Swing-Up of the Pendubot/Acrobot

Final Presentation for UR-19/20

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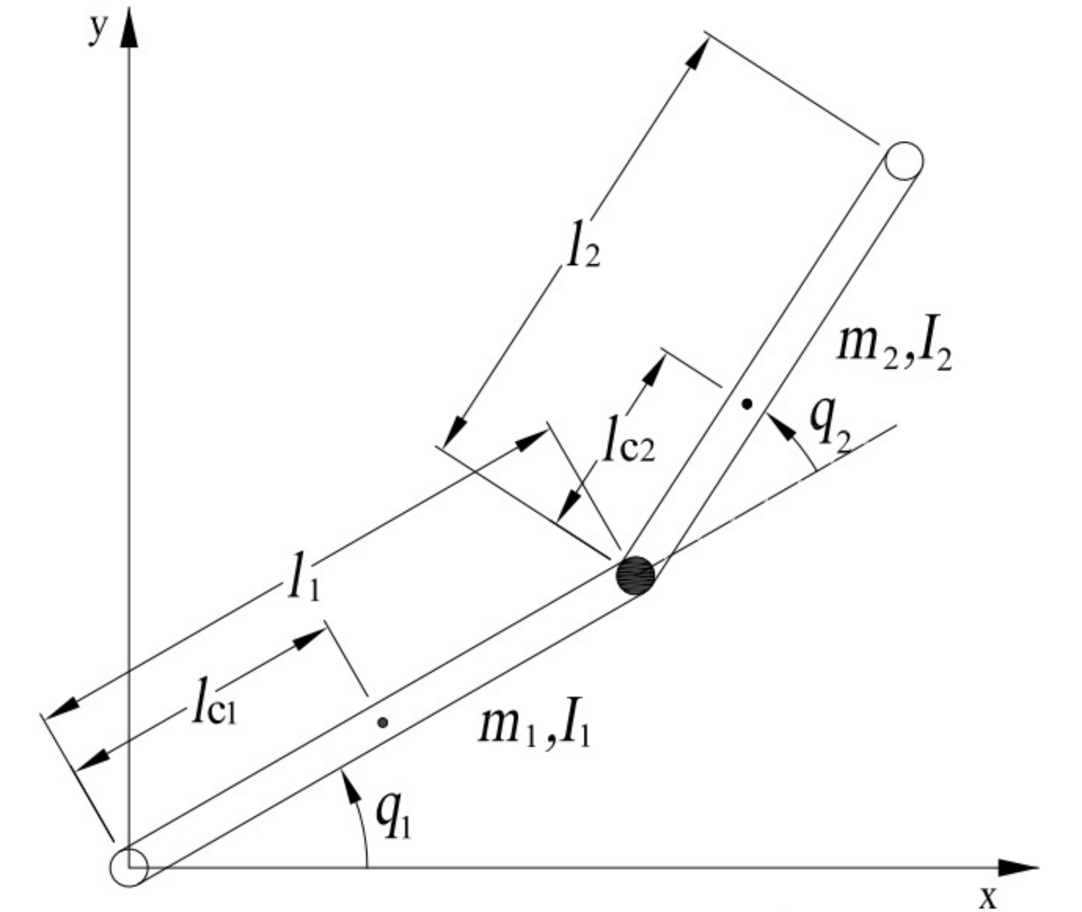
Introduction

- The objective is to perform a successful Swing-Up of the Acrobot / Pendubot underactuated mechanism
- First, the dynamic model for both robots will be introduced
- Two different approaches will be presented: Partial Feedback Linearization and Energy Approach
- Partial Feedback Linearization derives into 2 cases: Non-Collocated and Collocated Case
- Simulations for both the Acrobot and Pendubot will be demonstrated

Case Study: Acrobot / Pendubot



Pendubot

$$\tau = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 0 \\ \tau_2 \end{bmatrix}$$

Acrobot

Case Study: Acrobot / Pendubot

Dynamic Equations: Lagrange Formulation

$$M(q)\ddot{q} + C(q,\dot{q}) + g(q) = \tau$$

$$M(q) = \begin{bmatrix} a_1 + 2a_2cos(q_2) & a_3 + a_2cos(q_2) \\ a_3 + a_2cos(q_2) & a3 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} -a_2sin(q_2)\dot{q_2}(2\dot{q_1} + \dot{q_2}) \\ a_2sin(q_2)\dot{q_1}^2 \end{bmatrix}$$

$$g(q) = \begin{bmatrix} a_4cos(q_1) + a_5cos(q_1 + q_2) \\ a_5cos(q_1 + q_2) \end{bmatrix}$$

