Network Security Class
Lab Session 1

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1 Assignment

A C and Matlab codebase is given, containing the implementation of a linear and a nearly-linear Feistel ciphers, plus three sets of plaintext/ciphertext pairs for different cipher instances and keys. The assignment is as follows:

- implement the `feistel_decrypt` procedure;
- carry out a KPA on the linear Feistel cipher with 8 rounds and 32 bit key and plaintext;
- carry out a KPA on the nearly linear Feistel cipher with 4 rounds and 8 bit key and plaintext, and evaluate its success rate;
- carry out a KPA on the nearly linear Feistel cipher with 2 rounds and 32 bit key and plaintext, and evaluate its success rate.

2 Implementing the decryption procedure

Instead of copy/pasting the code, I slightly modified the original `feistel_encrypt` function so that it can work *both as an encryptor and as a decryptor* (thus renaming it `feistel_encdec`).
Since the only difference in encryption and decryption algorithms is the order in which subkeys are generated (one being the reverse of the other), it seemed almost natural to try to implement a single procedure with the capability of accepting a flag that can reverse the order of the subkey generation.

To do this, we need to manipulate the `subkey_cyclic_rotation` function, since its original implementation is *"round-agnostic"*: that is, its output depends only on the current input but not on the round number. This is a correct formulation that strictly resembles the general block diagram of a Feistel cipher, but it doesn't allow us to get keys in reversed order if not by generating all the keys upfront and iterating over them backwards. Looking at this function with a more software-oriented eye, though, we can see that it can be modified so that its output can be *"round-aware"*.
In fact, the output of this function is built by splitting the input array into two halves and by rotating them "outward" by 1 bit (1 bit `ROL` for 1st half, 1 bit `ROR` for 2nd half).
If we compare the output of this function at round j with the original key k, we see that the subkey k_j is just the original key k, splitted in two halves and outward-rotated by j bits (assuming j starts from 1). This tells us that if we parametrize the amount of bits rotated by the function, we can obtain a general subkey k_j with a single call.
The resulting function is called `half_outward_shift`; to use it in the new Feistel encryptor/decryptor procedure, we need to modify the code so that it passes the correct parameters to it.

If we just want to retain the original behavior of the `feistel_encrypt` function, we only need to eval at each round `subkey_generation` with the original key and round number as parameters.
However, we also need to use the same cycle to perform ciphertext decryption. This can be done by replacing the plain round number i passed to the subkey generation function with the function $abs(i-D)$, where D is a constant that evaluates to `0` if we want to encrypt the input, and to `nr_rounds` if we're going to decrypt the message.

As a recap, we:

- created a new subkey generation function `half_outward_shift` which can calculate subkey k_j with a single call starting from original key and round number

- renamed `feistel_encrypt` to `feistel_encdec` and modified it so that it can perform both encryption and decryption by setting a flag parameter
- wrapped calls to `feistel_encdec` into two helper functions `feistel_encrypt` and `feistel_decrypt`

To prove that the resulting code is correct, it is sufficient to perform the decryption of the encrypted message a certain number of times over a certain number of random messages and see that the result of the composition of the functions equates to the original message:

```
for i = 1:very_large_number
  msg = dec2hex(randi([0,2^32-1],1,1))
  assert(isequal(
    msg,
    feistel_decrypt(
      feistel_encrypt(msg, key, 4, ...
        @linear_round_function, @half_outward_shift),
      key, 4, @linear_round_function, @half_outward_shift),
    'u and u_hat are supposed to be equal'))
end
```

3 Linear cryptanalysis

Mathematical model

Since it's a given assumption that the cipher implemented using `@linear_round_function` is indeed linear, we can treat the encryptor as a linear system and analyze it with the binary version of a *pulse train*. In our case, given $k \in \mathbb{B}^{32}$, $u \in \mathbb{B}^{32}$, $x \in \mathbb{B}^{32}$ (respectively, the key, plaintext message and ciphertext), we can model the encryptor as:

$$x = \mathbf{A}k + \mathbf{B}u$$

where $\mathbf{A}$ and $\mathbf{B}$ are two (32×32) square matrices such that $\mathbf{A}, \mathbf{B} \in \mathbb{B}^{32} \times \mathbb{B}^{32}$.

That said, under the assumption that $\mathbf{A}$ is non-singular we can find the solution of the linear system for a given known plaintext-ciphertext as:

$$k = \mathbf{A}^{-1}(x - \mathbf{B}u)$$

In this case, since the system is linear, we need to solve this system only for one given plaintext-ciphertext pair. Solving the system for all the other pairs must give the same key.

