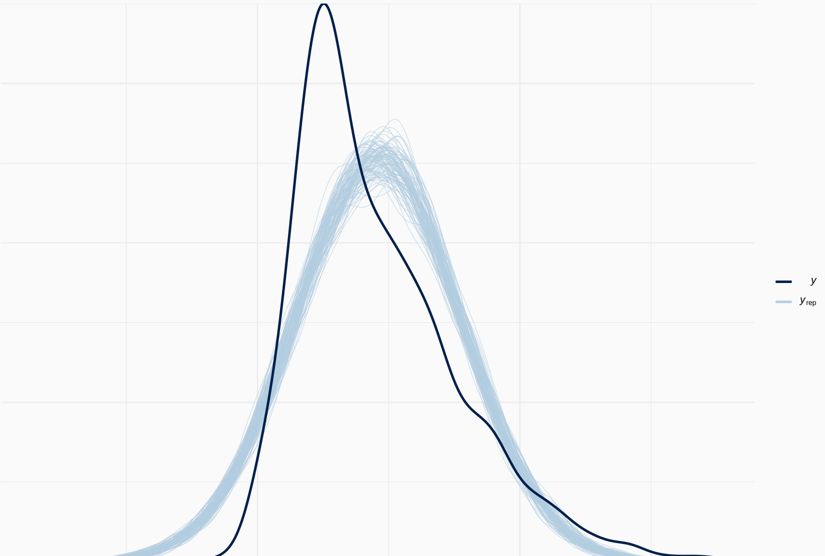
Plot model

```
plot(vot1)
```



Posterior predictive check

```r
pp_check(vot1, nsamples = 100)
```



Adding predictors

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times coronal_i + \beta_2 \times velar_i$$

$$\alpha \sim Normal(\mu_1, \sigma_1)$$

$$\beta_1 \sim Normal(\mu_2, \sigma_2)$$

$$\beta_2 \sim Normal(\mu_3, \sigma_3)$$

$$\sigma \sim HalfCauchy(x_0, \gamma)$$

Adding predictors

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \beta_1 \times coronal_i + \beta_2 \times velar_i$$

$$\alpha \sim Normal(25, 10)$$

$$\beta_1 \sim Normal(10, 10)$$

$$\beta_2 \sim Normal(20, 10)$$

$$\sigma \sim HalfCauchy(0, 10)$$

Adding predictors

```r
get_prior(
  vot ~ 1 + c1_place,
  family = gaussian(),
  data = ita_egg
)
```

```
##                   prior     class            coef group resp dpar nlpar bound
## 1                               b
## 2                               b c1_placecoronal
## 3                               b   c1_placevelar
## 4 student_t(3, 19, 14) Intercept
## 5  student_t(3, 0, 14)     sigma
```

Adding predictors

```
priors <- c(
  prior(normal(25, 10), class = Intercept),
  prior(cauchy(0, 10), class = sigma),
  prior(normal(10, 10), class = b, coef = "c1_placecoronal"),
  prior(normal(20, 10), class = b, coef = "c1_placevelar")
)
```

Adding predictors

```r
vot2 <- brm(
  vot ~ 1 + c1_place,
  family = gaussian(),
  prior = priors,
  data = ita_egg,
  chains = 4,
  iter = 2000,
  file = "./cache/vot2"
)
```

Random effects

$$vot_i \sim Normal(\mu_i, \sigma)$$

$$\mu_i = \alpha + \alpha_{speaker[i]} + (\beta_1 + \beta_{1speaker[i]}) \times coronal_i + (\beta_2 + \beta_{2speaker[i]}) \times velar_i$$

$$\begin{bmatrix} \alpha_{speaker} \\ \beta_{1speaker[i]} \\ \beta_{2speaker[i]} \end{bmatrix} \sim MVNormal(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, S)$$

Random effects

$$\alpha \sim Normal(25, 10)$$

$$\alpha_{speaker} \sim Normal(0, \sigma_{speaker})$$

$$\beta_1 \sim Normal(10, 10)$$

$$\beta_2 \sim Normal(20, 10)$$

$$\sigma_{\alpha speaker} \sim Normal(0, \sigma_{speaker})$$

$$\sigma_{\beta 1 speaker} \sim HalfCauchy(0, 10)$$

$$\sigma_{\beta 2 speaker} \sim HalfCauchy(0, 10)$$

$$\sigma_{speaker} \sim HalfCauchy(0, 10)$$

$$\sigma \sim HalfCauchy(0, 10)$$

Random effects

```r
get_prior(
  vot ~ 1 + c1_place + (1 + c1_place | speaker),
  family = gaussian(),
  data = ita_egg
)
```

```
##                   prior     class            coef   group resp dpar nlpar bound
## 1                               b
## 2                               b c1_placecoronal
## 3                               b   c1_placevelar
## 4                lkj(1)       cor
## 5                             cor                 speaker
## 6  student_t(3, 19, 14) Intercept
## 7   student_t(3, 0, 14)        sd
## 8                              sd                 speaker
## 9                              sd c1_placecoronal speaker
## 10                             sd   c1_placevelar speaker
## 11                             sd       Intercept speaker
## 12  student_t(3, 0, 14)     sigma
```

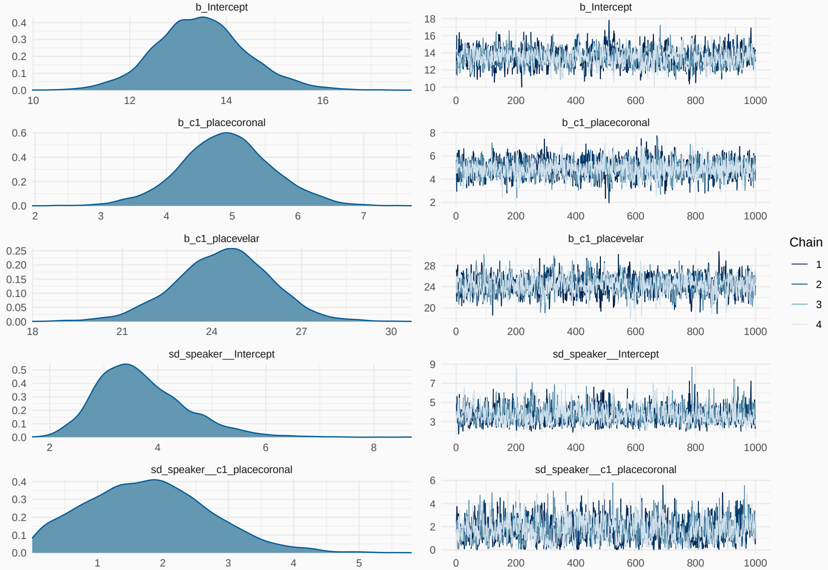
Random effects

```
priors <- c(
  prior(normal(40, 10), class = Intercept),
  prior(cauchy(0, 10), class = sigma),
  prior(normal(10, 10), class = b, coef = "c1_placecoronal"),
  prior(normal(20, 10), class = b, coef = "c1_placevelar"),
  prior(normal(0, 25), class = sd),
  prior(lkj(2), class = cor)
)
```

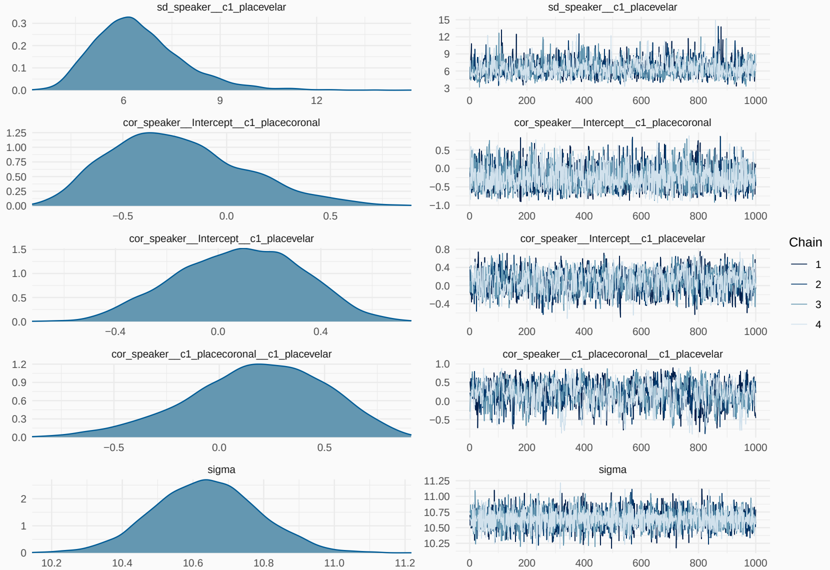
Random effects

```
vot3 <- brm(
  vot ~ 1 + c1_place + (1 + c1_place | speaker),
  family = gaussian(),
  prior = priors,
  data = ita_egg,
  chains = 4,
  iter = 2000,
  file = "./cache/vot3"
)
```