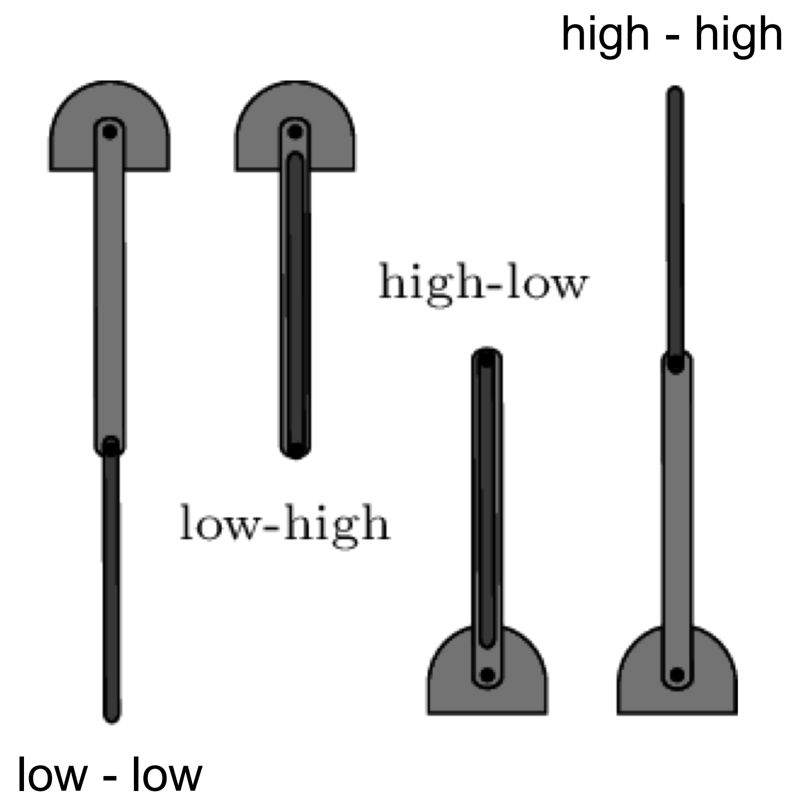
Where,

- M = Inertia Matrix
- C = Coriolis and Centrifugal effects
- g = gravity terms

- $a_1 = I_1 + m_1lc_1^2 + I_2 + m_2(l_1^2 + lc_2^2)$
- $a_2 = m_2l_1lc_2$
- $a_3 = I_2 + m_2lc_2^2$
- $a_4 = g(m_1lc_1 + m_2l_1)$
- $a_5 = gm_2lc_2$

Unforced Equilibrium Positions

- The equilibrium points are defined for $\dot{q} = \ddot{q} = 0$
- Moreover, for $\tau = 0$, there are 4 unforced equilibrium positions for both the Acrobot and the Pendubot
- The control objective is then to swing the Pendubot or Acrobot from the stable low-low position to the unstable top-top position
- After a successful Swing-Up, maintain the system in this unstable position by switching to a balancing controller
- During the Swing-Up, the other equilibrium points should be avoided



Balancing Controller

- Linearizing the Pendubot / Acrobot around the vertical equilibrium $x_e = (q_1, q_2, \dot{q_1}, \dot{q_2}) = (\frac{\pi}{2}, 0, 0, 0)$ results in the controllable linear system:

$$\dot{\tilde{x}} = A\tilde{x} + Bu$$

$$A = \begin{bmatrix} 0_{[2x2]} & I_{[2x2]} \\ -M^{-1}(x_e)(\frac{\partial g}{\partial q})|_{x_e} & 0_{[2x2]} \end{bmatrix} = \begin{bmatrix} 0_{[2x2]} & I_{[2x2]} \\ -M^e_{inv}.H^e & 0_{[2x2]} \end{bmatrix}$$

$$B = \begin{bmatrix} 0_{[2x2]} \\ M^e_{inv}.S \end{bmatrix}$$

- Where $\boldsymbol{S} = [\mathbf{0}\ \mathbf{1}]'$ for the Acrobot and $\boldsymbol{S} = [\mathbf{1}\ \mathbf{0}]'$ for the Pendubot

- There exists a Gain Matrix K such that in closed-loop the system is Asymptotically Stable (A.S)

Balancing Controller

Linear Quadratic Regulator (LQR)

- Proceeding by using the LQR method to find a Matrix of Gains, we minimize the cost function:

$$\min_{k} \int (\tilde{x}^T Q \tilde{x} + \tilde{u}^T R \tilde{u}) dt$$

- Where $Q = I_{[4x4]}$ is the Regulation Error Norm Matrix and $R = 1$ is the Control Effort Norm

- $\mathrm{K} = [-109.4468 \quad -109.3789 \quad -55.7054 \quad -37.4934]$ for the **Pendubot**
- $K = [-246.4813 \quad -98.6903 \quad -106.4640 \quad -50.1381]$ for the **Acrobot**

- Meaning that by doing $\boldsymbol{\tau} = -\boldsymbol{K}.\tilde{\boldsymbol{x}}$ and starting sufficiently close to x_e (inside the controllers basin of attraction) we are able to stabilize the system in the unstable Up-Up equilibrium position

Strong Inertial Coupling (SIC)

- For the derivations done next, one property needs to hold - the system has to be Strongly Inertially Coupled - meaning that:

$$M_{12} \neq 0$$

- It is possible to derive,

$$M_{12} = a_3 + a_2 cos(q_2) = I_2 + m_2 l_{c2}^2 + m_2 l_1 l_{c2} cos(q_2) = I_2 + m_2 l_{c2}(l_{c2} + l_1 cos(q_2))$$

- And for this expression to be non-zero for all values of q_2, the second link needs to be bigger than the first one:

$$l_{c2} > l_1$$

- This also imposes a restriction on the inertia parameters of the robot, thus the name:

$$I_2 > m_2 l_{c2}(l_1 - l_{c2})$$

Acrobot Swing-Up Control Problem
Partial Feedback Linearization

- Considering the dynamic equation of motion and by doing:

$$n(q) = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = C(q, \dot{q}) + g(q)$$

