Definition: A piecewise function is a function that is defined by two or more equations over a specific interval.

Example 1: $f(x) = \begin{cases} x + 1, & \text{if } x \le 1 \\ -3, & \text{if } x > 1 \end{cases}$

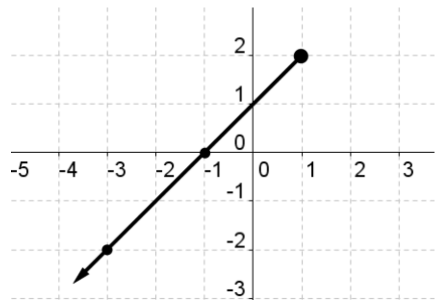
We have two different equations (or pieces) joined together under the function symbol f (x).

For x $\le$ 1, f (x) = x + 1.

Find the value of each of the following.

(a) f (- 3) $f(-3) = -3 + 1 = -2$
(b) f (- 1) $f(-1) = -1 + 1 = 0$
(c) f (1) $f(1) = 1 + 1 = 2$

If we graph this function by hand, we get



x	f (x)
- 3	- 2
- 1	0
1	2

If we graph this function using the TI calculator, then we write
$y_1 = (x + 1)(x \le 1)$.

To get $\le$ we press 2nd function, math, 6

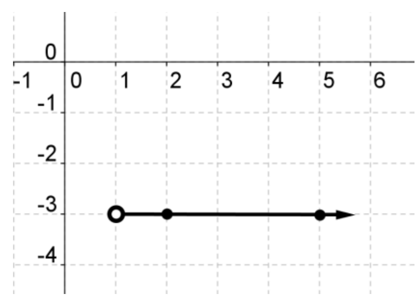
We are not finished because we have another part of f (x).

For x > 1, f (x) = - 3.

Find the value of each of the following.

(a) f (1.0001) $f(1.0001) = -3$
(b) f (2) $f(2) = -3$
(c) f (5) $f(5) = -3$

If we graph this function by hand, we get



x	f (x)
1.0001	- 3
2	- 3
5	- 3

Why do we need an open circle at (1, - 3)?

If we graph this function using the TI calculator, then we write $y_1 = -3\,(x > 1)$.

To get > we press 2nd function, math, 3

Notice that your calculator does not show the open circle for (1, - 3). *<u>You</u>* have to do that.

Now since f (x) is made up of two parts, we can put the two parts together.

Example 1: $f(x) = \begin{cases} x + 1, & \text{if } x \le 1 \\ -3, & \text{if } x > 1 \end{cases}$

For $x \le 1$, $f(x) = x + 1$.
For $x > 1$, $f(x) = -3$.

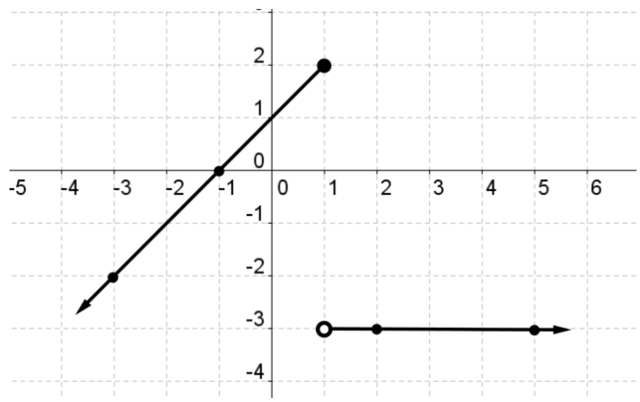
Find the value of each of the following.

(a) f (- 2)
(b) f (4)
(c) f (0)
(d) f (6)
(e) f (- 3)
(f) f (3)

Answers: (a) - 1, (b) - 3, (c) 1,
(d) - 3, (e) - 2, (f) - 3

Now let us put the two parts of the graph together.

	x	f (x)
For $x \le 1$, $f(x) = x + 1$	- 3	- 2
	- 2	- 1
	0	1
For $x > 1$, $f(x) = -3$	3	- 3
	4	- 3
	6	- 3



If we graph this function using the TI calculator, then we write

$y_1 = (x + 1)(x \le 1) + -3(x > 1)$.

Again your calculator does not show the open circle for (1, - 3). *__You__* have to do that.

__Domain and Range__

To find the domain and range we observe the following.

$f(x) = \begin{cases} \overset{\text{range}}{x + 1}, & \overset{\text{domain}}{\text{if } x \le 1} \\ -3, & \text{if } x > 1 \end{cases}$

The domain is $x \le 1$ and $x > 1$. So all real numbers are in the domain of f (x).
Therefore the domain of f (x) is $(-\infty, \infty)$.

The range can be found from the graph. Thus the range is from $(-\infty, 2]$.

Example 2: $f(x) = \begin{cases} 2x, & \text{if } x \leq 0 \\ 3x - 4, & \text{if } x \geq 3 \end{cases}$

We have two different equations (or pieces) joined together under the function symbol f (x).