System Analysis

With this structure in mind, we now have to derive the two matrices from the practical implementation to fill in the model and actually obtain the information we're looking for.
Deriving the two matrices can be done in two symmetric steps that apply the following idea to both the plaintext and the key:

1. We set one of the two inputs of the encryptor block to the 4-words binary representation of `0` (i.e. `00000000`)
2. We let the other input to be the binary representation of the number `1`, also as a 4-word message (i.e. `00000001`)
3. We run the encryptor with these particular inputs. It can be easily seen that the output is the value of the 31st column of either the $\mathbf{A}$ or $\mathbf{B}$ matrix, depending on how we initialized the inputs
4. We iterate all of the above over the inputs updated as follows:
    - The input set to `0` remains the same
    - The pulse input is left-shifted by 1 bit.

If we memorize the columns obtained as ciphertexts, after shifting the pulse array by its whole length we can obtain the full $\mathbf{A}$ or $\mathbf{B}$ matrix. Then, we can just perform the same procedure another time swapping the roles of the two inputs to obtain the other missing matrix.

Implementation Caveats

We developed our model under the assumption of matrix $\mathbf{A}$ being invertible. Unfortunately, it turns out that with the given Feistel cipher implementation, the key multiplication matrix doesn't have full rank but instead has

2 linearly dependent rows/columns. This prevents us to invert the **A** matrix. Still, we can get over this problem by using Gaussian elimination over a Galois field, provided by the `gflineq` function (see References). It is worth noting how this little complication in the resolution of the system eventually proves the cipher to be even weaker than supposed: in fact, since $rank(\mathbf{A}) = 30$, for each given key, there are $2^2 - 1$ that represent the same exact map between message and ciphertext spaces. This mean one could reduce the search space by a factor 4 in case of a bruteforce attack. This doesn't give a direct advantage to the cryptanalyst in this case, since solving the linear equation is a more efficient way to break the system than bruteforcing; however, it will be shown later how this helps in breaking the nearly-linear Feistel cipher.

Equivalent Keys

Given the last observation it's also worth noting how we can find the set of equivalent keys corresponding to a particular solution.
The focal point is the XORing of the key bits in the round function. Basically, we can see from the truth table of the XOR function that given two bits x and y, $x \oplus y = \bar{x} \oplus \bar{y}$. This in itself means that given any input, encrypting or decrypting with a key produces the same output of applying the same operation with its **bitwise complement**.
Furthermore, if we look at the cipher implementation, we notice that the subkey generation function treats the key as two independent halves (bits belonging to one half of the key never end up in the other half after subkey generation). At the same time, the round function relates each bit of the input with two **adjacent** bits of the key. Putting these two last observations together, we obtain that the composition of the round function and subkey generation is done independently on the two halves of the key and the input. This, combined with what already said, means that we can obtain equivalent keys by picking for each half of the key **its original form or its bitwise complement**. It's easy to see how this freedom partitions the key space in sets of equivalent keys each of size 4.

To better explain, let's look at an example. Given a key

```
12345678 == 0001001000110100|0101011001111000
```

its equivalent are:

```
EDCBA987 == 1110110111001011|1010100110000111
1234A987 == 0001001000110100|1010100110000111
```

```
EDCB5678 == 1110110111001011|0101011001111000
```

Attack Implementation

The impulse response of the system is calculated in the `impulse_response` function. It outputs the two matrices **A** and **B**. This function is then called by the `feistel_linear_cryptanalysis` script, which calculates the key for the first known-text pair and then compares it with the other known pairs as a proof of correctness.

The key resulting from running the cryptanalysis script with the KPA set #3 is:

```
C3F8177E
```

and its equivalent are:

```
3C07E881
3C07177E
C3F8E881
```

4 Nearly-linear cryptanalysis (4 rounds, 8 bits)

In this case the short length of the key makes quite feasible the simplest of the attacks: bruteforcing. Furthermore, we can rely on the properties described in the preceding section to even lower the number of keys we need to try.