Random effects



Random effects



Random effects

```r
cat(capture.output(summary(vot3))[1:16], sep = "\n")
```

```
##  Family: gaussian
##   Links: mu = identity; sigma = identity
## Formula: vot ~ 1 + c1_place + (1 + c1_place | speaker)
##    Data: ita_egg (Number of observations: 2624)
## Samples: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
##          total post-warmup samples = 4000
##
## Group-Level Effects:
## ~speaker (Number of levels: 18)
##                                   Estimate Est.Error l-95% CI u-95% CI Rhat
## sd(Intercept)                         3.70      0.83     2.38     5.57 1.00
## sd(c1_placecoronal)                   1.78      0.94     0.16     3.77 1.00
## sd(c1_placevelar)                     6.46      1.38     4.31     9.70 1.00
## cor(Intercept,c1_placecoronal)       -0.24      0.32    -0.75     0.47 1.00
## cor(Intercept,c1_placevelar)          0.08      0.24    -0.39     0.53 1.00
## cor(c1_placecoronal,c1_placevelar)    0.18      0.32    -0.51     0.74 1.01
```

Random effects

```r
cat(capture.output(summary(vot3))[18:30], sep = "\n")
```

```
## sd(Intercept)                          1379     1918
## sd(c1_placecoronal)                     729     1246
## sd(c1_placevelar)                      1375     1881
## cor(Intercept,c1_placecoronal)         2747     2459
## cor(Intercept,c1_placevelar)           1274     2002
## cor(c1_placecoronal,c1_placevelar)      582      862
## 
## Population-Level Effects: 
##                 Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## Intercept          13.48      0.95    11.64    15.46 1.00     1081     1672
## c1_placecoronal     4.91      0.70     3.50     6.32 1.00     2610     2153
## c1_placevelar      24.38      1.61    21.14    27.40 1.00     1218     1547
```

Binomial logistic regression

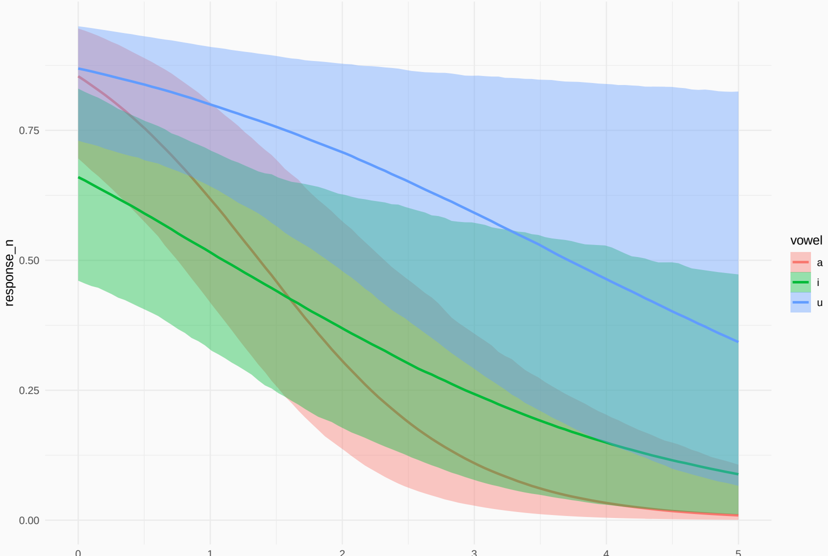
```
priors <- c(
  prior(student_t(3, 0, 2), class = Intercept),
  prior(student_t(3, 0, 2), class = b),
  prior(cauchy(0, 1), class = sd),
  prior(lkj(2), class = cor)
)
```

Binomial logistic regression

```r
burst1 <- brm(
  response_n ~
    burst *
    vowel +
    (1+burst|participant),
  data = burst,
  prior = priors,
  family = bernoulli,
  file = "./cache/burst1",
  control = list(adapt_delta = 0.999)
)
```

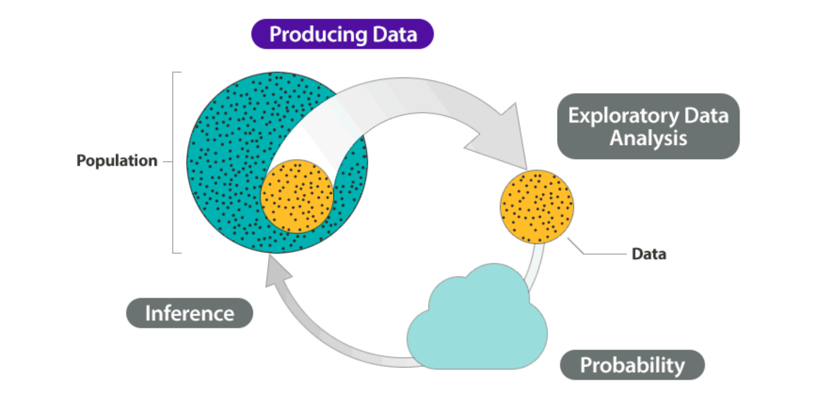
Binomial logistic regression

```
conditional_effects(burst1, effects = "burst:vowel")
```



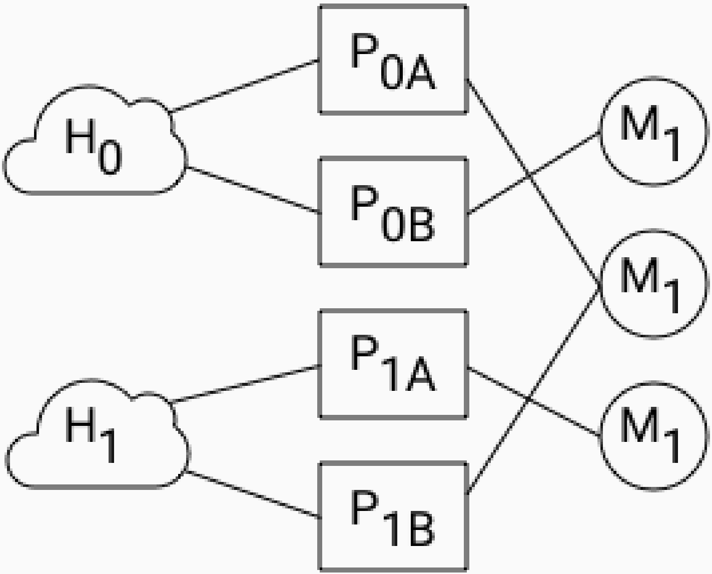
BAYESIAN INFERENCE

Inference



Open Learning Initiative, https://oli.cmu.edu/courses/concepts-of-statistics/ (CC BY-NC-SA 4.0)

Inference



Inference

- We want to know two things:
    - If there is **evidence for our hypothesis H** (or for the value of the parameter θ), and
    - What the **strength** of the evidence is.

Inference

Inferential statistics.

- We test H against empirical data (hypothesis testing).
    - It is important to decide in advance the details of the analysis (model and prior specification among other things).
- Inference is ultimately a long-term endaveour (via accumulation of knowledge).

Inference

Three ways of doing inference (hypothesis testing) with Bayesian statistics:

- Inference from the **posterior**.
- Inference using a **Region Of Practical Equivalence** (ROPE).
- Inference using the **Bayes factor**.