For $x \leq 0$, $f(x) = 2x$.
For $x \geq 3$, $f(x) = 3x - 4$

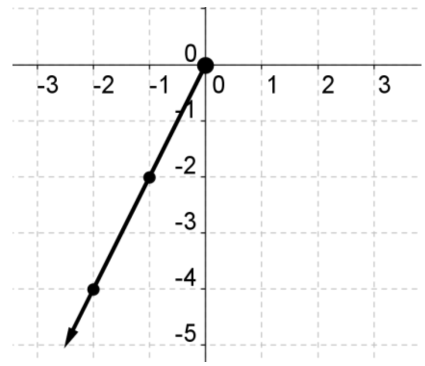
Find the value of each of the following.

(a) $f(-2)$
(b) $f(5)$
(c) $f(1)$
(d) $f(0)$
(e) $f(2)$
(f) $f(3)$
(g) $f(4)$
(h) $f(-1)$

Answers: (a) -4, (b) 11, (c) no solution
(d) 0, (e) no solution, (f) 5
(g) 8, (h) -2

Why is there no solution for f (1) and f (2)?

If we graph this function by hand for $x \leq 0$, **then** $f(x) = 2x$.

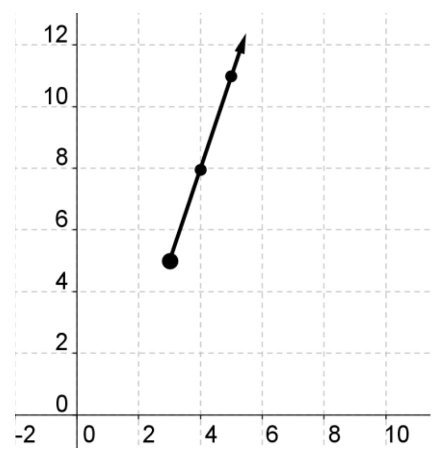


x	f (x)
- 2	- 4
- 1	- 2
0	0

If we graph this function using the TI calculator, then we write

$y_1 = 2x\,(x \leq 0)$.

If we graph this function by hand for $x \geq 3$, **then** $f(x) = 3x - 4$



x	f (x)
3	5
4	8
5	11

If we graph this function using the TI calculator, then we write

$y_1 = (3x - 4)(x \geq 3)$.

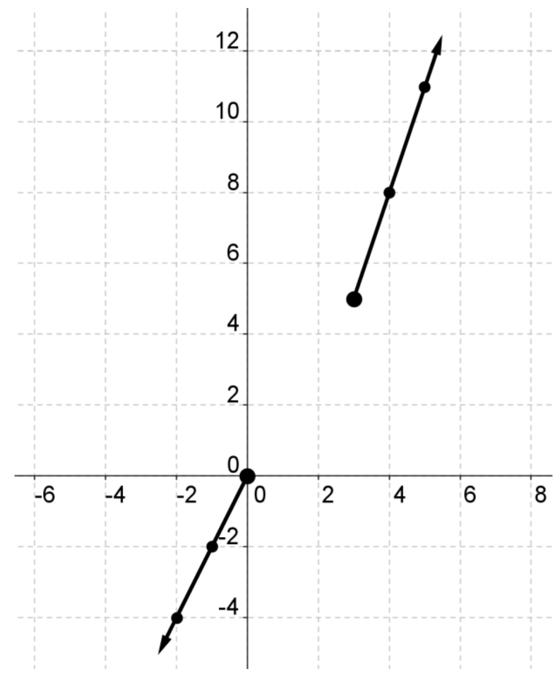
To get $\geq$ **we press** 2^{nd} **function, math, 4.**

Example 2: $f(x) = \begin{cases} 2x, & \text{if } x \le 0 \\ 3x - 4, & \text{if } x \ge 3 \end{cases}$

Now let us put the two parts of the graph together.

	x	f (x)
For $x \le 0$, $f(x) = 2x$	-2	-4
	-1	-2
	0	0
For $x \ge 3$, $f(x) = 3x - 4$	3	5
	4	8
	5	11

Our graph for all parts of f (x) is



If we graph this function using the TI calculator, then we write

$y_1 = 2x\,(x \le 0) + (3x - 4)\,(x \ge 3).$

Notice that 0 <u>is</u> in the domain of f (x). Your calculator shows (0, 0). However, 1 & 2 are <u>not in the domain</u> of f (x) , but the way your calculator indicates this is by showing (2, 0) and (1,0) in the table. Do not be fooled by this.

<u>**Domain and Range**</u>

Domain: Since $x \le 0$ or $x \ge 3$, the domain is $(-\infty, 0] \cup [3, \infty)$. The points between 0 and 3 are NOT in the domain of f (x).

Range: Since the range of f (x) has no values between 0 and 5, the range of f (x) is $(-\infty, 0] \cup [5, \infty)$.

Example 3: $f(x) = \begin{cases} x + 2, & \text{if } x < -2 \\ 5, & \text{if } -2 \le x < 4 \\ -2x + 7, & \text{if } x \ge 4 \end{cases}$

We have three equations joined together under the function symbol $f(x)$.

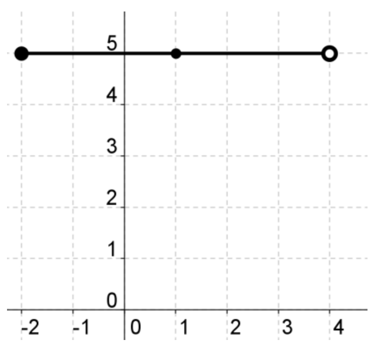
For $x < -2$, $f(x) = x + 2$.
For $-2 \le x < 4$, $f(x) = 5$
For $x \ge 4$, $f(x) = -2x + 7$

Find the value of each of the following.

