Linear Algebra Done Right – Chapter 3 Solutions

1 Exercises 3B

Exercise 1. Give an example of a linear map T widh dim null $T = 3$ and dim range $T = 2$.

The intuition behind the solution is that we can define a linear map just on the values on a basis of the linear space V we choose. I will choose V to be $\mathbb{R}^5$ and $T \in \mathcal{L}(\mathcal{V})$.

Therefore, I define $T : \mathbb{R}^5 \to \mathbb{R}^5$ with:

- T ((1,0,0,0,0))=0
- T ((0,1,0,0,0))=0
- T ((0,0,1,0,0))=0
- T ((0,0,0,1,0))= (0,0,0,1,0)
- T ((0,0,0,0,1))= (0,0,0,0,1)

It is clear that null T has a basis the first 3 basis vectors of $\mathbb{R}^5$ and the range of T has as a basis the last 2 basis vectors of $\mathbb{R}^5$.

Exercise 2. Suppose $S, T \in \mathcal{L}(V)$ such that range $S \subseteq$ null T. Prove that $(ST)^2$=0.

Solution. $(ST)^2(v) = (ST)(ST(v))$. We have that $ST(V) \in$ range S. Therefore $ST(V) \in$ null T.

Which means that $T(ST(v)) = 0, \forall v \in V$. Therefore $(ST)^2(v) = S(0) = 0$, because S is a linear map.

Exercise 3. Suppose $v_1, \cdots, v_m$ is a list of vectors in V. Define $T \in (\mathcal{F}^m, V)$ by:

$$T(z_1, \cdots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

- What property of T corresponds to $v_1, \ldots, v_m$ spanning V?
- What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

Solution. To answer the questions in order:

- If $v_1, \ldots, v_m$ span V, then T would be surjective as practically range T is just span V.
- If $v_1, \ldots, v_m$ are linearly independent, this means that $T(z_1, \ldots, z_m) = 0$ only if $z_1 = \cdots = z_m = 0$, therefore dim null T =0 , so it is injective.

Exercise 4. Show that X = { $T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$: dim null $T > 2$ } is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$

Solution. My intuition for this exercise was just to build two linear maps that had no overlap of values over the basis. So in one case, one of the maps would map a basis vector to 0 and in another it would map the basis vector to itself.

Therefore, let $v_1, \dots, v_5$ be a basis of $\mathbb{R}^{\ntriangleleft}$. I define the following maps T_1 and T_2:

- $T_1(v_1) = v_1$,$T_1(v_2) = v_2$,$T_1(v_3) = 0$,$T_1(v_4) = 0$,$T_1(v_5) = 0$
- $T_2(v_1) = 0$,$T_2(v_2) = 0$,$T_2(v_3) = v_3$,$T_2(v_4) = v_4$,$T_2(v_5) = 0$

Both of them clearly have dim null T_1 = dim null $T_2 = 3$.

However, when we calculate $T_1 + T_2$ we get:

- $(T_1+T_2)(v_1) = v_1$, $(T_1+T_2)(v_2) = v_2$, $(T_1+T_2)(v_3) = v_3$, $(T_1+T_2)(v_4) = v_4$, $(T_1+T_2)(v_5) = v_5$

Therefore, dim null $T_1 + T_2 = 1$, which means that $T_1 + T_2 \notin X$ which shows that the set X is not closed under addition, therefore it cannot be a subspace.

Exercise 5. Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that range T = null T.

Solution. My intuition behind this problem was that if range T = null T, then T(T(x))=0, $\forall x \in \mathbb{R}^4$. Now, I did "cheat" a little bit by thinking that T was a matrix and just looking for a matrix with $T^2 = 0$, where 0 here is the null 4×4 matrix.

Therefore, the final map that I have reached is: $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 - x_3, x_2 - x_3, x_1 - x_4) = (x_1 - x_4) * (1, 0, 0, 1) + (x_2 - x_3) * (0, 1, 1, 0)$. It is clear that the range is a linear combination between $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$ which are linearly independent, therefore dim range $T = 0$. We have that

$$T(T(x)) = 0$$

which means that range $T \subseteq$null T.

Now I need to prove that null $T \subseteq$ range T.

Let $y \in$ null T , be an arbitrary vector in $nullT$. Therefore, $T(y_1, y_2, y_3, y_4) = 0$ which means that $y_1 = y_4, y_2 = y_3$.

I will choose $x_1 = y_1, x_2 = y_2, x_3 = 0, x_4 = 0$, and get that $T(x_1, x_2, x_3, x_4) = (y_1, y_2, y_2, y_1) = (y_1, y_2, y_3, y_4)$. Therefore I have found x_1, x_2, x_3, x_4 such that $T(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, y_4)$ so $y \in$ range T. Therefore null $T \subseteq$ range T.

So null T = range T.

Exercise 6. Prove that there does not exist $T \in \mathcal{L}(\mathbb{R}^5)$ such that range T = null T.

Solution. We know that dim range T + dim null T = dim $\mathbb{R}^5$, which means 2*dim range $T = 5$ which is not possible as dim range T is an integer.

Exercise 7. Suppose V and W are finite-dimensional with $2 \leq$ dim V $\leq$ dim W. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution. I will build two maps that are not injective, but whose sum is. I will denote dim V = n, dim W = m. Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ be a basis of W. I can create the following mappings, using the fact that $2 \leq n \leq m$:

- $T_1(v_1) = w_1, T_1(v_2) = w_1, T_1(v_i) = w_i, \forall i \in \{3, \dots, n\}$
- $T_2(v_1) = w_2, T_2(v_2) = -w_2, T_2(v_i) = w_i, \forall i \in \{3, \dots, n\}$

They are both not injective as their nullspaces do not have dimension 0, clearly $T_1(v_1 - v_2) = 0$ and $T_2(v_1 + v_2) = 0$.

If we look at $T_1 + T_2$, we find that $(T_1 + T_2)(v_1) = w_1 + w_2, (T_1 + T_2)(v_2) = w_1 - w_2, \ldots, (T_1 + T_2)(v_i) = 2 * v_i, \forall i \in \{3, \ldots, n\}$. Let $x \in$ null $(T_1 + T_2)$. We have that $(T_1 + T_2)(x) = 0$, we can write x uniquely according to the basis $v_1, \ldots, v_n$ and get that: $(T_1 + T_2)(\alpha_1 v_1 + \cdots + \alpha_n v_n) = 0$. Therefore $\alpha_1(w_1 + w_2) + \alpha_2(w_1 - w_2) + \sum_{i=3}^{n} 2 * \alpha_i w_i = 0$.

So we get that $(\alpha_1 + \alpha_2)w_1 + (\alpha_1 - \alpha_2)w_2 + \sum_{i=3}^{n} w_i = 0$. We know that the w's are linearly independent, therefore $\alpha_i = 0, \forall i \in \{3, \ldots, n\}$ and $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 0$. This means that $\alpha_1 = \alpha_2 = 0$, so x = 0. Therefore, null $T_1 + T_2 = \{0\}$, so it is injective, therefore the set given in the hypothesis is not a subspace.