Università della Svizzera italiana | Institute of Computational Science ICS

Numerical Computing **2020**

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Solution for Project 6

Due date: Friday, December 18, 2020, 11:59 PM

Numerical Computing 2020 — Submission Instructions
(Please, notice that following instructions are mandatory: submissions that don't comply with, won't be considered)

- Assignments must be submitted to iCorsi (i.e. in electronic format).
- Provide both executable package and sources (e.g. C/C++ files, Matlab). If you are using libraries, please add them in the file. Sources must be organized in directories called:
  Project_number_lastname_firstname
  and the file must be called:
  project_number_lastname_firstname.zip
  project_number_lastname_firstname.pdf
- The TAs will grade your project by reviewing your project write-up, and looking at the implementation you attempted, and benchmarking your code's performance.
- You are allowed to discuss all questions with anyone you like; however: (i) your submission must list anyone you discussed problems with and (ii) you must write up your submission independently.

The purpose of this mini-project is to implement the Simplex Method to find the solution of linear programs, involving both the minimisation and the maximisation of the objective function.

1. Graphical Solution of Linear Programming Problems [25 points]

Please consider the following two problems:

(1)

$$\begin{aligned} \min \quad & z = 4x + y \\ \text{s.t.} \quad & x + 2y \leq 40 \\ & x + y \geq 30 \\ & 2x + 3y \geq 72 \\ & x,\, y \geq 0 \end{aligned}$$

SOLUTION We can solve the system of inequalities:

$$
\begin{aligned}
y &\leq 20 - \frac{x}{2} \\
y &\geq 30 - x \\
y &\geq 24 - \frac{2x}{3} \\
x, y &\geq 0
\end{aligned}
$$

This is the feasible region:

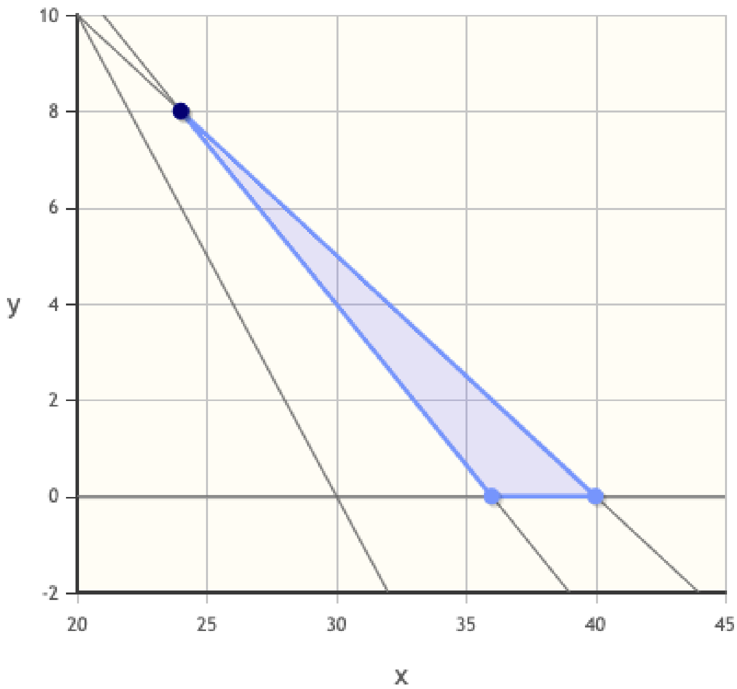

Figure 1: Feasible region

The optimal solution is at vertex (24,8) with $z^* = 4 \cdot 24 + 8 = 104$. For the other vertices we have: at vertex (36,0) $z = 4 \cdot 36 = 144$ and at vertex (40,0) $z = 4 \cdot 40 = 160$.

(2) A tailor plans to sell two types of trousers, with production costs of 25 CHF and 40 CHF, respectively. The former type can be sold for 85 CHF, while the latter for 110 CHF. The tailor estimates a total monthly demand of 265 trousers. Find the number of units of each type of trousers that should be produced in order to maximise the net profit of the tailor, if we assume that the he cannot spend more than 7000 CHF in raw materials.