(a) $f(-4)$	(b) $f(1)$
(c) $f(-2.0001)$	(d) $f(3.9999)$
(e) $f(6)$	(f) $f(-2)$
(g) $f(4)$	(h) $f(5)$
(i) $f(-3)$	

Answers: (a) -2, (b) 5, (c) ≈ 0
(d) 5, (e) -5, (f) 5
(g) -1, (h) -3, (i) -1

If we graph this function, by hand for $-2 \le x < 4$, then $f(x) = 5$



x	f (x)
- 2	5
1	5
3.9999	5

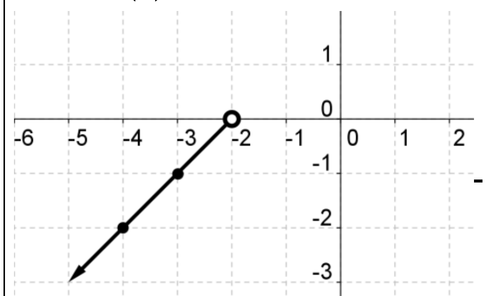
If we graph this function using the TI calculator, then we cannot write $-2 \le x < 4$.

We must do the following.

$y_1 = 5(-2 \le x)(x < 4)$.

Notice that your calculator does not show the open circle for (4, 5). *<u>You</u>* have to do that.

If we graph this function by hand for $x < 2$, then $f(x) = x + 2$



x	f (x)
- 4	- 2
- 3	- 1
- 2.0001	≈ 0

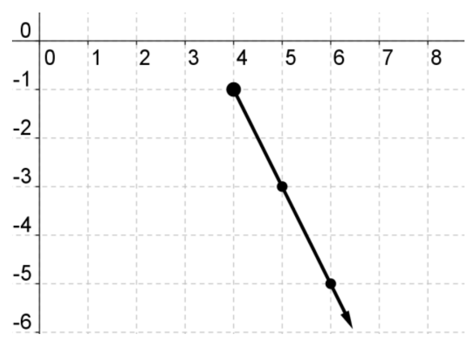
If we graph this function using the TI calculator, then we write

$y_1 = (x + 2)(x < -2)$.

To get < we press 2nd function, math, 5

Notice that your calculator does not show the open circle for (- 2, 0). *<u>You</u>* have to do that.

If we graph this function, by hand for $x \ge 4$, then $f(x) = -2x + 7$



x	f (x)
4	- 1
5	- 3
6	- 5

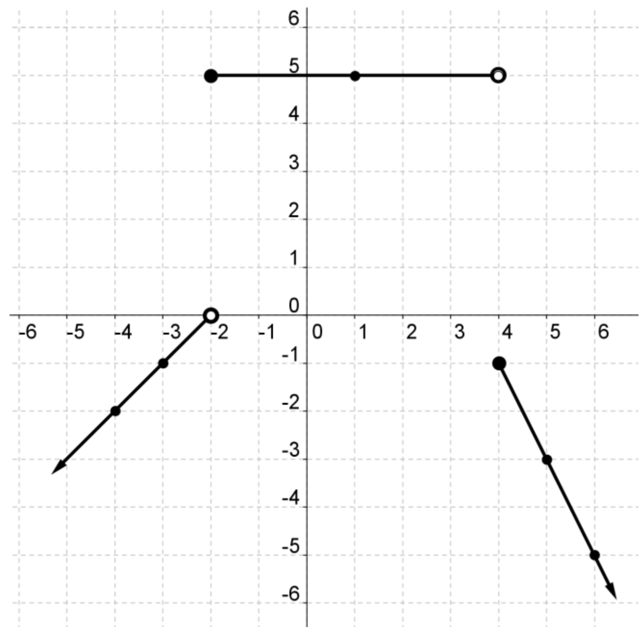
If we graph this function using the TI calculator, then we write

$y_1 = (-2x + 7)(x \ge 4)$.

Example 3: $f(x) = \begin{cases} x + 2, & \text{if } x < -2 \\ 5, & \text{if } -2 \le x < 4 \\ -2x + 7, & \text{if } x \ge 4 \end{cases}$

Now let us put the three parts of the graph together.

	x	f (x)
For $x < -2$, $f(x) = x + 2$	- 4	- 2
	- 3	- 1
	2.0001	≈ 0
For $-2 \le x < 4$, $f(x) = 5$	- 2	5
	1	5
	3.9999	5
For $x \ge 4$, $f(x) = -2x + 7$	4	- 1
	5	- 3
	6	- 5



If we graph this function using the TI calculator, then we write

$y_1 = (x + 2)(x < -2) + 5(-2 \le x)(x < 4) + (-2x + 7)(x \ge 4)$.

Remember that your calculator does not show the open circle for (- 2, 0) or (4, 5). ***<u>You</u>*** **have to do that.**

<u>Domain and Range</u>

The domain is $x < -2$, $-2 \le x < 4$, $x \ge 4$**. So all real numbers are in the domain of** $f(x)$**. Therefore the domain of** $f(x)$ **is** $(-\infty, \infty)$.

