# Switching and PI Control of Walking Motions of Planar Biped Walkers

E. R. Westervelt, J. W. Grizzle, and C. Canudas de Wit

***Abstract*—A companion paper has addressed the problem of designing controllers that induce exponentially stable, periodic walking motions at a fixed walking rate for a planar, biped robot with one degree of underactuation. This note provides two additional control features: 1) the ability to compose such controllers to obtain walking at several discrete walking rates with guaranteed stability during the transitions; and 2) the ability to regulate the average walking rate to a continuum of values.**

***Index Terms*—Bézier curves, hybrid systems, poincaré sections, walking, zero dynamics.**

## I. Introduction

This note builds on the results in [8] and [9], which developed the notion of the hybrid zero dynamics for the walking motion of an $N$-link, planar, biped robot with one less degree of actuation than degree of motion freedom (DOF) during the single support phase. This two-dimensional, invariant sub-dynamics of the complete hybrid model of the biped robot was shown to be key to designing exponentially stabilizing controllers for walking motions. In particular, exponentially stable orbits of the hybrid zero dynamics can be rendered exponentially stable in the complete hybrid model. The Poincaré map of the hybrid zero dynamics was proven to be diffeomorphic to a scalar, linear time-invariant (LTI) system, rendering transparent the existence and stability properties of periodic orbits of the hybrid zero dynamics. A special class of output functions based on Bézier polynomials was used to simplify the computation of the hybrid zero dynamics, while at the same time inducing a convenient, finite parameterization of these dynamics. Parameter optimization was then applied to the hybrid zero dynamics to directly design a provably stable, closed-loop system which satisfied design constraints, such as walking at a given average walking rate and the forces on the support leg lying in the allowed friction cone. All of the results were illustrated on a five-link walker.

This note provides two additional features: 1) the ability to compose the above controllers to obtain walking at several discrete walking rates with guaranteed stability during the transitions; and 2) the ability to regulate the robot's average walking rate to a continuum of values.

Section III presents a method for serially composing two controllers so as to transition the robot from walking at a given fixed walking rate to another, without loss of stability. The controller design is motivated by a switching idea presented in [3]: controllers were first designed to accomplish the individual tasks of juggling, catching, and palming a ping-pong ball by a robot arm; these controllers were then sequentially composed via switching to accomplish the complex task of maneuvering the ping-pong ball in a three-dimensional workspace with an obstacle. The regions of attraction of each controller were first empirically estimated within the full state–space of the robot. Switching from one controller to another without loss of stability was then accomplished by comparing the current state of the robot to the region of attraction of the controller for the next desired task. The problem faced in this note is more challenging in that the domains of attraction of any two of the individual controllers may have empty intersection, and hence a transition controller will be required to steer the robot from the region of attraction of one controller into the region of attraction of a second, "nearby" controller.

Section IV develops an event-based PI controller to regulate walking rate to a continuum of values. The controller uses integral action to adjust the parameters in a controller that, for fixed parameter values, induces an exponentially stable, periodic orbit. Parameter adjustment takes place just after impact (swing leg touching the ground). The analysis of the controller is based on the restricted Poincaré map of the hybrid zero dynamics.

Section V illustrates how the results of Sections III and IV, when taken together, afford the construction of a feedback controller that steers the robot from a standing position, through a range of walking rates, and back to a standing position, while providing stabilization and a modest amount of robustness to disturbances and parameter mismatch between the design model and the actual robot. The results are illustrated via simulation on the five-link model studied in [4] and [6]–[8]; an animation of the robot's walking motion is available in [1].

## II. Notation and Basic Facts

This section summarizes some notation and results from[1] [8]. The reader is encouraged to read [8] for further interpretation, context and supporting diagrams. The configuration coordinates of the robot in single support (also commonly called the swing phase) are denoted by $q = (q_1, \ldots, q_N)'$, the state space is denoted by $T\mathcal{Q}$, and a control is applied at each connection of two links, but not at the contact point with the ground (i.e., no ankle torque), for a total of $(N-1)$ controls. The detailed assumptions on the robot (bipedal, planar and one less degree of actuation than degrees of freedom, point feet, rigid contact model) and the walking gait (instantaneous double support phase, no slipping nor rebound at impact, motion from left to right, symmetric gait) are given in [8, Sec. II].

The hybrid model of the robot (single support phase Lagrangian dynamics plus impact map) is expressed as a nonlinear system with impulse effects

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \qquad x^- \notin S \\ x^+ &= \Delta(x^-) \qquad x^- \in S. \end{aligned} \tag{1}$$

The impact or walking surface $S$ is defined as

$$S := \{(q, \dot{q}) \in T\mathcal{Q} \mid p_2^v(q) = 0,\ p_2^h(q) > 0\} \tag{2}$$

where $p_2^v$ and $p_2^h$ are the Cartesian coordinates of the swing leg end. The impact map $\Delta : S \to S$ computes the value of the state just after impact with $S$, $x^+ = (q^+, \dot{q}^+)$, from the value of the state just before impact, $x^- = (q^-, \dot{q}^-)$. Since the configuration coordinates necessarily involve the specification of which of the two the legs is in contact with the ground, the coordinates must be relabeled after each step to take into account the successive changing of the support leg. This is reflected in the impact map via a constant, invertible matrix $R$

$$q^+ := Rq^-. \tag{3}$$

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[1]Preliminary versions of [8] have been presented in [7] and [9].

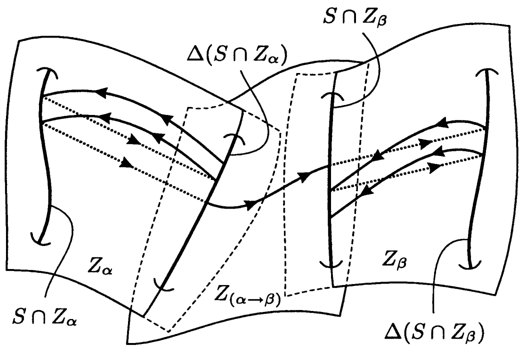


Fig. 1. Abstraction of the composition of two controllers $\Gamma_\alpha$ and $\Gamma_\beta$ via transition controller $\Gamma_{(\alpha\to\beta)}$. Under the action of $\Gamma_\alpha$ the dynamics evolve on $Z_\alpha$. Switching to $\Gamma_{(\alpha\to\beta)}$ when the state enters $\Delta(S \cap Z_\alpha)$ causes the dynamics to evolve along $Z_{(\alpha\to\beta)}$ to $S \cap Z_\beta$. Switching to $\Gamma_\beta$ when the state enters $S \cap Z_\beta$ causes the dynamics to evolve on $Z_\beta$.

The control design involves the choice of a set of holonomic constraints that are asymptotically imposed on the robot via feedback control. This is accomplished by interpreting the constraints as output functions depending only on the configuration variables of the robot, and then combining ideas from finite-time stabilization and computed torque. The outputs $y \in I\!R^{N-1}$ are chosen as

$$y = h(q,\alpha) = H_0 q - h_d(\theta(q),\alpha) \tag{4}$$

with terms defined as follows.

1) $H_0$ is an $(N-1)\times N$ matrix of real coefficients specifying what is to be controlled.
2) $\theta(q) := cq$, where $c$ is a $1 \times N$ row vector of real coefficients, is a scalar function of the configuration variables and should be chosen so that it is monotonically increasing along a step of the robot ($\theta(q)$ is playing the role of time). Define $\theta^+ = cq^+$ and $\theta^- = cq^-$ to be the initial and final values of $\theta$, respectively, along a step.
3) Normalization of $\theta$ to take values between zero and one

$$s(q) := \frac{\theta(q) - \theta^+}{\theta^- - \theta^+}. \tag{5}$$

4) Bézier polynomials of order $M \geq 3$

$$b_i(s) := \sum_{k=0}^{M} \alpha_k^i \frac{M!}{k!(M-k)!} s^k (1-s)^{M-k}. \tag{6}$$

5) For $\alpha_k^i$ as before define the $(N-1)\times 1$ column vector $\alpha_k := \left(\alpha_k^1, \ldots, \alpha_k^{N-1}\right)'$ and the $(N-1)\times(M+1)$ matrix $\alpha := [\alpha_0, \ldots, \alpha_M]$.
6)

$$h_d(\theta(q),\alpha) := \begin{bmatrix} b_1 \circ s(q) \\ \vdots \\ b_{N-1} \circ s(q) \end{bmatrix}. \tag{7}$$

The matrix of parameters $\alpha$ is said to be a regular parameter of output (4) if the output satisfies [8, Sect. III.A, HH1–HH4] and [8, Sect. III.B, HH5], which together imply the invertibility of the decoupling matrix and the existence of a two-dimensional, smooth, zero dynamics associated with the single support phase of the robot. Let $Z_\alpha$ be the (swing phase) zero dynamics manifold. Let $\Gamma_\alpha$ be any feedback satisfying [8, Sec. III.C, CH2–CH5] so that $Z_\alpha$ is invariant under the swing phase dynamics in closed loop with $\Gamma_\alpha$ and is locally finite-time attractive otherwise. Note that standard results imply that $\Gamma_\alpha|_{Z_\alpha} = -(L_g L_f h)^{-1} L_f^2 h$ [5] and, thus: 1) $\Gamma_\alpha|_{Z_\alpha}$ is uniquely determined by the choice of parameters used in the output and is completely independent of the choice of feedback used to drive the constraints to zero in finite time; and 2) even though $\Gamma_\alpha$ is necessarily not smooth, $\Gamma_\alpha|_{Z_\alpha}$ is as smooth as the robot model.

