

# Linear Algebra Done Right – Chapter 3 Solutions

## 1 Exercises 3B

**Exercise 1.** Give an example of a linear map  $T$  with  $\dim \text{null } T = 3$  and  $\dim \text{range } T = 2$ .

The intuition behind the solution is that we can define a linear map just on the values on a basis of the linear space  $V$  we choose. I will choose  $V$  to be  $\mathbb{R}^5$  and  $T \in \mathcal{L}(V)$ .

Therefore, I define  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$  with:

- $T((1,0,0,0,0))=0$
- $T((0,1,0,0,0))=0$
- $T((0,0,1,0,0))=0$
- $T((0,0,0,1,0))=(0,0,0,1,0)$
- $T((0,0,0,0,1))=(0,0,0,0,1)$

It is clear that  $\text{null } T$  has a basis the first 3 basis vectors of  $\mathbb{R}^5$  and the range of  $T$  has as a basis the last 2 basis vectors of  $\mathbb{R}^5$ .

**Exercise 2.** Suppose  $S, T \in \mathcal{L}(V)$  such that  $\text{range } S \subseteq \text{null } T$ . Prove that  $(ST)^2=0$ .

*Solution.*  $(ST)^2(v) = (ST)(ST(v))$ . We have that  $ST(V) \in \text{range } S$ . Therefore  $ST(V) \in \text{null } T$ .

Which means that  $T(ST(v)) = 0, \forall v \in V$ . Therefore  $(ST)^2(v) = S(0) = 0$ , because  $S$  is a linear map.

**Exercise 3.** Suppose  $v_1, \dots, v_m$  is a list of vectors in  $V$ . Define  $T \in (\mathcal{F}^m, V)$  by:

$$T(z_1, \dots, z_m) = z_1v_1 + \dots + z_mv_m.$$

- What property of  $T$  corresponds to  $v_1, \dots, v_m$  spanning  $V$ ?
- What property of  $T$  corresponds to the list  $v_1, \dots, v_m$  being linearly independent?

*Solution.* To answer the questions in order:

- If  $v_1, \dots, v_m$  span  $V$ , then  $T$  would be surjective as practically range  $T$  is just span  $V$ .
- If  $v_1, \dots, v_m$  are linearly independent, this means that  $T(z_1, \dots, z_m) = 0$  only if  $z_1 = \dots = z_m = 0$ , therefore  $\dim \text{null } T = 0$ , so it is injective.