SOLUTION Use x and y as the number of trousers of the two types. The net profit for each type is 60 CHF and 70 CHF. We also know that the number of trousers estimated by the tailor is 265. I choose to put the greater or equal because as the tailor wants to be prepared to accommodate every client we need to provide at least that number of trousers. For this reason we have the following linear programming problem:

$$
\begin{aligned}
\max \quad & z = 60x + 70y \\
\text{s.t.} \quad & x + y \geq 265 \\
& 25x + 40y \leq 7000 \\
& x,\, y \geq 0
\end{aligned}
$$

We can solve the system of inequalities:

$$y \geq 265 - x$$
$$y \leq 175 - \frac{5x}{8}$$
$$x, y \geq 0$$

This is the feasible region:

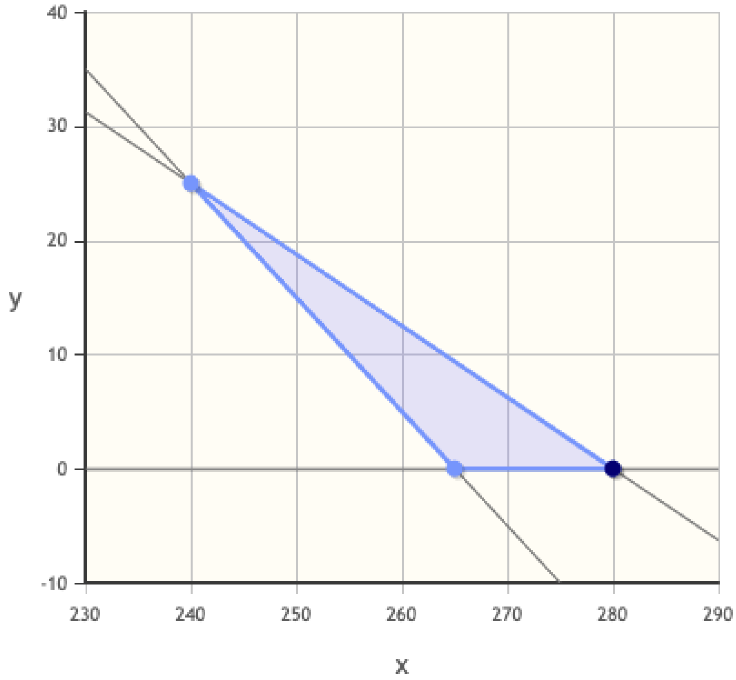

Figure 2: Feasible region

The optimal solution is at vertex (280,0) with $z^* = 60 \cdot 280 = 16800$. For the other vertices we have: at vertex (265,0) $z = 60 \cdot 265 = 15900$ and at vertex (240,25) $z = 60 \cdot 240 + 70 \cdot 25 = 16150$. Hence the tailor should produce 280 units of the first trouser type and enjoy 16800 CHF of profit.

2. Implementation of the Simplex Method [35 points]

In this first part of the assignment, you are required to complete 2 functions which are part of a dummy implementation of the simplex method. Specifically you have to complete the TODOs in:

- *standardise.m*, which writes a maximisation or minimisation input problem in standard form;
- *simplexSolve.m*, which solves a maximisation or minimisation problem using the simplex method.

You are given also some already-implemented functions to help you in your task: *simplex.m* is a wrapper which calls all the functions necessary to find a solution to the linear program; *auxiliary.m* solves the auxiliary problem to find a feasible starting basic solution of the linear program; *printSol.m* is a function which prints the optimal solution found by the simplex algorithm. Finally, *testSimplex.m* presents a series of 6 problems to check if your implementation is correct, before moving to the next part of the assignment. Additional details to aid you in your implementation can be found in the comments inside the code.

3. Applications to Real-Life Example: Cargo Aircraft [25 points]

In this second part of the assignment, you are required to use the simplex method implementation to solve a real-life problem taken from economics (constrained profit maximisation).