Notice that the range of $f(x)$ **has no value that will give you 0 and no values between 0 and 5. Therefore the range of** $f(x)$ **is** $(-\infty, 0) \cup \{5\}$ **where 5 is <u>a single number</u> and is not an interval. The symbols { } are needed to show that 5 is a single number.**

One function you may be familiar with is the absolute value function, $f(x) = |x|$. This is one example of a function which has a piecewise definition.

Recall, the absolute value of a negative number is the opposite sign of the number, where the absolute value of a positive number is unchanged. The piecewise equivalent of this description is:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

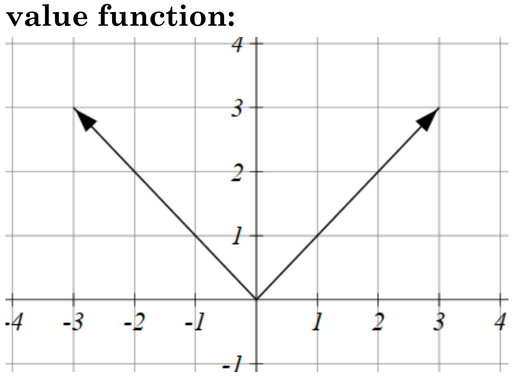
For $x < 0$, $f(x) = -x$.
For $x \geq 0$, $f(x) = x$.

Find the value of each of the following.

(a) $f(-2)$
(b) $f(5)$
(c) $f(0)$
(d) $f(-9)$

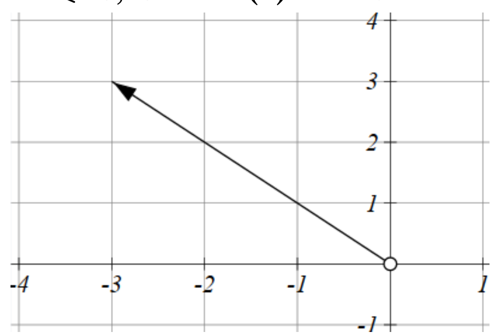
Answers: (a) 2, (b) 5, (c) 0 (d) 9

If you combine both the graphs on the right on the same coordinate system, you get the familiar graph of the absolute value function:



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

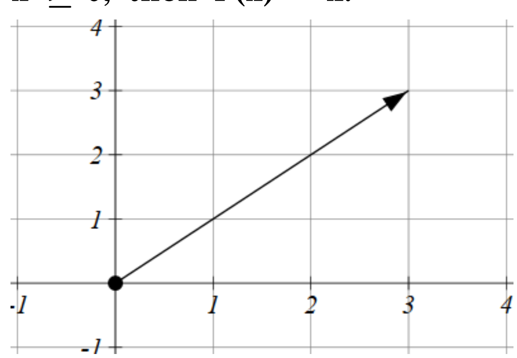
If we graph this function by hand for $x < 0$, then $f(x) = -x$.



If we graph this function using the TI calculator, then we write

$y_1 = -x\,(x < 0)$.

If we graph this function by hand for $x \geq 0$, then $f(x) = x$.



If we graph this function using the TI calculator, then we write

$y_1 = x\,(x \geq 0)$.

To get $\geq$ we press 2nd function, math, 4.

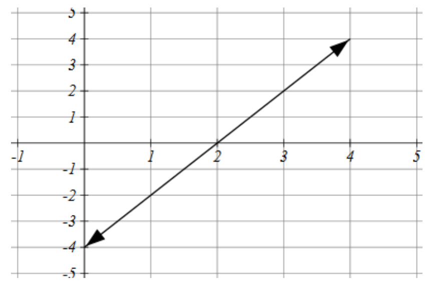
Example 4: $f(x) = |2x - 4|$

Here is a slightly more complicated absolute value function. To find its graph, we use the definition of absolute value. When the expression inside the absolute value symbols is positive, we would like it to remain the same, and when the expression in negative, we want use the opposite of the expression, or its negative.

The expression will change signs when:

$$2x - 4 = 0 \qquad \text{or} \qquad x = 2$$

By looking at a graph of y = 2 x - 4, we can see it has negative outputs when x is less than two, and positive outputs when x is greater than 2.



So, for $x \geq 2$, $f(x) = 2x - 4$

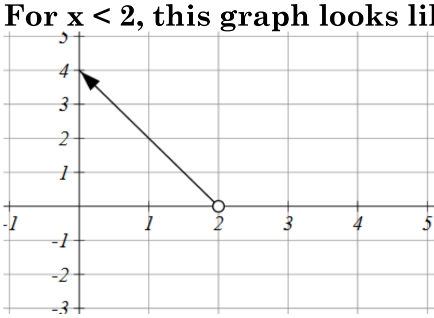
(since $2x - 4$ has only positive values when $x \geq 2$)

And for $x < 2$, $f(x) = -(2x - 4)$
$= -2x + 4$

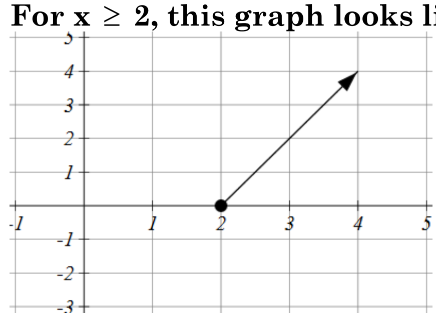
(since $2x - 4$ has negative values when $x < 2$)