With a plain bruteforce attack, we would need to scan $2^{l_k} = 2^8 = 256$ keys. But, if we consider that the key space is partitioned into subsets of cardinality 4, then we need to try just one key per subset to decide if it works or not. To do this, we divide the key in two halves, each of length $l_t = l_x/2$. Then it's easy to see that given a binary representation of a number $n \leq 2^{l_t}/2 - 1$, for its complement $\bar{n}$ holds $\bar{n} \geq 2^{l_t}/2$. So, for every key half, we just need to count up to $2^{l_t}/2 - 1 = 2^{l_t - 1} - 1 = 7$. Combining the two halves, we need to scan a total of $2^{l_t-1} \cdot 2^{l_t-1} = 2^{2l_t-2} = 2^{l_x-2} = 2^6 = 64$ keys, as expected given the cardinality of the subsets.
If we find a matching key, we can then rely on the structure of equivalent keys depicted in the preceding section to derive the other 3 matching keys belonging to the same subset as the key found by trial.

This refined bruteforce attack is implemented in the `feistel_nearly_linear_bruteforce` function, which can be called as:

```
keys = feistel_nearly_linear_bruteforce('KPAnonlinear8.hex', 8, 4);
```

Then `keys` contains all the keys that successfully encrypted known plaintexts into their respective known ciphertexts.

In our case, the set of matching keys is the following:

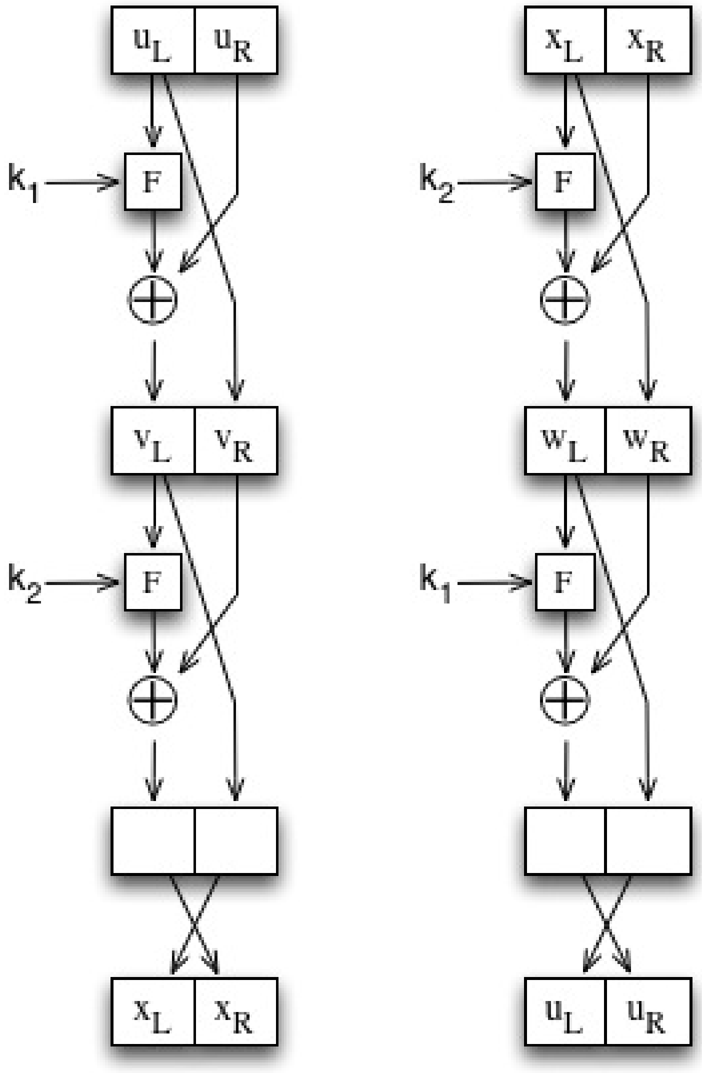
```
45
BA
4B
B4
```

which means that there is a single subset of equivalent keys that correctly encrypt *all* plaintexts. That said, we can state that the success probability of the attack is 1.

5 Nearly-linear cryptanalysis (2 rounds, 32 bits)

Since only 2 rounds are performed here, we can use a *meet in the middle* technique.
That is, we can exploit the relation between plaintext and ciphertext and try to invert this relation in a fashion similar to what we've done with the linear cipher.

Let's start by analyzing the structure of the particular Feistel cipher under the circumstances given by the assignment.



By looking at the figure, we can setup the following equations (2 pairs for the encryptor, 2 pairs for the decryptor):

$$v_L = F(u_L, k_1) \oplus u_R \qquad v_R = u_L \qquad x_R = F(v_L, k_2) \oplus v_R \qquad x_L = v_L$$

$$w_L = F(x_L, k_2) \oplus x_R \qquad w_R = x_L \qquad u_R = F(w_L, k_1) \oplus w_R \qquad u_L = w_L$$

Then, by expanding the first 4 equations we see that:

$$x_R = F(v_L, k_2) \oplus u_L \qquad x_L = F(u_L, k_1) \oplus u_R$$

$$x_R = F(x_L, k_2) \oplus u_L \qquad x_L = F(u_L, k_1) \oplus u_R$$

Then, by XORing on both sides of the equations we obtain:

$$x_L \oplus u_R = F(u_L, k_1)$$

$$x_R \oplus u_L = F(x_L, k_2)$$

The same result can be obtained by exploting the relationships in the last 2 pairs of equations coming from the decryptor's model.

We can see how the left term of both equations is well-known; if we had a way to calculate the inverse of the round function with respect to the two subkeys, then we could simply retrieve the key given one known text pair. This, however, can't be done in this case in a deterministic way since the round function is non-linear. Nonetheless, we can make the relation between the round function's arguments explicit and see that we can still gather few bits of information given a known text pair. We begin by writing down the bitwise expression for F. Given:

$$w = F(k, y)$$

and

$$w = w_{15}\ w_{14}\ ...\ w_1\ w_0$$

$$y = y_{15}\ y_{14}\ \cdots\ y_1\ y_0$$

$$k = k_{31}\ k_{30}\ ...\ k_1\ k_0$$

then

$$w_{2j} = y_{2j} \vee (k_{2j} \oplus k_{2j+1})$$

(that is, each output bit is the OR between the input bit at same position and the XOR of a pair of adjacent bits of the key).

That said, we can look at the truth table of the OR function and infer the following:

$$\forall j : y_{2j} = 0 \Rightarrow k_{2j} \oplus k_{2j+1} = w_{2j}$$

so, we have a way to infer valid linear equations between bits of the key (note that we have to account for different subkeys being used in different rounds; this is easy to handle since we know how the bits positions change between subkeys).

We can then collect each valid equation and put it in a system; in the ideal (from the attacker standpoint) case, we then obtain a full rank system of linear equations that can be easily solved to recover all key bits.

However, since there's no guarantee of obtaining a good amount of linearly independent equations, we need to account for a certain amount of freedom when solving the system.

In fact, in the given case, applying this technique holds a matrix of rank **29** instead of the expected **30**. In should be noted that, whatever the rank obtained, any particular solution of the given system will correctly encrypt all known plaintexts into their corresponding known ciphertexts (and viceversa); this result holds by construction of the linear equation system itself. However, unless the rank is full, we still can't be sure that a particular key solving the system is the *right* key we're looking for.

A step further can be done in this direction by trying to list *all* keys satisfying the system. This can be done by collecting those expressions that didn't allow us to infer valid equations for the system during the discovery phase; then, they can be appended to the matrix by manually setting the corresponding constant each time at either 0 or 1. This way we can perform a brute force attack just on the part of the problem we can't gather information about. This allows to reduce the search space from 2^{32} to 2 keys.

This process has been implemented in two functions: `find_valid_lineq_for_mitm` and `feistel_mitm_cryptanalysis`. The first discovers all equations for a given plain-ciphertext pair, returning two matrices, one containing valid equations and one containing the expressions representing the degree of freedom.
The second function, instead, iterates over all given text pairs building the overall system of linear equations, then applying those that have free constants and solving it iteratively, each time setting the free constants to a different value.

In the particular case solved with this technique, `find_valid_lineq_for_mitm` were able to find 75 equations over the key bits, put into a matrix of rank 29, and 2 equations with unkown constant value. For 2 instances of the 4 possible systems obtainable by setting the unknown constants, there were no solution; the other two, instead, provided the two expected keys; these keys, in turn, given the considerations done in the preceding sections, form a set containing them, their complements and combinations of the two. As a result, we can say that the key used to produce the given text pairs belong to the following set:

```
E176A1CE
1E895E31
E1765E31
```

```
1E89A1CE
1E76A1CE
E1895E31
1E765E31
E189A1CE
```

Furthermore, given the considerations about equivalent keys, we can state that the set is composed of 2 partitions, each containing 4 equivalent keys, so with just the given set of known text pairs if we pick a key to decrypt further transmissions the success rate is 50% (or 100% if we consider decrypting with two distinct keys and under the assumption we can discriminate between good and bad decrypted messages).

6 References

- Since in Octave there isn't a function to solve systems of linear equations over a Galois Field, a function `gflineq` has been added, whose original implementation has been found at:
  `http://read.pudn.com/downloads64/sourcecode/others/224341/ldpc_toolkit/gflineq.m__.htm`.