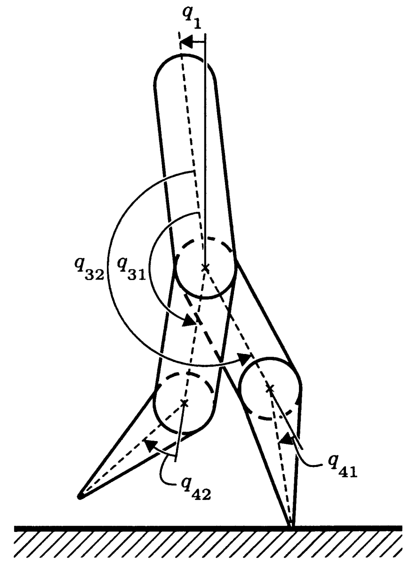


Fig. 2. Schematic of five-link model studied.

For a regular parameter value of output (4), $\alpha$, the definition of the outputs and basic properties of Bézier polynomials yield a very simple characterization of $S \cap Z_\alpha$, the configuration and velocity of the robot at the end of a phase of single support. Define

$$q_\alpha^- = H^{-1} \begin{bmatrix} \alpha_M \\ \theta_\alpha^- \end{bmatrix} \tag{8}$$

$$\omega_\alpha^- = H^{-1} \begin{bmatrix} \frac{M}{\theta_\alpha^- - \theta_\alpha^+}(\alpha_M - \alpha_{M-1}) \\ 1 \end{bmatrix} \tag{9}$$

where $H := [H_0' \quad c]'$, and the initial and final values of $\theta$ corresponding to this output are denoted by $\theta_\alpha^+$ and $\theta_\alpha^-$, respectively. Then $S \cap Z_\alpha = \left\{(q_\alpha^-, \dot{q}_\alpha^-) \mid \dot{q}_\alpha^- = a\,\omega_\alpha^-, a \in I\!R\right\}$ and is determined by the *last two columns* of the parameter matrix $\alpha$. In a similar fashion $\Delta(S \cap Z_\alpha)$, the configuration, $q_\alpha^+$, and velocity, $\dot{q}_\alpha^+$, of the robot at the beginning of a subsequent phase of single support, may be simply characterized and are determined by the *first two columns* of the parameter matrix $\alpha$.

Let $\beta$ also be a regular parameter value of output (4). Then, using arguments almost identical to those in the proof of [8, Th. 4], it follows that

$$\begin{bmatrix} \beta_0 \\ \theta_\beta^+ \end{bmatrix} = HRH^{-1} \begin{bmatrix} a_M \\ \theta_\alpha^- \end{bmatrix} \tag{10}$$

implies $h(\cdot,\beta) \circ \Delta|_{(S\cap Z_\alpha)} = 0$, while, if $\dot{q}_\alpha^+ := \Delta_{\dot{q}}(q_\alpha^-)\,\omega_\alpha^-$, results in $c\dot{q}_\alpha^+ \neq 0$, then

$$\beta_1 = \frac{\theta_\beta^- - \theta_\alpha^+}{Mc\dot{q}_\alpha^+} H_0 \dot{q}_\alpha^+ + \alpha_0 \tag{11}$$

implies $L_f h(\cdot,\beta) \circ \Delta|_{(S\cap Z_\alpha)} = 0$. The key thing to note is that these two conditions involve, once again, only the *first two columns* of the parameter matrix $\beta$. In a similar fashion the *last two columns* of the parameter matrix $\beta$ may be chosen so that $h(\cdot,\beta)|_{(S\cap Z_\alpha)} = 0$, and $L_f h(\cdot,\beta)|_{(S\cap Z_\alpha)} = 0$.

