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An Unavoidable Set of Configurations in Planar Triangulations*

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Let T be a normal triangulation (considered in the context of the four-color problem). Assume that no two vertices of degree 5 are adjacent. Then T contains at least 1 of the 47 configurations in Table I all of which are likely to be 4-color reducible.

INTRODUCTION

Algorithms for establishing the 4-color reducibility of configurations in planar triangulations were introduced by Birkhoff [5] and further developed by many investigators (see [1, 3, 4, 7, 9] and the literature quoted therein). To use four-color reducibility to establish the four-color conjecture it appears necessary to show that every planar triangulation contains a reducible configuration, that is, that there exists an unavoidable set of reducible configurations.

In [7, pp. 11, 216] Heesch stated the conjecture that such an unavoidable set of reducible configurations exists and actually can be constructed. He supports this conjecture by several partial results in [7, 8] and others which are still unpublished; for a more detailed discussion see the first two introductory sections of [2]. The work which led to [6, 2] and this paper was inspired by the work of Heesch.

Since it is often quite time consuming to show that a configuration is reducible (the time required for certain configurations which must be attacked may be measured in hours even on the most advanced computers), we have chosen to construct unavoidable sets of "likely to the reducible" configurations and then test members of such sets for reducibility later.

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Our concept of what configurations are "likely to be reducible" has been derived from (i) private communications of Heesch on certain apparent "reduction obstacles," (ii) the work of Stromquist [10], who developed a more general theory of reduction obstacles using the methods of Tutte and Whitney [11], and (iii) probability considerations of our own.

Our method of constructiong unavoidable sets of likely-to-be-reducible configurations is described in detail in [2]. It is rather complicated and certainly involves considerable use of computers. In this paper we study a restricted class of triangulations to provide an illustration of the method and many of the techniques of [2] which is comparatively easy to check. We consider the case of triangulations T which do not contain any pair of adjacent vertices of degree 5, i.e., all 5-vertices are isolated. The restriction appears to reduce the number of members of the unavoidable set obtained by a factor of about 50. Furthermore, certain theoretical tools developed in [2] may be greatly simplified for this case. We intended to make this paper readable independent of [2]. However, some concepts are only intuitively described; for formal definitions the reader is referred to [2].

All configurations are described by drawings. They have degree specifications assigned to their vertices (see Table I). We use the coding introduced by Heesch [7] to indicate degree specifications. For a boundary vertex V of a configuration $\mathscr{C}$, we define the number of "legs" on V to be the difference between the specified degree of V and the number of edges of $\mathscr{C}$ on V. These legs may be thought of as edges leading from V to vertices of T neighboring $\mathscr{C}$ but not belonging to $\mathscr{C}$. The set of all such neighboring vertices form an n-ring about $\mathscr{C}$. We use m for the number of vertices of $\mathscr{C}$ and n for the number of neighboring vertices. A configuration is called geographically good if no vertex of the configuration has three or more nonconsecutive neighbors outside the configuration. A geographically good configuration $\mathscr{C}$ is defined to be "likely to be (4-color) reducible (if contained in a planar triangulation)" if

(1) $\mathscr{C}$ does not contain any "hanging 5–5 pair" (i.e., no pair of adjacent 5-vertices which are adjacent to only one other vertex of $\mathscr{C}$; in this paper the condition is automatically fulfilled by the assumption of isolated 5-vertices).

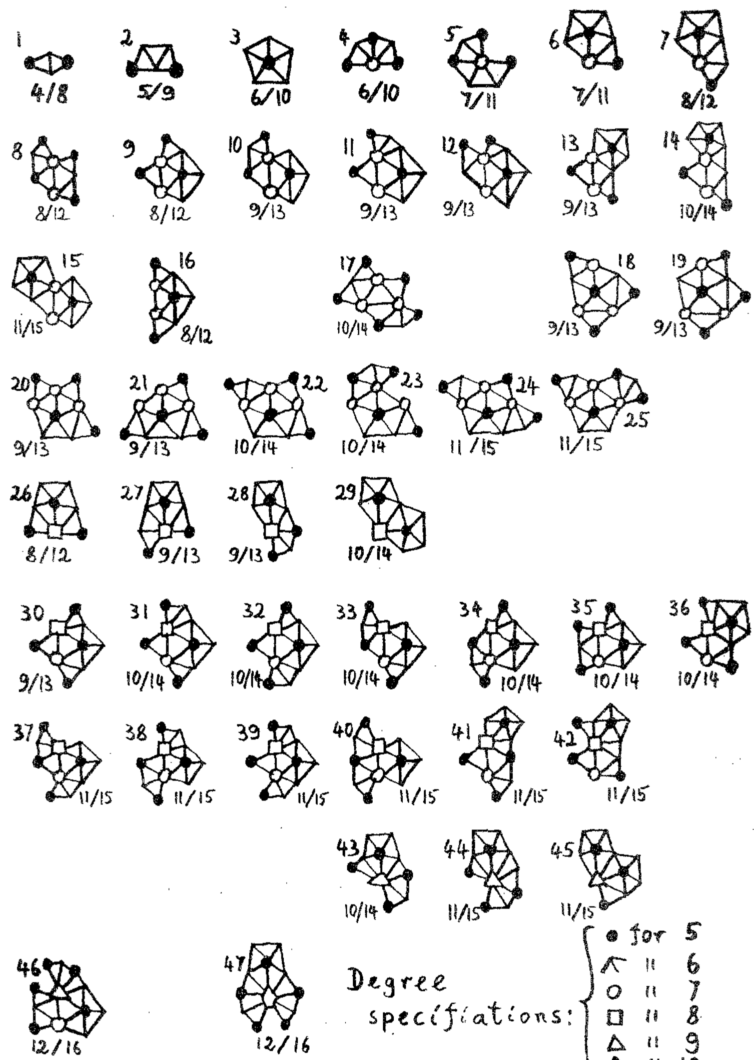
(2) $n - m \leqslant 4$ (i.e., in the terminology of [2], $\varphi = n - m - 3 \leqslant 1$).

A geographically good configuration without hanging pairs avoids the "reduction obstacles" of Heesch mentioned above which are also the most important obstacles found by Stromquist in [10].

Configurations 1,..., 6 (with $n \leqslant 11$) in Table 1 have been proved reducible by various investigators (see [3] for 1, [4] for 2, [12] for 3, and [1] for 4, 5, and 6). We believe that most of the configurations 7,..., 47 are reducible and could be proved so with sufficient computational effort. If a few of them are

TABLE I

The 47 Likely-To-Be-Reducible Configurations in the Unavoidable Set $\mathscr{S}$



irreducible then the methods used would enable us to find a new unavoidable set in which the irreducible configurations are replaced by different, likely-to-be-reducible, configurations.

The principal result of this paper improves a result obtained in the introduction to [2]. There the discharging method of Heesch [7] was used to exhibit an unavoidable set $\mathscr{S}_3^*$ whose only irreducible configurations are the 5–5 edge (which contains two 4-legger vertices, $\varphi = 1$) and the 5–6–6 triangle (which has two 4-legger vertices of degree 6, $\varphi = 2$). The result of this paper may be regarded as replacing the irreducible 5–6–6 triangle of $\mathscr{S}_3^*$ by the 47 likely to be reducible configurations of Table I (but not replacing the 5–5 edge) and thus obtaining an unavoidable set $\mathscr{S}$.

Statement and Proof of the Theorem.

THEOREM. *Let T be a triangulation of the 2-sphere S^2 or the projective plane P^2 such that every vertex of T has degree at least 5 and such that T contains no 1- or 2-circuits nor any 3-, 4-, or 5-circuits other than the boundary circuits of disks in S^2 or P^2 the interiors of which contain at most one vertex of T. Further assume that T has no pair of adjacent vertices of degree five. Then T contains at least 1 of the 47 configurations of Table I.*

Remark. By the statement that T "contains" a configuration $\mathscr{C}$ we mean that there exists a simplicial immersion $f: \mathscr{C} \to T$ which respects (in the obvious sense) the degree specifications of $\mathscr{C}$. But if the diameter of $\mathscr{C}$ (the number of edges required to join any two vertices of $\mathscr{C}$ by an edge path) is not greater than four then the assumptions on excluded circuits in T imply that f must be a proper embedding. Since none of the configurations in Table I has diameter greater than four, the theorem implies that 1 of the 47 configurations is properly embedded in T (but it is still possible that the n-ring of its neighbors in T is improper).

Proof of the theorem. Assume that T' is a triangulation of an arbitrary 2-manifold M^2 which fulfills the above hypothesis (with T' for T and M^2 for "S^2 or P^2"). We must prove that if the Euler characteristic χ of T' is positive (i.e., if M^2 is S^2 or P^2) -hen T' contains at least 1 of the 47 configurations of Table I.

We define a charge function q_0 on the verices V of T' by

$$q_0(V_k) = 60\,(6 - k),$$

where k is the degree of V_k in T'. Then Euler's formula can be written

$$\sum q_0(V) = 360\chi,$$

where the sum is taken over all vertices V of T' (see [2]).

We define a "discharging procedure," i.e., we move the positive charges of the 5-vertices ($q_0(V_5) = 60$) to their negative neighbors ($q_0(V_k) < 0$ if $k \geqslant 7$) in a way which does not change the sum of the charges. This procedure may or may not "completely discharge" T', i.e., it may or may not lead to a new charge function which assigns each vertex V of T' nonpositive charge. Of course, if $\chi > 0$ then no procedure can completely discharge T'. We prove that if our procedure fails to completely discharge T' then T' contains at least 1 of the 47 configurations of Table I.

A. An Intermediate Result Obtained by Fractional Discharging Alone

Here we explore a rather simple discharging procedure, called fractional discharging, which is almost, but not quite, sufficient to exhibit an unavoidable set of likely to be reducible configurations in T.

A.1. *The Fractional Discharging Algorithm*

Let V_5 be a positive 5-vertex with at least two negative neighbors, say $V^{(1)},\dots, V^{(\mu)}$, $2 \leqslant \mu \leqslant 5$, in T'. Then we distribute the positive charge, $q_0(V_5) = 60$, of V_5 to $V^{(1)},\dots, V^{(\mu)}$ in such a way that the portion which is transferred to $V^{(i)}$ depends on the negative charge $q_0(V^{(i)}) = 60$ (degree of $V^{(i)}$ minus 6), of $V^{(i)}$ and the number, $\nu(V^{(i)})$, of positive V_5-neighbors of $V^{(i)}$ in T'.

We call the number

$$L(V^{(i)}) = \nu(V^{(i)}) - (1/60)\, q_0(V^{(i)})$$

the load of $V^{(i)}$ in T', and the quotient

$$\operatorname{Cap}(V^{(i)}) = \frac{-q_0(V^{(i)})}{\nu(V^{(i)})}, \qquad \text{provided that } \nu(V^{(i)}) > 0,$$

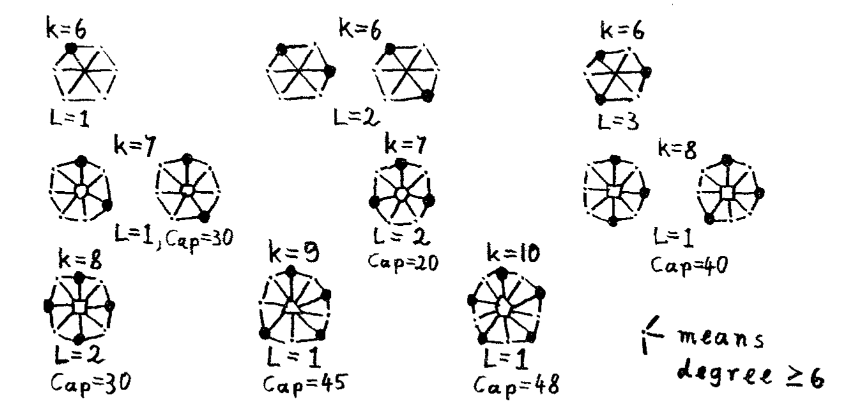
the capacity of $V^{(i)}$ in T'.

Table II shows all possible first neighborhoods of a vertex V_k ("k-wheels"), where V_k has degree $k \geqslant 6$ and $L(V_k) > 0$. Where $k \geqslant 7$ the capacities are also given in Table II. Note that Table II contains all cases in which Cap $(V_k) < 60$ and that there is only one case in which $\operatorname{Cap}(V_k) < 30$ (which is the case $k = 7$, $L = 2$, $\operatorname{Cap} = 20$).

Now the fractional discharging of V_5 is carried out in two steps according to the following "discharging algorithm." First, every neighbor $V^{(i)}$ $(1 \leqslant i \leqslant \mu)$ of V_5 with $\operatorname{Cap}(V^{(i)}) \geqslant 30$ (if there are such $V^{(i)}$'s) receives a charge from V_5 equal to its capacity (after this step the charge of V_5 may be negative, zero, or positive). If the charge of V_5 is still positive after the first step then in the second step we distribute the remaining charge of V_5 in equal fractions to those $V^{(i)}$'s with $\operatorname{Cap}(V^{(i)}) = 20$.

TABLE II

The 12 k-Wheels with Load $L > 0$



This procedure is carried out simultaneously for all 5-vertices of T' which fulfill the above condition, i.e., have at least two major neighbors in T'. The charge function so obtained is denoted q^*.

A.2. *The Possible Failures of Fractional Discharging*

Now we consider the case that the fractional discharging does not completely discharge T' (which case must occur if M^2 is the 2-sphere or the projective plane), i.e., that there is at least one vertex V of T' with $q^*(V) > 0$. We claim that the following theorem holds.

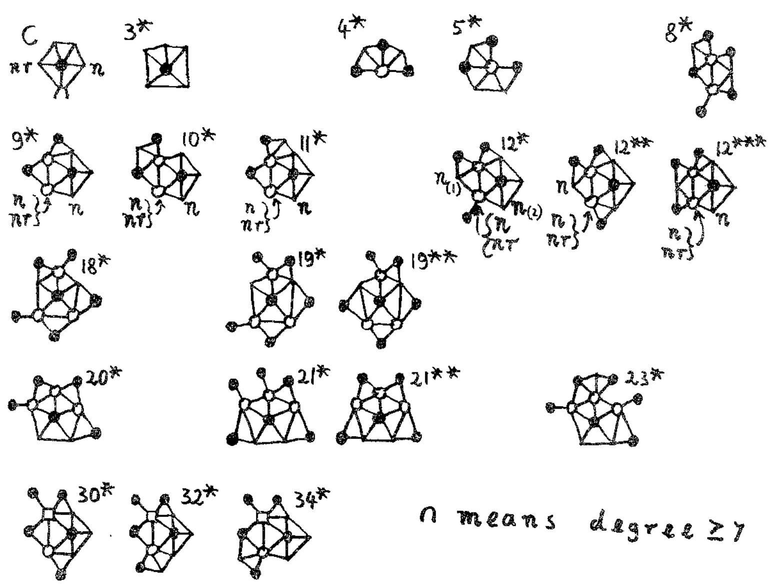
THEOREM A. *If after the fractional discharging (as described above) there is some vertex V of T' such that $q^*(V) > 0$ then T' contains at least* 1 *of the* 21 *configurations of Table III.*

COROLLARY. *The* 21 *configurations in Table III, together with the* 5–5 *edge (and with the reducible* 2-, 3-, 4-, *and* 5-*circuits and the vertices of degrees smaller than* 5) *form a set $\mathscr{S}^*$ which is unavoidable (in planar triangulations).*

Remark. Only one configuration in Table III does not contain a likely-to-be-reducible subconfiguration, and that is Configuration C (which is not geographically good). The numbers assigned to the configurations in Table III correspond to those in Table I. Configuration 4* is same as Configuration 4; Configuration 8* is Configuration 8 with one additional ("hanging") 5-vertex; Configurations 12*, 12**, and 12*** each contain Configuration 12 with one additional 5-vertex, etc. The additional 5-vertices increase the loads

TABLE III

The 21 Configurations (Besides V_2, V_3, V_4, etc.) in the Unavoidable Set $\mathscr{S}^*$



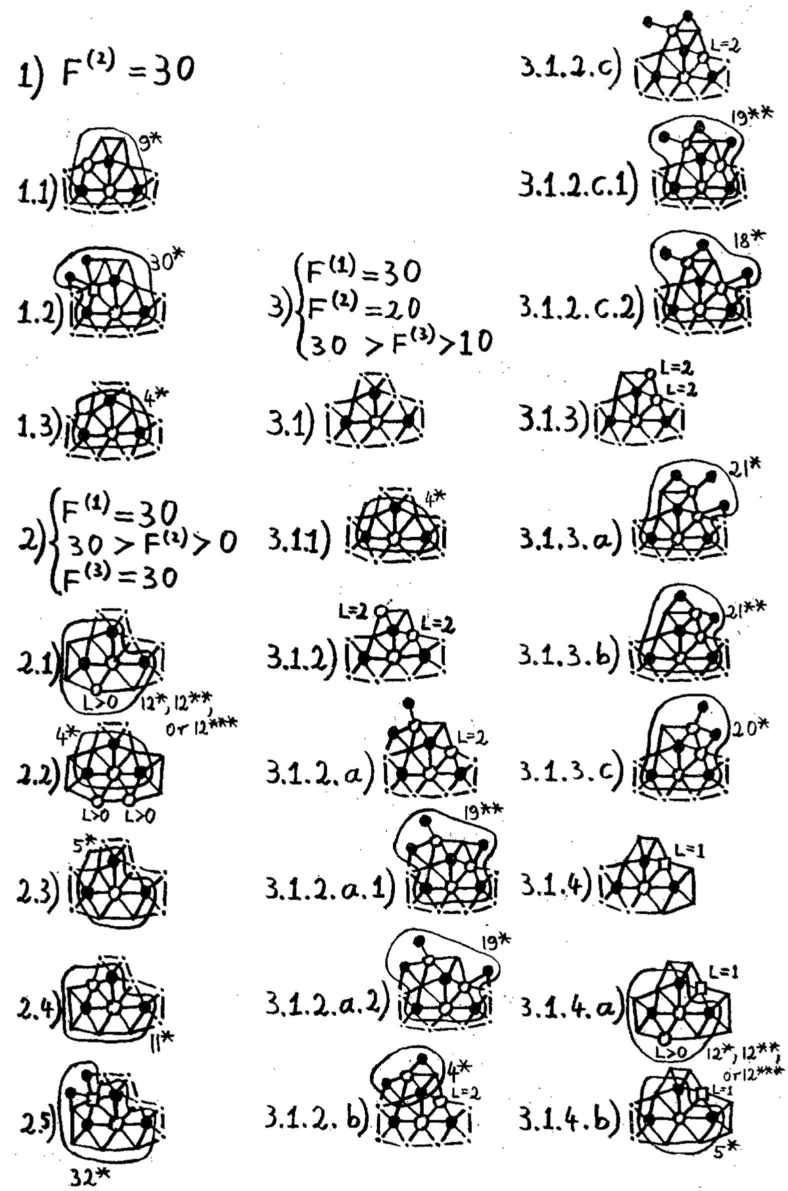
of some vertices in the configurations of Table III; these higher loads will decrease the number of "modifications" which we have to consider in Section B.

Proof of Theorem A. It follows immediately from the definition of the fractional discharging that the vertex V in the statement of the theorem must be either a 5-vertex with fewer than two major neighbors or a major vertex with capacity smaller than 30. In the first case (because of our hypothesis forbidding adjacent 5-vertices) T' contains either Configuration 3* or C of Table III with V as central vertex. In the second case, Table II shows that V is a 7-vertex with three V_5-neighbors, say $V_5^{(1)}$, $V_5^{(2)}$, $V_5^{(3)}$, so that the sum of the charges $F^{(1)}$, $F^{(2)}$, $F^{(3)}$ which are transferred from these V_5's to V (in the second step of the discharging procedure) is greater than 60; we call this case "q^*-overcharging of V."

All possible cases in which q^*-overcharging can occur are presented in Table IV. The notation is chosen so that $V_5^{(1)}$, $V_5^{(2)}$, $V_5^{(3)}$ lie in clockwise order about V and so that the adjacent pair of non-5-neighbors of V lie between $V_5^{(3)}$ and $V_5^{(1)}$. (In the drawings $V_5^{(2)}$ is above V.) By symmetry we may assume that $F^{(1)} \geqslant F^{(3)}$. Note that none of the contributions $F^{(1)}$, $F^{(2)}$, $F^{(3)}$ can be greater than 30, and that consequently none can be zero; hence none of $V_5^{(1)}$, $V_5^{(2)}$, $V_5^{(3)}$ can have a major neighbor of load $L < 1$ (since otherwise it

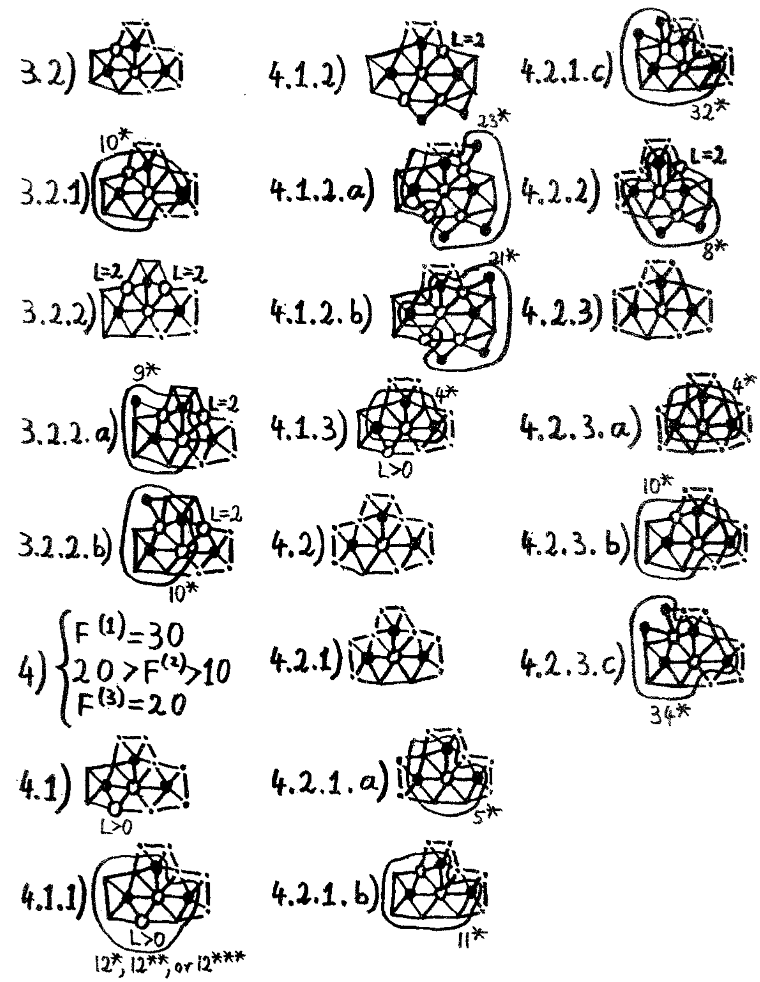
TABLE IV

The Case of q^*-Overcharging



(table continued)

TABLE IV—*Continued*



would be emptied of positive charge in the first step of the procedure). It is easy to determine the charge contribution $F^{(i)}$, $i = 1, 2, 3$, from the degrees and loads of those major neighbors of $V_5^{(i)}$ which are different from V. In particular, using Table II, we see that $F^{(i)} = 30$ if and only if $V_5^{(i)}$ has precisely one major neighbor besides V and that neighbor either is a V_7 or is a V_8 with load $L = 2$. If $F^{(i)} < 30$ then the largest possible contribution is $F^{(i)} = 20$ which occurs if and only if $V^{(i)}$ either has precisely one major neighbor, a V_8 with $L = 1$, besides V or has precisely two V_7-neighbors with $L = 2$ besides V.

In Table IV, four major cases, (1), (2), (3), (4), are distinguished corresponding to choices of $F^{(i)}$-values. In the tree of subcases, a branch terminates when it arrives at a subcase which is a configuration containing a subconfiguration which appears in Table III. These subconfigurations are circled in Table IV and the number of the subconfiguration (as used in Table III) is shown.

To illustrate the argument we explain Case (3) in some detail. Note that if $F^{(1)} = 30$ then at least one of subcases 3.1, 3.2 must occur (but these are certainly incomplete descriptions since the degrees of several vertices are not fully specified). Now if we specialize 3.1 by specifying the vertex between $V_5^{(2)}$ and $V_5^{(3)}$ to be a V_6 we force an instance of 4* (subcase 3.1.1). If however we make it a V_7 then not only must it have load $L = 2$ but also we are forced to one of two choices for the remaining neighbors of $V_5^{(2)}$ and their loads (subcases 3.1.2 and 3.1.3). If we make it a V_8 we obtain subcase 3.1.4. We always attempt to reduce the size of the decision tree by making as few choices as feasible at each node. