Decidable Reasoning About Time in Finite-Domain Situation Calculus Theories

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Abstract

Representing time is crucial for cyber-physical systems and has been studied extensively in the situation calculus. The most commonly used approach represents time by adding a real-valued fluent $\text{time}(a)$ that attaches a time point to each action and consequently to each situation. We show that in this approach, checking whether there is a reachable situation that satisfies a given formula is undecidable, even if the domain of discourse is restricted to a finite set of objects. We present an alternative approach based on well-established results from timed automata theory by introducing clocks as real-valued fluents with restricted successor state axioms and comparison operators. With this restriction, we can show that the reachability problem for finite-domain basic action theories is decidable. Finally, we apply our results on Golog program realization by presenting a decidable procedure for determining an action sequence that is a successful execution of a given program.

1 Introduction

Time is a ubiquitous concept in various applications, especially in cases where digital systems interact with the real world such as applications in robotics or generally in cyber-physical systems. Early approaches model discrete time, which allows to reason about and model system states at discrete points in time. This approach gives rise to the problem of choosing an appropriate granularity (resolution) of time to facilitate useful reasoning. The introduction of models which incorporate continuous and dense notions of time circumvents this problem and allows for more expressive modeling and analysis approaches. Dense time has been under extensive investigation in the formal methods community, e.g., in the analysis of hybrid systems which started with the introduction of timed automata (TAs) (Alur and Dill 1994).

The situation calculus (SC) is a well-known first-order formalism for reasoning about dynamically changing worlds, where each change in the world is caused by an *action*. Among others, it permits a representation of continuous time by attaching a time of occurrence to each action and hence also each situation (Pinto and Reiter 1995; Reiter 1996). However, while these approaches are quite expressive, we show that they are problematic if we are concerned about decidability. In this paper, we focus on *reachability*, i.e., checking whether there is a reachable situation where some formula ϕ holds. We show that with the established model of

time in the SC, the reachability problem is undecidable. To overcome this, we present a restricted representation of time, which is inspired from timed automata theory. Rather than allowing arbitrary real-valued fluents with the usual operators $+, \times, <$, we introduce real-valued *clocks* that may only be compared against natural numbers and which can be reset to zero by an action, but not changed otherwise. We show that with these restrictions, the reachability problem becomes decidable, at least for domains with a finite number of objects. We then extend this result to GOLOG programs and investigate *program realization*. Given a GOLOG program, we present a decidable method for determining a corresponding action sequence that results in a final program configuration.

This paper is organized as follows: We discuss related work in Section 2 and present the foundations of the SC in Section 3. We discuss reachability in Section 4, where we show undecidability for the existing approaches and then show a decidable approach in detail. In Section 6, we apply these results to program realization and conclude in Section 7.

2 Related Work

Already McCarthy (1963) proposed a fluent function $time$ that gives the value of time in a situation. Elaborating on this idea, Gelfond, Lifschitz, and Rabinov (1991) described an extension where time is a fluent with values from the integers or real numbers and where each action has a duration. Miller (1996) attached time to each action and allowed overlapping actions with durations, along with partial ordering of actions that allows to describe two concurrent sequences of actions. Pinto and Reiter (1995) modeled durative actions with instantaneous *start* and *end* actions and continuous time to describe actions and events. By distinguishing actual and possible situations, they define a total ordering on situations allowing to express linear-time temporal properties within the SC. Several approaches (Miller 1996; Reiter 1996) allow modeling continuous processes, where a fluent continuously changes its value between two situations. The *hybrid situation calculus* (Batusov, De Giacomo, and Soutchanski 2019) combines the SC with hybrid systems by embedding *hybrid automata* (Alur et al. 1993). Finzi and Pirri (2005) combine the SC with temporal constraint reasoning.

The *event calculus* (Kowalski and Sergot 1986; Mueller 2008) uses an explicit representation of discrete or continu-

ous time, where *events* occur at certain points on the timeline. In contrast to the SC, formulas do not refer to situations, but to points in time, and it uses circumscription to solve the frame problem. A comparison of the two formalisms can be found in (Van Belleghem, Denecker, and De Schreye 1997). As there is no action programing language such as GOLOG in the event calculus, we focus on the SC.

Timed automata (TAs) (Alur and Dill 1994) extend finite automata with continuous time. While the reachability problem for the original model is decidable, several extensions render it undecidable (Alur and Madhusudan 2004). For TAs with *diagonal* constraints of the form $x - y \bowtie c$ (in contrast to $x \bowtie c$), reachability remains decidable (Alur and Dill 1994), while it is undecidable with *additive constraints* $(x + y \bowtie c)$ (Bérard and Dufourd 2000). Allowing to reset clocks to non-zero values also affects decidability: In the presence of diagonal-free clock constraints, resets of clocks such as $x := x + 1$ maintain decidability, while in the presence of diagonal clock constraints, reachability is undecidable. On the other hand, updates of the form $x := x - 1$ always result in undecidability (Bouyer et al. 2004). As the boundary of decidability for timed automata is well-studied, it seems reasonable to adopt the underlying model of time in the SC.

3   The Situation Calculus

The situation calculus (SC) (McCarthy 1963; Reiter 2001) is a sorted predicate logic for reasoning about dynamically changing worlds. All changes in the world are caused by *actions*, which are terms in the language. A *situation* is a possible history of the world, where the distinguished function symbol $\mathrm{do}(\alpha, \sigma)$ describes the resulting situation of doing action α in situation σ and the constant S_0 denotes the initial situation where no actions have yet occurred. Besides actions and situations, the sort *objects* is used for terms describing objects of the world. *Fluents* are relational or functional symbols that describe possibly changing properties of the world and that have a situation as their last argument. Relational and functional symbols that remain unchanged between situations are called *rigid* and do not have a situation argument. We denote relational symbols with uppercase letters, e.g., $IsFull(p, s)$ and functional symbols with lower-case letters, e.g., $counter(s)$. Apart from the situation argument, all arguments of a fluent are of type object. We assume that there is a finite set of fluent symbols $\mathcal{F}$ and a finite number of action types. In contrast to the usual definition, we also assume a *finite set of objects* $\mathcal{O}$, for which we require the unique name assumption and domain closure. For an action type A, we write $|A|$ for the number of arguments of A. In the following, s denotes situation variables, σ ground situation terms, a action variables, α ground action terms, A action types, p object variables, ρ ground object terms, and $\vec{x}$ a tuple of variables $x_1, \ldots, x_k$ of the corresponding sort (similarly for terms). We also write $\phi_1 \supset \phi_2$ for $\neg\phi_1 \vee \phi_2$ and $\phi_1 \equiv \phi_2$ for $\phi_1 \supset \phi_2 \wedge \phi_2 \supset \phi_1$, where $\equiv$ has lowest operator precedence and free variables are assumed to be universally quantified from the outside, e.g., $F(\vec{o}, s) \equiv \phi_1 \vee \phi_2$ stands for $\forall \vec{o}, s.(F(\vec{o}, s) \equiv (\phi_1 \vee \phi_2))$. Finally, we write $\mathrm{do}([\alpha_1; \ldots; \alpha_k], S_0)$ for $\mathrm{do}(\alpha_k, \mathrm{do}(\ldots, \mathrm{do}(\alpha_1, S_0)))$.

A formula is *uniform in* s if it does not mention any situation terms other than s and does not mention Poss (see (Reiter 2001) for a formal definition). A *situation-suppressed formula* is a formula where the situation argument of each fluent is omitted. For a formula ϕ and a situation term σ, we write $\phi[\sigma]$ for the formula obtained by replacing each occurrence of s with σ. Similarly, for a situation-suppressed formula ϕ, $\phi[\sigma]$ is like ϕ with every fluent augmented by the situation argument σ.

The SC has been extended to represent time by adding an additional real-valued sort *time* with the standard interpretation of the real numbers and their operations (addition, subtraction, etc.) and relations $(<, \leq$, etc.) and augmenting each action term $A(\vec{x}, t)$ with an additional time argument t, which specifies the time point when the action occurs (Pinto and Reiter 1995; Reiter 2001). An additional function symbol $\mathrm{time}(\alpha)$ is used to describe the starting time of the action situation, i.e., $\mathrm{time}(A(\vec{x}, t)) := t$. This allows axiomatizing the starting time of a situation with $\mathrm{start}(\mathrm{do}(A, s)) = \mathrm{time}(A)$. With real numbers as an additional sort and independent of whether actions have an execution timepoint, we may also allow *real-valued fluent functions*. As we will see in Section 4, having either action timepoints or real-valued fluent functions without any constraints renders the reachability problem undecidable.

Basic Action Theories

A basic action theory (BAT) (Pirri and Reiter 1999) is an axiomatization of a domain and describes all available actions with precondition axioms and successor state axioms. In addition to the domain-specific axioms, a BAT also contains the foundational axioms of the SC. Similar to (De Giacomo, Lespérance, and Patrizi 2012), we include domain closure axioms for action types and objects. Here, we axiomatize a finite set of objects.

Definition 1. *A basic action theory (BAT) is a set of sentences $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_{poss} \cup \mathcal{D}_{ssa} \cup \mathcal{D}_{ca} \cup \mathcal{D}_{co} \cup \Sigma$ where:*

- $\mathcal{D}_0$ *is a description of the initial situation with a set of sentences uniform in S_0. Here, we assume to have complete information, i.e., for every atomic (fluent or rigid) formula ϕ uniform in S_0, either $\mathcal{D} \models \phi$ or $\mathcal{D} \models \neg\phi$.*

- $\mathcal{D}_{poss}$ *are precondition axioms of the form $\mathrm{Poss}(A(\vec{o}), s) \equiv \phi_A(\vec{o}, s)$, one per action type $A \in \mathcal{A}$ and where $\phi_A(\vec{o}, s)$ is a formula uniform in s that describes when the action $A(\vec{o})$ can be performed.*

- $\mathcal{D}_{ssa}$ *are successor state axioms (SSAs) describing action effects. For each relational fluent $F \in \mathcal{F}$, $\mathcal{D}_{ssa}$ contains an axiom of the form $F(\vec{o}, \mathrm{do}(a, s)) \equiv \phi_F(\vec{o}, a, s)$, where $\phi_F(\vec{o}, a, s)$ is a formula uniform in s that describes how the fluent $F(\vec{o})$ changes if some action a is performed. For each functional fluent $f \in \mathcal{F}$, $\mathcal{D}_{ssa}$ contains an axiom of the form $f(\vec{o}, \mathrm{do}(a, s)) = y \equiv \phi_f(\vec{o}, y, a, s)$, where $\phi_f(\vec{o}, y, a, s)$ is a formula uniform in s describing how the fluent $f(\vec{o})$ changes if an action a is performed.*

- $\mathcal{D}_{ca}$ *are domain closure and unique name axioms (UNAs)*

on actions $\mathcal{A}$, e.g.,

$$\forall a(\exists \vec{x}_1(a = A_1(\vec{x}_1)) \vee \ldots \vee \exists \vec{x}_n(a = A_n(\vec{x}_n)))$$
$$\forall \vec{x}_i \vec{x}_j (A_i(\vec{x}_i) \neq A_j(\vec{x}_j)) \, for \, all \, i \neq j$$
$$\forall \vec{x}_1 \vec{x}_1' (A_1(\vec{x}_1) = A_1(\vec{x}_1') \supset \vec{x}_1 = \vec{x}_1') \wedge \ldots$$
$$\wedge \, \forall \vec{x}_n \vec{x}_n'(A_n(\vec{x}_n) = A(\vec{x}_n') \supset \vec{x}_n = \vec{x}_n')$$

- $\mathcal{D}_{co}$ *are domain closure and UNAs on object constants* $\mathcal{O}$, *e.g.,* $\forall o(o = \rho_1 \vee o = \rho_2 \vee \ldots) \wedge \rho_1 \neq \rho_2 \wedge \ldots$.
- Σ*: foundational, domain-independent axioms, possibly including* time *and* start *from above and including the relation* $\sqsubset$ *that provides an ordering relation on situations.*

We call $\mathcal{D}$ a *timed BAT* if it includes axioms for time, start, etc. from above and we say $\mathcal{D}$ is *real-valued* if it includes a fluent of sort *time*. Note that these are two independent properties: We may have occurrence times for actions without having any real-valued fluents and we may use real-valued fluents without assigning any time values to actions. In the following, we assume that situation terms only mention action types from $\mathcal{A}$ and object terms from $\mathcal{O}$. Following (Reiter 2001), a situation s is *executable*, written $\mathrm{Exec}(s)$, if every action performed in reaching s is possible in the situation where it occurred: $\mathrm{Exec}(s) := \mathrm{do}(a, s') \sqsubseteq s \supset \mathrm{Poss}(a, s')$.

Regression A fundamental reasoning task in the SC is *projection*, i.e., determining whether a sentence ϕ is true in a future situation. A common method to solve projection is *regression* (Pirri and Reiter 1999), which reduces the formula ϕ to an equivalent formula $\mathcal{R}[\phi]$ which only refers to the initial situation. Given a BAT, regression substitutes each fluent mentioned in ϕ by the right-hand side of the corresponding SSA until every fluent only mentions S_0. Regression is sound and complete and thus may be used for reducing queries about a future situation to the initial situation. We introduce a variant of regression in Section 5.

4 Reachability

Generally speaking, reachability asks whether there is a reachable situation where some property holds. Formally, given a BAT $\mathcal{D}$ and a formula ϕ, ϕ is *reachable* if there is an executable situation where ϕ holds, i.e., if $\mathcal{D} \models \exists s \, \mathrm{Exec}(s) \wedge \phi[s]$. Many problems such as planning or verification of safety properties can be formulated as a reachability problem. It is therefore desirable to have a sound and complete method to decide reachability. However, depending on the expressivity of the underlying framework, the reachability problem may become undecidable. It is easy to see that for a given BAT over a finite set of objects that does not make use of time, the reachability problem is decidable. On the other hand, if we include fluents of sort *time* and allow arbitrary arithmetic operations on those fluents, then the problem is no longer decidable:[1]

Theorem 1. *The reachability problem for a real-valued BAT is undecidable.*

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[1]Full proofs can be found in the appendix.

Proof Idea. By reduction of the halting problem for 2-counter machines (2CMs). Given a 2CM $\mathcal{M}$, we can construct a BAT $\mathcal{D}_{\mathcal{M}}$ such that $\mathcal{D}_{\mathcal{M}} \models \exists s. \, \mathrm{Exec}(s) \wedge Halt(s)$ iff $\mathcal{M}$ halts. $\square$

Intuitively, the problem is that we may use time fluents as counters for 2CMs. These counters may be unbounded because time fluents are not bounded. A natural restriction is thus adding an upper bound b to all fluents of sort *time*, resulting in the *time-bounded reachability problem*: Given a bound b, a BAT $\mathcal{D}$, and a property ϕ, does $\mathcal{D} \cup \{\forall t. \, t \leq b\} \models \exists s \, \mathrm{Exec}(s) \wedge \phi[s]$ hold? Unfortunately, this restriction is not sufficient to accomplish decidability:

Theorem 2. *The time-bounded reachability problem for a real-valued BAT and a time bound $b \geq 2$ is undecidable.*

Proof Idea. Again by reduction of the halting problem for 2CMs. A counter value u is encoded by a fluent value 2^{1-u}. A counter is decremented by doubling and incremented by halving the corresponding fluent and tested for zero by comparing the corresponding fluent to 2. $\square$

These results suggest that we need to restrict the arithmetic operators on time fluents such that time may not change its value arbitrarily, but is monotonically increasing. However, this is not sufficient; even if we do not allow real-valued fluents but instead only consider actions with time arguments, the reachability problem remains undecidable:

Theorem 3. *The reachability problem for timed BATs without any fluents of sort* time *is undecidable.*

Proof Idea. By reduction of the language emptiness problem for TAs with 4 clocks $c_1, \ldots, c_4$ extended with additive clock constraints of the form $c_1 + c_2 = 1$, which is undecidable (Bérard and Dufourd 2000). For a TA $\mathcal{A}$, we can construct a BAT and a formula ϕ such that $\mathcal{D} \models \exists s. \, \mathrm{Exec}(s) \wedge \phi[s]$ iff the language of $\mathcal{A}$ is non-empty. $\square$

5 Clocks in the Situation Calculus

In this section, we propose an alternative approach for modeling time in the SC that maintains decidability. We adopt methods from timed automata theory by defining *clocked BATs* and show that the reachability problem for clocked BATs is decidable. In particular, we introduce *clocks* as real-valued functional fluents with restricted SSAs that guarantee uniform passing of time and which allow an action to reset a clock to zero. Furthermore, we require that the BAT uses a distinguished action $wait$ that causes time to pass. Apart from $wait$, all actions are instantaneous, i.e., they do not have a time argument and there is no delay between the previous and the current action. This allows to use clocks both in action preconditions and successor state axioms. We restrict formulas mentioning clocks to comparisons to natural numbers:

Definition 2. *We distinguish the following types of formulas:*

1. *A formula ϕ is a* clock comparison *if it is of the form $f(\vec{x}, s) \bowtie v$ or $v \bowtie v'$, where $f \in \mathcal{F}$ is some functional fluent, $v, v' \in \mathbb{N}$, and $\bowtie \in \{<, \leq, =, \geq, >\}$.*

2. *A formula ϕ is called* clocked *if every atomic sub-formula mentioning a term of sort* time *is a clock comparison.*
3. *A formula ϕ is called* time-independent *if it does not mention any term of sort* time.

As we will see later, these restrictions will allow us to apply *regionalization*, which defines an equivalence relation on clock values. Based on regionalization, we will define a *time-abstract bisimulation*, resulting in a *finite* abstraction of the otherwise infinite tree of situations, which permits a decidable procedure for the reachability problem. We start by introducing clocks to BATs:

Definition 3. *A BAT $\mathcal{D}$ is called a* clocked BAT *if:*

1. *Every func. fluent f is real-valued, i.e., of domain* time;
2. *For each func. fluent f, $\mathcal{D}_0$ contains an axiom $f(\vec{o}) = 0$;*
3. *For each func. fluent f, $\mathcal{D}_{ssa}$ contains an axiom of the form*

$$f(\vec{o}, \mathrm{do}(a, s)) = y \equiv \exists t.\, a = wait(t) \wedge y = f(\vec{o}, s) + t$$
$$\vee\, a \neq wait(t) \wedge (\phi_f(\vec{o}, a, s) \wedge y = 0 \vee \neg \phi_f(\vec{o}, a, s) \wedge y = f(\vec{o}, s))$$

where $\phi_f(\vec{o}, a, s)$ is a time-independent formula uniform in s describing if the action a resets the value of $f(\vec{o})$;
4. *For every relational fluent R, $\mathcal{D}_{ssa}$ contains an axiom of the form $R(\vec{o}, \mathrm{do}(a, s)) \equiv \phi_R(\vec{o}, a, s)$ where $\phi_R(\vec{o}, a, s)$ is a clocked formula uniform in s.*
5. *Every precondition axiom in $\mathcal{D}_{poss}$ is of the form $\mathrm{Poss}(A(\vec{x})) \equiv \phi_A(\vec{x}, s)$ such that $\phi_A(\vec{x}, s)$ is a clocked formula uniform in s;*
6. *Waiting is always possible: $\mathrm{Poss}(wait(t), s) \equiv \top$;*

For the sake of simplicity, we assume that all functional fluents are of domain *time* and are initialized to zero. While this is not strictly necessary, it simplifies the treatment of functional fluents later on. The restricted form of the SSAs of functional fluents enforce a behavior similar to clocks in timed automata: An action $wait(t)$ increments each clock value by the value t; all other actions either leave the value unchanged or reset it to zero.[2] Hence, we will also call these fluents *clocks*. We write $\mathcal{C}$ for the set of all ground situation-suppressed clock terms, we denote clock variables with $c, c_1, \ldots$ and (possibly situation-suppressed) ground clock terms with $\omega, \omega_1, \ldots$. Finally, we also restrict the usage of clocks in all axioms of a clocked a BAT: Both precondition and successor state axioms may only use clocked formulas, which restrict any operations on clocks to clock comparisons. As we will see later on, this is an essential restriction for allowing regionalization. However, it is easy to show that clocked BATs are expressive enough to represent TAs.

We use a variant of the coffee delivery robot (Reiter 2001) as running example:

Example 1. *We model a coffee delivery service that has the durative actions* brew *and* pour *and objects* Pot, Mug_1, *and* Mug_2. *The robot needs to brew coffee and then pour*

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it into the mugs. The fluents Brew *and* Pour *indicate that the robot is currently brewing (respectively pouring) coffee. After brewing, the pot* IsFull *and after pouring into a mug, the mug* IsFull. *The clock c_{brew} tracks the duration of* brew, *while c_{glob} tracks global time. Brewing takes at least 1 time unit. If the robot brews coffee for at least 2 time units, the coffee is* Strong.

- *Initially, the robot is doing nothing, nothing is filled, the coffee is not strong, and all clocks are initialized to zero. Also, only Mug_1 should be filled with strong coffee:*

$$\neg Brew(p, S_0), \neg Pour(p, m, S_0), \neg IsFull(x, S_0), \neg Strong(x, S_0),$$
$$c_{brew}(p, S_0) = 0, c_{glob}(S_0) = 0, \forall o.\, WantStrong(o) \supset o = Mug_1$$

- *The robot may start a durative action if it is not doing anything and end it if it has started it before. Brewing takes at least 1 time unit. In order to pour the coffee, there must be coffee in the pot:*

$$\mathrm{Poss}(sBrew(p), s) \equiv \neg Brew(p, s) \wedge \neg Pour(p, m, s)$$
$$\mathrm{Poss}(eBrew(p), s) \equiv Brew(p, s) \wedge c_{brew}(p, s) \geq 1$$
$$\mathrm{Poss}(sPour(p, m), s) \equiv IsFull(p, s)$$
$$\qquad \wedge \neg Pour(p', m', s) \wedge \neg Brew(p, s)$$
$$\mathrm{Poss}(ePour(p, m), s) \equiv Pour(p, m, s)$$
$$\mathrm{Poss}(wait(t), s) \equiv \top$$

- *The SSAs state that* brew *fills the pot with coffee and makes it strong if it took at least 2 time units:*

$$IsFull(o, \mathrm{do}(a, s)) \equiv a = eBrew(o)$$
$$\qquad \vee \exists p.\, a = ePour(p, o) \vee IsFull(p, s)$$
$$Strong(o, \mathrm{do}(a, s)) \equiv a = eBrew(o) \wedge c_{brew}(o, s) \geq 2$$
$$\qquad \vee \exists p.\, a = ePour(p, o) \wedge Strong(p, s) \vee Strong(o, s)$$
$$Brew(p, \mathrm{do}(a, s)) \equiv a = sBrew(p)$$
$$\qquad \vee Brew(p, s) \wedge a \neq eBrew(p)$$
$$Pour(p, m, \mathrm{do}(a, s)) \equiv \exists a = sPour(p, m)$$
$$\qquad \vee Pour(p, m, s) \wedge a \neq ePour(p, m)$$
$$c_{brew}(p, \mathrm{do}(a, s)) = y \equiv \exists t(a = wait(t) \wedge y = c_{brew}(p) + t)$$
$$\qquad \vee a \neq wait(t) \wedge (a = sBrew(p) \wedge y = 0$$
$$\qquad \vee a \neq sBrew(p) \wedge y = c_{brew}(p, s))$$
$$c_{glob}(\mathrm{do}(a, s)) = y \equiv \exists t.\, a = wait(t) \wedge y = c_{glob}(s) + t$$
$$\qquad \vee a \neq wait(t) \wedge y = c_{glob}(s)$$

- *There are no other objects or actions:*

$$\forall o(o = Pot \vee o = Mug_1 \vee \ldots) \wedge Mug_1 \neq Pot \wedge \ldots$$
$$\forall a(\exists o.a = sBrew(o) \vee \ldots \vee \exists t.\, a = wait(t))$$
$$\qquad \wedge \forall o_1, o_2.\, sBrew(o_1) \neq eBrew(o_2) \wedge \ldots$$
$$\qquad \wedge \forall o_1, o_2.\, sBrew(o_1) = sBrew(o_2) \supset o_1 = o_2 \wedge \ldots$$

Regression

Regression allows us to reduce queries about a future situation to an equivalent query about the initial situation. Here, we use a simplified variant of regression, where we require that all action terms are ground:

Definition 4. *A formula ϕ is* regressable *iff*

1. *ϕ is first order.*
2. *Every term of sort* situation *mentioned in ϕ has the form $\mathrm{do}([\alpha_1, \ldots, \alpha_n], S_0)$, where $\alpha_1, \ldots, \alpha_n$ are ground terms of sort* action.

3. *Every atom mentioning a clock f is of the form $f(\vec{o}, s) \bowtie \tau$, where $\vec{o}$ are terms of sort object and $\tau \in \mathbb{R}$.*[3]
4. *For every atom of the form $\mathrm{Poss}(a, s)$ mentioned in ϕ, a has the form $A(\vec{x})$ for some action type A.*
5. *ϕ does not quantify over situations.*
6. *ϕ does not mention the symbol $\sqsubset$, nor does it mention any equality atom $s = s'$ for situation terms s, s'.*

We extend the standard regression operator $\mathcal{R}$ with a regression rule for clock comparisons:

Definition 5. *Let $\phi = f(\rho, \mathrm{do}(\alpha, \sigma)) \bowtie r$ where $f \in \mathcal{F}$ is a clock and $r \in \mathbb{R}$. By Definition 3, there is an SSA of the following form:*

$$f(\vec{x}, \mathrm{do}(a, s)) = y \equiv \exists d.\, a = wait(t) \wedge y = f(\vec{x}, s) + t$$
$$\vee\, a \neq wait(t) \wedge (\phi_f(\vec{x}, a, s) \wedge y = 0 \vee \neg\phi_f(\vec{x}, a, s) \wedge y = f(\vec{x}, s))$$

Then $\mathcal{R}[\phi]$ is defined as follows:

1. *If $\alpha = wait(d)$ for some $t \in \mathbb{R}_{\geq 0}$, then $\mathcal{R}[\phi] := \mathcal{R}[f(\vec{\rho}, \sigma) \bowtie r']$ with $r' = r - t$.*
2. *Otherwise:*
   $\mathcal{R}[\phi] := \mathcal{R}[\phi_f(\vec{\rho}, \alpha, \sigma) \wedge 0 \bowtie r \vee \neg\phi_f(\vec{\rho}, \alpha, \sigma) \wedge f(\vec{\rho}, \sigma) \bowtie r]$

We obtain a variant of the regression theorem:

Theorem 4. *Let ϕ be a regressable formula. Then $\mathcal{R}[\phi]$ is uniform in S_0 and $\mathcal{D} \models \phi$ iff $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\phi]$.*

With this variant of regression, deciding whether a BAT entails a regressable clocked formula is decidable:

Theorem 5. *Let ϕ be a regressable clocked formula and $\mathcal{D}$ a clocked BAT (over finite objects and with complete information). Then $\mathcal{D} \models \phi$ is decidable.*

Regionalization

To show decidability of the reachability problem for clocked BATs, we use an approach based on abstraction for timed automata inspired by (Alur and Dill 1994; Ouaknine and Worrell 2007). We construct a finite abstraction of the otherwise infinite tree of situations of a clocked BAT $\mathcal{D}$ using a time-abstract bisimulation relation over situations:

Definition 6 (Time-abstract Bisimulation). *An equivalence relation $R \subseteq \mathcal{S} \times \mathcal{S}$ is a time-abstract bisimulation on $\mathcal{S}$ if $(\sigma_1, \sigma_2) \in R$ implies*

1. *For every clocked formula ϕ uniform in s: $\mathcal{D} \models \phi[\sigma_1]$ iff $\mathcal{D} \models \phi[\sigma_2]$.*
2. *For every action type $A \neq wait$ and ground tuple of arguments $\vec{\rho}$, $(\mathrm{do}(A(\vec{\rho}), \sigma_1), \mathrm{do}(A(\vec{\rho}), \sigma_2)) \in R$.*
3. *For every $\tau_1 \in \mathbb{R}_{\geq 0}$ there exists a $\tau_2 \in \mathbb{R}_{\geq 0}$ such that $(\mathrm{do}(wait(\tau_1), \sigma_1), \mathrm{do}(wait(\tau_2), \sigma_2)) \in R$.*

Intuitively, two situations are bisimilar if they represent the same state of the world (i.e., they satisfy the same formulas) and if applying the same action in both situations results again in bisimilar situations.

In the following, we construct such a bisimulation by applying regionalization to situations. Generally, a *clock valuation over clocks C* is a function $\nu : C \to \mathbb{R}_{\geq 0}$. For situations, we can define clock valuations as follows:

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[3]In particular, we disallow quantifying over terms of sort *time*.

Definition 7 (Clock Valuation). *Let $\mathcal{D}$ be a clocked BAT with (situation-suppressed) clocks $\mathcal{C}$ and σ a situation. The clock valuation $\nu_\sigma^\mathcal{D}$ of σ is the clock valuation over $\mathcal{C}$ with $\nu_\sigma^\mathcal{D}(\omega) = \tau$ iff $\mathcal{D} \models \omega[\sigma] = \tau$ for a ground clock term ω.*

We will omit the superscript $\mathcal{D}$ if $\mathcal{D}$ is clear from context. It is sometimes convenient to write a clock valuation ν as a set of pairs $\{(c, v) \mid c \in C, v = \nu(c)\} \subseteq C \times \mathbb{R}_{\geq 0}$. We also write $\nu + \tau$ for $\{(c, v + \tau) \mid c \in C, v = \nu(c)\}$ where all clocks are incremented by τ. We can show that every situation has a unique clock valuation:

Lemma 1. *If $\mathcal{D}$ is a clocked BAT and σ a situation, then ν_σ always exists and is unique.*

We use *regionalization* to obtain an abstraction of clock valuations. We briefly outline the intuition behind this idea: Given a constant $\mathcal{K} \in \mathbb{N}$ that is the maximal constant of sort *time* mentioned in $\mathcal{D}$ and in the reachability query ϕ, we can collect all clock valuations with a value larger than $\mathcal{K}$ in a single class $\top$. Furthermore, as clocks are only compared to natural numbers, for a single clock it suffices to observe only integer parts and whether fractional parts differ from zero to derive a finite set of equivalence classes for clock valuations $v \in [0, \mathcal{K}]$. In the presence of multiple clocks, the order between the fractional parts needs to be taken into account as well (Alur 1999). We define the *fractional part* $\mathrm{fract}(v)$ of $v \in [0, \mathcal{K}] \cup \{\top\}$ as $v - \lfloor v \rfloor$ if $v \in [0, \mathcal{K}]$ and zero otherwise. This leads to the definition of clock regions:

Definition 8 (Clock Regions). *Given a maximal constant $\mathcal{K}$, let $V = [0, \mathcal{K}] \cup \{\top\}$. We define the region equivalence as the equivalence relation $\sim_\mathcal{K}$ on V such that $u \sim_\mathcal{K} v$ if*

- *$u = v = \top$, or*
- *$u, v \neq \top$, $\lceil u \rceil = \lceil v \rceil$, and $\lfloor u \rfloor = \lfloor v \rfloor$.*

A region is an equivalence class of $\sim_\mathcal{K}$. We extend region equivalence to clock valuations. Let ν, ν' be two clock valuations over $\mathcal{C}$, then ν and ν' are region-equivalent, written $\nu \cong_\mathcal{K} \nu'$ iff (1) for every $c \in \mathcal{C}$, $\nu(c) \sim_\mathcal{K} \nu'(c)$, (2) for every $c, c' \in \mathcal{C}$, $\mathrm{fract}(\nu(c)) \leq \mathrm{fract}(\nu(c'))$ iff $\mathrm{fract}(\nu'(c)) \leq \mathrm{fract}(\nu'(c'))$.

Note that there is only a finite number of regions. Based on region equivalence wrt. a maximal constant $\mathcal{K}$, we can now define an equivalence relation on situations:

Definition 9. *Let $\approx_{\mathcal{D}\mathcal{K}} \subseteq \mathcal{S} \times \mathcal{S}$ be an equivalence relation such that $\sigma_1 \approx_{\mathcal{D}\mathcal{K}} \sigma_2$ iff*

1. *for every fluent $R \in \mathcal{F}$, $\mathcal{D} \models R(\vec{\rho}, \sigma_1)$ iff $\mathcal{D} \models R(\vec{\rho}, \sigma_2)$,*
2. *there is a bijection $f : \nu_{\sigma_1} \to \nu_{\sigma_2}$ such that*
   (a) *if $f(c_1, v_1) = (c_2, v_2)$, then $c_1 = c_2$ and $v_1 \sim_\mathcal{K} v_2$*
   (b) *if $f(c_1, v_1) = (c_2, v_2)$ and $f(c_1', v_1') = (c_2', v_2')$, then $\mathrm{fract}(v_1) \leq \mathrm{fract}(v_1')$ iff $\mathrm{fract}(v_2) \leq \mathrm{fract}(v_2')$.*

Two situations are equivalent if they satisfy the same fluents and agree on the clock regions. For a situation σ, we write $[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ for the equivalence class of σ wrt. $\approx_{\mathcal{D}\mathcal{K}}$. We denote the set of equivalence classes resulting from $\approx_{\mathcal{D}\mathcal{K}}$ as $\mathcal{S}/\approx_{\mathcal{D}\mathcal{K}}$. Equivalence directly extends to clocked formulas:

Lemma 2. *Let $\sigma_1 \approx_{\mathcal{D}\mathcal{K}} \sigma_2$ and let ϕ be a clocked formula uniform in s with maximal constant $\leq \mathcal{K}$. Then $\mathcal{D} \models \phi[\sigma_1]$ iff $\mathcal{D} \models \phi[\sigma_2]$.*

Algorithm 1 Computation of the time-abstract transition system $\mathcal{T}_\mathcal{D}$ for a clocked BAT $\mathcal{D}$ and a maximal constant $\mathcal{K}$.

$Open \leftarrow \{S_0\}; V \leftarrow \{[S_0]_{\approx_{\mathcal{D}\mathcal{K}}}\}; E \leftarrow \emptyset$
while $Open \neq \emptyset$ **do**
$\quad \sigma \leftarrow \textsc{pop}(Open)$
$\quad Acts \leftarrow \{A(\vec{\rho}) \mid A \in \mathcal{A} \setminus \{wait\}, \vec{\rho} \in \mathcal{O}^{|A|}\}$
$\quad Acts \leftarrow Acts \cup \{wait(\tau) \mid \tau \in \textsc{TSuccs}(\nu_\sigma, \mathcal{K})\}$
$\quad$ **for all** $\alpha \in Acts$ **do**
$\quad\quad$ **if** $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\text{Poss}(\alpha, \sigma)]$ **then**
$\quad\quad\quad$ **if** $[\text{do}(\alpha, \sigma)]_{\approx_{\mathcal{D}\mathcal{K}}} \notin V$ **then**
$\quad\quad\quad\quad V \leftarrow V \cup \{[\text{do}(\alpha, \sigma)]_{\approx_{\mathcal{D}\mathcal{K}}}\}$
$\quad\quad\quad\quad Open \leftarrow Open \cup \{\text{do}(\alpha, \sigma)\}$
$\quad\quad\quad E \leftarrow E \cup \{([\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}, [\text{do}(\alpha, \sigma)]_{\approx_{\mathcal{D}\mathcal{K}}})\}$
return $(V, [S_0]_{\approx_{\mathcal{D}\mathcal{K}}}, E)$

We use a well-known result from (Alur and Dill 1994) which states that it is always possible to maintain equivalence wrt. $\cong_\mathcal{K}$ between clock valuations for a given time increment:

Proposition 1. *Let ν_1 and ν_2 be two clock valuations over a set of clocks $\mathcal{C}$ such that $\nu_1 \cong_\mathcal{K} \nu_2$. Then for all $\tau \in \mathbb{R}_{\geq 0}$ there exists a $\tau' \in \mathbb{R}_{\geq 0}$ such that $\nu_1 + \tau \cong_\mathcal{K} \nu_2 + \tau'$.*

Therefore, for two situations with $\sigma_1 \approx_{\mathcal{D}\mathcal{K}} \sigma_2$ and any action $wait(\tau_1)$, it is always possible to find a τ_2 such that $\text{do}(wait(\tau_1), \sigma_1) \approx_{\mathcal{D}\mathcal{K}} \text{do}(wait(\tau_2), \sigma_2)$. Based on this, we can show that $\approx_{\mathcal{D}\mathcal{K}}$ satisfies all criteria of Definition 6:

Theorem 6. $\approx_{\mathcal{D}\mathcal{K}}$ *is a time-abstract bisimulation.*

Hence, it suffices to argue about the resulting equivalence classes to determine entailment with respect to a given $\mathcal{D}$. This enables us to construct a time-abstract transition system as a finite abstraction for the otherwise infinite tree of situations, where states collect $\approx_{\mathcal{D}\mathcal{K}}$-equivalent situations connected by executable actions from $\mathcal{D}$:

Definition 10 (Time-abstract Transition System)**.** *Let $\mathcal{D}$ be a clocked BAT. We define the TS $\mathcal{T}_\mathcal{D} = (Q, q_0, \rightarrow)$ as follows:*

- *The set of states Q consists of equivalence classes of $\approx_{\mathcal{D}\mathcal{K}}$, i.e., $Q \subseteq \mathcal{S}/\approx_{\mathcal{D}\mathcal{K}}$;*
- *The initial state is the equivalence class $q_0 = [S_0]_{\approx_{\mathcal{D}\mathcal{K}}}$;*
- *$[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}} \rightarrow [\sigma']_{\approx_{\mathcal{D}\mathcal{K}}}$ if σ' is an executable successor situation of σ, i.e., if $\sigma' = \text{do}(\alpha, \sigma)$ and $\mathcal{D} \models \text{Exec}(\sigma')$.*

It is easy to see that it is sufficient to consider $\mathcal{T}_\mathcal{D}$ for checking reachable situations:

Lemma 3. $\mathcal{D} \models \text{Exec}(\sigma)$ *iff $[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ is reachable in $\mathcal{T}_\mathcal{D}$.*

In contrast to the tree of situations, the time-abstract transition system $\mathcal{T}_\mathcal{D}$ is finite:

Lemma 4. *Let $\mathcal{D}$ be a clocked BAT. Then the time-abstract transition system $\mathcal{T}_\mathcal{D}$ has finitely many states.*

Hence, the time-abstract transition system $\mathcal{T}_\mathcal{D}$ is useful for deciding reachability: every reachable situation is also reachable in $\mathcal{T}_\mathcal{D}$, which is a finite transition system and can therefore be explored exhaustively. It remains to be shown that we can actually construct $\mathcal{T}_\mathcal{D}$.