Taking $\beta = \alpha$, conditions (10) and (11) imply that $\Delta(S \cap Z_\alpha) \subset Z_\alpha$, in which case $Z_\alpha$ is then controlled-invariant for the full hybrid model of the robot. The resulting restriction dynamics is called the *hybrid zero dynamics*. Necessary and sufficient conditions can be given for the hybrid zero dynamics to admit an exponentially[2] stable, periodic orbit, $\mathcal{O}_\alpha$, [8]. When these conditions are met, the matrix of parameters $\alpha$ is said to give rise to an exponentially stable walking motion. Under controller $\Gamma_\alpha$, the exponentially stable orbit in the hybrid zero dynamics is also exponentially stable in the full-order model, (1). The domain of attraction of $\mathcal{O}_\alpha$ in the full dimensional model cannot be easily estimated; however, its domain of attraction intersected with $S \cap Z_\alpha$, that is, the domain of attraction of the associated fixed-point of the restricted Poincaré map, $\rho_\alpha : S \cap Z_\alpha \to S \cap Z_\alpha$, is computed analytically in [8, Sec. IV]; denote this set by $\mathcal{D}_\alpha$, which is a subset of $S \cap Z_\alpha$.

Finally, define the average walking rate over a step,[3] $\bar{\nu}$ (m/s), to be step length (m) divided by the elapsed time of a step (s). Since the controllers employed are not smooth, $\bar{\nu} : S \to I\!R$ is not a smooth function of the states. However, if $\alpha$ is a regular parameter value of output (4) giving rise to a hybrid zero dynamics, $\Delta(S \cap Z_\alpha) \subset Z_\alpha$, then $\bar{\nu}$ restricted to $S \cap Z_\alpha$ depends smoothly on the states and the parameter values $\alpha$ used to define the outputs, (4); an explicit formula for $\bar{\nu}$ is given in [8, Sec. VI].

## III. Provably Stable Composition of Walking Motions

Let $\alpha$ and $\beta$ be two regular sets of parameters of output (4) with corresponding zero dynamics manifolds, $Z_\alpha$ and $Z_\beta$. Suppose that $\Delta(S \cap Z_\alpha) \subset Z_\alpha$ and $\Delta(S \cap Z_\beta) \subset Z_\beta$, and that there exist exponentially stable periodic orbits,[4] $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$; denote the corresponding controllers by $\Gamma_\alpha$ and $\Gamma_\beta$. The goal is to be able to transition from $\mathcal{O}_\alpha$ to $\mathcal{O}_\beta$ without the robot falling (i.e., with stability guaranteed). If it were known that the domains of attraction of the two orbits had a nonempty intersection, then the method of [3] could be applied directly. Numerically evaluating the domains of attraction on the full-order model is unpleasant, so another means of assuring a stable transition is sought that is based on easily computable quantities, the domains of attraction of the restricted Poincaré maps associated with $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$.

Since, in general, $Z_\alpha \cap Z_\beta = \emptyset$, the method for providing a stable transition from $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ will be to introduce a one-step transition controller $\Gamma_{(\alpha\to\beta)}$ whose (swing phase) zero dynamics manifold $Z_{(\alpha\to\beta)}$ connects the zero dynamics manifolds $Z_\alpha$ and $Z_\beta$ (see Fig. 1). More precisely, switching will be synchronized with impact events and the zero dynamics manifold $Z_{(\alpha\to\beta)}$ will be chosen to map exactly from the one-dimensional manifold $\Delta(S \cap Z_\alpha)$ (i.e., the state of the robot just after impact with $S$ under controller $\Gamma_\alpha$) to the one-dimensional manifold $S \cap Z_\beta$ (i.e., the state of the robot just before impact with $S$ under controller $\Gamma_\beta$). The one-step transition controller $\Gamma_{(\alpha\to\beta)}$ differs from a deadbeat controller in that $\Gamma_{(\alpha\to\beta)}$ takes *all points* in a subset of manifold $\Delta(S \cap Z_\alpha)$ into a subset of the manifold $S \cap Z_\beta$ as opposed to a deadbeat controller that would map a subset of $\Delta(S \cap Z_\alpha)$ to a *point* in $S \cap Z_\beta$. The design of multistep transition controllers is also possible but not addressed here.

[2]Note that finite-time stabilization is used only to constrain $(N-1)$ of the $N$ degrees of freedom while the stability properties of the uncontrolled degree of freedom is determined by the resulting closed-loop system dynamics.

[3]A step starts with the swing leg on the ground and behind the robot and ends with the swing leg on the ground and in front of the robot.

[4]Typically, these would correspond to walking at different average walking rates.