A cargo aircraft has 4 compartments (indicated simply as $S_1, \ldots, S_4$) used to store the goods to be transported. Details about the weight capacity and storage capacity of the different compartments can be inferred from the data reported in the following table:

Compartment	Weight Capacity (t)	Storage Capacity (m^3)
S_1	18	11930
S_2	32	22552
S_3	25	11209
S_4	17	5870

The following four cargos are available for shipment during the next flight:

Cargo	Weight (t)	Volume (m^3/t)	Profit (CHF/t)
C_1	16	320	135
C_2	32	510	200
C_3	40	630	410
C_4	28	125	520

Any proportion of the four cargos can be accepted, and the profit obtained for each cargo is increased by 10% if it is put in S_2, by 20% if it is put in S_3 and by 30% if it is put in S_4, due to the better storage conditions. The objective of this problem is to determine which amount of the different cargos will be transported and how to allocate it among the different compartments, while maximising the profit of the owner of the cargo plane. Specifically you have to:

1. Formulate the problem above as a linear program: what is the objective function? What are the constraints? Write down all equations, with comments explaining what you are doing.

   SOLUTION In order to formulate this problem as a linear problem we have to choose the variables. We want to maximize the profit while determining which amount of the different cargos will be transported and where to allocate it among the different compartments. Then we have w_{ij} as the weight in Compartment S_i of the cargo C_j. The objective function is given by the the distribution of the cargos across the compartments:

$$\begin{aligned} z = \quad & (135w_{11} + 200w_{12} + 410w_{13} + 520w_{14}) + \\ 1.1 \cdot & (135w_{21} + 200w_{22} + 410w_{23} + 520w_{24}) + \\ 1.2 \cdot & (135w_{31} + 200w_{32} + 410w_{33} + 520w_{34}) + \\ 1.3 \cdot & (135w_{41} + 200w_{42} + 410w_{43} + 520w_{44}). \end{aligned}$$

   We add the multiplication factors to increase the profit of the compartments as given.
   Let's now consider the constraints. The first constraint is the limit given by the weight capacity of the compartments:

$$\begin{aligned} w_{11} + w_{12} + w_{13} + w_{14} &\leq 18 \\ w_{21} + w_{22} + w_{23} + w_{24} &\leq 32 \\ w_{31} + w_{32} + w_{33} + w_{34} &\leq 25 \\ w_{41} + w_{42} + w_{43} + w_{44} &\leq 17. \end{aligned}$$

   The second constraint is given by the storage capacity of the compartments:

$$\begin{aligned} 320w_{11} + 510w_{12} + 630w_{13} + 125w_{14} &\leq 11930 \\ 320w_{21} + 510w_{22} + 630w_{23} + 125w_{24} &\leq 22552 \\ 320w_{31} + 510w_{32} + 630w_{33} + 125w_{34} &\leq 11209 \\ 320w_{41} + 510w_{42} + 630w_{43} + 125w_{44} &\leq 5870. \end{aligned}$$

The third constraint is given by the fact that the total sum of the partial weights of a cargo across the compartments is limited by the weight of the cargo itself:

$$\begin{aligned}
w_{11} + w_{21} + w_{31} + w_{41} &\leq 16 \\
w_{12} + w_{22} + w_{32} + w_{42} &\leq 32 \\
w_{13} + w_{23} + w_{33} + w_{43} &\leq 40 \\
w_{14} + w_{24} + w_{34} + w_{44} &\leq 28.
\end{aligned}$$

Then the last constrain is:

$$w_{ij} \geq 0, \qquad\qquad \text{for } i, j = 1, 2, 3, 4.$$

All of this with the goal to maximize the profit.
Putting all together we have the following linear program:

$$\begin{aligned}
\max \quad & z = (135w_{11} + 200w_{12} + 410w_{13} + 520w_{14}) + \\
& 1.1 \cdot (135w_{21} + 200w_{22} + 410w_{23} + 520w_{24}) + \\
& 1.2 \cdot (135w_{31} + 200w_{32} + 410w_{33} + 520w_{34}) + \\
& 1.3 \cdot (135w_{41} + 200w_{42} + 410w_{43} + 520w_{44}) \\
\text{s.t.} \quad & w_{11} + w_{12} + w_{13} + w_{14} \leq 18 \\
& w_{21} + w_{22} + w_{23} + w_{24} \leq 32 \\
& w_{31} + w_{32} + w_{33} + w_{34} \leq 25 \\
& w_{41} + w_{42} + w_{43} + w_{44} \leq 17 \\
& 320w_{11} + 510w_{12} + 630w_{13} + 125w_{14} \leq 11930 \\
& 320w_{21} + 510w_{22} + 630w_{23} + 125w_{24} \leq 22552 \\
& 320w_{31} + 510w_{32} + 630w_{33} + 125w_{34} \leq 11209 \\
& 320w_{41} + 510w_{42} + 630w_{43} + 125w_{44} \leq 5870 \\
& w_{11} + w_{21} + w_{31} + w_{41} \leq 16 \\
& w_{12} + w_{22} + w_{32} + w_{42} \leq 32 \\
& w_{13} + w_{23} + w_{33} + w_{43} \leq 40 \\
& w_{14} + w_{24} + w_{34} + w_{44} \leq 28 \\
& w_{ij} \geq 0, \quad \text{for } i, j = 1,2,3,4.
\end{aligned}$$

2. Create a script *exercise2.m* which uses the simplex method implemented in the previous exercise to solve the problem. What is the optimal solution? Visualise it graphically and briefly comment the results obtained (are you surprised of this outcome on the basis of your data?).

   SOLUTION The optimal value resulting from the program written in *exercise2.m* is 41890. That means that the maximum profit of the owner of the cargo plane is 41890 CHF. The *Figure 3* shows the visual representation of the distribution of the cargos across the compartments. It comes to no surprise that the best strategy is to put the highest profitable cargo (C_4) in the compartment with the most increase in profit (S_4). Once S_4 is filled up, we put the rest of C_4 in S_3 (compartment with the second highest increase). We than proceed to add the cargo C_3 to fill S_3, the rest of it goes in S_2 with the first part of C_2. The second part goes in S_1 and fills this compartment up. This is the best way to maximize the profit: we put the highest profitable available cargo in the highest increasing available compartment. With this reasoning the first cargo (C_1) is not transported as the aircraft doesn't have enough capacity and any proportion of the cargos can be accepted ($w_{ij} \geq 0$).

