Linear Algebra Done Right – Chapter 3 Solutions

1 Exercises 3B

Exercise 1. Give an example of a linear map T widh dim null $T = 3$ and dim range $T = 2$.

The intuition behind the solution is that we can define a linear map just on the values on a basis of the linear space V we choose. I will choose V to be $\mathbb{R}^5$ and $T \in \mathcal{L}(\mathcal{V})$.

Therefore, I define $T : \mathbb{R}^5 \to \mathbb{R}^5$ with:

- T ((1,0,0,0,0))=0
- T ((0,1,0,0,0))=0
- T ((0,0,1,0,0))=0
- T ((0,0,0,1,0))= (0,0,0,1,0)
- T ((0,0,0,0,1))= (0,0,0,0,1)

It is clear that null T has a basis the first 3 basis vectors of $\mathbb{R}^5$ and the range of T has as a basis the last 2 basis vectors of $\mathbb{R}^5$.

Exercise 2. Suppose $S, T \in \mathcal{L}(V)$ such that range $S \subseteq$ null T. Prove that $(ST)^2$=0.

Solution. $(ST)^2(v) = (ST)(ST(v))$. We have that $ST(V) \in$ range S. Therefore $ST(V) \in$ null T.

Which means that $T(ST(v)) = 0, \forall v \in V$. Therefore $(ST)^2(v) = S(0) = 0$, because S is a linear map.

Exercise 3. Suppose $v_1, \cdots, v_m$ is a list of vectors in V. Define $T \in (\mathcal{F}^m, V)$ by:

$$T(z_1, \cdots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

- What property of T corresponds to $v_1, \ldots, v_m$ spanning V?
- What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

Solution. To answer the questions in order:

- If $v_1, \ldots, v_m$ span V, then T would be surjective as practically range T is just span V.
- If $v_1, \ldots, v_m$ are linearly independent, this means that $T(z_1, \ldots, z_m) = 0$ only if $z_1 = \cdots = z_m = 0$, therefore dim null T =0 , so it is injective.

Exercise 4. Show that X = { $T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$: dim null $T > 2$ } is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$

Solution. My intuition for this exercise was just to build two linear maps that had no overlap of values over the basis. So in one case, one of the maps would map a basis vector to 0 and in another it would map the basis vector to itself.

Therefore, let $v_1, \dots, v_5$ be a basis of $\mathbb{R}^{\not 4}$. I define the following maps T_1 and T_2:

- $T_1(v_1) = v_1$,$T_1(v_2) = v_2$,$T_1(v_3) = 0$,$T_1(v_4) = 0$,$T_1(v_5) = 0$
- $T_2(v_1) = 0$,$T_2(v_2) = 0$,$T_2(v_3) = v_3$,$T_2(v_4) = v_4$,$T_2(v_5) = 0$

Both of them clearly have dim null T_1 = dim null $T_2 = 3$.

However, when we calculate $T_1 + T_2$ we get:

- $(T_1+T_2)(v_1) = v_1$, $(T_1+T_2)(v_2) = v_2$, $(T_1+T_2)(v_3) = v_3$, $(T_1+T_2)(v_4) = v_4$, $(T_1+T_2)(v_5) = v_5$

Therefore, dim null $T_1 + T_2 = 1$, which means that $T_1 + T_2 \notin X$ which shows that the set X is not closed under addition, therefore it cannot be a subspace.

Exercise 5. Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that range T = null T.

Solution. My intuition behind this problem was that if range T = null T, then T(T(x))=0, $\forall x \in \mathbb{R}^4$. Now, I did "cheat" a little bit by thinking that T was a matrix and just looking for a matrix with $T^2 = 0$, where 0 here is the null 4×4 matrix.

Therefore, the final map that I have reached is: $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 - x_3, x_2 - x_3, x_1 - x_4) = (x_1 - x_4) * (1, 0, 0, 1) + (x_2 - x_3) * (0, 1, 1, 0)$. It is clear that the range is a linear combination between $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$ which are linearly independent, therefore dim range $T = 0$. We have that

$$T(T(x)) = 0$$

which means that range T $\subseteq$null T.

Now I need to prove that null $T \subseteq$ range T.

Let $y \in$ null T , be an arbitrary vector in $nullT$. Therefore, $T(y_1, y_2, y_3, y_4) = 0$ which means that $y_1 = y_4, y_2 = y_3$.

I will choose $x_1 = y_1, x_2 = y_2, x_3 = 0, x_4 = 0$, and get that $T(x_1, x_2, x_3, x_4) = (y_1, y_2, y_2, y_1) = (y_1, y_2, y_3, y_4)$. Therefore I have found x_1, x_2, x_3, x_4 such that $T(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, y_4)$ so $y \in$ range T. Therefore null $T \subseteq$ range T.

So null T = range T.

Exercise 6. Prove that there does not exist $T \in \mathcal{L}(\mathbb{R}^5)$ such that range T = null T.

Solution. We know that dim range T + dim null T = dim $\mathbb{R}^5$, which means 2*dim range $T = 5$ which is not possible as dim range T is an integer.

Exercise 7. Suppose V and W are finite-dimensional with $2 \leq$ dim V $\leq$ dim W. Show that $\{T \in \mathcal{L}(V, W) : T$ is not injective$\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution. I will build two maps that are not injective, but whose sum is. I will denote dim V = n, dim W = m. Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ be a basis of W. I can create the following mappings, using the fact that $2 \leq n \leq m$:

- $T_1(v_1) = w_1, T_1(v_2) = w_1, T_1(v_i) = w_i, \forall i \in \{3, \dots, n\}$
- $T_2(v_1) = w_2, T_2(v_2) = -w_2, T_2(v_i) = w_i, \forall i \in \{3, \dots, n\}$

They are both not injective as their nullspaces do not have dimension 0, clearly $T_1(v_1 - v_2) = 0$ and $T_2(v_1 + v_2) = 0$.

If we look at $T_1 + T_2$, we find that $(T_1+T_2)(v_1) = w_1 + w_2, (T_1+T_2)(v_2) = w_1 - w_2, \ldots, (T_1 + T_2)(v_i) = 2 * v_i, \forall i \in \{3, \ldots, n\}$. Let $x \in$ null $(T_1 + T_2)$. We have that $(T_1 + T_2)(x) = 0$, we can write x uniquely according to the basis $v_1, \ldots, v_n$ and get that: $(T_1 + T_2)(\alpha_1 v_1 + \cdots + \alpha_n v_n) = 0$. Therefore $\alpha_1(w_1 + w_2) + \alpha_2(w_1 - w_2) + \sum_{i=3}^{n} 2\alpha_i w_i = 0$.

So we get that $(\alpha_1+\alpha_2)w_1 + (\alpha_1 - \alpha_2)w_2 + \sum_{i=3}^{n} 2\alpha_i w_i = 0$. We know that the w's are linearly independent, therefore $\alpha_i = 0, \forall i \in \{3, \ldots, n\}$ and $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 0$. This means that $\alpha_1 = \alpha_2 = 0$, so x = 0. Therefore, null $T_1 + T_2 = \{0\}$, so it is injective, therefore the set given in the hypothesis is not a subspace.

Exercise 8. Suppose V and W are finite-dimensional with dim $V \geq$ dim $W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution. I will construct two linear maps that are not surjective, but whose sum is surjective. Let dim $V = n$ and dim $W = m$, where $n \geq m \geq 2$.

Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W.

Define the following two maps:

- $T_1(v_1) = w_1, T_1(v_2) = 0, T_1(v_i) = w_i$ for all $i \in \{3, \ldots, m\}$, and $T_1(v_j) = 0$ for all $j \in \{m+1, \ldots, n\}$ (if $n > m$)
- $T_2(v_1) = 0, T_2(v_2) = w_2, T_2(v_i) = 0$ for all $i \in \{3, \ldots, n\}$

Both T_1 and T_2 are not surjective because:

- For T_1: range $T_1 \subseteq \text{span}\{w_1, w_3, \ldots, w_m\}$, which has dimension $m - 1 < m$
- For T_2: range $T_2 \subseteq \text{span}\{w_2\}$, which has dimension $1 < m$

Now consider $T_1 + T_2$:

- $(T_1 + T_2)(v_1) = w_1$
- $(T_1 + T_2)(v_2) = w_2$
- $(T_1 + T_2)(v_i) = w_i$ for all $i \in \{3, \ldots, m\}$
- $(T_1 + T_2)(v_j) = 0$ for all $j \in \{m+1, \ldots, n\}$ (if $n > m$)

Therefore $(T_1 + T_2)(v_i) = w_i, \forall i \in \{1, \ldots m\}$ and 0 otherwise.

If $x \in W$, let $\alpha_1, \ldots, \alpha_m$ be the coefficients such that $x = \alpha_1 w_1 + \cdots + \alpha_m w_m$ because $w_1, \ldots, w_m$ is a basis. We will have that $(T_1 + T_2)(\alpha_1 v_1 + \cdots + \alpha_m v_m) = \alpha_1 w_1 + \cdots + \alpha_m w_m = x$, therefore there is $y \in V$ such that $(T_1 + T_2)(y) = x \in W$, $\forall x \in W$. So $(T_1 + T_2)$ is surjective, therefore the given subspace is not closed under addition.

Exercise 9. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Provate that $Tv_1, ..., Tv_n$ is linearly independent in W.

Solution. Let $\alpha_1, \ldots, \alpha_n$ be scalars such that $\alpha_1 T v_1 + \cdots + \alpha_n T v_n = 0$. We will have that $T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = 0$. We know that T is injective, therefore null $T = \{0\}$, so $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$. $v_1, ..., v_n$ are linearly independent =¿ $\alpha_1 = \cdots = \alpha_n = 0$. Therefore $Tv_1, ..., Tv_n$ are also linearly independent, as if $\alpha_1 T v_1 + \cdots + \alpha_n T v_n = 0 => \alpha_1 = \cdots = \alpha_n = 0$

Exercise 10. Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Show that $Tv_1, \ldots, Tv_n$ spans range T.

Solution. Let $x \in$ range T, this means that $\exists y \in V$ such that $T(y) = x$. $v_1, ..., v_n$ spans V, therefore $\exists \alpha_1, ..., \alpha_n$ scalars such that $\alpha_1 v_1 + \cdots + \alpha_n v_n$=y. This means that $\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = x$, due to T's linearity. Therefore, $\forall x \in$ range T, $\exists \alpha_1, ..., \alpha_n$ such that $\alpha_1 Tv_1 + \cdots + \alpha_n Tv_n = x$. Therefore, $Tv_1, \ldots, Tv_n$ span range T.

Exercise 11. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that:

$$U \cap \text{ null } T = \{0\} \text{ and range } T = \{Tu : u \in U\}$$

Solution. Let $t_1, \ldots, t_n$ be a basis of range T. We know that there must exist $u_1, ..., u_n \in V$ such that $T(u_1) = t_1, ..., T(u_n) = t_n$. $T(u_1), ..., T(u_n)$ are linearly independent, we can use exercise 4 from 3A and get that $u_1, ..., u_n$ are also linearly independent.

I will denote my subspace U to be span $\{u_1, ..., u_n\}$. Let $u \in U \cup$ null T. We have that $T(u) = 0$. However, u is a linear combination of $\{u_1, ..., u_n\}$ so we get that $\exists \alpha_1, ..., \alpha_n$ scalars such that $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$. Therefore, $T(\alpha_1 u_1 + ... + \alpha_n u_n) = 0 <=> \alpha_1 t_1 + ... + \alpha_n t_n = 0$, but $t_1, ..., t_n$ are linearly independent, therefore $\alpha_1 = \cdots = \alpha_n = 0$. This means that $u = 0$. Therefore $U \cup$ null $T = \{0\}$.

Finally, I need to prove that range $T = \{Tu : u \in U\}$. I will prove this by double inclusion.

Let $x \in$ range T. This means that $\exists \alpha_1, ..., \alpha_n$ such that $\alpha_1 t_1 + ... + \alpha_n t_n = x$. If we choose $u = \alpha_1 u_1 + ... + \alpha_n u_n$, it will be in U because $U =$ span $\{u_1, ..., u_n\}$ and we will have that $T(u) = \alpha_1 T(u_1) + ... + \alpha_n T(u_n) = \alpha_1 t_1 + ... + \alpha_n t_n = x$. Therefore, $x \in \{Tu : u \in U\}$. Therefore, range $T \subseteq \{Tu : u \in U\}$.

Let $x \in \{Tu : u \in U\}$. This means that $\exists u \in U \subseteq V$ such that $Tu = x$. Therefore, $x \in$ range T. So we have that $\{Tu : u \in U\} \subseteq T$.

Finally, $\{Tu : u \in U\} =$ range T.

Exercise 12. Suppose T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$$

Prove that T is surjective.

Solution. null $T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$.

null $T = \{(5x_2, x_2, 7x_4, x_4) \in \mathbb{F}^4 | x_2, x_4 \in \mathbb{F}^2\}$

null $T = \{x_2(5, 1, 0, 0) + x_4(0, 0, 7, 1) | x_2, x_4 \in \mathbb{F}^2\}$.

null $T =$ span $\{(5, 1, 0, 0), (0, 0, 7, 1)\}$. Also, it is clear that (5,1,0,0) and (0,0,7,1) are linearly independent, therefore $\{(5, 1, 0, 0), (0, 0, 7, 1)\}$ is a basis for null T. Therefore dim null $T = 2$. From the Fundamental Theorem of Linear Maps we have that dim range T+dim null T = dim $\mathbb{F}^4$=4. Therefore, dim range T =2. But range T is a subspace of $\mathbb{F}^2$ which also has dim $\mathbb{F}^2$=2. So we will have that range $T = \mathbb{F}^2$ which means that T is surjective.

Exercise 13. Suppose U is a three-dimensional subspace of $\mathbb{R}^8$ and that T is a linear map from R^8 to R^5 such that null $T = U$. Prove that T is surjective.

Solution. U is a three-dimensional subspace of $\mathbb{R}^8$, therefore dim $U = 3$. We have that null $T = U$, so dim null T=3. From the Fundamental Theorem of Linear Maps we will have that dim range T = 5. But range T is a subspace of $\mathbb{R}^5$ which also has dimension 5, therefore range $T = \mathbb{R}^5$ so we will have that T is surjective.