- We can re-write the expression in the following form,

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + n_1 = 0 \tag{1}$$
$$M_{12}\ddot{q}_1 + M_{22}\ddot{q}_2 + n_2 = \tau_2 \tag{2}$$

- It is now possible to proceed with an Input / Output Linearization in order to achieve a linear response from either DOF. Depending on the output equation we arrive to 2 different scenarios:

- **NON-COLLOCATED Case** → the output corresponds to the passive joint (non-collocated with the input): $\boldsymbol{y} = \boldsymbol{q_1}$

- **COLLOCATED Case –>** the output corresponds to the active joint (collocated with the input): $\boldsymbol{y} = \boldsymbol{q_2}$

Acrobot Swing-Up Control Problem

Partial Feedback Linearization: Non-Collocated Case

- Choosing the output to be $\boldsymbol{y} = \boldsymbol{q_1}$, we differentiate until the the input appears. Taking the second derivative, $\ddot{\boldsymbol{y}} = \ddot{\boldsymbol{q_1}}$, we need to isolate $\ddot{\boldsymbol{q_2}}$ from (1):

$$\ddot{q}_2 = -\frac{1}{M_{12}}(n_1 + M_{11}\ddot{q}_1)$$

- Replacing in (2) yields,

$$\alpha(q)\ddot{q}_1 + \beta(q) = \tau_2$$

$$\alpha(q) = M_{12} - \frac{M_{22}M_{11}}{M_{12}}$$

$$\beta(q) = n_2 - \frac{M_{22}}{M_{12}}n_1$$

- Meaning that conditions $\alpha(q) \neq 0$ and $\mathrm{M}_{12} \neq 0$ need to be verified. This is guaranteed by the Strong Inertial Coupling (SIC) property

Acrobot Swing-Up Control Problem
Partial Feedback Linearization: Non-Collocated Case

- Defining an additional outer control term v_1 we can choose the control input to be:

$$\tau_2 = \alpha(q)v_1 + \beta(q)$$

$$v_1 = \ddot{q_1}^d + k_p(q_1^d - q_1) + k_d(\dot{q_1}^d - \dot{q_1})$$

- To achieve the input- output linearized system,

$$\begin{cases} \ddot{q_1} = v_1 \\ \ddot{q_2} = -\frac{1}{M_{12}}(n_1 + M_{11}\ddot{q_1}) \end{cases}$$

- Since we wish to arrive at the Up-Up position, we simply set $q_1^d = \left(\frac{\pi}{2}\right)$, $\dot{q}_1^d = 0 = \ddot{q}_1^d$, meaning:

$$v_1 = k_p\left(\frac{\pi}{2} - q_1\right) - k_d\dot{q_1}$$

Acrobot Swing-Up Control Problem

Non-Collocated Case: Zero Dynamic analysis

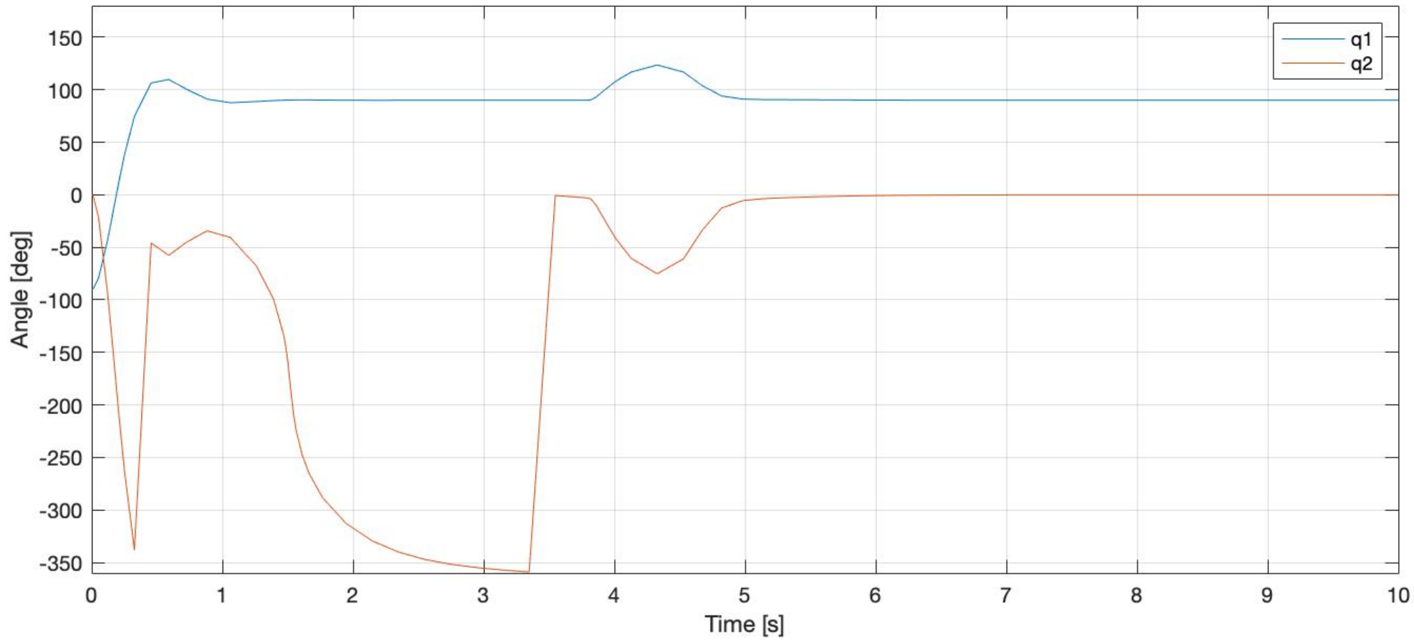
- The second joint acceleration $\ddot{q_2}$ will assume values corresponding to a periodic orbit, that will depend on the first joint motion
- Since at steady-state $q_1 = \left(\frac{\pi}{2}\right), \dot{q_1} = 0 = \ddot{q_1}$ we can replace these values on the dynamics of $\ddot{q_2}$, to obtain the residual dynamics when the first joint has reached the desired position, called the **Zero Dynamics of the System:**

