## Dot product

$\vec{v}\cdot\vec{u} = \|\vec{v}\|\cdot\|\vec{u}\|\cdot\cos\theta$

$\text{comp}_{\vec{a}}\vec{v} = \frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|}$ (scalar comp)

$\text{proj}_{\vec{a}}\vec{v} = \frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}$ (vect. proj. of $\vec{v}$ onto $\vec{a}$)

$\vec{v} = \text{proj}_{\vec{a}}\vec{v} + \vec{w}$

$(\vec{v} - \text{proj}_{\vec{a}}\vec{v}) + \text{proj}_{\vec{a}}\vec{v} = \vec{v}$

WORK $= \vec{F}\cdot\vec{d}$

## CROSS PRODUCT

$\|\vec{u}\times\vec{w}\| = \pm\|\vec{v}\|\cdot\|\vec{w}\|\cdot\sin\alpha$ $= |\text{AREA of } \square|$

(i) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

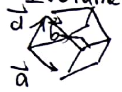
(ii) $c(\vec{a}\times\vec{b}) = (c\vec{a})\times\vec{b} = \vec{a}\times(c\vec{b})$

(iii) $\vec{a}\times(\vec{b}+\vec{d}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{d}$

(iv) $\vec{a}\times(\vec{b}\times\vec{d}) = (\vec{a}\cdot\vec{d})\vec{b} - (\vec{a}\cdot\vec{b})\vec{d}$

(v). $\vec{a}\cdot(\vec{b}\times\vec{d}) = (\vec{a}\times\vec{b})\cdot\vec{d}$

$= \pm$Volume of box $\vec{a}\cdot\vec{b}\cdot\vec{d}$



"scalar triple prod"

## Planes:

General (implicit) Eqn: $Ax+By+Cz = D$

$(x,y,z)\cdot\langle A,B,C\rangle = D$

Standard form: $A(x-x_0)+B(y-y_0)+C(z-z_0) = 0$

Vector (parametric) form: $\langle x,y,z\rangle = \vec{OP} + T_0\vec{PQ} + T_1\vec{PR}$

## Lines:

parametric: $x = x_0+aT,\ y = y_0+bT,\ z = z_0+cT$

Symmetric: $(T=)\ \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

| | // | X |
|---|---|---|
| MEET | SAME | INCIDENT |
| X MEET | PARALLEL | SKEW |

① **Distance: point to plane:**

1° random pt at $Q$

2° take $\|\text{proj}_{\vec{N}}\vec{PQ}\| = \frac{|\vec{PQ}\cdot\vec{N}|}{\|\vec{N}\|}$

② **Dist: point to line:**

$\|\vec{PQ} - \text{proj}_{\vec{d}}\vec{PQ}\|$

$= \|\vec{PQ}\|\cdot\sin\theta = \frac{\|\vec{PQ}\times\vec{d}\|}{\|\vec{d}\|}$

③ **Dist: line to line (skew)**

1° Create plane $H$ // to $l_1$ contain $l_2$

2° take dist of $l_1$ to $H$.

## QUADRATIC SURFACES:

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$. ellipsoid

2. $\frac{z}{c} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$ elliptic paraboloid.

3. $\frac{z}{c} = \frac{x^2}{a^2}-\frac{y^2}{b^2}$ hyperbolic paraboloid

4. $\frac{z^2}{c^2} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$ cone

5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$. 1 sheet hyperbolid

6. $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ 2 sheet

## Partial derivatives

**Linearization** of $f(x,y)$ at $(a,b)$:

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

graph $z = f(a,b)+f_x(a-b)\cdots f_y(y-b)$ is the tangent plane to the graph of the function $f(x,y)$ at $(a,b)$

**Total differentiation:**

$df = f_x(a,b)dx + f_y(a,b)dy$

3VA: $\Delta f = f_x(p)\Delta x + f_y(p)\Delta y + f_z(p)\Delta z$

$f(q)-f(p)$

$f(q)-f(p)$: difference on tangent plane

**IMPLICIT DIFFERENTIATION**

$F(x,y,z) = 0$.

① $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z};\ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

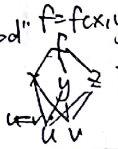
② Differentiate both side w.r.t to $x$ & $y$; treat $z$ as $f(x,y)$.

**CHAIN RULE**

$x = x(u,v),\ y = y(u,v),\ z = z(u,v)$

$f = f(x,y,z)$.

$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}\cdots$



**Directional Derivatives**

$D_{\vec{u}}f = \vec{\nabla}f\cdot\hat{u}$: rate of change of $f$ in direction of $\vec{u}$

$\vec{\nabla}(f) = \langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$.

$\vec{\nabla}f$ is $\perp$ to tangent plane & level curves

& fastest increase.

Find min/max/0 Rate of change:

$D_{\vec{u}}f = \vec{\nabla}f\cdot\vec{u} = \cos\theta\|\vec{\nabla}f\|$;

$\theta$ = Angle between $\vec{u}$ & $\vec{\nabla}f$.

**2nd derivative test**

$D(a,b) = \begin{vmatrix} f_{xx}|(a,b) & f_{xy}|(a,b) \\ f_{yx}|(a,b) & f_{yy}|(a,b)\end{vmatrix}$

$D>0$. $\begin{cases} f_{xx}<0 & \text{loc. Max} \\ f_{xx}>0 & \text{loc. min}\end{cases}$

$D<0$. Saddle

$D=0$ undetermined

**Lagrange Multiplier**

Boundary: $g(x,y)=0$, function: $f(x,y)$.