Exercise 14. Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

Solution. Let T be a map with the null space from the hypothesis. We can use the same logical steps as in Exercise 13 and prove that dim null $T = 2$. Therefore, using the Fundamental Theorem of Linear Maps we will have that dim range $T = 3$. However, range T is a subspace of $\mathbb{F}^2$ which means that dim range $T \leq$ dim $\mathbb{F}^2$=2. This is a contradiction, therefore there exists no linear map defined from $\mathbb{F}^5$ to $\mathbb{F}^2$ with the given null space.

Exercise 15. Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Solution. Let $T \in \mathcal{L}(V, V)$ be the linear map such that dim null $T =$ m $\in \mathbb{N}$ and dim range $T =$ n $\in \mathbb{N}$. Let $b_1, ..., b_n$ be a basis of range $T =¿ \exists u_1, ..., u_n$ such that $T(u_1) = b_1, \ldots, T(u_n) = b_n$.

Let $v \in V$, this means that $T(v) \in$ range T. Therefore, $\exists \alpha_1, ..., \alpha_n$ such that $T(v) = \alpha_1 b_1 + ... + \alpha_n b_n$. This means that $T(v) = T(\alpha_1 u_1 + \cdots + \alpha_n u_n) <=> T(v - \alpha_1 u_1 - \cdots - \alpha_n u_n) = 0$. So, $v - \alpha_1 u_1 - ... - \alpha_n u_n \in$ null T.

The last part tells us that $\exists k_1, ..., k_m$ such that $v - \alpha_1 u_1 - \cdots - \alpha_n u_n = k_1 n_1 + \cdots k_m n_m$ where $n_1, ..., n_m$ is a basis of null T. We have thus proved that for any $v \in V$ there exists n+m scalars such that $v = \alpha_1 u_1 + \cdots + \alpha_n u_n + k_1 n_1 + \cdots + k_m n_m$. Therefore, $V =$ span $\{u_1, ..., u_n, n_1, ..., n_m\}$. So V is spanned by a finite list of vectors , therefore V is finite-dimensional.

Exercise 16. Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if dim $V \leq$ dim W.

Solution. To prove the if and only if, we need to prove both ways of the implication.

The left to right implication is that if there exists an injective linear map from V to W then dim $V \leq$ dim W. This is equivalent to If dim V ¿ dim W then there exists no injective linear map from V to W and this was proved in the text.

Now the right to left implication is that if dim $V \leq$ dim W then there exists an injective linear map from V to W. To build this linear map we take $v_1, ..., v_n$ the basis of V, where dim $V =$ n and $w_1, ..., w_m$ the basis of W where dim $W =$ m. We then create the linear map T such that $T(v_i) = w_i, \forall i \in W$. If $v \in$ null T , we will write v in terms of the basis vectors and get $T(\alpha_1 v_1 + ... + \alpha_n v_n) = 0 <=> \alpha_1 w_1 + ... + \alpha_n w_n = 0$, but $w_1, ..., w_m$ are linearly independent, therefore $\alpha_1 = \cdots = \alpha_n$, so v=0. This means that null $T = \{0\}$, therefore T is injective.

Exercise 17. Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V to W if and only if dim $V \geq$ dim W.

Solution. The left to right implication is that if there exists a surjective linear map from V to W then dim $V \geq$ dim W. This is equivalent to If dim V ¡ dim W then there exists no surjective linear map from V to W and this was proved in the text.

Now for the right to the left implication, we denote dim $W =$ m, dim $V =$ n. Then we construct the linear map by taking $T(v_i) = w_i, \forall i \in \{1, ..., m\}$ and $T(v_i) = 0 \forall m+1, ..., n$. Then, it is trivial to prove that this is surjective.

Exercise 18. Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = U$ if and only if dim $U \geq$ dim $V -$ dim W.

Solution. The left to right implication says that if there exists $T \in \mathcal{L}(V, W)$ such that null T=U then dim $U \geq$ dim $V-$ dim W. Well we can use the Fundamental Theorem of Linear Maps and get that dim null T + dim range T = dim V, which means that dim U+dim range T = dim V. dim range $T \leq$ dim W, therefore $-$ dim range $T \geq$ - dim W, so we have that dim $U \geq$ dim $V-$ dim W.

For the right to left implication we are told that if dim $U \geq$ dim V - dim W then there exists $T \in \mathcal{L}(V, W)$. We can prove the contrapositive, that if dim $U <$ dim $V-$ dim W there is no linear map $T \in \mathcal{L}(V, W)$ such that null $T = U$.

To prove this contrapositive we do it by a proof of contradiction. Let there be a linear map $T \in \mathcal{L}(V, W)$ such that null $T = U$. We know that dim $U <$ dim $V-$ dim W. However, dim range T + dim null T = dim range T + dim U = dim V. Therefore dim U = dim $V-$ dim range T. This means that dim $V-$ dim range $T <$ dim $V-$ dim W, therefore dim $W <$ dim range T which is a contradiction because range T is a subspace of W.