Decidability

We now describe an algorithm for constructing $\mathcal{T}_\mathcal{D}$, which is shown in Algorithm 1. Starting with the initial state $[S_0]_{\approx_{\mathcal{D}\mathcal{K}}}$, the algorithm iteratively adds a state $[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ to $\mathcal{T}_\mathcal{D}$ whenever σ is reachable by a possible single action from an existing state, i.e., if $\sigma = \text{do}(\alpha, \sigma')$ and $\mathcal{D} \models \text{Poss}(\alpha, \sigma')$ for some state $[\sigma']_{\approx_{\mathcal{D}\mathcal{K}}}$ of $\mathcal{T}_\mathcal{D}$. It does so in a breadth-first search manner by keeping a list of *open* states $Open$, which consists of states that need to be expanded. The algorithm computes all possible successors by enumerating all possible ground action terms. For the action $wait$, it includes a canonical timepoint τ as representative for the equivalence class $[\nu_\sigma + \tau]_{\cong_\mathcal{K}}$. These canonical time successors $\textsc{TSuccs}(\nu_\sigma, \mathcal{K})$ are computed based on a well-known method (Ouaknine and Worrell 2007).

We can now show that the reachability problem is indeed decidable. We start with $\approx_{\mathcal{D}\mathcal{K}}$:

Lemma 5. *Let σ_1 and σ_2 be two situations. Then it is decidable whether $\sigma_1 \approx_{\mathcal{D}\mathcal{K}} \sigma_2$.*

Algorithm 1 indeed computes the time-abstract transition system $\mathcal{T}_\mathcal{D}$. Moreover, the algorithm terminates:

Lemma 6. *For a clocked BAT $\mathcal{D}$, Algorithm 1 terminates and returns the time-abstract transition system $\mathcal{T}_\mathcal{D}$.*

Combining these results, we obtain:

Theorem 7. *Reachability for clocked BATs is decidable.*

Example 2. *We can check the following reachability queries:*

$$\phi_1 := \forall o.IsFull(o) \qquad \phi_2 := \forall o.Strong(o) \equiv WantStrong(o)$$
$$\phi_3 := c_{glob}() \leq 1 \wedge Strong(Mug_1)$$

It is easy to see that ϕ_1 is reachable, e.g., with

$$\sigma_1 = \text{do}([sBrew(Pot); wait(1); eBrew(Pot); sPour(Pot, Mug_1);$$
$$ePour(Pot, Mug_1); sPour(Pot, Mug_2); ePour(Pot, Mug_2)], S_0)$$

For ϕ_2, a different action sequence is required, as the pot may not be filled with strong and weak coffee at the same time. Instead, we can brew coffee directly in one of the mugs:

$$\sigma_2 = \text{do}([sBrew(Pot); sBrew(Mug_1); wait(1); eBrew(Pot); wait(2);$$
$$eBrew(Mug_1); sPour(Pot, Mug_2); wait(1); ePour(Pot, Mug_2)], S_0)$$

Finally, there is no reachable situation that satisfies ϕ_3, as it requires to provide strong coffee after one time unit, while it takes at least two time units to prepare strong coffee.

6 Realizing GOLOG Programs

In this section, we show how we can utilize reachability for clocked BATs to decide realizability for a GOLOG program Δ. The goal is to check whether the non-determinisms in Δ can be resolved and action timepoints can be determined such that the resulting action sequence is executable and results in a final configuration of the program. Similar to the above, we create a transition system $\mathcal{T}_\Delta$ from a given program Δ and check whether a final program state is reachable in $\mathcal{T}_\Delta$.

Golog Transition Semantics

We consider programs in CONGOLOG (De Giacomo, Lespérance, and Levesque 2000) based on clocked BATs. A *program* is a pair $\Delta = (\mathcal{D}, \delta)$, where $\mathcal{D}$ is a BAT and δ is a program expression built from the following grammar:

$$\delta ::= \alpha \mid \phi? \mid \delta;\delta \mid \delta \mid \delta \mid \pi x.\,\delta \mid \delta^* \mid \delta\|\delta$$

Program expressions consist of actions α, tests $\phi?$, sequence of actions $\delta_1;\delta_2$, non-deterministic branching $\delta_1|\delta_2$, non-deterministic choice of argument $\pi x.\,\delta$, non-deterministic iteration δ^*, and concurrent execution $\delta_1\|\delta_2$. We require that each action term is time-suppressed (i.e., $wait()$ has no argument) and each test $\phi?$ tests a situation-suppressed clocked formula ϕ. We write $\mathrm{nil} := \mathrm{TRUE}?$ for the empty program that always succeeds. As usual, conditionals and loops can be defined as macros, i.e., **if** ϕ **do** δ_1 **else** δ_2 **fi** $:=$ $[\phi?;\delta_1]|[\neg\phi?;\delta_2]$ and **while** ϕ **do** δ **done** $:=$ $[\phi?;\delta]^*;\neg\phi?$. We call $\Delta = (\mathcal{D}, \delta)$ a *clocked program* if $\mathcal{D}$ is a clocked BAT and we assume that $\mathcal{K} \in \mathbb{N}$ is the maximal constant mentioned in $\mathcal{D}$. We adapt the GOLOG transition semantics from CONGOLOG such that each transition step follows a single primitive action, corresponding to *synchronized conditionals* known from INDIGOLOG (De Giacomo et al. 2009). We also extend the relation Trans to include the action such that $\mathrm{Trans}(\delta, s, a, \delta', s')$ is true if there is a single-step transition with action a in the program δ starting in situation s and resulting in situation s' with remaining program δ'. Finally, we assume that the program δ does not fix the time of occurrence for *wait* and instead determine the time in the transition semantics. Therefore, *wait* actions in δ do not have a time argument t and the timepoint is instead fixed in the transition rule for a *wait* action.[4] We also write $\mathrm{Trans}^*(\delta, s, \vec{a}, \delta', s')$ for the transitive closure of Trans such that $\vec{a}$ is the sequence of actions resulting from following the transitions according to Trans.

Example 3. *We consider the following program δ:*

$(sBrew(Pot); \exists m(WantStrong(m)) \equiv c_{brew}(Pot) \geq 2?; eBrew(Pot);$
$\quad$ **while** $(\exists m\neg IsFull(m))$ **do** $\pi m.\, sPour(Pot, m); ePour(Pot, m);$
$\quad$ **done**) $\| wait()^*$

The program first brews coffee. If there is a mug that should be filled with strong coffee, it waits until $c_{brew}(Pot) \geq 2$, otherwise it stops brewing earlier. Afterwards, it fills all mugs.

Program Realization

We describe how to construct a transition system $\mathcal{T}_\Delta$ to determine a realization of a given program Δ similar to $\mathcal{T}_\mathcal{D}$ from above. We first formally define program realization:

Definition 11 (Program Realization)**.** *Let $\mathcal{D}$ be a BAT and δ a program. An action sequence $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle$ is a realization of δ wrt. $\mathcal{D}$ if $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha}, \delta', \sigma') \wedge \mathrm{Final}(\delta', \sigma')$. The corresponding decision problem is to decide whether such a realization exists.*

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[4]We may still enforce a certain time delay by using clock formulas in tests $\phi?$, as demonstrated in Example 3.

Algorithm 2 Construction of the time-abstract transition system $\mathcal{T}_\Delta$ to determine program realizability.

$Open \leftarrow \{(S_0, \delta_0)\};\ V \leftarrow \emptyset;\ E \leftarrow \emptyset$
while $Open \neq \emptyset$ **do**
$\quad (\sigma, \delta) \leftarrow \mathrm{POP}(Open)$
$\quad Acts \leftarrow \{A(\vec{\rho}) \mid A \in \mathcal{A} \setminus \{wait\}, \vec{\rho} \in \mathcal{O}^{|A|}\}$
$\quad Acts \leftarrow Acts \cup \{wait(\tau) \mid \tau \in \mathrm{TSUCCS}(\nu_\sigma, \mathcal{K})\}$
$\quad$ **for all** $\alpha \in Acts$ **do**
$\quad\quad$ **if** $\mathcal{D} \models \mathrm{Trans}(\delta, \sigma, \alpha, \delta', \sigma')$ **then**
$\quad\quad\quad$ **if** $([\sigma']_{\approx_{\mathcal{DK}}}, \delta') \notin V$ **then**
$\quad\quad\quad\quad V \leftarrow V \cup \{([\sigma']_{\approx_{\mathcal{DK}}}, \delta')\}$
$\quad\quad\quad\quad Open \leftarrow Open \cup \{(\sigma', \delta')\}$
$\quad\quad\quad E \leftarrow E \cup \{(([\sigma]_{\approx_{\mathcal{DK}}}, \delta), ([\sigma']_{\approx_{\mathcal{DK}}}, \delta'))\}$
return $(V, ([S_0]_{\approx_{\mathcal{DK}}}, \delta_0), E)$

First, we can show that the equivalence relation $\approx_{\mathcal{DK}}$ can also be applied to the transition semantics, i.e., bisimilar situations cannot be distinguished by possible program transitions:

Lemma 7. *Let σ_1, σ_2 be two situations with $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$ and such that $\mathcal{D} \models \mathrm{Trans}(\delta, \sigma_1, \alpha, \delta^*, \sigma_1')$. Then:*

1. *If $\alpha = wait(\tau_1)$ for some $\tau_1 \in \mathbb{R}_{>0}$, then there is $\tau_2 \in \mathrm{TSUCCS}(\sigma_2, K)$ such that $\mathrm{Trans}(\delta, \sigma_2, wait(\tau_2), \delta^*, \sigma_2')$ and $\sigma_1' \approx_{\mathcal{DK}} \sigma_2'$.*

2. *If $\alpha = A(\vec{\rho})$ for some action type $A \neq wait$, then $\mathrm{Trans}(\delta, \sigma_2, \alpha, \delta^*, \sigma_2')$ and $\sigma_1' \approx_{\mathcal{DK}} \sigma_2'$.*

This allows us to use time-abstract bisimulations for determining realizability. We adapt Algorithm 1 such that states also contain the remaining program. The resulting algorithm is shown in Algorithm 2. The algorithm is correct, i.e., it contains a final program configuration iff a realization exists:

Lemma 8. *Let $\Delta = (\mathcal{D}, \delta)$ be a program over a clocked BAT $\mathcal{D}$ and $\mathcal{T}_\Delta$ the transition system returned by Algorithm 2. Then $\mathcal{T}_\Delta$ contains a state $([\sigma']_{\approx_{\mathcal{DK}}}, \delta')$ reachable from $([S_0]_{\approx_{\mathcal{DK}}}, \delta)$ with $\mathcal{D} \models \mathrm{Final}(\delta', \sigma')$ iff there is a realization of program Δ.*

Moreover, the transition system is finite:

Lemma 9. *$\mathcal{T}_\Delta$ has finitely many states.*

Combining the results, we obtain:

Theorem 8. *The realization problem for a program $\Delta = (\mathcal{D}, \delta)$ over a clocked BAT $\mathcal{D}$ is decidable.*

We conclude by presenting a realization of the example:

Example 4. *The following sequence is a realization of the program from Example 3:*

$sBrew(Pot); wait(2); eBrew(Pot); sPour(Pot, Mug_1);$
$ePour(Pot, Mug_1); sPour(Pot, Mug_2); ePour(Pot, Mug_2);$

However, the program does not fill every mug with the correct kind of coffee. In fact, let $\delta' := \delta; \forall m(Strong(m) \equiv WantStrong(m))?$ be the program δ augmented with a check whether every mug was filled with the right kind of coffee. Then, no realization of δ' exists: As the program brews coffee only once, it fills all mugs with the same coffee.

7 Conclusion

We investigated how continuous time can be modeled in the situation calculus (SC), focusing on *reachability*, which asks whether there is a reachable situation that satisfies some given formula ϕ, and *program realization*, which is the task to determine an action sequence corresponding to a successful execution of a given GOLOG program. For these problems, previous approaches for modeling time in the SC are unsuitable, because reachability (and thus also realizability) is undecidable. We presented an alternative approach based on *clocks*, real-valued fluents with restricted successor state axioms and described sound and complete algorithms for determining reachability and realizability, hence showing that both problems are decidable.

Our approach is restricted to basic action theories with a finite set of objects. For future work, it may be interesting to extend this approach towards more expressive formalisms that retain decidability, e.g., *bounded action theories* (De Giacomo, Lespérance, and Patrizi 2016), or the two-variable fragment of the SC (Gu and Soutchanski 2010), which has been used for verifying GOLOG programs over local-effect (Claßen et al. 2014) and acyclic (Zarrieß and Claßen 2016) BATs.

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A Appendix

In the following, we provide complete definitions for TAs, 2-counter machines (2CMs), and the GOLOG transition semantics, we describe an algorithm for computing time successors, and we include all proofs in full detail.

Timed Automata

TAs extend classical transition systems with time by adding a finite number of clocks that allow measuring time. Clocks increase their valuations *continuously* while the system is in a certain location (mode) and may be reset to zero when switching locations via possibly guarded transitions. Guard conditions are specified as clock constraints.

Definition 12 (Clock Constraint). *Let X be a set of clocks. The set $\Phi(X)$ of clock constraints g is defined by the grammar:*

$$g ::= x \bowtie c \mid g_1 \wedge g_2$$

where $\bowtie \in \{<, \leq, =, \geq, >\}$, $x \in X$ is a clock, and $c \in \mathbb{N}$ is a constant.

Definition 13 (Timed Automata). *A timed automaton (c.f. (Alur 1999)) $\mathcal{A} = (L, l^0, L^F, \Sigma, X, E)$ is a tuple with*

- *a finite set of locations L;*
- *an initial location l^0;*
- *a set of final locations $L^F \subseteq L$;*
- *a finite set of labels Σ;*
- *a finite set of clocks X;*
- *a finite set of switches $E \subseteq L \times \Sigma \times \Phi(X) \times 2^X \times L$ which defines possible switches (l, k, g, Y, l') from location l to location l', labelled with k, only possible if the guard condition g is satisfied. All clocks in Y are reset to zero upon taking the switch.*

The behavior of a TA can be described by means of a transition system $\mathcal{S}_{\mathcal{A}}$ where a state combines a location and a clock valuation. The transition relation between states of $\mathcal{S}_{\mathcal{A}}$ combines a continuous step (letting a specific amount of time pass) and a discrete switch of locations.

Definition 14. *Let $\mathcal{A} = (L, l_0, L_F, \Sigma, X, E)$ be a TA. The corresponding labeled transition system (LTS) $\mathcal{S}_{\mathcal{A}} = (Q_{\mathcal{A}}, q^0_{\mathcal{A}}, Q^F_{\mathcal{A}}, \Sigma_{\mathcal{A}}, \rightarrow)$ is defined as follows:*

- *A state $q_{\mathcal{A}} \in Q_{\mathcal{A}}$ of $\mathcal{S}_{\mathcal{A}}$ is a pair (l, ν) such that*
  1. *l is a location of $\mathcal{A}$,*
  2. *ν is a clock valuation for the clocks X of $\mathcal{A}$*

- *The initial state (l^0, ν_0) consists of the initial location l_0 and a clock valuation ν_0 where all clocks are zero-initialized,*
- *The final states $Q^F_{\mathcal{A}} = \{(l, \nu) \mid l \in L^F\}$ are those states that contain a final location of $\mathcal{A}$,*
- *The labels $\Sigma_{\mathcal{A}}$ consist of the labels of the TA and time increments, i.e., $\Sigma_{\mathcal{A}} = \Sigma \times \mathbb{R}$.*
- *A transition $(l, \nu) \xrightarrow[\delta]{a} (l', \nu')$ consists of two steps:*
  1. Elapse of time: *All clocks are incremented by some time increment $\delta \in \mathbb{R}_{\geq 0}$*
  2. Switch of location: *The location changes based on a switch $(l, a, \varphi, Y, l') \in E$, where the clock constraint φ must be satisfied by the incremented clocks ν^* and Y specifies which clocks are reset after the transition, i.e., $\nu^* \models \varphi$ and $\nu' = \nu^*[Y := 0]$.*

A *path* from $q_{\mathcal{A}}$ to $q'_{\mathcal{A}}$ is a finite sequence of transitions $q_{\mathcal{A}} \rightarrow q^1_{\mathcal{A}} \rightarrow \ldots \rightarrow q'_{\mathcal{A}}$. A *run* on $\mathcal{A}$ is a path starting in the initial state $q^0_{\mathcal{A}}$. We denote the set of all finite runs of $\mathcal{A}$ with $\mathrm{Runs}^*(\mathcal{A})$. A finite run is *accepting* if it ends in an accepting state $q_{\mathcal{A}} \in Q^F_{\mathcal{A}}$.

Definition 15 (Language of a TA). *Given a (finite) run*

$$p = (l_0, \nu_0) \xrightarrow[\delta_1]{a_1} \ldots \xrightarrow[\delta_n]{a_n} (l_{n+1}, \nu_{n+1}) \ldots \in \mathrm{Runs}^*(\mathcal{A})$$

on a TA $\mathcal{A}$. The timed word induced by p is the timed word

$$\mathrm{tw}(p) = (a_1, \delta_1)(a_2, \delta_1 + \delta_2) \ldots (a_n, \sum_1^n \delta_i) \ldots$$

The language of finite words of $\mathcal{A}$ is the set

$$\mathcal{L}^*(\mathcal{A}) = \{\mathrm{tw}(p) \mid p \in \mathrm{Runs}^*_F(\mathcal{A})\}$$

2-Counter Machines

We use 2-counter machines (2CMs) as defined by (Bouyer et al. 2004):

Definition 16 (2-Counter Machines). *A 2-counter machine (2CM) is a finite set of labeled instructions over two counters c_1 and c_2. There are two types of instructions:*

1. *An* incrementation instruction *of counter $x \in \{c_1, c_2\}$:*

$$p: \ x := x + 1; \ \textbf{goto } q$$

   The instruction increments counter x by one and then goes to the next instruction q.

2. *A* decrementation instruction *of counter $x \in \{c_1, c_2\}$:*

$$p: \ \textbf{if } x > 0 \begin{cases} \textbf{then } x := x - 1; \ \textbf{goto } q \\ \textbf{else goto } r \end{cases}$$

   The instruction branches on x: If x is larger than 0, then it decrements x and goes to instruction q. Otherwise, it does not change x and directly goes to instruction r.

*The machine starts with instruction s_0 and with counter values $c_1 = c_2 = 0$ and stops at a special instruction **HALT**. The* halting problem *for a 2CM is to decide whether a machine reaches the instruction **HALT**.*

Theorem 9. *The halting problem for 2CMs is undecidable (Minsky 1967).*

Full GOLOG Transition Semantics

$\text{Trans}(a, s, a, \delta', s') \equiv$
$\quad\quad \text{Poss}(a, s) \wedge \delta' = \text{nil} \wedge s' = \text{do}(a, s)$

$\text{Trans}(wait(), s, a, \delta', s') \equiv$
$\quad\quad \exists t. \, a = wait(t) \wedge \delta' = \text{nil} \wedge s' = \text{do}(a, s)$

$\text{Trans}(\phi?, s, a, \delta', s') \equiv \text{FALSE}$

$\text{Trans}(\delta_1; \delta_2, s, a, \delta', s') \equiv$
$\quad\quad \exists \gamma. \, \delta' = (\gamma; \delta_2) \wedge \text{Trans}(\delta_1, s, a, \gamma, s')$
$\quad\quad\quad \vee \, \text{Final}(\delta_1, s) \wedge \text{Trans}(\delta_2, s, a, \delta', s')$

$\text{Trans}(\delta_1 | \delta_2, s, a, \delta', s') \equiv$
$\quad\quad \text{Trans}(\delta_1, s, a, \delta', s') \vee \text{Trans}(\delta_2, s, a, \delta', s')$

$\text{Trans}(\pi x.\delta, s, a, \delta', s') \equiv \exists v. \, \text{Trans}(\delta_v^x, s, a, \delta', s')$

$\text{Trans}(\delta^*, s, a, \delta', s') \equiv$
$\quad\quad \exists \gamma. \, (\delta' = \gamma; \delta^*) \wedge \text{Trans}(\delta, s, a, \gamma, s')$

$\text{Trans}(\delta_1 \| \delta_2, s, a, \delta', s') \equiv$
$\quad\quad \exists \gamma. \, \delta' = (\gamma \| \delta_2) \wedge \text{Trans}(\delta_1, s, a, \gamma, s')$
$\quad\quad\quad \vee \, \exists \gamma. \, \delta' = (\delta_1 \| \gamma) \wedge \text{Trans}(\delta_2, s, a, \gamma, s')$

$\text{Final}(\delta, s)$ is true if program δ may terminate in situation s:

$$\text{Final}(a, s) \equiv \text{FALSE}$$
$$\text{Final}(\phi?, s) \equiv \phi[s]$$
$$\text{Final}(\delta_1; \delta_2, s) \equiv \text{Final}(\delta_1) \wedge \text{Final}(\delta_2)$$
$$\text{Final}(\delta_1 | \delta_2, s) \equiv \text{Final}(\delta_1) \vee \text{Final}(\delta_2)$$
$$\text{Final}(\pi x.\delta, s) \equiv \exists v. \, \text{Final}(\delta_v^x, s)$$
$$\text{Final}(\delta^*, s) \equiv \text{TRUE}$$
$$\text{Final}(\delta_1 \| \delta_2, s) \equiv \text{Final}(\delta_1, s) \wedge \text{Final}(\delta_2, s)$$

Time Successor Computation

Algorithm 3 Time successor computation for a situation σ.

---
function TSUCCS(σ, K)
   | $\nu \leftarrow \nu_\sigma$; $last \leftarrow \text{start}(\sigma)$; $Succs \leftarrow \{last\}$
   | **while** $\exists (c, v) \in \nu : v \leq K$ **do**
   | | $\mu \leftarrow \max_{(c,v) \in \nu} \text{fract}(v)$
   | | **if** $\exists (c, v) \in \nu : \text{fract}(v) = 0$ **then** $incr \leftarrow \frac{1 - \mu}{2}$
   | | **else** $incr \leftarrow 1 - \mu$
   | | $last \leftarrow last + incr$
   | | $Succs \leftarrow Succs \cup \{last\}$
   | | $\nu \leftarrow \nu + incr$
   | **return** $Succs$

Algorithm 3 is a variant of a well-known method to compute time successors (e.g., (Ouaknine and Worrell 2007)) with the following property:

Proposition 2. *Let ν be a clock valuation, $\tau \in \mathbb{R}_{\geq 0}$, and Succs the time successors computed by Algorithm 3. Then there is a $\tau' \in Succs$ such that $\nu + \tau \cong_{\mathcal{K}} \nu + \tau'$.*

Proofs

Theorem 1. *The reachability problem for a real-valued BAT is undecidable.*

Proof. Let $Incrs = \{(p, c, q)_i\}_i$ be the finite set of increment instructions, where p is the instruction label, $c \in \{1, 2\}$ is the index of the counter to be incremented, and q is the next instruction label. Similarly, let $Decrs = \{(p, c, q, r)_i\}_i$ be the finite set of decrement instructions, where p is the instruction label, $c \in \{1, 2\}$ is the index of the counter to be decremented, q is the jump instruction if the condition is true, and r is the jump instruction otherwise. We define a BAT $\mathcal{D}_\mathcal{M}$ corresponding to a 2CM $\mathcal{M}$ as follows: We use the unary relational fluent $Next(i)$ to describe the next instruction i, two nullary functional fluents f_1, f_2 for counter values (we omit parentheses and write f_1 instead of $f_1()$), and the nullary relational fluent $Halt$ which is true iff $\mathcal{M}$ halted. We define the BAT $\mathcal{D}_\mathcal{M}$ as follows:

- Initially, both counting fluents are set to zero and the next instruction is s_0:

$$\mathcal{D}_0 = \{f_1 = 0, f_2 = 0, Next(i) \equiv i = s_0\}$$

- An action is possible if it is the next instruction:

$$\text{Poss}(i) \equiv Next(i)$$

- The SSA for each counting fluent $f_i \in \{f_1, f_2\}$ works as follows: If the last instruction was an increment instruction from $Incrs$ and it incremented c_i, then the fluent f_i is incremented as well. Similarly, if the last instruction was a decrement instruction and the counter value was larger than zero, then the fluent is decremented, otherwise it remains unchanged:

$$f_i(\text{do}(a, s)) = n \equiv$$
$$\bigvee_{(p,c,q) \in Incrs} a = p \wedge c = i \wedge n = f_i + 1$$
$$\vee \bigvee_{(p,c,q,r) \in Decrs} a = p \wedge c = i$$
$$\wedge (f_i > 0 \wedge n = f_i - 1 \vee f_i = 0 \wedge n = 0)$$
$$\vee f_i = n \wedge \bigwedge_{(p,c,q) \in Incrs} (a \neq p \vee c \neq i)$$
$$\wedge \bigwedge_{(p,c,q,r) \in Decrs} (a \neq p \vee c \neq i)$$

- The SSA for $Next$ works as follows: If the last instruction was an increment instruction that sets the next instruction to q, then set Q accordingly. Similarly, if the last instruction was a decrement instruction, then branch on the value of the corresponding counter (i.e., whether $f_i > 0$) and set the next instruction accordingly:

$$Next(i) \equiv$$
$$\bigvee_{(p,c,q) \in Incrs} a = p \wedge i = q$$
$$\vee \bigvee_{(p,c,q,r) \in Decrs} a = p$$
$$\wedge ((c = 1 \wedge f_1 > 0 \vee c = 2 \wedge f_2 > 0) \wedge i = q$$
$$\vee (c = 1 \wedge f_1 = 0 \vee c = 2 \wedge f_2 = 0) \wedge i = r)$$

By construction, $\mathcal{D} \models \exists s\, \mathrm{Exec}(s) \wedge Halt(s)$ iff the 2CM reaches the instruction **HALT**. $\square$

Theorem 2. *The time-bounded reachability problem for a real-valued BAT and a time bound $b \geq 2$ is undecidable.*

Proof. The construction is the same as in Theorem 1, except that counting fluents work differently: A counter value u is encoded by a fluent value 2^{1-u}. Hence, every increment of f_i is replaced by an update to $\frac{f_i}{2}$ and every decrement of f_i is replaced by an update to $2f_i$. Furthermore, a comparison $f_i = 0$ is replaced by $f_i = 2$ and $f_i > 0$ is replaced by $f_i < 2$. We obtain:

- $\mathcal{D}_0 = \{f_1 = 2, f_2 = 2, Next(i) \equiv i = s_0\}$
- precondition axiom: $\mathrm{Poss}(i) \equiv Next(i)$
- SSA for each counting fluent $f_i \in \{f_1, f_2\}$:

$$f_i(\,\mathrm{do}(a,s)) = n \ \equiv$$
$$\bigvee_{(p,c,q) \in Incrs} a = p \wedge c = i \wedge n = \frac{f_i}{2}$$
$$\vee \bigvee_{(p,c,q,r) \in Decrs} a = p \wedge c = i$$
$$\wedge\,(f_i < 2 \wedge n = 2f_i \vee f_i = 2 \wedge n = 2)$$
$$\vee\, f_i = n \wedge \bigwedge_{(p,c,q) \in Incrs}(a \neq p \vee c \neq i)$$
$$\wedge \bigwedge_{(p,c,q,r) \in Decrs}(a \neq p \vee c \neq i)$$

- SSA for $Next$:

$$Next(i) \ \equiv$$
$$\bigvee_{(p,c,q) \in Incrs} a = p \wedge i = q$$
$$\vee \bigvee_{(p,c,q,r) \in Decrs} a = p$$
$$\wedge\,((c = 1 \wedge f_1 < 2 \vee c = 2 \wedge f_2 < 2) \wedge i = q$$
$$\vee\,(c = 1 \wedge f_1 = 2 \vee c = 2 \wedge f_2 = 2) \wedge i = r)$$

$\square$

Theorem 3. *The reachability problem for timed BATs without any fluents of sort* time *is undecidable.*

Proof. By reduction of the language emptiness problem for TAs with 4 clocks extended with additive clock constraints of the form $c_1 + c_2 = 1$. We first define an *actual path of situations*, which is a sequence of situations that have actually occurred (Pinto and Reiter 1995):

$$\mathrm{Actual}(S_0)$$
$$\mathrm{Actual}(\mathrm{do}(a,s)) \supset \mathrm{Actual}(s) \wedge \mathrm{Poss}(a,s)$$
$$\mathrm{Actual}(\mathrm{do}(a_1,s)) \wedge \mathrm{Actual}(\mathrm{do}(a_2,s)) \supset a_1 = a_2$$

Given an actual path of situations, $\mathrm{Occurs}(a,s)$ describes that a is an occurring action and $\mathrm{Occurs}_{\mathrm{T}}(a,t)$ gives the timepoint t when the action a occurs:

$$\mathrm{Occurs}(a,s) \ \equiv\ \mathrm{Actual}(\mathrm{do}(a,s))$$
$$\mathrm{Occurs}_{\mathrm{T}}(a,t) \ \equiv\ \exists s.\ \mathrm{Occurs}(a,s) \wedge \mathrm{start}(\mathrm{do}(a,s)) = t$$

We can now define $\mathrm{Last}(a,t)$ as the timepoint t of the *last* occurrence of a (and zero if a has never occurred):

$$\mathrm{Last}(a,t) \equiv \mathrm{Occurs}_{\mathrm{T}}(a,t) \wedge \neg\exists t'.t' > t \wedge \mathrm{Occurs}_{\mathrm{T}}(a,t')$$
$$\vee\, \forall t' \neg\, \mathrm{Occurs}_{\mathrm{T}}(a,t') \wedge t = 0$$

Given a TA $\mathcal{A}$, we construct a BAT $\mathcal{D}$ such that the language of $\mathcal{A}$ is non-empty if and only if $\mathcal{D} \models \exists s\, \mathrm{Exec}(s) \wedge \bigvee_{l \in L_F} Loc(l,s)$. Let $\mathcal{A} = (L, l_0, L_F, \sigma, X, E)$ be a TA. For each switch $(l, k, g, Y, l') \in E$, we first translate the clock constraint formula g to a SC formula $\mu(g)$ with free variable t as follows:

$$c \bowtie r \mapsto \exists t'\, \mathrm{Last}(reset(c,t'),t') \wedge (t - t') \bowtie r$$
$$g_1 \wedge g_2 \mapsto \mu(g_1) \wedge \mu(g_2)$$
$$c_1 + c_2 = 1 \mapsto \exists t_1, t_2\, \mathrm{Last}(reset(c_1,t_1),t_1)$$
$$\wedge\, \mathrm{Last}(reset(c_2,t_2),t_2)$$
$$\wedge\, (t - t_1) + (t - t_2) = 1$$

Now, we can define $\mathcal{D}_{\mathcal{A}}$ as follows. Initially, the TA is in location l_0 and all clocks are initialized to zero and do not need to be reset:

$$Loc(l) \equiv l = l_0, c_1() = 0, \ldots c_4() = 0, \forall c \neg NeedsReset(c)$$

A switch action at timepoint t is possible if no action needs a reset, if the TA is in the correct source location, and if the clock constraint g is satisfied at timepoint t. Hence, for each $(l, k, g, Y, l') \in E$:

$$\mathrm{Poss}(switch_{l,k,g,Y,l'}(t),s) \equiv$$
$$\forall c \neg NeedsReset(c) \wedge Loc(l) \wedge \mu(g)$$

A reset action for clock c is possible iff c needs a reset:

$$\mathrm{Poss}(reset(c,t),s) \equiv NeedsReset(c) \wedge \mathrm{start}(s) = t$$

$NeedsReset(c)$ is true, if the last action was a discrete switch which requires clock c to be reset or if it was true before and the last action did not reset c:

$$NeedsReset(c, \mathrm{do}(a,s)) \ \equiv$$
$$\bigvee_{(l,k,g,Y,l') \in E} \exists t\, a = switch_{l,k,g,Y,l'}(t) \wedge c \in Y$$
$$\vee\, NeedsReset(c,s) \wedge \forall t\, a \neq reset(c,t)$$

The fluent Loc is set to l' if the last action was a discrete switch to location l' or keeps its value if no discrete switch took place:

$$Loc(l^*, \mathrm{do}(a,s)) \ \equiv$$
$$\bigvee_{(l,k,g,Y,l') \in E} \exists t\, a = switch_{l,k,g,Y,l'}(t) \wedge l^* = l'$$
$$\vee\, Loc(l,s) \wedge \neg \bigvee_{(l,k,g,Y,l') \in E} \exists t\, a = switch_{l,k,g,Y,l'}(t) \wedge l^* = l$$

By construction, $\mathcal{D} \models \exists s \, \mathrm{Exec}(s) \wedge \bigvee_{l \in L_F} \mathrm{Loc}(l, s)$ iff there is some accepting path in the LTS $\mathcal{S}_{\mathcal{A}}$ corresponding to $\mathcal{A}$. $\square$

Theorem 4. *Let ϕ be a regressable formula. Then $\mathcal{R}[\phi]$ is uniform in S_0 and $\mathcal{D} \models \phi$ iff $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\phi]$.*

Proof. By induction on the maximal length of all situation terms in ϕ and structural sub-induction on ϕ.
Base case. Let $\sigma = S_0$. Then $\mathcal{R}[\phi] = \phi$ and $\mathcal{D} \models \phi$ iff $\mathcal{D}_0 \cup \mathcal{D}_c \models \phi$ because ϕ is uniform in S_0.
Induction step. Let $\sigma = \mathrm{do}(\alpha, \sigma')$. Here, we only consider the case of formulas of the form $\phi = f(\rho, \sigma) \bowtie r$ and refer to (Pirri and Reiter 1999) for the other cases. If $\alpha = \mathit{wait}(d)$ for some $d \in \mathbb{R}_{\geq 0}$, then $\mathcal{R}[f(\vec{\rho}, \sigma) \bowtie r] = \mathcal{R}[f(\vec{\rho}, \sigma') \bowtie r']$ with $r' = r - d$. By induction, $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\phi_f(\vec{\rho}, \alpha, \sigma) \bowtie r']$ iff $\mathcal{D} \models \phi_f(\vec{\rho}, \alpha, \sigma) \bowtie r'$. Otherwise (if α is not wait), by the definition of the SSA for f, $\mathcal{D} \models f(\vec{\rho}, \mathrm{do}(\alpha, \sigma)) = y \equiv \phi_f(\vec{\rho}, \alpha, \sigma) \wedge y = 0 \vee \neg \phi_f(\vec{\rho}, \alpha, \sigma) \wedge y = f(\vec{\rho}, \sigma)$ and so $\mathcal{D} \models f(\vec{\rho}, \mathrm{do}(\alpha, \sigma)) \bowtie r \equiv \phi_f(\vec{\rho}, \alpha, \sigma) \wedge 0 \bowtie r \vee \neg \phi_f(\vec{\rho}, \alpha, \sigma) \wedge f(\vec{\rho}, \sigma) \bowtie r$. By induction, this holds iff $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\phi_f(\vec{\rho}, \alpha, \sigma) \wedge 0 \bowtie r \vee \neg \phi_f(\vec{\rho}, \alpha, \sigma) \wedge f(\vec{\rho}, \sigma) \bowtie r]$ and so $\mathcal{D} \models f(\vec{\rho}, \mathrm{do}(\alpha, \sigma)) \bowtie r$ iff $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[f(\vec{\rho}, \mathrm{do}(\alpha, \sigma)) \bowtie r]$. $\square$

Theorem 5. *Let ϕ be a regressable clocked formula and $\mathcal{D}$ a clocked BAT (over finite objects and with complete information). Then $\mathcal{D} \models \phi$ is decidable.*

Proof. As regression is sound and complete, it is sufficient to consider $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\phi]$, which can be reduced to entailment in propositional logic: First, as $\mathcal{O}$ is finite, we can eliminate each quantifier in $\mathcal{D}$ and ϕ by substitution, e.g., by substituting $\exists o. \phi'$ with $\bigvee_{o \in \mathcal{O}} \phi'$. Second, note that regressing a clock comparison $f(\vec{\rho}, \sigma) \bowtie r$ results in a formula $f(\vec{\rho}, S_0) \bowtie r'$, where $r' \in \mathbb{R}$. As all clock values are initialized to 0, we can directly replace each such comparison by $\top$ or $\bot$, e.g., $f(\vec{\rho}, S_0) > 5$ is simplified to $\bot$. Finally, with the unique name assumption, we can interpret predicate symbols as propositional variables, thereby obtaining a propositional set of sentences $\mathcal{D}_0'$ and a propositional formula ϕ' such that $\mathcal{D}_0' \models \phi'$ iff $\mathcal{D}_0 \cup \mathcal{D}_c \models \phi$. $\square$

Lemma 1. *If $\mathcal{D}$ is a clocked BAT and σ a situation, then ν_σ always exists and is unique.*

Proof. By definition of clocked BATs, each clock is uniquely initialized to 0. Furthermore, once the initial value is fixed, the SSAs uniquely define each clock value in all successor situations. $\square$

Lemma 2. *Let $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$ and let ϕ be a clocked formula uniform in s with maximal constant $\leq \mathcal{K}$. Then $\mathcal{D} \models \phi[\sigma_1]$ iff $\mathcal{D} \models \phi[\sigma_2]$.*

Proof. Let $f : \nu_{\sigma_1} \to \nu_{\sigma_2}$ be a bijection witnessing $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$. We show that $\mathcal{D} \models \phi[\sigma_1]$ iff $\mathcal{D} \models \phi[\sigma_2]$ by structural induction on ϕ.

- Let $\phi = R(\vec{\rho}, s)$ for some fluent predicate R. By Definition 6, it directly follows that $\mathcal{D} \models \phi[\sigma_1]$ iff $\mathcal{D} \models \phi[\sigma_2]$.

- Let $\phi = \omega \bowtie \tau$ for some ground situation-suppressed clock term $\omega \in \mathcal{C}$ and $\tau \in \mathbb{N}$. Let $\nu_{\sigma_1}(\omega) = v_1$ and $\nu_{\sigma_2}(\omega) = v_2$. As f is a witness for $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$, it follows that $f(\omega, v_1) = (\omega, v_2)$ and $v_1 \sim_{\mathcal{K}} v_2$. By Definition 8, $v_1 \bowtie \tau$ iff $v_2 \bowtie \tau$ and so $\mathcal{D} \models \omega[\sigma_1] \bowtie \tau$ iff $\mathcal{D} \models \omega[\sigma_2] \bowtie \tau$.

- For the Boolean connectives as well as existential and universal quantification, the claim follows by induction. $\square$

Theorem 6. $\approx_{\mathcal{DK}}$ *is a time-abstract bisimulation.*

Proof. We show that $\approx_{\mathcal{DK}}$ satisfies the criteria from Definition 6. Let $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$ such that $f : \nu_{\sigma_1} \to \nu_{\sigma_2}$ is a bijection witnessing $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$.

1. Let ϕ be a clocked formula uniform in s. It directly follows from Lemma 2 that $\mathcal{D} \models \phi[\sigma_1]$ iff $\mathcal{D} \models \phi[\sigma_2]$.

2. Let $A \neq \mathit{wait}$ be some action type, $\vec{\rho}$ a ground tuple of arguments. Let $\sigma_1' = \mathrm{do}(A(\vec{\rho}), \sigma_1)$ and $\sigma_2' = \mathrm{do}(A(\vec{\rho}), \sigma_2)$. We show that $\sigma_1' \approx_{\mathcal{DK}} \sigma_2'$:

  (a) Let $R \in \mathcal{F}$ be a relational fluent and $\vec{\rho}'$ a ground tuple of arguments. We show that $\mathcal{D} \models R(\vec{\rho}', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}', \sigma_2')$. There is a SSA for R of the form $R(\vec{x}, \mathrm{do}(a, s)) \equiv \phi_R(\vec{x}, a, s)$, where $\phi_R(\vec{x}, a, s)$ is a clocked formula uniform in s. As $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$, it directly follows from Lemma 2 that $\mathcal{D} \models \phi_R(\vec{\rho}', A(\vec{\rho}), \sigma_1)$ iff $\mathcal{D} \models \phi_R(\vec{\rho}', A(\vec{\rho}), \sigma_2)$ and therefore $\mathcal{D} \models R(\vec{\rho}', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}', \sigma_2')$.

  (b) We construct a bijection $f' : \nu_{\sigma_1'} \to \nu_{\sigma_2'}$ witnessing $\approx_{\mathcal{DK}}$. Let c be a functional fluent and $\omega = c(\vec{\rho}')$. There is a SSA of the form $c(\vec{x}, \mathrm{do}(a, s)) = y \equiv \exists d. a = \mathit{wait}(d) \wedge y = c(\vec{x}, s) + d \vee a \neq \mathit{wait}(d) \wedge (\phi_c(\vec{x}, a, s) \wedge y = 0 \vee \neg \phi_c(\vec{x}, a, s) \wedge y = c(\vec{x}, s))$. Thus, $\omega[\sigma_1'] = 0$ if $\mathcal{D} \models \phi_c(\vec{\rho}, \alpha, \sigma_1)$ and $\omega[\sigma_1'] = \omega[\sigma_1]$ otherwise, similarly for $\omega[\sigma_2']$. Now, as $\phi_c(\rho, a, s)$ is a clocked formula uniform in s, by Lemma 2, $\mathcal{D} \models \phi_c(\rho, \alpha, \sigma_1)$ iff $\mathcal{D} \models \phi_c(\rho, \alpha, \sigma_2)$. Hence, for each ω, we can set $f'(\omega, 0) = (\omega, 0)$ if $\mathcal{D} \models \phi_c(\rho, \alpha, \sigma_1)$ and $f'(\omega, \nu_{\sigma_1}(\omega)) = (\omega, \nu_{\sigma_2}(\omega))$ otherwise.

3. Let $\tau_1 \in \mathbb{R}_{\geq 0}$ and $\sigma_1' = \mathrm{do}(\mathit{wait}(\tau_1), \sigma_1)$. As $\nu_{\sigma_1} \cong_{\mathcal{K}} \nu_{\sigma_2}$, it follows with Proposition 1 that there is some $\tau_2 \in \mathbb{R}_{\geq 0}$ such that $\nu_{\sigma_1} + \tau_1 \cong_{\mathcal{K}} \nu_{\sigma_2} + \tau_2$. Let $\sigma_2' = \mathrm{do}(\mathit{wait}(\tau_2), \sigma_2)$. We show that $\sigma_1' \approx_{\mathcal{DK}} \sigma_2'$:

  (a) Let $R \in \mathcal{F}$ be a relational fluent and $\vec{\rho}'$ a ground tuple of arguments. We show that $\mathcal{D} \models R(\vec{\rho}', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}', \sigma_2')$. There is a SSA for R of the form $R(\vec{x}, \mathrm{do}(a, s)) \equiv \phi_R(\vec{x}, a, s)$, where $\phi_R(\vec{x}, a, s)$ is a clocked formula uniform in s. As $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$, it directly follows from Lemma 2 that $\mathcal{D} \models \phi_R(\vec{\rho}', \mathit{wait}(\tau_1), \sigma_1)$ iff $\mathcal{D} \models \phi_R(\vec{\rho}', \mathit{wait}(\tau_2), \sigma_2)$ and therefore $\mathcal{D} \models R(\vec{\rho}', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}', \sigma_2')$.

  (b) Let c be a functional fluent and $\omega = c(\vec{\rho}')$. By definition of the SSA for c, it directly follows that $\nu_{\sigma_1'}(\omega) = \nu_{\sigma_1}(\omega) + \tau_1$ and $\nu_{\sigma_2'}(\omega) = \nu_{\sigma_2}(\omega) + \tau_2$. Thus, we can set $f'(\omega, \nu_{\sigma_1'}(\omega)) = (\omega, \nu_{\sigma_2'}(\omega))$. As $\nu_{\sigma_1} + \tau_1 \cong_{\mathcal{K}} \nu_{\sigma_2} + \tau_2$, it follows that f' is a bijection witnessing $\approx_{\mathcal{DK}}$. $\square$

Lemma 3. $\mathcal{D} \models \mathrm{Exec}(\sigma)$ *iff* $[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ *is reachable in* $\mathcal{T}_{\mathcal{D}}$.

Proof.
$\Rightarrow$: By induction on the number n of do operators in σ.
Base case. Let $n = 0$. Therefore, $\sigma = S_0$. By definition, $[S_0]_{\approx_{\mathcal{D}\mathcal{K}}}$ is reachable in $\mathcal{T}_{\mathcal{D}}$.
Induction step. Let $\sigma = \mathrm{do}(\alpha, \sigma')$ such that $\mathcal{D} \models \mathrm{Exec}(\sigma)$. By induction, $[\sigma']_{\approx_{\mathcal{D}\mathcal{K}}}$ is reachable from $[S_0]_{\approx_{\mathcal{D}\mathcal{K}}}$ in $\mathcal{T}_{\mathcal{D}}$. By definition of $\mathcal{T}_{\mathcal{D}}$, $[\sigma']_{\approx_{\mathcal{D}\mathcal{K}}} \to [\sigma']_{\approx_{\mathcal{D}\mathcal{K}}}$ is a transition in $\mathcal{T}_{\mathcal{D}}$ and so $[\sigma']_{\approx_{\mathcal{D}\mathcal{K}}}$ is also reachable in $\mathcal{T}_{\mathcal{D}}$.
$\Leftarrow$: Follows immediately by the definition of $\mathcal{T}_{\mathcal{D}}$. $\qquad \square$

Lemma 4. *Let $\mathcal{D}$ be a clocked BAT. Then the time-abstract transition system $\mathcal{T}_{\mathcal{D}}$ has finitely many states.*

Proof. As there are only finitely many fluents $\mathcal{F}$, objects $\mathcal{O}$, and clock regions, $\approx_{\mathcal{D}\mathcal{K}}$ has finitely many equivalence classes. Also, as $\mathcal{O}$ and $\mathcal{A}$ are finite, each state only has finitely many successors. Therefore, the time-abstract transition system $\mathcal{T}_{\mathcal{D}}$ has finitely many states. $\qquad \square$

Lemma 5. *Let σ_1 and σ_2 be two situations. Then it is decidable whether $\sigma_1 \approx_{\mathcal{D}\mathcal{K}} \sigma_2$.*

Proof. For every fluent $R \in \mathcal{F}$ and parameters $\vec{\rho}$, $\mathcal{D} \models R(\vec{\rho}, \sigma_1)$ iff $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[R(\vec{\rho}, \sigma_1)]$, similarly for σ_2. Following Theorem 5, checking whether $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[R(\vec{\rho}), \sigma]$ is decidable. Furthermore, because $\mathcal{D}$ has finitely many objects and finitely many fluents, there are only finitely many fluent atoms that need to be checked. Finally, checking for the existence of a bijection $f : \nu_{\sigma_1} \to \nu_{\sigma_2}$ can be done by enumerating all possible bijections (of which there are finitely many) and checking whether all requirements are satisfied. $\qquad \square$

Lemma 6. *For a clocked BAT $\mathcal{D}$, Algorithm 1 terminates and returns the time-abstract transition system $\mathcal{T}_{\mathcal{D}}$.*

Proof. Let $\mathcal{T}_{\mathcal{D}}$ be the time-abstraction transition system of a BAT $\mathcal{D}$ and T the transition system returned by Algorithm 1. We need to show for every path p starting in $[S_0]_{\approx_{\mathcal{D}\mathcal{K}}}$ that p is a path in $\mathcal{T}_{\mathcal{D}}$ iff p is a path in T.

$\Rightarrow$: Let $p = [S_0]_{\approx_{\mathcal{D}\mathcal{K}}} \to^* [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ be a path in $\mathcal{T}_{\mathcal{D}}$. We show by induction on the length n of p that p is a path in T.
Base case. Let $n = 0$, i.e., p is the empty path. Clearly, the empty path is also a path in T.
Induction step. Let $p = [S_0]_{\approx_{\mathcal{D}\mathcal{K}}} \to^* [\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}} \to [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$. By induction, $[S_0]_{\approx_{\mathcal{D}\mathcal{K}}} \to^* [\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}}$ is a path in T. It remains to be shown that $[\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}} \to [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ is a transition in T. Clearly, $\sigma = \mathrm{do}(\alpha, \sigma^*)$ for some α. We distinguish two cases:

1. Assume $\alpha = A(\vec{\rho})$ for some action type $A \in \mathcal{A} \setminus \{wait\}$. Clearly, α is added to $Acts$ in line 4. By definition of $\mathcal{T}_{\mathcal{D}}$, $\mathcal{D} \models \mathrm{Exec}(\sigma)$ and therefore $\mathcal{D} \models \mathrm{Poss}(\alpha, \sigma^*)$. Therefore, $[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}} \in V$ (either it is added in line 9 or it was already in V before) and $([\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}}, [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}})$ is added to E in line 11.

2. Otherwise, $\alpha = wait(\tau)$ for some $\tau \in \mathbb{R}_{\geq 0}$. By Lemma 2, there is a $\tau' \in Succs$ such that $\nu_{\sigma^*} + \tau \cong_{\mathcal{K}} \nu_{\sigma^*} + \tau'$ and so $\alpha' = wait(\tau')$ is added to $Acts$ in line 5. Also, $\mathrm{do}(\alpha, \sigma^*) \approx_{\mathcal{D}\mathcal{K}} \mathrm{do}(\alpha', \sigma^*)$, and so $[\mathrm{do}(\alpha, \sigma^*)]_{\approx_{\mathcal{D}\mathcal{K}}} = [\mathrm{do}(\alpha', \sigma^*)]_{\approx_{\mathcal{D}\mathcal{K}}} = [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$. Furthermore, as the precondition axiom of $wait$ is $\mathrm{Poss}(wait(d), s) \equiv \top$, it is clear that $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\mathrm{Poss}(\alpha', \sigma)]$. Therefore, $[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}} \in V$ (either it is added in line 9 or it was already in V before) and $([\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}}, [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}})$ is added to E in line 11.

$\Leftarrow$: Let $p = [S_0]_{\approx_{\mathcal{D}\mathcal{K}}} \to^* [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ be a path in T. We show by induction on the length $n = |p|$ of p that (1) $\mathcal{D} \models \mathrm{Exec}(\sigma)$, (2) p is a path in $\mathcal{T}_{\mathcal{D}}$.
Base case. Let $n = 0$, i.e., p is the empty path. Clearly, the empty path is also a path in $\mathcal{T}_{\mathcal{D}}$. Also, by definition, $\mathcal{D} \models \mathrm{Exec}(S_0)$.
Induction step. Let $p = [S_0]_{\approx_{\mathcal{D}\mathcal{K}}} \to^* [\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}} \to [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$. By induction, $\mathcal{D} \models \mathrm{Exec}(\sigma^*)$ and $[S_0]_{\approx_{\mathcal{D}\mathcal{K}}} \to^* [\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}}$ is a path in $\mathcal{T}_{\mathcal{D}}$. It remains to be shown that $\mathcal{D} \models \mathrm{Exec}(\sigma)$ and $[\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}} \to [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ is a transition in $\mathcal{T}_{\mathcal{D}}$. Clearly, $\sigma = \mathrm{do}(\alpha, \sigma^*)$ for some ground action term α. From the definition of Algorithm 1, it directly follows that $\mathcal{D} \models \mathrm{Poss}(\alpha, \sigma^*)$ and so $\mathcal{D} \models \mathrm{Exec}(\sigma)$. As $\mathcal{D} \models \mathrm{Exec}(\sigma)$, by definition of $\mathcal{T}_{\mathcal{D}}$, $[\sigma^*]_{\approx_{\mathcal{D}\mathcal{K}}} \to [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ is a transition in $\mathcal{T}_{\mathcal{D}}$.

We now show that the algorithm terminates. By Theorem 5 checking whether $\mathcal{D}_0 \cup \mathcal{D}_c \models \mathcal{R}[\mathrm{Poss}(\alpha, \sigma)]$ is decidable. Also, by Lemma 5, checking for each new situation $\mathrm{do}(\alpha, \sigma)$ whether there is an existing state $[\sigma]_{\approx_{\mathcal{D}\mathcal{K}}} \in V$ (line 8) is decidable. Furthermore, by Lemma 4, $\mathcal{T}_{\mathcal{D}}$ is finite, and so the loop eventually terminates. $\qquad \square$

Theorem 7. *Reachability for clocked BATs is decidable.*

Proof. By Lemma 3, it is sufficient to consider the time-abstract transition system $\mathcal{T}_{\mathcal{D}}$. By Lemma 6, Algorithm 1 terminates and computes $\mathcal{T}_{\mathcal{D}}$. We now show how to use $\mathcal{T}_{\mathcal{D}}$ for deciding the reachability problem of some property ϕ with maximal constant $\leq \mathcal{K}$: By construction, every state of $\mathcal{T}_{\mathcal{D}}$ is an equivalence class wrt. $\approx_{\mathcal{D}\mathcal{K}}$ of executable situations, of which there are finitely many. By Lemma 2, it is sufficient to consider one situation $\sigma^* \in [\sigma]_{\approx_{\mathcal{D}\mathcal{K}}}$ for each such state. Furthermore, by Theorem 4, we can check whether $\mathcal{D} \models \phi[\sigma^*]$ by regression. If such a state exists, return "reachable", otherwise return "not reachable". $\qquad \square$

Lemma 7. *Let σ_1, σ_2 be two situations with $\sigma_1 \approx_{\mathcal{D}\mathcal{K}} \sigma_2$ and such that $\mathcal{D} \models \mathrm{Trans}(\delta, \sigma_1, \alpha, \delta^*, \sigma_1')$. Then:*

1. *If $\alpha = wait(\tau_1)$ for some $\tau_1 \in \mathbb{R}_{\geq 0}$, then there is $\tau_2 \in \mathrm{TSuccs}(\sigma_2, K)$ such that $\mathrm{Trans}(\delta, \sigma_2, wait(\tau_2), \delta^*, \sigma_2')$ and $\sigma_1' \approx_{\mathcal{D}\mathcal{K}} \sigma_2'$.*

2. *If $\alpha = A(\vec{\rho})$ for some action type $A \neq wait$, then $\mathrm{Trans}(\delta, \sigma_2, \alpha, \delta^*, \sigma_2')$ and $\sigma_1' \approx_{\mathcal{D}\mathcal{K}} \sigma_2'$.*

Proof. First, note that by Lemma 2, for every clocked formula ϕ uniform in s, $\mathcal{D} \models \phi[\sigma_1]$ iff $\mathcal{D} \models \phi[\sigma_2]$ and so it directly follows that $\mathcal{D} \models \mathrm{Final}(\delta, \sigma_1)$ iff $\mathcal{D} \models \mathrm{Final}(\delta, \sigma_2)$.

1. Let $\alpha = wait(\tau_1)$ for some $\tau_1 \in \mathbb{R}_{\geq 0}$. By Lemma 2, there is a $\tau_2 \mathrm{TSUCCS}(\sigma_2, K)$ such that $\nu_{\sigma_1} + \tau_1 \cong_K \nu_{\sigma_2} + \tau_2$. Let $\sigma_2' = \mathrm{do}(wait(\tau_2), \sigma_2)$. With $\mathcal{D} \models \mathrm{Poss}(wait(t), s) \equiv \top$, it follows that $\mathrm{Trans}(\delta, \sigma_2, wait(\tau_2), \delta^*, \sigma_2')$. It remains to be shown that $\sigma_1 \approx_{\mathcal{DK}} \sigma_2'$. Analogously to Theorem 6:

   (a) Let $R \in \mathcal{F}$ be a relational fluent and $\vec{\rho}\,'$ a ground tuple of arguments. We show that $\mathcal{D} \models R(\vec{\rho}\,', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}\,', \sigma_2')$. There is a SSA for R of the form $R(\vec{x}, \mathrm{do}(a, s)) \equiv \phi_R(\vec{x}, a, s)$, where $\phi_R(\vec{x}, a, s)$ is a clocked formula uniform in s. As $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$, it directly follows from Lemma 2 that $\mathcal{D} \models \phi_R(\vec{\rho}\,', wait(\tau_1), \sigma_1)$ iff $\mathcal{D} \models \phi_R(\vec{\rho}\,', wait(\tau_2), \sigma_2)$ and therefore $\mathcal{D} \models R(\vec{\rho}\,', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}\,', \sigma_2')$.

   (b) Let c be a functional fluent and $\omega = c(\vec{\rho}\,')$. By definition of the SSA for c, it directly follows that $\nu_{\sigma_1'}(\omega) = \nu_{\sigma_1}(\omega) + \tau_1$ and $\nu_{\sigma_2'}(\omega) = \nu_{\sigma_2}(\omega) + \tau_2$. Thus, we can set $f'(\omega, \nu_{\sigma_1'}(\omega)) = (\omega, \nu_{\sigma_2'}(\omega))$. As $\nu_{\sigma_1} + \tau_1 \cong_K \nu_{\sigma_2} + \tau_2$, it follows that f' is a bijection witnessing $\approx_{\mathcal{DK}}$.

2. Let $\alpha = A(\rho)$ for some action type $A \neq wait$ and so $\sigma_1' = \mathrm{do}(\alpha, \sigma_1)$. There is a precondition axiom of the form $\mathrm{Poss}(A(\vec{x}, s)) \equiv \phi_A(\vec{x}, s)$ where $\phi_A(\vec{x}, s)$ is a clocked formula uniform in s. As $\mathrm{Trans}(\delta, \sigma_1, \alpha, \delta^*, \sigma_1')$, it is clear that $\mathcal{D} \models \phi_A(\vec{\rho}, \sigma_1)$ and with $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$, also $\mathcal{D} \models \phi_A(\vec{\rho}, \sigma_2)$. Therefore, $\mathcal{D} \models \mathrm{Poss}(A(\rho), \sigma_2)$ and so $\mathcal{D} \models \mathrm{Trans}(\delta, \sigma_2, \alpha, \delta^*, \sigma_2')$ with $\sigma_2' = \mathrm{do}(\alpha, \sigma_2)$. It remains to be shown that $\sigma_1' \approx_{\mathcal{DK}} \sigma_2'$. Analogously to Theorem 6:

   (a) Let $R \in \mathcal{F}$ be a relational fluent and $\vec{\rho}\,'$ a ground tuple of arguments. We show that $\mathcal{D} \models R(\vec{\rho}\,', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}\,', \sigma_2')$. There is a SSA for R of the form $R(\vec{x}, \mathrm{do}(a, s)) \equiv \phi_R(\vec{x}, a, s)$, where $\phi_R(\vec{x}, a, s)$ is a clocked formula uniform in s. As $\sigma_1 \approx_{\mathcal{DK}} \sigma_2$, it directly follows from Lemma 2 that $\mathcal{D} \models \phi_R(\vec{\rho}\,', A(\vec{\rho}), \sigma_1)$ iff $\mathcal{D} \models \phi_R(\vec{\rho}\,', A(\vec{\rho}), \sigma_2)$ and therefore $\mathcal{D} \models R(\vec{\rho}\,', \sigma_1')$ iff $\mathcal{D} \models R(\vec{\rho}\,', \sigma_2')$.

   (b) We construct a bijection $f' : \nu_{\sigma_1'} \to \nu_{\sigma_2'}$ witnessing $\approx_{\mathcal{DK}}$. Let c be a functional fluent and $\omega = c(\vec{\rho}\,')$. There is a SSA of the form $c(\vec{x}, \mathrm{do}(a, s)) = y \equiv \exists d.\, a = wait(d) \wedge y = c(\vec{x}, s) + d \vee a \neq wait(d) \wedge (\phi_c(\vec{x}, a, s) \wedge y = 0 \vee \neg\phi_c(\vec{x}, a, s) \wedge y = c(\vec{x}, s))$. Thus, $\omega[\sigma_1'] = 0$ if $\mathcal{D} \models \phi_c(\vec{\rho}, \alpha, \sigma_1)$ and $\omega[\sigma_1'] = \omega[\sigma_1]$ otherwise, similarly for $\omega[\sigma_2']$. Now, as $\phi_c(\rho, a, s)$ is a clocked formula uniform in s, by Lemma 2, $\mathcal{D} \models \phi_c(\rho, \alpha, \sigma_1)$ iff $\mathcal{D} \models \phi_c(\rho, \alpha, \sigma_2)$. Hence, for each ω, we can set $f'(\omega, 0) = (\omega, 0)$ if $\mathcal{D} \models \phi_c(\rho, \alpha, \sigma_1)$ and $f'(\omega, \nu_{\sigma_1}(\omega)) = (\omega, \nu_{\sigma_2}(\omega))$ otherwise. $\square$

Lemma 8. *Let $\Delta = (\mathcal{D}, \delta)$ be a program over a clocked BAT $\mathcal{D}$ and $\mathcal{T}_\Delta$ the transition system returned by Algorithm 2. Then $\mathcal{T}_\Delta$ contains a state $([\sigma']_{\approx_{\mathcal{DK}}}, \delta')$ reachable from $([S_0]_{\approx_{\mathcal{DK}}}, \delta)$ with $\mathcal{D} \models \mathrm{Final}(\delta', \sigma')$ iff there is a realization of program Δ.*

Proof. We first show that $([\sigma]_{\approx_{\mathcal{DK}}}, \delta')$ is reachable in $\mathcal{T}_\Delta$ iff $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha}, \delta', \sigma)$.
$\Rightarrow$: Let $p = ([S_0]_{\approx_{\mathcal{DK}}}, \delta) \to^* ([\sigma']_{\approx_{\mathcal{DK}}}, \delta')$ be a path

in $\mathcal{T}_\Delta$. We show by induction on the length n of p that $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha}, \delta', \sigma')$ for some $\vec{\alpha}$ with $|\vec{\alpha}| = n$:
Base case. Let $n = 0$ and therefore $\sigma' = S_0$ and $\delta' = \delta$. Clearly, $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \langle\rangle, \delta, S_0)$.
Induction step. Let $p = ([S_0]_{\approx_{\mathcal{DK}}}, \delta) \to^* ([\sigma^*]_{\approx_{\mathcal{DK}}}, \delta^*) \to ([\sigma']_{\approx_{\mathcal{DK}}}, \delta')$. By induction, $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha}, \delta^*, \sigma^*)$ for some $\vec{\alpha}$ with $|\vec{\alpha}| = n$. By construction of $\mathcal{T}_\Delta$, $\mathcal{D} \models \mathrm{Trans}(\delta^*, \sigma^*, \alpha, \delta', \sigma')$ for some α and therefore also $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha} \cdot \alpha, \delta', \sigma')$.
$\Leftarrow$: Let $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha}, \delta', \sigma')$. We show by induction on the length n of $\vec{\alpha}$ that there is a path $p = ([S_0]_{\approx_{\mathcal{DK}}}, \delta) \to^* ([\sigma']_{\approx_{\mathcal{DK}}}, \delta')$ in $\mathcal{T}_\Delta$ with length n.
Base case. Let $n = 0$ and therefore $\sigma' = S_0$ and $\delta' = \delta$. Clearly, there is an empty path in $\mathcal{T}_\Delta$ starting in $([S_0]_{\approx_{\mathcal{DK}}}, \delta)$.
Induction step. Let $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha}, \delta^*, \sigma^*)$ for some $\vec{\alpha}$ with $|\vec{\alpha}| = n$ and let $\mathcal{D} \models \mathrm{Trans}(\delta^*, \sigma^*, \alpha, \delta', \sigma')$. By induction, there is a path $p = ([S_0]_{\approx_{\mathcal{DK}}}, \delta) \to^* ([\sigma^*]_{\approx_{\mathcal{DK}}}, \delta^*)$ with length n. By Lemma 7, there is a α' such that $\mathcal{D} \models \mathrm{Trans}(\delta^*, \sigma^*, \alpha', \delta', \sigma'')$ and $\sigma'' \approx_{\mathcal{DK}} \sigma'$. Hence, $([\sigma^*]_{\approx_{\mathcal{DK}}}, \delta) \to ([\sigma']_{\approx_{\mathcal{DK}}}, \delta)$ is a transition in $\mathcal{T}_\Delta$.

Now, as $([\sigma]_{\approx_{\mathcal{DK}}}, \delta')$ is reachable in $\mathcal{T}_\Delta$ iff $\mathcal{D} \models \mathrm{Trans}^*(\delta, S_0, \vec{\alpha}, \delta', \sigma)$ for some $\vec{\alpha}$, it follows with Lemma 2 that $\mathcal{D} \models \mathrm{Final}(\delta', \sigma)$ iff $\vec{\alpha}$ is a realization of Δ. $\square$

Lemma 9. *$\mathcal{T}_\Delta$ has finitely many states.*

Proof. As $\mathcal{O}$ and $\mathcal{A}$ are finite, each state only has finitely many successors. Furthermore, as there only finitely many fluents $\mathcal{F}$ and objects $\mathcal{O}$, $\approx_{\mathcal{DK}}$ has finitely many equivalence classes. Finally, δ has finitely many sub-program expressions. Therefore, the time-abstract transition system $\mathcal{T}_\mathcal{D}$ has finitely many states. $\square$

Theorem 8. *The realization problem for a program $\Delta = (\mathcal{D}, \delta)$ over a clocked BAT $\mathcal{D}$ is decidable.*

Proof. By Lemma 8, it is sufficient to check whether there is a reachable state in $\mathcal{T}_\Delta$ that corresponds to a final program configuration. By Lemma 9, $\mathcal{T}_\Delta$ is finite and so explicitly constructing it is feasible. Finally, checking whether $\mathcal{D} \models \mathrm{Trans}(\delta, \sigma, \alpha, \delta', \sigma')$ reduces to checking entailment of clocked formulas uniform in σ (in particular, checking $\mathcal{D} \models \mathrm{Poss}(\alpha, \sigma)$ and $\mathcal{D} \models \phi[\sigma]$ for tests ϕ?). By Theorem 5, this is decidable. $\square$