Inference from the posterior

1. Formulate **hypothesis**:
   - H: Condition B decreases reaction times relative to Condition A.
2. Choose **priors**:
   - $\beta = HalfNormal(0, -500)$ (plus other relevant priors).
3. Collect data.
4. Calculate **posterior**:
   - Condition B 95% CI = [-80, -15] ms.
5. **Inference**:
   - The posterior suggests that Condition B decreases reaction times by 15 to 80 ms at 95% confidence.

Inference from the posterior

1. Formulate **hypothesis**:
   - H: Condition B decreases reaction times relative to Condition A *by 100 ms*.
2. Choose **priors**:
   - $\beta = HalfNormal(47.5, 16.25)$ (plus other relevant priors).
3. Collect data.
4. Calculate **posterior**:
   - Condition B 95% CI = [-80, -15] ms.
5. **Inference**:
   - The posterior suggests that Condition B decreases reaction times by a smaller amount than expected from H (15 to 80 ms at 95% confidence).

Inference with a ROPE

H0 vs H1

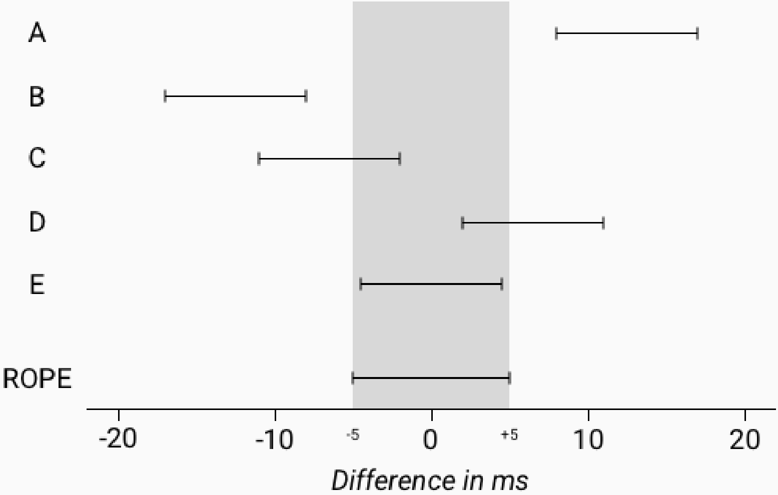
- H1 states that Condition B increases segment duration (alternative hypothesis), while H0 states that Condition B does not increase segment duration (null hypthesis, null effect).
    - $H_1 : \beta > 0$
    - $H_0 : \beta = 0$

Inference with a ROPE

Region of Practical Equivalence (ROPE):

- Define a region around $\beta = 0$ that practically correponds to a null effect.
  - For example: [-5, $+5$] ms ($-5 \geq \beta \leq +5 =$ null effect).
  - This ROPE has a width of 10 ms.
- Choose a minimal sample size (ideally based on prospective power analyses).
- Collect data until the 95% CI of β has width equal to or smaller than the width of the ROPE.
  - We focus on the estimate **precision** of β.
  - Independent from the value of β.
  - Higher precision means greater confidence in the estimated value of β.

Inference with a ROPE



Bayes Factor

The Bayes factor is the ratio of the likelihood of H1 to the likelihood of H2.

$BF_{12} = \mathcal{L}(H_1)/\mathcal{L}(H_2)$

Bayes Factor

BF	$p(M1\|D)$	evidence
1–3	0.5–0.75	weak
3–20	0.75–0.95	positive
20–150	0.95–0.99	strong
> 150	> 0.99	very strong

Bayes Factor

```r
vot3_bf <- brm(
  vot ~ 1 + c1_place + (1 + c1_place | speaker),
  family = gaussian(),
  prior = priors,
  data = ita_egg,
  chains = 4,
  iter = 20000,
  file = "./cache/vot3_bf",
  save_all_pars = TRUE
)

vot3_bf_null <- brm(
  vot ~ 1 + c1_place + (1 | speaker),
  family = gaussian(),
```

Bayes Factor

```
bayes_factor(vot3_bf, vot3_bf_null)
```

Sensitivity analysis

References

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