$$\ddot{q_2} = \frac{sin(q_2)}{M_{12}}(a_2\dot{q_2}^2 + a_5) \tag{3}$$

- That have 2 equilibrium points, $q_2 = \pi, q_2 = 0$
- The trajectories of the system will be mostly periodic, the typical behaviour being the first link to converge exponentially to $q_1 = \left(\frac{\pi}{2}\right)$ and the second link to oscillate around one of the equilibrium points of (3)
- The strategy for the Swing-Up will then to choose a pair of gains (k_p, k_d) that swings the second link on the homoclinic orbit of the saddle equilibrium point.
- When the trajectory enters the Basin of Attraction of the LQR controller, the balancing controller is switched on to keep the Acrobot at the desired position

Acrobot Swing-Up Control Problem

Non-Collocated Case: Simulation Results



m_1	m_2	l_1	l_2	l_{c1}	l_{c2}	I_1	I_2	g	k_p	k_d
1	1	1	2	0.5	1	0.083	0.33	9.8	50	8

Table 1: Parameters used for simulation of Acrobot

Acrobot Swing-Up Control Problem
Partial Feedback Linearization: Collocated Case

- Choosing the output to be $\boldsymbol{y} = \boldsymbol{q_2}$, we differentiate until the the input appears. Taking the second derivative, $\ddot{\boldsymbol{y}} = \ddot{\boldsymbol{q_2}}$, we now isolate $\ddot{\boldsymbol{q_1}}$ from (1):

$$\ddot{q}_1 = -\frac{1}{M_{11}}(n_1 + M_{12}\ddot{q}_2)$$

- Replacing in (2) yields,

$$\alpha(q)\ddot{q}_2 + \beta(q) = \tau_2$$

$$\alpha(q) = M_{22} - \frac{M_{12}^2}{M_{11}}$$

$$\beta(q) = n_2 - \frac{M_{12}}{M_{11}}n_1$$

- Proceeding similarly as before, we can arrive to an input-output linearized system by choosing:

$$\tau_2 = \alpha(q)v_2 + \beta(q)$$

Acrobot Swing-Up Control Problem

Collocated Case: Energy Approach

- This time, however, we choose q_2^d such that it excites the internal dynamics of the system, which produce the motion of the first link
- The idea will then be to swing the second link between fixed values $\pm a$ in order to pump energy into the system by swinging link 2 ‘in phase’ with with the motion of link 1, so that energy is transferred from link 2 to link 1:

$$q_2^d = \frac{2a}{\pi} arctan(\dot{q_1})$$

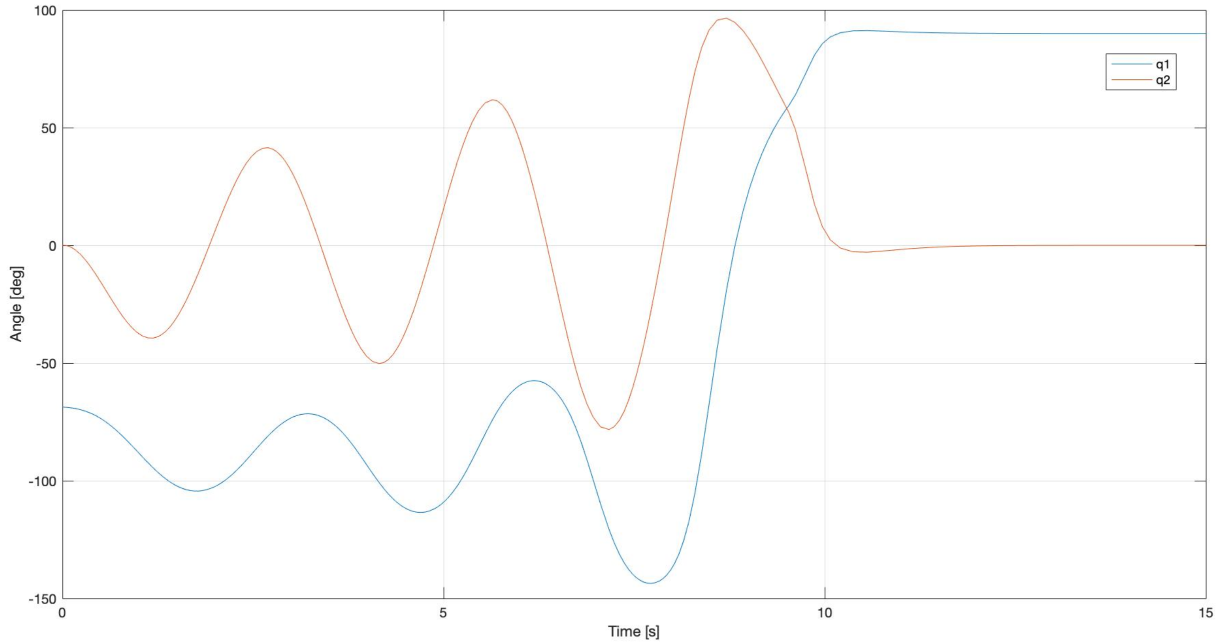
- By choosing the outer control loop term to be,

$$v_2 = k_p(q_2^d - q_2) - k_d\dot{q_2}$$

- The amplitude of the first link will be increased with each swing until it reaches the basin of attraction of the balancing controller, being captured by it to remain in the unstable equilibrium position

Acrobot Swing-Up Control Problem

Collocated Case: Simulation Results



m_1	m_2	l_1	l_2	l_{c1}	l_{c2}	I_1	I_2	g	k_p	k_d	a
1	1	1	2	0.5	1	0.083	0.33	9.8	78	13	2

Table 2: Parameters used for simulation of Acrobot

Pendubot Swing-Up Control Problem
Partial Feedback Linearization

- We can re-write the expression for the dynamics of the Pendubot in the form,

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + n_1 = \tau_1 \tag{4}$$

$$M_{12}\ddot{q}_1 + M_{22}\ddot{q}_2 + n_2 = 0 \tag{5}$$

- Proceeding like for the case of the Acrobot we can design a controller for 2 different scenarios

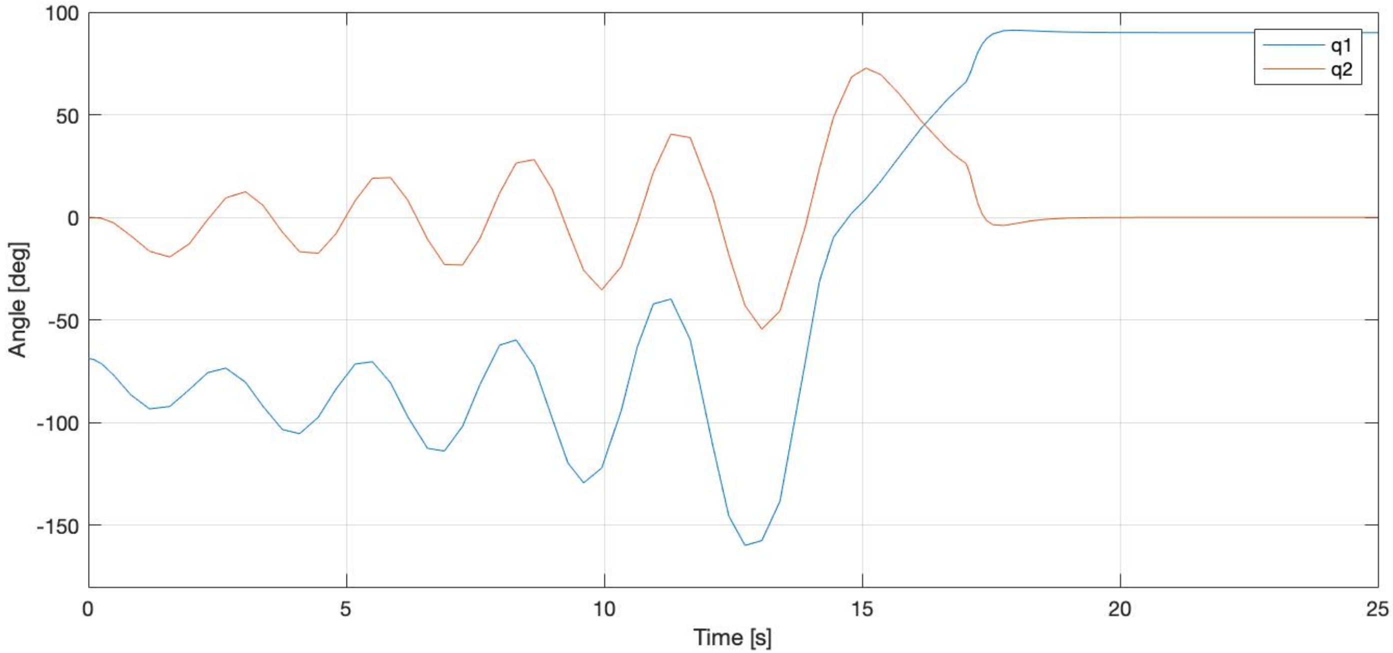
- **NON-COLLOCATED Case →** the output corresponds to the passive joint: $\boldsymbol{y} = \boldsymbol{q_2}$

- **COLLOCATED Case –>** the output corresponds to the active joint: $\boldsymbol{y} = \boldsymbol{q_1}$