This makes the piecewise definition $f(x) = \begin{cases} -2x + 4, & \text{if } x < 2 \\ 2x - 4, & \text{if } x \geq 2 \end{cases}$

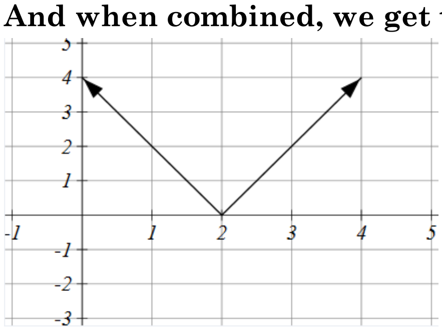
For x < 2, this graph looks like:



For $x \geq 2$, this graph looks like



And when combined, we get the final graph:



Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

This same process can be used whenever dealing with an expression inside absolute value symbols.

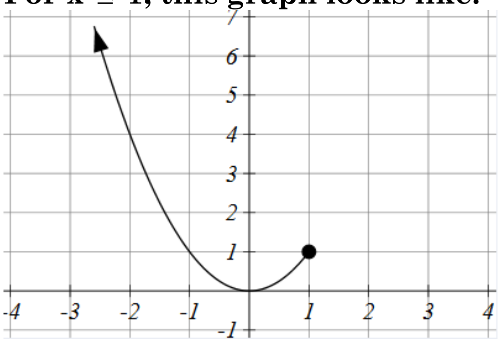
Example 5: $f(x) = \begin{cases} x^2, & \text{if } x \le 1 \\ 2x+1, & \text{if } x > 1 \end{cases}$

This example involves both a quadratic and a linear function.

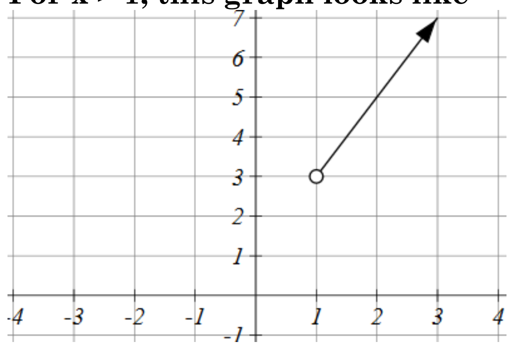
For $x \le 1$, $f(x) = x^2$.

For $x > 1$, $f(x) = 2x+1$

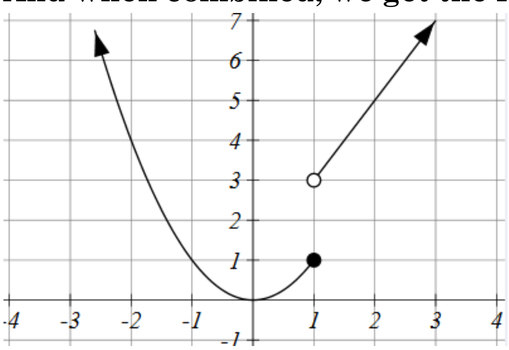
For $x \le 1$**, this graph looks like:**



For $x > 1$**, this graph looks like**



And when combined, we get the final graph:



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Challenge:
How could you change the definition of the function in this example so that the two pieces combine to give a graph with no breaks or gaps?

Homework Exercises

Find the answer of each value of f (x). If there is no answer for f (x), write none. Graph each piecewise function. Then find the domain and range of each piecewise function.

1.

$$f(x) = \begin{cases} 2 - x, & \text{if } x \le 0 \\ x - 5, & \text{if } x > 0 \end{cases}$$

(a) f (- 3) (b) f (- 1)
(c) f (0) (d) f (1)
(e) f (3) (f) f (4)

2.

$$f(x) = \begin{cases} -3, & \text{if } x \le -3 \\ x, & \text{if } -3 < x < 2 \\ 2, & \text{if } x \ge 2 \end{cases}$$

a) f (- 5) (b) f (- 3)
(c) f (- 1) (d) f (1)
(e) f (2) (f) f (5)

3.

$$f(x) = \begin{cases} -x - 5, & \text{if } x < 0 \\ x - 5, & \text{if } x \ge 0 \end{cases}$$

(a) f (- 4) (b) f (- 2)
(c) f (- 1) (d) f (0)
(e) f (2) (f) f (4)

4.

$$f(x) = \begin{cases} -x, & \text{if } x < 1 \\ 3x + 1, & \text{if } x \ge 1 \end{cases}$$

(a) f (- 3) (b) f (- 2)
(c) f (- 1) (d) f (0)
(e) f (1) (f) f (2)

5.

$$f(x) = \begin{cases} 3x, & \text{if } x < -1 \\ x, & \text{if } -1 \le x < 3 \\ 2 - x, & \text{if } x \ge 3 \end{cases}$$

(a) f (- 2) (b) f (- 1)
(c) f (1) (d) f (2)
(e) f (3) (f) f (4)

6.

$$f(x) = \begin{cases} -4, & \text{if } x < -2 \\ 3 - 2x, & \text{if } -2 \le x < 3 \\ x - 4, & \text{if } x \ge 3 \end{cases}$$