$\begin{cases}\vec{\nabla}f = \lambda\vec{\nabla}g \\ g(x,y)=0\end{cases}$. $<\ \begin{matrix} f_x = \lambda g_x \\ f_y = \lambda g_y\end{matrix}$

AVG of function:

1-VAR: $\text{AVG}(a,b) f(x) = \frac{1}{b-a}\int_a^b f(x)dx$

2-VAR: $\text{AVG}_D f(x,y) = \frac{1}{\text{AREA}_D}\iint_D f(x,y)dA$

C1:

if $f(x,y) = h(x)g(y)$. then

$\int_c^d\int_a^b f(x,y)\,dy\,dx = \left(\int_c^d h(x)dx\right)\left(\int_a^b g(y)dy\right)$

**POLAR COORDINATES**

$\int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r,\theta)\,r\,dr\,d\theta$ = reverse order.

**AREA. DENSITY. MASS:**

$\iint_D 1\,dx\,dy$ = AREA of D.

MASS: $\iint_D \rho(x,y)\,dA$

C.O.M $= (\bar{x}, \bar{y})$

$\bar{x} = \frac{1}{M}\cdot\iint_D x\rho(x,y)\,dA$ ← Moment y

$\bar{y} = \frac{1}{M}\cdot\iint_D y\rho(x,y)\,dA$. ← moment x.

## Triple Integrals

① If f is continuous, order doesn't matter

General Domain:

$\int_a^b\int_{c(x)}^{d(x)}\int_{e(x,y)}^{g(x,y)} f\,dz\,dy\,dx$

e.g. triangular pyramid (tetrahedron), $3x+2y+z=6$ & coordinate plane:

$\int_0^2\int_0^{-\frac{3}{2}x+3}\int_0^{6-3x-2y} f\,dz\,dy\,dx$

1. Choose coord system:

· Rectangular

· Cylindrical. Same as polar + z there.

· Spherical: $z = \rho\cos\phi$

$r = \rho\sin\phi$

$\rho^2 = x^2+y^2+z^2$

2. Evaluate Integral:

· Rect:

$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$

· Cylindrical:

$\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

· Spherical:

$\int_c^d\int_\alpha^\beta\int_a^b f(x,y,z)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$

→ Write eqn of tangent plane to a surface given by some equation in x, y, z-variables.

① $z = f(x,y)$, linearization of f

② $z = f(x,y)$, $F(x,y,z) = f(x,y)-z$. which is level surface of $F(x,y,z)=0$,

Gradient of F: normal vector to the tangent plane at $(a,b,f(a,b))$

Normal line: $\langle f_x, f_y, -1\rangle$ in $f(x,y)$

tangent line: $\langle 1, 0, f_x(x_0,y_0)\rangle$

& $\langle 0, 1, f_y(x_0,y_0)\rangle$

$(u_1, u_2, D_{\vec{u}}f(x_0,y_0))$ is the tangent line to f in the direction of $\vec{u}$ at $(x_0, y_0)$.

$\int\frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

## Common Integrals

$\int\tan u\,du = \ln|\sec u| + C$

$\int\sec u\,du = \ln|\sec u + \tan u| + C$

$\int\sec^2 u\,du = \tan u + C$

$\int\sec u\tan u\,du = \sec u + C$

$\int\csc u\cot u\,du = -\csc u + C$

$\int\csc^2 u\,du = -\cot u + C$

$\int\frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

$\int\frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + C$

$\int x^{-1}dx = \ln|x| + C$

$\int\frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

$\int\ln x\,dx = x\ln x - x + C$

$\int e^x dx = e^x + C$

IBP: $\int u\,dv = uv - \int v\,du$

$\int\sin^n x\cos^m x\,dx$:

1. n odd: $\sin x(\sin^{n-1}x)\cos^m x$ ← to cos

2. m odd: same

3. n & m odd: 1./2.

4. n & m even.:

$\sin(2x) = 2\sin x\cos x$

$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1-2\sin^2 x$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$\int\tan^n x\sec^m x\,dx$:

1. n odd: use $\tan^2 x = \sec^2 x - 1$

$u = \sec x$, $du = \sec x\tan x\,dx$

2. m even: strip $\sec^2 x$ out; Convert rest to tangent. $u = \tan x$

3. n odd & m even: 1./2.

4. n even & m odd: dealt with differently

$\sqrt{a^2-b^2x^2} \Rightarrow x = \frac{a}{b}\sin\theta$

$\cos^2\theta = 1-\sin^2\theta$

$\sqrt{b^2x^2-a^2} \Rightarrow x = \frac{a}{b}\sec\theta$

$\tan^2\theta = \sec^2\theta - 1$

$\sqrt{a^2+b^2x^2} \Rightarrow x = \frac{a}{b}\tan\theta$

$\sec^2\theta = 1+\tan^2\theta$

$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

pt & Distance between Planes.

$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Derivatives:

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$\frac{d}{dx}(a^x) = a^x\ln(a)$

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$. $x>0$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$, $x\neq 0$

$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln a}$, $x>0$

$\int\frac{1}{\sqrt{x^2-a^2}}dx = \ln|x+\sqrt{x^2-a^2}| + C$

$\int\frac{1}{\sqrt{x^2+a^2}}dx = \ln|x+\sqrt{x^2+a^2}| + C$