- The results are very similar as for the case of the Acrobot and are reported next. Notice that for the Pendubot we will use a Zero Dynamic Analysis for the Collocated Case and an Energy Approach for the Non-Collocated Case

Pendubot Swing-Up Control Problem

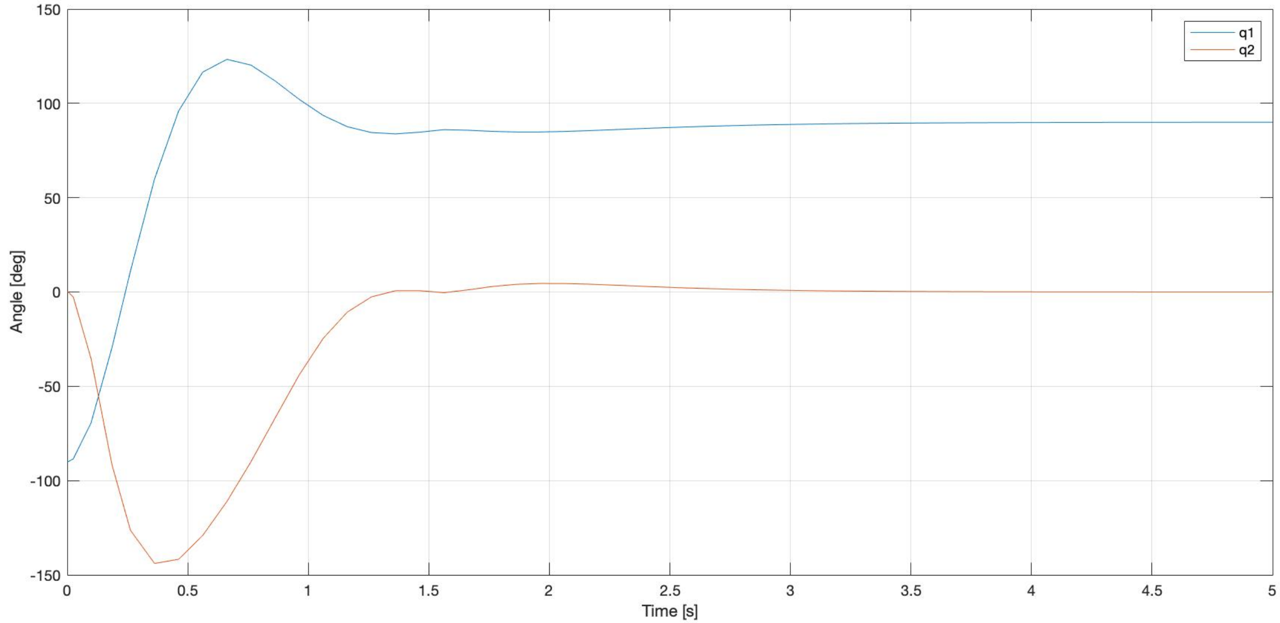
Non-Collocated Case: Simulation Results



m_1	m_2	l_1	l_2	l_{c1}	l_{c2}	I_1	I_2	g	k_p	k_d	a
1	1	1	2	0.5	1	0.083	0.33	9.8	2.1	1.4	1.76

Table 3: Parameters used for simulation of Pendubot

Pendubot Swing-Up Control Problem
Collocated Case: Simulation Results



m_1	m_2	l_1	l_2	l_{c1}	l_{c2}	I_1	I_2	g	k_p	k_d
1	1	1	2	0.5	1	0.083	0.33	9.8	28	5

Table 4: Parameters used for simulation of Pendubot

Energy Based Control of the Pendubot

Passivity Property

- The Potential Energy of the Pendubot can be defined as:

$$P(q) = a_4 sin(q_1) + a_5 sin(q_1 + q_2)$$

- Which is related to g(q) as follows:

$$g(q) = \frac{\partial P(q)}{\partial q}$$

- The Total Energy of the system is thus:

$$E(q) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$$

- We can differentiate to obtain:

$$\dot{E} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T g(q)$$

Energy Based Control of the Pendubot
Passivity Property

- Using the Skew-Symmetry property of the matrix:

$$z^T[\dot{M}(q) - 2C(q,\dot{q})]z = 0, \forall z$$

- We can derive,

$$\dot{E} = \dot{q}\tau = \dot{q}_1\tau_1$$

- Finally, integrating on both sides,

$$\int_0^t \dot{q}_1\tau_1 dt = E(t) - E(0)$$

- It is possible to conclude that the system having τ_1 as input and $\dot{q}_1$ as output is Passive

- This Passivity property of the system suggests using the Total Energy in the controller design

Energy Based Control of the Pendubot

Total Energy of the Equilibrium Points

- The Total Energy of the system is different for every Equilibrium Point:

$$E_{Up-Up} = E(\frac{\pi}{2}, 0, 0, 0) = a_4 + a_5$$

$$E_{Down-Up} = E(-\frac{\pi}{2}, 0, \pi, 0) = -a_4 + a_5$$

$$E_{Up-Down} = E(\frac{\pi}{2}, 0, \pi, 0) = a_4 - a_5$$

$$E_{Down-Down} = E(-\frac{\pi}{2}, 0, 0, 0) = -a_4 - a_5$$

- The objective is again to reach the Up-Up unstable position from the initial Down-Down position, while avoiding the other equilibrium points.