(a) f (- 3) (b) f (- 2)
(c) f (0) (d) f (1)
(e) f (3) (f) f (4)

7.

$$f(x) = \begin{cases} x + 3, & \text{if } x < -3 \\ -x, & \text{if } x \ge 2 \end{cases}$$

(a) f (- 6) (b) f (- 3)
(c) f (- 1) (d) f (0)
(e) f (2) (f) f (5)

8.

$$f(x) = \begin{cases} -1, & \text{if } x \le 1 \\ 3, & \text{if } x \ge 4 \end{cases}$$

(a) f (- 3) (b) f (1)
(c) f (2) (d) f (3)
(e) f (4) (f) f (5)

9.

$$f(x) = \begin{cases} x, & \text{if } x < 2 \\ -x + 5, & \text{if } x > 2 \end{cases}$$

(a) f (- 4) (b) f (- 3)
(c) f (0) (d) f (2)
(e) f (3) (f) f (5)

10.

$$f(x) = \begin{cases} 2x, & \text{if } x < -1 \\ 1, & \text{if } -1 \le x < 1 \\ x - 1, & \text{if } x \ge 2 \end{cases}$$

(a) f (- 3) (b) f (- 2)
(c) f (- 1) (d) f (1)
(e) f (2) (f) f (4)

11.

$f(x) = |3x + 6|$

(a) f (- 3) **(b)** f (- 2)
(c) f (0) **(d)** f (1)

12.

$$f(x) = \begin{cases} -2, \text{ if } x \le -3 \\ |x|, \text{ if } -3 < x < 2 \\ 3, \text{ if } x \ge 2 \end{cases}$$

a) f (- 5) **(b)** f (- 3)
(c) f (- 1) **(d)** f (1)
(e) f (2) **(f)** f (5)

13.

$$f(x) = \begin{cases} 2, \text{ if } x \le -1 \\ x^2, \text{ if } x > 1 \end{cases}$$

(a) f (- 4) **(b)** f (- 2)
(c) f (- 1) **(d)** f (0)
(e) f (1) **(f)** f (2)

14.

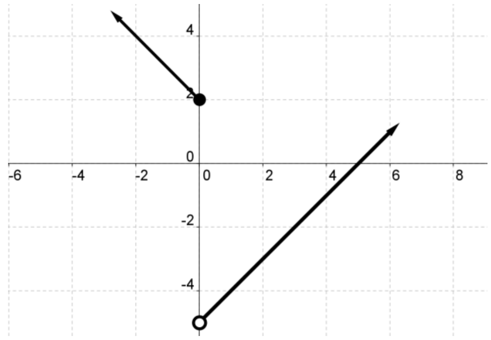
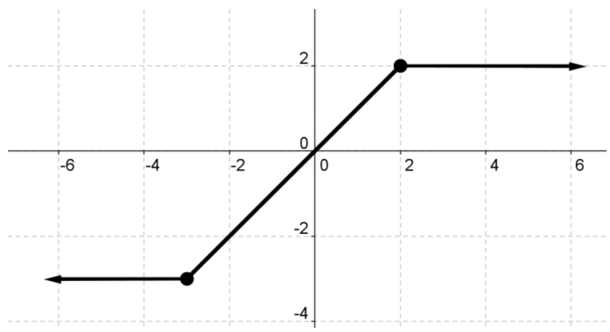
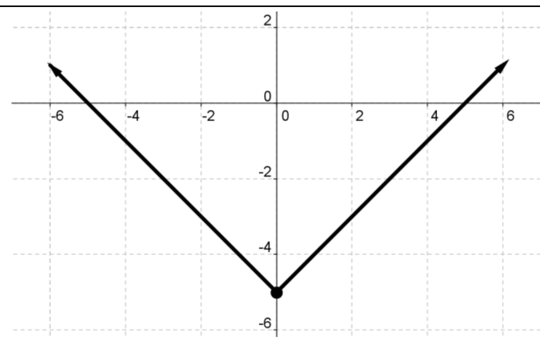
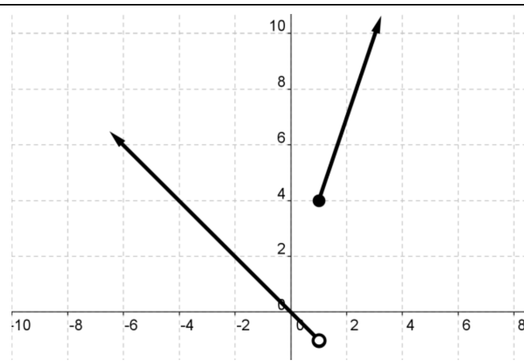
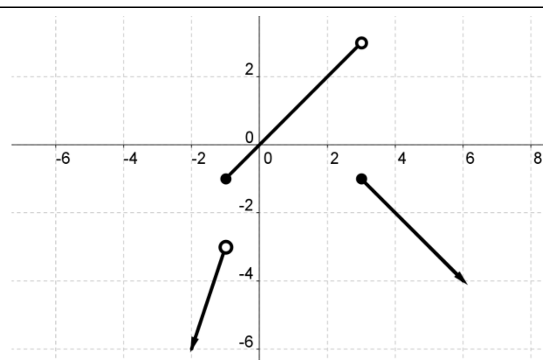
$$f(x) = \begin{cases} 1 - x^2, \text{ if } x \le 2 \\ x, \text{ if } x > 2 \end{cases}$$