From (8)–(11), any zero dynamics manifold $Z_{(\alpha\to\beta)}$ with parameters

$$\begin{aligned}
(\alpha\to\beta)_0 &= \alpha_0\\
(\alpha\to\beta)_1 &= \alpha_0 - \frac{\theta_\beta^- - \theta_\alpha^+}{\theta_\alpha^- - \theta_\alpha^+}(\alpha_0 - \alpha_1)\\
(\alpha\to\beta)_{M-1} &= \beta_M + \frac{\theta_\beta^- - \theta_\alpha^+}{\theta_\beta^- - \theta_\beta^+}(\beta_{M-1} - \beta_M)\\
(\alpha\to\beta)_M &= \beta_M\\
\theta^+_{(\alpha\to\beta)} &= \theta^+_\alpha\\
\theta^-_{(\alpha\to\beta)} &= \theta^-_\beta
\end{aligned}\tag{12}$$

satisfies $Z_{(\alpha\to\beta)} \cap \Delta(S \cap Z_\alpha) = \Delta(S \cap Z_\alpha)$ and $\Delta(S \cap Z_{(\alpha\to\beta)}) = \Delta(S \cap Z_\beta)$. The choice of the intermediate parameter values, $(\alpha\to\beta)_i$, $i = 2$ to $M-2$ affects the walking motion, and one could choose their values through optimization, for example, to minimize the torques required to evolve along the surface $Z_{(\alpha\to\beta)}$. However, the simple choice

$$(\alpha\to\beta)_i = \frac{\alpha_i + \beta_i}{2}, \qquad i = 2 \text{ to } M-2 \tag{13}$$

has proven effective in examples worked by the authors. The reason for this seems to be intimately linked the use of Bézier polynomials; see [8, Sec. V.B].

Assume that the parameter matrix given in (12) and (13) is regular and let $\Gamma_{(\alpha\to\beta)}$ be an associated controller; then $\Gamma_{(\alpha\to\beta)}|_{Z_{(\alpha\to\beta)}}$ is uniquely determined by the matrix of parameters $(\alpha\to\beta)$. The goal now is to determine under what conditions $\Gamma_{(\alpha\to\beta)}$ will effect a transition from the region of attraction of $\mathcal{O}_\alpha$ to the region of attraction of $\mathcal{O}_\beta$.

Let $P_{(\alpha\to\beta)} : S \to S$ be the Poincaré return map of the model (1) in closed loop with $\Gamma_{(\alpha\to\beta)}$ and consider $P_{(\alpha\to\beta)}|_{(S\cap Z_\alpha)}$. By construction of $Z_{(\alpha\to\beta)}$, $\Delta(S \cap Z_\alpha) \subset Z_{(\alpha\to\beta)}$. Since $Z_{(\alpha\to\beta)}$ is invariant under $\Gamma_{(\alpha\to\beta)}$, it follows that $P_{(\alpha\to\beta)}(S \cap Z_\alpha) \subset S \cap Z_{(\alpha\to\beta)}$. But by construction, $S \cap Z_{(\alpha\to\beta)} = S \cap Z_\beta$. Thus, the restriction of the Poincaré return map to $S \cap Z_\alpha$ induces a (partial) map

$$\rho_{(\alpha\to\beta)} : S \cap Z_\alpha \to S \cap Z_\beta. \tag{14}$$

In [8, Sec. IV.A], a closed-form expression for $\rho_{(\alpha\to\beta)}$ is computed on the basis of the two-dimensional zero dynamics associated with $Z_{(\alpha\to\beta)}$.

Let $\mathcal{D}_\alpha \subset S \cap Z_\alpha$ and $\mathcal{D}_\beta \subset S \cap Z_\beta$ be the domains of attraction of the restricted Poincaré maps $\rho_\alpha : S\cap Z_\alpha \to S\cap Z_\alpha$ and $\rho_\beta : S\cap Z_\beta \to S \cap Z_\beta$ associated with the orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, respectively. (Since the existence of exponentially stable, periodic orbits has been assumed, these domains are nonempty and open). It follows that $\rho^{-1}_{(\alpha\to\beta)}(\mathcal{D}_\beta)$ is precisely the set of states in $S \cap Z_\alpha$ that can be steered into the domain of attraction of $\mathcal{O}_\beta$ under the control law $\Gamma_{(\alpha\to\beta)}$. In general, from stability considerations, one is more interested in $\mathcal{D}_\alpha \cap \rho^{-1}_{(\alpha\to\beta)}(\mathcal{D}_\beta)$, the set of states in the domain of attraction of $\mathcal{O}_\alpha$ that can be steered into the domain of attraction of $\mathcal{O}_\beta$ in one step under the control law $\Gamma_{(\alpha\to\beta)}$.