Energy Based Control of the Pendubot

System's Energy

- If the objective is to reach the top position, the final energy of the system will be:

$$E(q,\dot{q}) = \frac{1}{2}a_3\dot{q_2}^2 + P(q) = a_4 + a_5$$

- Supposing also that $q_1 = \pi/2$ and $\dot{q_1} = 0$ then the above expression would become:

$$\frac{1}{2}a_3\dot{q_2}^2 = a_5(1 - cos(q_2))$$

- This equality defines a trajectory for $q_2, \dot{q_2}$ that corresponds to a homoclinic orbit, meaning it will move until reaching the equilibrium $(q_2, \dot{q_2}) = (0, 0)$

- Therefore, the Up-Up position can be reached if we can bring the system to the homoclinic orbit defined above for $q_1 = \pi/2$ and $\dot{q_1} = 0$

Energy Based Control of the Pendubot
Lyapunov Function

- Considering,

$$\tilde{q_1} = (q_1 - \frac{\pi}{2})$$

$$\tilde{E} = (E - E_{Up-Up})$$

- It is wished to bring to zero $\widetilde{q_1}$, $\dot{q}_1$ and $\tilde{E}$. Thus, the semi-definite positive Lyapunov Candidate is considered:

$$V(q, \dot{q}) = \frac{K_E}{2}\tilde{E}^2 + \frac{K_D}{2}\dot{q}_1^2 + \frac{K_P}{2}\tilde{q_1}^2 \tag{6}$$

- Differentiating,

$$\begin{aligned} \dot{V} &= K_E\tilde{E}\dot{E} + K_D\dot{q_1}\ddot{q_1} + K_P\tilde{q_1}\dot{q_1} \\ &= \dot{q_1}(K_E\tilde{E}\tau_1 + K_D\ddot{q_1} + K_P\tilde{q_1}) \end{aligned} \tag{7}$$

Energy Based Control of the Pendubot
Control Law

- By solving (4)-(5) for $\ddot{q_1}$ we can find the expression:

$$\ddot{q_1} = \frac{1}{a_1 a_3 - a_3^2 - a_2^2 cos^2(q_2)}(a_3 \tau_1 + F(q)) \tag{8}$$

$$F(q) = a_2 a_3 sin(q_2)(\dot{q_1} + \dot{q_2})^2 + a_2^2 cos(q_2) sin(q_2) \dot{q_1}^2 \\ -a_3 a_4 cos(q_1) + a_2 a_5 cos(q_2) cos(q_1 + q_2)$$

- By choosing the control law:

$$\tau_1 = \frac{K_D F - (a_1 a_3 - a_3^2 - a_2^2 cos^2(q_2))(\dot{q_1} + K_P \tilde{q_1})}{(a_1 a_3 - a_3^2 - a_2^2 cos^2(q_2)) K_E \tilde{E} + K_D a_3} \tag{9}$$

- Substituting (8) and (9) in (7) yields,

$$\dot{V} = -\dot{q_1}^2 \leq 0$$

Energy Based Control of the Pendubot

Required Conditions

- The control law will have no singularities provided that:

$$[(a_1 a_3 - a_3^2 - a_2^2 cos^2(q_2))K_E \tilde{E} + K_D a_3] \neq 0$$

- Taking into account the above condition and since we also want to avoid all other equilibrium points except for the Up-Up position, we require that:

$$|\tilde{E}| < c = min(|E_{Up-Up} - E_{Up-Down}|,$$
$$|E_{Up-Up} - E_{Down-Down}|, |E_{Up-Up} - E_{Down-Up}|, \frac{k_D - \epsilon}{k_E(a_1 - a_3)})$$
$$= min(2a_4, 2a_5, \frac{k_D - \epsilon}{k_E(a_1 - a_3)})$$

- Since V is a nonincreasing function, the above will hold if the initial conditions are such that:

$$V(0) \leq \frac{c^2}{2}$$

Energy Based Control of the Pendubot

Stability Analysis

- The stability of the system can be analysed using **LaSalle's Invariance Theorem**
- By redefining the state for the closed-loop system as:

$$z = (q_1, sin(q_1), sin(q_1 + q_2), \dot{q_1}, cos(q_2), sin(q_2), \dot{q_2})$$