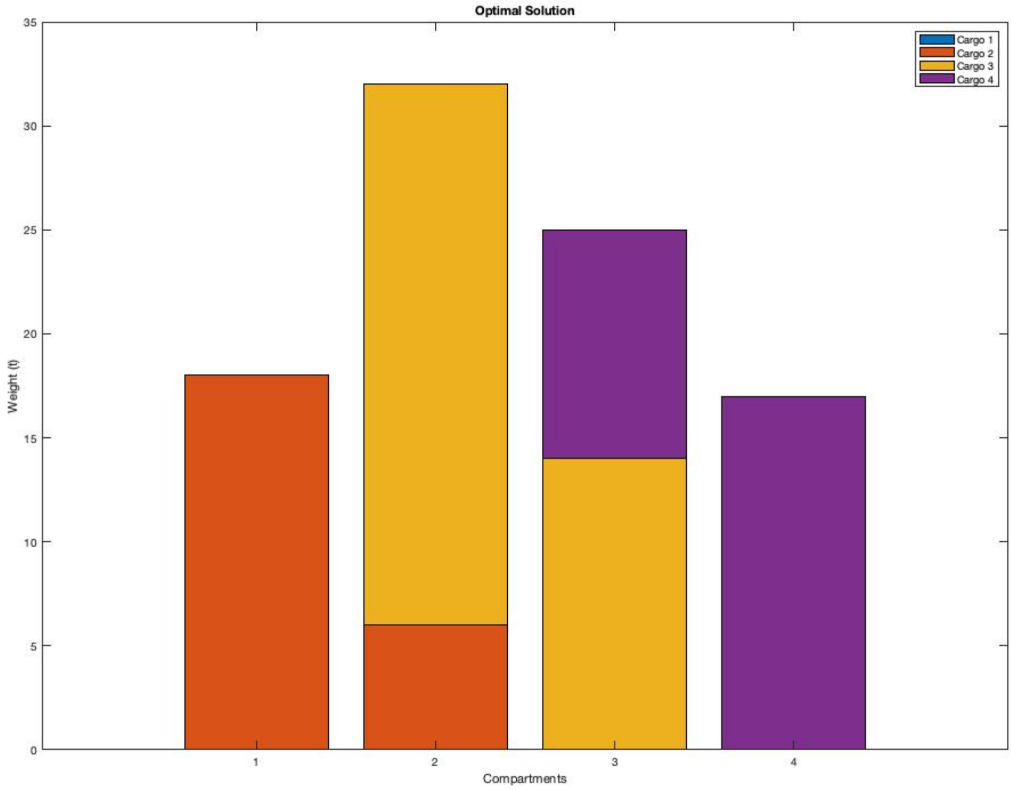

Figure 3: Cargo Aircraft

4. Cycling and Degeneracy [15 points]

Consider now the following simple problem:

$$\begin{aligned} \max \ & z = 3x_1 + 4x_2, \\ \text{s.t.} \ & 4x_1 + 3x_2 \leq 12 \\ & 4x_1 + x_2 \leq 8 \\ & 4x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0. \end{aligned}$$

1. Create a script *exercise3.m* which uses the simplex method implemented above to solve this problem. Do you achieve convergence within the maximum number of iterations (given by the maximum number of possible basic solutions)? Do you notice any strange behaviour? (*hint:* check, e.g., the indices of the entering and departing variables)

   SOLUTION When executing the program I get this message:
   Incorrect loop, more iterations than the number of basic solutions.
   This means the we don't achieve convergence within the maximum number of iterations. This can be caused by the presence of a cycle through the iterations of the algorithm. If we look after the first iteration we have a tie when choosing a basic solution as departing variables (namely S_1 and S_3). Choosing either one will produce a situation where b has a zero.

2. Look at the number of constraints and at the number of unknowns: what can you notice about the underlying system of equations? Represent them graphically and try to use this

information to explain the behaviour of your solver in the previous point.

SOLUTION We can see that we have 2 unknowns and 3 constraints, that means 3 inequalities and 2 unknowns producing an overdetermined system. Let's take a look at the feasible region:

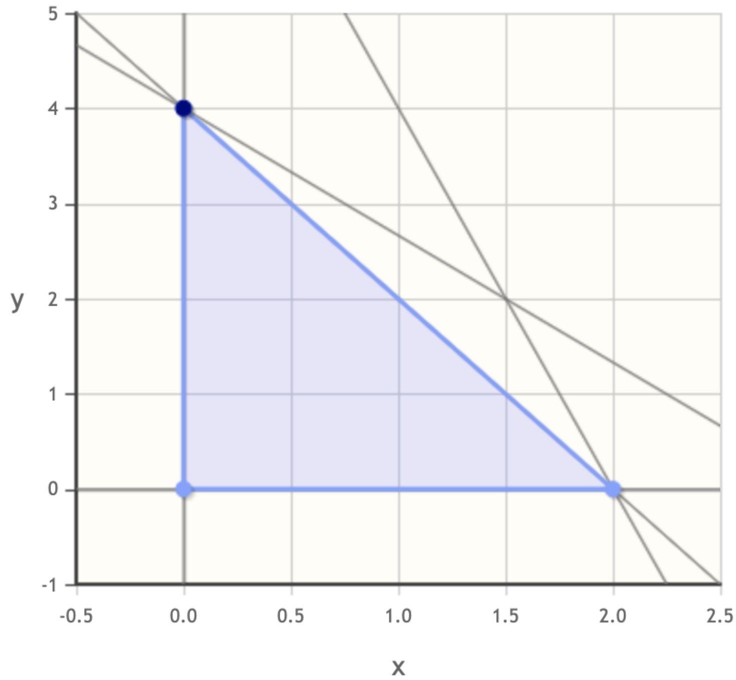

Figure 4: Feasible region

If we look at the intersections we have that the point (0,4) is the optimal solution:

a) (0,0): z = 0

b) (0,4): z = 4*4 = 16

c) (2,0): z = 2*3 = 6

We can see we have three lines crossing this point, this redundant information may be the cause of the cycling in the simplex method as normally we just need two constraints to determine a point, here we have three.