*Theorem 1 (Serial Composition of Stable Walking Motions):* Assume that $\alpha$ and $\beta$ are regular parameters of output (4) and that $(\alpha\to\beta)$ defined by (12) and (13) is also regular. Suppose furthermore that

1) $\Delta(S \cap Z_\alpha) \subset Z_\alpha$ and $\Delta(S \cap Z_\beta) \subset Z_\beta$;

TABLE I

APPLIED CONTROL (NOTES: S.P. IS STANDING POSITION, AND $A \rightarrow B$ INDICATES A TRANSITION CONTROLLER FROM A CONTROLLER AT AVERAGE WALKING RATE $A$ TO AVERAGE WALKING RATE $B$)

| Step num. | Transition Controller |
|---|---|
| 1 | S.P. $\rightarrow$ 0.45 |
| 4 | 0.45 $\rightarrow$ 0.75 |
| 17 | 0.75 $\rightarrow$ 0.55 |
| 22 | 0.55 $\rightarrow$ 0.35 |
| 27 | 0.35 $\rightarrow$ 0.25 |

| Step nums. | Controller | $\eta^*$ $(m/s)$ |
|---|---|---|
| 2–3 | 0.45 | 0.45 |
| 5–16 | 0.75 | 0.8 |
| 18–21 | 0.55 | 0.55 |
| 23–26 | 0.35 | 0.35 |
| 28 | 0.25 | 0.00 |

TABLE II

EXAMPLE COMPOSITION STEP STATISTICS: STARTING FROM A STANDING POSITION, INCREASING AVERAGE WALKING RATE, $\bar{\nu}$, TO 0.8 M/S, THEN DECREASING $\bar{\nu}$ TO 0 M/S, A STOP

| Step num. | $T_I$ $(s)$ | $\bar{\nu}$ $(m/s)$ |
|---|---|---|
| 1 | 0.619 | 0.516 |
| 2 | 1.199 | 0.266 |
| 3 | 0.875 | 0.365 |
| 4 | 0.915 | 0.425 |
| 5 | 0.692 | 0.562 |
| 6 | 0.594 | 0.655 |
| 7 | 0.535 | 0.727 |
| 8 | 0.510 | 0.763 |
| 9 | 0.498 | 0.781 |
| 10 | 0.492 | 0.791 |
| 11 | 0.489 | 0.796 |
| 12 | 0.487 | 0.798 |
| 13 | 0.486 | 0.800 |
| 14 | 0.486 | 0.800 |
| 15 | 0.486 | 0.801 |

| Step num. | $T_I$ $(s)$ | $\bar{\nu}$ $(m/s)$ |
|---|---|---|
| 16 | 0.486 | 0.801 |
| 17 | 0.441 | 0.775 |
| 18 | 0.478 | 0.715 |
| 19 | 0.520 | 0.657 |
| 20 | 0.569 | 0.600 |
| 21 | 0.609 | 0.561 |
| 22 | 0.528 | 0.499 |
| 23 | 0.535 | 0.492 |
| 24 | 0.584 | 0.450 |
| 25 | 0.656 | 0.401 |
| 26 | 0.726 | 0.362 |
| 27 | 0.659 | 0.318 |
| 28 | 0.642 | 0.327 |
| 29 | 0.690 | 0.272 |
| 30 | 0.875 | 0.000 |

2) there exist exponentially stable, periodic orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ in $Z_\alpha$ and $Z_\beta$, respectively, so that the domains of attraction $\mathcal{D}_\alpha \subset S \cap Z_\alpha$ and $\mathcal{D}_\beta \subset S \cap Z_\beta$ of the associated restricted Poincaré maps are nonempty and open.

Then the set of states in $\mathcal{D}_\alpha$ that can be steered into $\mathcal{D}_\beta$ in one step under any control law $\Gamma_{(\alpha\rightarrow\beta)}$ satisfying [8, Sec. IV.C, CH2–CH5] is equal to $\mathcal{D}_\alpha \cap \rho^{-1}_{(\alpha\rightarrow\beta)}(\mathcal{D}_\beta)$. □

*Proof:* This follows directly from the definition of $\rho_{(\alpha\rightarrow\beta)}$. ■

An example is given in Section V.

## IV. EVENT-BASED PI CONTROL OF THE AVERAGE WALKING RATE

Section III demonstrated how to achieve walking at several *discrete* walking rates through a switching law and a one-step transition controller. The goal here is to design an event-based controller[5] that adjusts the parameters in the output (4) to achieve walking at a *continuum* of rates. The controller design is based, once again, on the hybrid zero dynamics.

Let $\alpha$ be a regular parameter value of output (4) for which there exists an exponentially stable periodic orbit. Let $z^*_\alpha$ be the corresponding fixed point of the restricted Poincaré map, $\rho_\alpha : S \cap Z_\alpha \rightarrow S \cap Z_\alpha$. To emphasize the dependence on the parameter value, for $z \in S \cap Z_\alpha$, let $\rho(z,\alpha) := \rho_\alpha(z)$.

Suppose there exists at least one pair of indices $(k,i)$ such that $\partial\bar{\nu}\circ\rho(z^*_\alpha,\alpha)/\partial\alpha^i_k \neq 0$, with $2 \leq k \leq M-2$. Then there exists $\delta\alpha \in I\!R^{(N-1)\times(M+1)}$ such that

$$(\delta\alpha)_0 = (\delta\alpha)_1 = (\delta\alpha)_{M-1} = (\delta\alpha)_M = 0 \tag{15}$$

[5]That is, one that acts step-to-step with updates occurring at impacts.

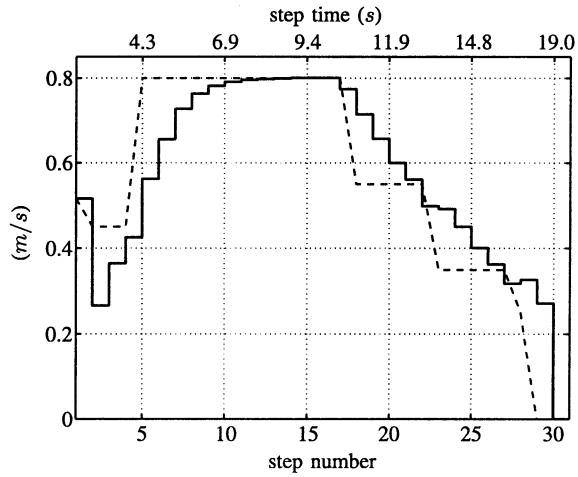


Fig. 3. Command (dashed) versus actual (solid) average walking rate (Note: the sloped portions of the command curve correspond to transition controllers, and average walking rate is computed at the end of each step).

$$\sum_{i=1}^{N-1}\sum_{k=0}^{M}\delta\alpha^i_k \frac{\partial\bar{\nu}\circ\rho(z^*_\alpha,\alpha)}{\partial\alpha^i_k} \neq 0. \tag{16}$$

For $w \in I\!R$ sufficiently small in magnitude, $\alpha + w\delta\alpha$ is also regular. From (15)

$$\begin{aligned} S\cap Z_{\alpha+w\delta\alpha} &= S\cap Z_\alpha \\ \Delta(S\cap Z_{\alpha+w\delta\alpha}) &= \Delta(S\cap Z_\alpha). \end{aligned} \tag{17}$$

Thus, $\rho_{\alpha+w\delta\alpha} : S\cap Z_\alpha \rightarrow S\cap Z_\alpha$, and the following dynamic system can be defined:

$$\begin{aligned} z(k+1) &= \rho(z(k), \alpha + w(k)\delta\alpha) \\ \eta(k) &= \bar{\nu}(z(k)) \end{aligned} \tag{18}$$

with one-dimensional state–space $S\cap Z_\alpha$, input $w \in I\!R$ and output equal to average walking rate, $\eta = \bar{\nu} \in I\!R$. Moreover, the equilibrium point $z^*_\alpha$ for $w = 0$ is exponentially stable since the periodic orbit is exponentially stable, while the condition (16) implies that the linearization of (18) about the equilibrium has a well-defined relative degree equal to one. Therefore, by standard arguments, the following holds.

*Theorem 2 (Event-Based PI Control of Average Walking Rate):* Let $\alpha$ be a regular parameter value for which there exists an exponentially stable periodic orbit in $Z_\alpha$. Assume there exists $\delta\alpha$ satisfying (15) and (16). Then, average walking rate can be regulated via PI control. In particular, there exist $\epsilon > 0$, and scalars $K_P$ and $K_I$ such that for all $\eta^*$ such that $|\eta^* - \bar{\nu}(z^*_\alpha)| < \epsilon$, the system with proportional plus integral control

$$\begin{aligned} z(k+1) &= \rho(z(k), \alpha + w(k)\delta\alpha) \\ \eta(k) &= \bar{\nu}(z(k)) \\ e(k+1) &= e(k) + (\eta^* - \eta(k)) \\ w(k) &= K_p(\eta^* - \eta(k)) + K_I e(k) \end{aligned} \tag{19}$$

has an exponentially stable equilibrium, and thus, when initialized sufficiently near the equilibrium, $\lim_{k\rightarrow\infty}(\eta^* - \eta(k)) = 0$. □