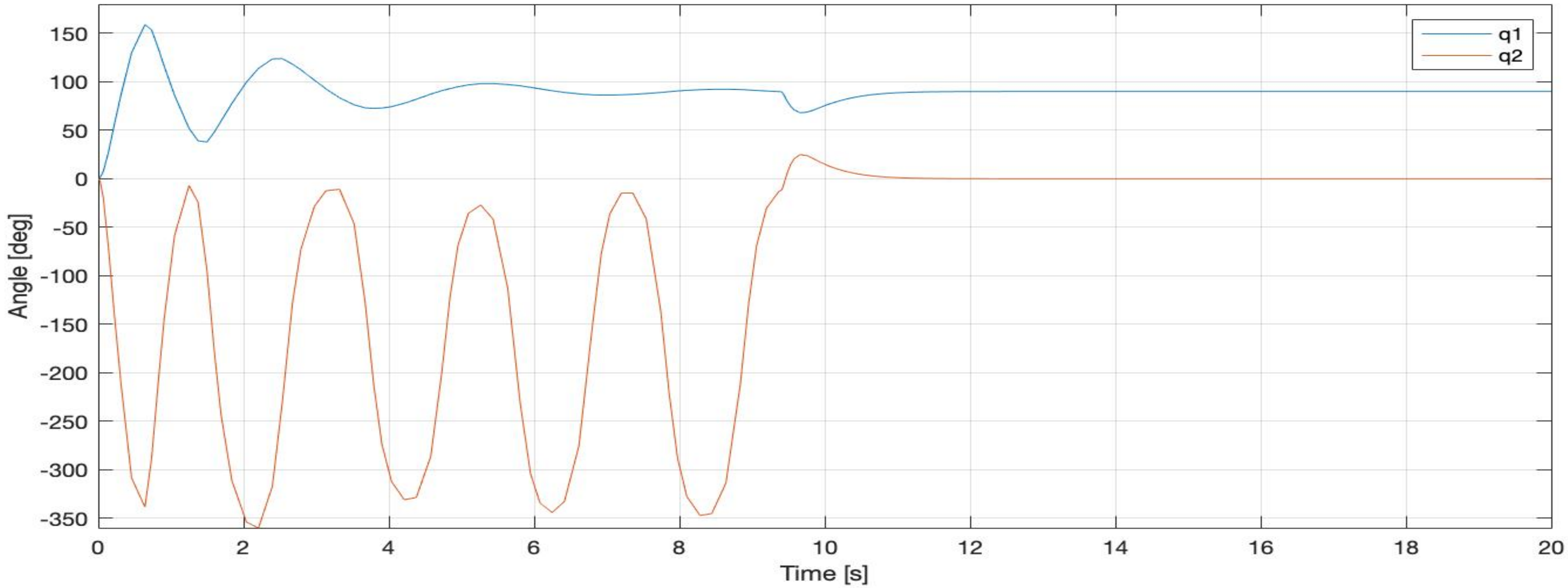
- We can assume the solution belongs inside a compact set ***Ω***
- Defining also ***Γ*** as the set of pairs in Ω such that $\dot{V}(z) = 0$ and **M** the largest invariant set in Γ, then LaSalle's assures that every solution starting in *Ω* approaches M as $t \rightarrow \infty$
- By computing set M, it can be shown that it is restricted to the homoclinic orbit in the interval:

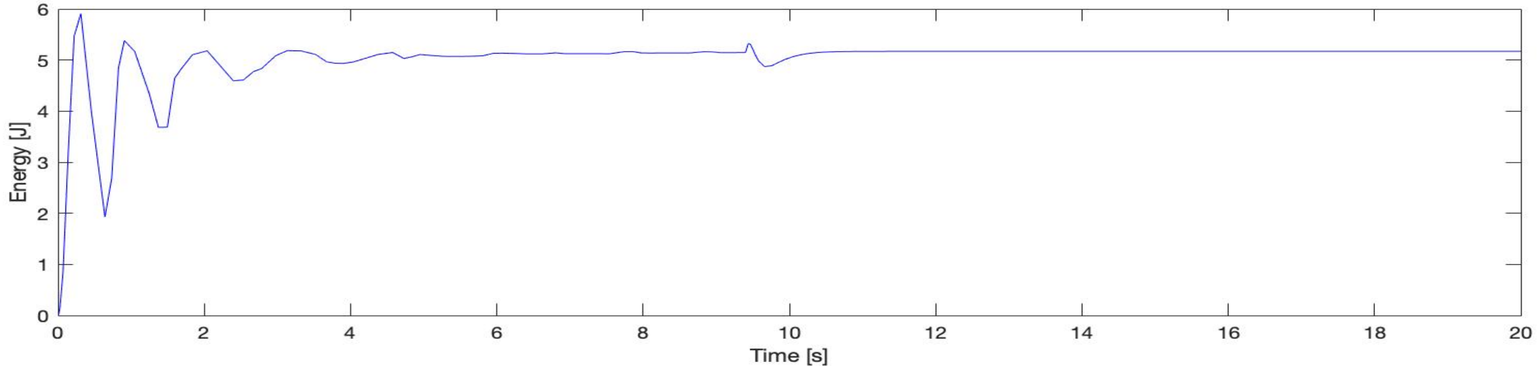
$$(q_1, \dot{q_1}, q_2, \dot{q_2}) = (\frac{\pi}{2} - \epsilon, 0, \epsilon, 0)$$

- Provided $k_p > 0$ is sufficiently small

Energy Based Control of the Pendubot

Simulation Results





Conclusion

- In Partial feedback Linearization the time needed for the system to enter the basin of attraction was smaller than that of the Energy Approach, at least when the controlled output was q_1

- When the output was chosen to be q_2, the system needs to build enough energy to enter the basin of attraction of the balancing controller, since the first link needs to swing to the Up-position, taking a longer time

- When we are controlling q_2 it was also necessary to provide an initial input in order to make q_1 oscillate, otherwise the system never leaves the Down-Down configuration

- For the Energy Based controller we have a clear stability analysis of the convergence of the whole system close to the Up-Up configuration, however, there are certain restrictions for the initial conditions to guarantee the system stability

- The Partial Feedback Linearization doesn't present these restrictions

References

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