Q.E.D

Exercise 19. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, W)$ such that ST is the identity operator on V.

Solution. Let's prove the right to left implication. We know there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V. Let $u, v \in V$ such that $T(u) = T(v)$. If $T(u) = T(v)$, this means that $ST(u) = ST(v)$, but ST is the identity operator on V, therefore $u = v$. So we obtained that $\forall u, v \in V$, if $T(u) = T(v) => u = v$. Therefore T is injective.

Now for the left to right implication, we know that T is injective, we want to find $S \in \mathcal{L}(V, W)$ such that ST is the identity operator on V. I will first define the linear map $S' \in$ range $\mathcal{T}, \mathcal{V}$, such that $S'(x) = T^{-1}(x), \forall x in$range T. Where with $T^{-1}(x)$ I have denoted the pre-image $v' \in V$ such that $T(v') = x$. We know it is unique because T is injective. Therefore, we can write that $S'(T(v')) = v', \forall v' in V$, so $S'(T)$ acts as the identity operator on V. Now, we can use exercise 13 from 3A, because range T is a subspace of W we can extend S' to $S \in \updownarrow(W, V)$ such that $S'(T(v)) = S(T(V)), \forall v \in V$ and we have now found the necessary map.

Exercise 20. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W.

Solution. Let's prove the right to left implication. If there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W, then we have that $TS(w) = w, \forall w \in W$. This means that $\forall w \in W$, there exists $v = S(w) \in V$ such that $T(v) = w$. Therefore, T is surjective.

Now for the left to right implication. Let T be a surjective linear map from $\mathcal{L}(V, W)$. This means that $\forall w \in W, \exists v' \in V$ such that $T(v') = w$. Let $w_1, \ldots, w_n$ be a basis of W where $n =$ dim W. We will have that $S(w_i) = v_i$, where v_i is a vector such that $T(v_i) = w_i$. We know v_i exists because T is surjective. S is a properly defined map from W to V as we defined it on a basis of W. With this construction we have that $TS(w_i) = w_i, \forall i \in \{1, \ldots, n\}$. Now, we will have that $TS(w) = T(S(\alpha_1 w_1 + \cdots + \alpha_n w_n)) = \alpha_1 TS(w_1) + \cdots + \alpha_n TS(w_n) = \alpha_1 w_1 + ... + \alpha w_n = w$,therefore TS acts as the identity operator on W.

Exercise 21. Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \text{ null } T + \dim\ (U \cap \text{ range } T).$$

Solution. First we will prove that $A = \{v \in V : Tv \in U\}$ is a subpsace of V:

- $T(0) = 0 \in U$, therefore $0 \in A$.

- If $v_1, v_2 \in A$, we have that Tv_1 is in U and Tv_2 is in U, U is a subspace, therefore $T(v_1+v_2) \in U$, so $v_1 + v_2 \in A$. A is closed under addition
- If $v \in A$, then Tv is in U, which means that $\alpha Tv \in U$, so $T(\alpha v) \in U$ by T's linearity, so $\alpha v \in A$. A is closed under scalar multiplication.

Therefore, A is a subspace of V.

Now if $v \in$ null T, we have that $Tv = 0 \in U$, therefore null $T \subseteq A$. We know that null T is a vector space therefore it is also a subspace of A. We take a basis $b_1, \dots, b_m$ of null T where dim null T = m. We can extend the basis over null T to be a basis over A as A is a finite-dimensional space, let's denote the extension basis vectors $c_{m+1}, \dots c_n$ where $n =$ dim A.

The conclusion of the problem now is equivalent to proving that $n - m =$ dim ($U\cap$ range T).

Let $v \in A$, we can write $v = \alpha_1 b_1 + \cdots + \alpha_m b_m + \alpha_{m_1} c_{m+1} + \cdots + \alpha_n c_n$. We will have that $Tv = T(\alpha_{m+1}c_{m+1} + \cdots + \alpha_n c_n) = \alpha_{m+1}T(c_{m+1}) + \cdots + \alpha_n T(c_n)$. Therefore if $Tv \in U$, then we can write it as a linear combination of $T(c_{m+1}), \dots, T(c_n)$, so $U\cup$ range $T =$ span $\{T(c_{m+1}), \dots, T(c_n)\}$.

Now we need to prove that $T(c_{m+1}), \dots, T(c_n)$ are linearly independent. If $\alpha_{m+1}T(c_{m+1}) + \cdots + \alpha_n T(c_n) = 0$, we will have that $T(\alpha_{m+1}c_{m+1} + ... + \alpha_n c_n) = 0$, which means $\alpha_{m+1}c_{m+1} + ... + \alpha_n c_n \in$ null T. This means that there exist $\alpha_1, ..., \alpha_m$ such that $\alpha_{m+1}c_{m+1} + ... + \alpha_n c_n = \alpha_1 b_1 + ... + \alpha_m b_n$. But we know that $b_1, ..., b_m, c_{n+1}, ..., c_n$ are linearly independent, so $\alpha_{m+1} = ... = \alpha_n$ so we get that $T(c_{m+1}), ..., T(c_n)$ are linearly independent.

With the result from above we get that $T(c_{m+1}), ..., T(c_n)$ is a basis of $U\cap$ range T, so dim ($U\cap$ range T) $= n - (m+1) + 1 = n - m$. Q.E.D.

Exercise 22. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\text{dim null } ST \leq \text{dim null } S + \text{dim null } T.$$

Solution. null $ST = \{v \in U : Tv \in \text{null } S\}$. We know that null S is a subspace of V. Using exercise 21 we find out that:

$$\text{dim null } ST = \text{dim null } T + \text{dim (null } S \cap \text{range T)}$$

Clearly, dim (null $S\cap$ range T) $\leq$ dim null S. So we have proved that:

$$\text{dim null } ST \leq \text{dim null } T + \text{dim null } S$$

Exercise 23. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that:

$$\text{dim range } ST \leq \min\{\text{dim range } S, \text{dim range } T\}$$

Solution. range ST is a vector space and range $ST \subseteq$ range S. This means that range ST is a subspace of range S, so we have that dim range $ST \leq$ dim range S.

Now, let $x \in$ range ST, this means that $\exists v'$ such that x=$ST(v')$. Let's take a basis of range T, let it be $b_1, ..., b_n$ where $n =$ dim range T. Therefore, $\exists \alpha_1, ..., \alpha_n$ such that $x = S(\alpha_1 b_1 + ... + \alpha_n b_n)$. $x = \alpha_1 S(b_1) + ... + \alpha_n S(b_n)$. So, range ST is the span of the vectors $S(b_1), ..., S(b_n)$ which from theory means that we can reduce this spanning set into a basis for ST, so dim range $ST \leq n =$ dim range T.

We have that dim range $ST \leq$ dim range T and dim range $ST \leq$ dim range S. In conclusion,

$$\text{dim range } ST \leq \min\{\text{dim range } S, \text{dim range } T\}$$

Exercise 24.

- Suppose dim $V = 5$ and $S, T \in \mathcal{V}$ are such that $ST = 0$. Prove that dim range $TS \leq 2$.
- Given an example of $S, T \in \mathbb{F}^5$ with $ST = 0$ and dim range $TS = 2$.

Solution. For the first point, we have that $ST = 0 =>$ dim range $ST = 0 =>$ dim null $ST = 5$. This means that $5 \leq$ dim null S + dim null T. We know that $S, T \in \mathcal{L}(V)$, therefore dim range T + dim null T + dim range S+dim null S =5. Which means that $5 \leq 5-$ dim null S + $5 -$ dim null T $<=>$ dim range S + dim range $T \leq 5$. With the last inequality we have that $2 \min\{\text{dim range } S, \text{dim range } T\} \leq 5$. Which means that $\min \text{dim range } S + \min \text{dim range } T \leq 2$ because that minimum is an integer.

Finally, we know that dim range $TS \leq \min \text{dim range } S + \min \text{dim range } T$ due to symmetry, so dim range $TS \leq 2$.

For the second point, let $v_1, ..., v_5$ be a basis of $\mathbb{F}^5$. I will define S and T at the basis vectors:

- $S(v_1) = v_1, S(v_2) = v_4, S(v_3) = S(v_4) = S(v_5) = 0$
- $T(v_1) = T(v_2) = T(v_3) = v_3, T(v_4) = T(v_5) = v_4$.

Therefore we will have:

- $ST(v_1) = ST(v_2) = ST(v_3) = S(v_3) = 0, ST(v_4) = ST(v_5) = S(v_4) = 0$
- $TS(v_1) = T(v_1) = v_3, TS(v_2) = T(v_4) = v_4, TS(v_3) = TS(v_4) = TS(v_5) = 0$

It is clear from here that $ST = 0$ and dim range $TS = 2$ as a basis for range TS is $\{v_3, v_4\}$.

Exercise 25. Suppose that V is finite-dimensional and $S, T \in \mathcal{V}, \mathcal{W}$. Prove that null $S \subseteq$ null T if and only if there exists $E \in \mathcal{W}$ such that $T = ES$.

Solution. Let's prove the assertion from left to right. Let null $S \subseteq$ null T. I will prove that there exists $E \in \mathcal{V}$ such that $T = ES$.

Let $v_1, ..., v_n \in V$ be a basis of V with dim $V = n$. Therefore, $S(v_1), ..., S(v_n)$ is a basis for range S. I can define E' to be the map on range $\mathcal{S}, \mathcal{W}$ such that $E'(w) = T(v')$, for $w \in$ range S and $Sv' = v$. Now, for each w we just choose one of the v' such that $Sv' = v$. What I now have to show is that $E'S(v_i) = T(v_i), \forall v_i$. Let v_i' be the pre-image chosen when building E' for $S(v_i)$. This means that $E'S(v_i) = E'(v_i') = T(v_i')$, however we know that $S(v_i') = S(v_i)$, therefore $v_i' - v_i \in$ null $S \subseteq$ null T therefore $v_i' - v_i \subseteq$ null T which means that $T(v_i') = T(v_i)$, so $E'S(v_i) = T(v_i') = T(v_i)$. So we now have this map $E' \in$ range $\mathcal{S}, \mathcal{W}$ such that $E'S(v_i) = T(v_i)$ for all $v_i \in V$. I cna extend E' to obtain $E \in \mathcal{W}, \mathcal{W}$ such that $E(w) = E'(w), \forall w \in$ range S.

So now, we have that $ES(v_i) = T(v_i), \forall v_i \in V$. This completely defines the ES map, and it is clear that $T = ES$. Q.E.D.

Now for the assertion from right to left, if there exists the map $E \in \mathcal{W}$ such that $T = ES$. If $x \in$ null S, we have that $ES(x) = E(0) = 0 = T(x)$, so $x \in$ null T. Therefore null $S \subseteq$ null T.

Exercise 26. Suppose that V is finite-dimensional and $S, T \in \mathcal{V}, \mathcal{W}$. Prove that range $S \subseteq$ range T if and only if there exists $E \in \mathcal{V}$ such that $S = TE$.

Solution. Let's prove the assertion from left to right: Let range $S \subseteq$ range T. Let $v_1, ..., v_n$ be a basis of V. As the range of S is included in the range of T, for each $S(v_i)$ there exists a u_i such that $T(u_i) = S(v_i)$. I will uniquely define $E \in \mathcal{V}$ such that $E(v_i) = u_i$. This means that $\forall v_i$, $TE(v_i) = T(u_i) = S(v_i)$, so TE and S are the same map as they have the same unique representation over the basis vectors.

For the assertion from right to left: There exists a map $E \in \mathcal{V}$ such that $S = TE$. Let $x \in$ range S. This means that $\exists v' \in V$ such that $S(v') = x$. But $S(v') = TE(v') = x$, so $x \in$ range T with one of the possible pre-images being $E(v')$. Therefore, range $S \subseteq$ range T.

Exercise 27. Suppose $P \in \mathcal{V}$ and $P^2 = P$. Prove that $V =$ null $P \oplus$ range P.

Solution. First, let's look at the intersection between null P and range P. Let $x \in$ (null $P \cap$ range P). We have that $P(x) = 0$ and $\exists x' \in V$ s.t $P(x') = x$. This means that $P^2(x') = P(x)$, so $P(x') = 0$, but $P(x') = x$, so $x = 0$. Therefore null $P \cap$ range $P = \{0\}$. This means that null P + range P = null $P \oplus$ range P.

From theory we also know that dim (null $P \oplus$ range P) = dim null P + dim range P = dim V. null $P \oplus$ range P is actually a subspace of V, so we will have that null P $\oplus$ range $P = V$.

Exercise 28. Suppose $D \in \mathcal{P}(\mathbb{R})$ is such that deg Dp = (deg p) - 1 for every non-constant polynomial $p \in \mathcal{P}(R)$. Prove that D is surjective.

Solution. To prove that D is surjective, I will show that $\forall n \in N, x^n \in$ range D. Then, it is easy to show that any polynomial of finite degree is in range T.

To prove the above, I will use strong induction. I define the proposition as being $P(n) : x^n \in$ range D.

First we solve the base case. We know that $D(x)$ is a polynomial of degree 0. This means that $\exists t \in \mathbb{R}, t \neq 0$ such that $D(x) = t$. Because D is linear, we have that $D(\frac{x}{t}) = 1$ and the division is possible because t is not 0. Therefore, $P(0)$ is true. Now we will solve $P(0), ..., P(n-1) - True =>$ $P(n) - True$. Given the properties of D we know that $D(x^{n+1})$ must be a polynomial of degree n, $D(x^{n+1}) = a_n x^n + \cdots + a_0$, where a_n needs to be different than 0 and the others not necessarily.

Now we know that $P(0), \ldots, P(n-1)$ are all true, so we denote the pre-image (which exists) of x^i with p_i such that $D(p_i) = x^i$. Therefore we will find that $D(x^{n+1} - a_{n-1}p_{n-1} - a_{n-2}p_{n-2} - \cdots - a_0 p_0) = D(x^{n+1}) - a_{n-1}x^{n-1} - ... - a_0 = x^n$. As such, $P(n)$ is also true.

Thus, $P(n) - True \forall n \in \mathbb{N}$ so we have that D is surjective.

Exercise 29. Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.

Solution. We can just define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that $D(q) = 5q'' + 3q'$. It is clear that it has the property from exercise 28, so we can use the result and say that D is surjective, therefore for every $p \in \mathcal{P}(\mathbb{R})$, there exists q such that $D(q) = p$, Q.E.D.

Exercise 30. Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in null φ. Prove that

$$V = \text{null } \varphi \oplus \{au : a \in \mathbb{F}\}$$

Solution. range φ is a subspace of F, therefore dim range $\varphi \in \{0, 1\}$. However, $\varphi \neq 0$, which means that dim range $\varphi = 1$. Therefore, dim V = dim null φ + 1.

Now, let's analyze what null $\varphi \cup \{au : a \in \mathbb{F}\}$ looks like. If $x \in$ null φ, $\varphi(x) = 0$. If $x \in \{au : a \in \mathbb{F}\}$, then there exists a' such that $x = a'u$. Putting them together, we get that $\varphi(a'u) = 0$, so $a'\varphi(u) = 0$. We know that $\varphi(u)$ cannot be 0, so $a' = 0$, so $x = 0$. Thus, null $\varphi \cup \{au : a \in \mathbb{F}\} = \{0\}$. This means null φ + $\{au : a \in \mathbb{F}\}$ = null $\varphi \oplus \{au : a \in \mathbb{F}\}$.

From theory this means that dim (null $\varphi \oplus \{au : a \in \mathbb{F}\}$) = dim null phi + dim $\{au : a \in \mathbb{F}\}$ = dim null phi + 1. But dim V = dim null phi + 1 and the direct sum is a subspace of V , therefore they must be equal Q.E.D.

Exercise 31. Suppose V is finite-dimensional, X is a subspace of V, and Y is a finite-dimensional subspace of W. Prove that there exists $T \in \mathcal{V}, \mathcal{W}$ such that null $T = X$ and range $T = Y$ if and only if dim X + dim Y = dim V.

Solution. The left to right proposition is: There exists $T \in \mathcal{L}(V, W)$ such that null T = X and range T = Y. Prove that dim X + dim Y = dim V. This comes easily from the Fundamental Theorem of Linear Maps, dim null T + dim range T = dim X + dim Y = dim V.

The right to left proposition is: If dim X+ dim Y = dim V then there exists $T \in \mathcal{L}(V, W)$ such that null $T = X$ and range $T = Y$. We will get the basis of X to be $x_1, ..., x_n$ with n= dim X. Then we are going to extend this basis with the vectors $b_{n+1}, ..., b_m$ where m = dim V. Now we know that dim Y = m-n. We can construct the following linear map T: $T(x_i) = 0, \forall i \in \{1, ..., n\}$ and $T(b_j) = y_{j-n}, \forall j \in n+1, ..., m$, where $y_1, ..., y_{m-n}$ is a basis of Y. With this construction, $x_1, ..., x_n$ will be the basis of null T and $T(b_j) = y_{j-n}$ will be the basis of range T. This means that the basis of X is the basis of null T and the basis of Y is the basis of range T, so null T = X, range T = Y. Q.E.D.

Exercise 32. Suppose V is finite-dimensional with dim $V > 1$. Show that if $\varphi : \mathcal{L}(V) \to \mathbb{F}$ is a linear map such that $\varphi(ST) = \varphi(S)\varphi(T)$ for all $S, T \in \mathcal{V}$, then $\varphi = 0$.

Solution. Let $x \in$ null φ. Let $E \in \mathcal{V}$ be a random element of that set. We will have:

- $\varphi(xE) = \varphi(x)\varphi(E) = 0$, so $xE \in$ null φ.
- $\varphi(Ex) = \varphi(E)\varphi(x) = 0$, so $Ex \in$ null φ.

Therefore null φ is a two-sided ideal of $\mathcal{L}(V)$. Using the hint, this means that null φ is either $\{0\}$ or $\mathcal{L}(V)$.

We need to show that there is a non-zero map in null φ and then that means null $\varphi = \mathcal{L}(V)$, so $\varphi = 0$.

We need to create S and T such that $ST = 0$ but $S \neq 0$ and $T \neq 0$. We can do this by taking a basis of V, $v_1, ..., v_n$ with n = dim V ¿ 1 and then we define:

- $S(v_1) = v_1, S(v_2) = S(v_3) = ... = S(v_n) = 0$
- $T(v_1) = T(v_2) = ... = T(v_n) = 0$

We are guaranteed that $v_1 \neq v_2$ because $n \geq 2$. Therefore $ST(v_i) = 0. \forall i \in \{1, ..., n\}$ so we have that $\varphi(ST) = 0 = \varphi(S) * \varphi(T)$ which means that either $S \in$ null φ or $T \in$ null φ. This means that there is a non trivial element in null φ Q.E.D.

Exercise 33. This exercise will be added later.