The previous controller is realized on the full-hybrid model as follows:

$$\left.\begin{aligned} \dot{x} &= f(x) + g(x)\Gamma_{\alpha+w\delta\alpha} \\ \dot{e} &= 0 \\ \dot{w} &= 0 \end{aligned}\right\} \quad x^- \notin S$$
$$\left.\begin{aligned} x^+ &= \Delta(x^-) \\ e^+ &= e^- + (\eta^* - \bar{\nu}(x^-)) \\ w^+ &= K_p(\eta^* - \bar{\nu}(x^-)) + K_I e^- \end{aligned}\right\} \quad x^- \in S. \tag{20}$$

## V. Example: Starting and Ending From a Standing Position

Sections III and IV demonstrated a means to achieve walking at several discrete walking rates and at a continuum of rates. This section summarizes the application of these techniques to the five-link model studied in [4], [6], and [8] (see Fig. 2) and gives a simple technique for starting and another for stopping the robot from a standing position. For reasons of space the details of the model are omitted and the reader is referred to [8]. This example will illustrate the following: the robot will start from a standing position, increase in walking rate to approximately 0.8 m/s, and then slow to a stop.

Initiating walking from a stable standing position—defined as a configuration where $\dot{q} = 0$, both leg ends are on the ground, and where the projection of the robot's center of mass is between the end of the legs—was accomplished by moving the projection of the robot's center of mass in front of what will become the stance leg—thus, moving the robot to an unstable standing position—while not violating the no-slip condition for that leg. This task can be accomplished using traditional control techniques as the system is fully actuated (in fact, it is over actuated). To that end, simple, hand-crafted joint trajectories that move the torso and hips forward were tracked at each joint. When the ground reaction forces became zero on what will become the swing leg, the control was switched from joint trajectory tracking to a transition controller of the kind designed in Section III. The transition controller steers the robot from the unstable standing position to the domain of attraction of an exponentially stable controller.

During the application of each exponentially stable controller, event-based PI control was applied to hasten convergence to the desired walking rate. The transition from one exponentially stable controller to another was done with a transition controller designed according to (12) and (13).

To transition from walking to a stable standing position, event-based PI control was used on a controller with a slow average walking rate, 0.25 m/s, with a set-point of $\eta^* = 0$. Using this technique slowed the robot until it did not have enough energy to make a step, thus stopping the robot.[6]

Table I lists which controllers were applied during which steps. Note that 1) in both Tables I and II, the step-initiation controller (based on tracking) that was applied just prior to the first transition controller is not listed; 2) in Table I, the exponentially stable controllers are identified by their average walking rate fixed points in m/s; and 3) the event-based PI control was only used during the application of exponentially stable controllers and not during the application of transition controllers. Adding event-based PI control increased the rate of convergence to the respective controller's fixed point while preserving exponential stability. This idea is similar to that presented in [2], but is more general in its approach.

The left half of Table I lists the transition controllers that were applied while the right half lists the exponentially stable controllers and the desired fixed points, $\eta^*$'s, used for the PI control. In each application of PI control, the gains were chosen to be $K_p = -1.7$ and $K_I = 1$. Finite differences were used to compute $\partial\bar{\nu} \circ \rho(z_\alpha^*, \alpha)/\partial\alpha_k^i$ for several values of $i$ and $k$. In this way, it was determined that adjusting the angle of the swing leg femur during mid-step would have a sufficiently strong effect on the average walking speed (this corresponded to $i = 2$ and $k = 3$). Hence, $\delta\alpha$ was chosen to be all zeros with the exception of $\delta\alpha_3^2$ which was set to 1. Each exponentially stable controller was initialized with $e^-$ of (20) set to zero (i.e., $e^- = 0$ on steps 2, 5, 18, etc.).

Table II lists the time to impact (step duration), $T_I$, for each step and average velocity, $\bar{\nu}$, for each step. The robot begins in an unstable standing position at the start of step one; increases in average walking rate to a maximum of approximately 0.8 m/s on the 16th step; and then is slowed to a stop on the thirtieth step, where the robot begins to step forward but then rocks backward to a stop. The peak torque for this example is 63 Nm, less than half of the 150 Nm that is possible with the motors and gearing of the robot studied in [4], [6], [8]. Fig. 3 gives the commanded versus actual average walking rate. For an animation of this example as well as additional supporting plots, see [1].

[6]In fact, due to rigid impacts, the robot will continue to rock back and forth, alternating impacts with each leg, and decreasing the robot's kinetic energy with each impact.

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