

Linear Algebra Done Right – Chapter 3 Solutions

1 Exercises 3B

Exercise 1. Give an example of a linear map T with $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$.

The intuition behind the solution is that we can define a linear map just on the values on a basis of the linear space V we choose. I will choose V to be \mathbb{R}^5 and $T \in \mathcal{L}(V)$.

Therefore, I define $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ with:

- $T((1,0,0,0,0))=0$
- $T((0,1,0,0,0))=0$
- $T((0,0,1,0,0))=0$
- $T((0,0,0,1,0))= (0,0,0,1,0)$
- $T((0,0,0,0,1))= (0,0,0,0,1)$

It is clear that $\text{null } T$ has a basis the first 3 basis vectors of \mathbb{R}^5 and the range of T has as a basis the last 2 basis vectors of \mathbb{R}^5 .

Exercise 2. Suppose $S, T \in \mathcal{L}(V)$ such that $\text{range } S \subseteq \text{null } T$. Prove that $(ST)^2=0$.

Solution. $(ST)^2(v) = (ST)(ST(v))$. We have that $ST(V) \in \text{range } S$. Therefore $ST(V) \in \text{null } T$.

Which means that $T(ST(v)) = 0, \forall v \in V$. Therefore $(ST)^2(v) = S(0) = 0$, because S is a linear map.

Exercise 3. Suppose v_1, \dots, v_m is a list of vectors in V . Define $T \in (\mathcal{F}^m, V)$ by:

$$T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m.$$

- What property of T corresponds to v_1, \dots, v_m spanning V ?
- What property of T corresponds to the list v_1, \dots, v_m being linearly independent?

Solution. To answer the questions in order:

- If v_1, \dots, v_m span V , then T would be surjective as practically $\text{range } T$ is just $\text{span } V$.
- If v_1, \dots, v_m are linearly independent, this means that $T(z_1, \dots, z_m) = 0$ only if $z_1 = \dots = z_m = 0$, therefore $\dim \text{null } T = 0$, so it is injective.