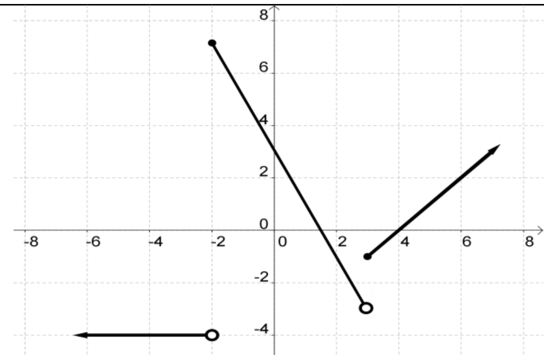
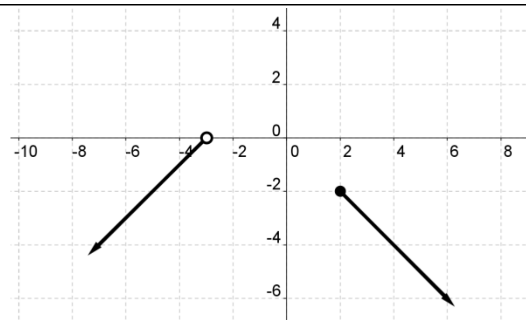
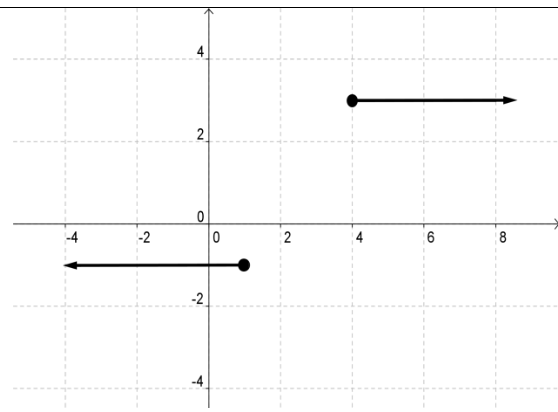
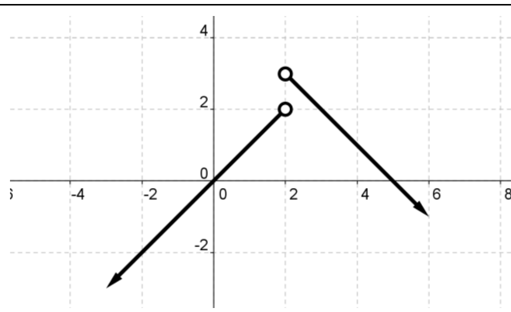
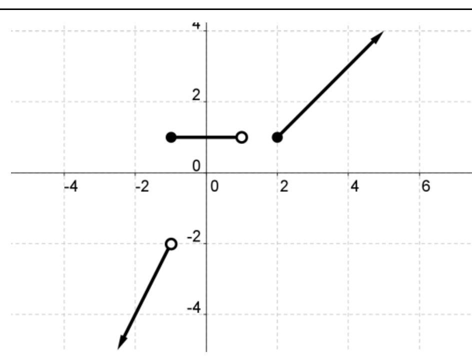
(a) f (- 2) **(b)** f (- 1)
(c) f (0) **(d)** f (1)
(e) f (2) **(f)** f (4)

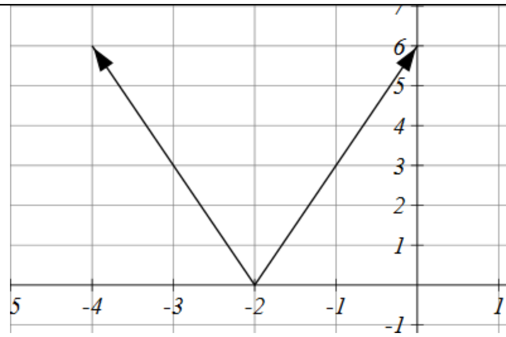
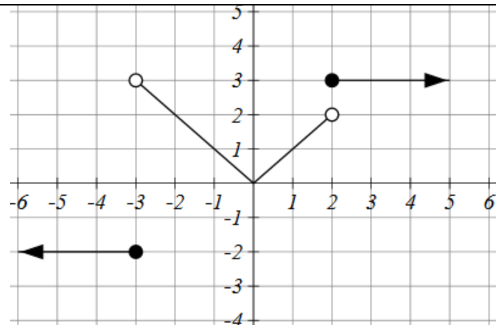
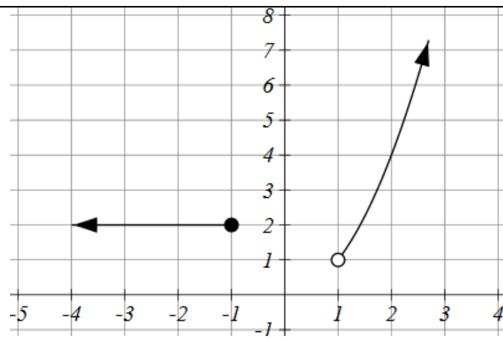
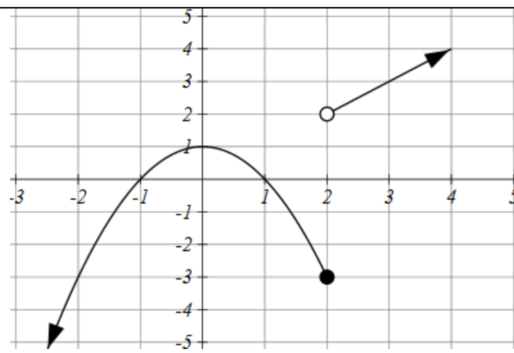
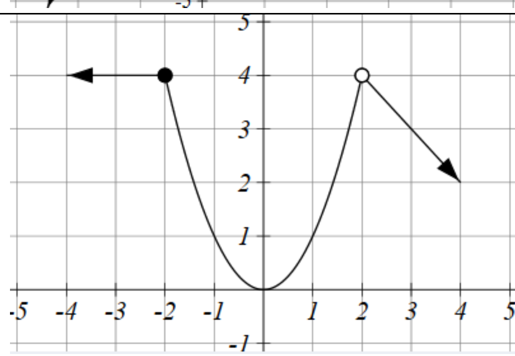
15.

$$f(x) = \begin{cases} 4, \quad \text{if } x < -2 \\ x^2, \quad \text{if } -2 \le x < 2 \\ -x + 6, \text{ if } x > 2 \end{cases}$$

(a) f (- 3) **(b)** f (- 2)
(c) f (0) **(d)** f (2)
(e) f (3) **(f)** f (4)

Ans. to f (x)	Graph of f (x)	Domain of f (x)	Range of f (x)
1. (a) 5 (b) 3 (c) 2 (d) - 4 (e) - 2 (f) - 1		$(-\infty, \infty)$	$(-5, \infty)$
2. (a) - 3 (b) - 3 (c) - 1 (d) 1 (e) 2 (f) 2		$(-\infty, \infty)$	$[-3, 2]$
3. (a) - 1 (b) - 3 (c) - 4 (d) - 5 (e) - 3 (f) - 1		$(-\infty, \infty)$	$[-5, \infty)$
4. (a) 3 (b) 2 (c) 1 (d) 0 (e) 4 (f) 7		$(-\infty, \infty)$	$(-1, \infty)$
5. (a) - 6 (b) - 1 (c) 1 (d) 2 (e) - 1 (f) - 2		$(-\infty, \infty)$	$(-\infty, 3)$

Ans. to f (x)	Graph of f (x)	Domain of f (x)	Range of f (x)
6. (a) - 4 (b) 7 (c) 3 (d) 1 (e) - 1 (f) 0		$(-\infty, \infty)$	$\{-4\} \cup$ $(-3, \infty)$ - 4 is in the range of f (x). The numbers between - 4 and - 3 are <u>not</u> in the range of f(x).
7. (a) - 3 (b) none (c) none (d) none (e) - 2 (f) - 5		$(-\infty, -3) \cup [2, \infty)$ The numbers between - 3 and 2 are <u>not</u> in the domain of f (x).	$(-\infty, 0)$
8. (a) - 1 (b) - 1 (c) none (d) none (e) 3 (f) 3		$(-\infty, 1] \cup$ $[4, \infty)$ The numbers between 1 and 4 are <u>not</u> in the domain of f (x).	$\{-1, 3\}$ Only - 1 and 3 are in the range of f (x). This is why we need the { } symbols. Note: [- 1, 3] is <u>NOT</u> correct here. Why?
9. (a) - 4 (b) - 3 (c) 0 (d) none (e) 2 (f) 0		$(-\infty, 2) \cup$ $(2, \infty)$ 2 is <u>not</u> in the domain of f (x).	$(-\infty, 3)$ 3 is <u>not</u> in the range of f (x).
10. (a) - 6 (b) - 4 (c) 1 (d) none (e) 1 (f) 3		$(-\infty, 1) \cup$ $[2, \infty)$ 1 and the numbers between 1 and 2 are <u>not</u> in the domain of f (x), but 2 <u>is</u> in the domain of f (x).	$(-\infty, -2) \cup$ $[1, \infty)$ - 2 and the numbers between - 2 and 1 are <u>not</u> in the range of f (x), but 1 <u>is</u> in the range of f (x).

Ans. to f (x)	Graph of f (x)	Domain of f (x)	Range of f (x)
11. **(a) 3** **(b) 0** **(c) 6** **(d) 9**		$(-\infty, \infty)$	$[0, \infty)$
12. **(a) - 2** **(b) - 2** **(c) 1** **(d) 1** **(e) 3** **(f) 3**		$(-\infty, \infty)$	$\{-2\}\cup[0,3]$
13. **(a) 2** **(b) 2** **(c) 2** **(d) none** **(e) none** **(f) 4**		$(-\infty, -1]\cup(1,\infty)$	$(1, \infty)$
14. **(a) - 3** **(b) 0** **(c) 1** **(d) 0** **(e) -3** **(f) 4**		$(-\infty,\infty)$	$(-\infty, 1]\cup(2,\infty)$
15. **(a) 4** **(b) 4** **(c) 0** **(d) none** **(e) 3** **(f) 2**		$(-\infty, 2)\cup(2,\infty)$	$(-\infty,4]$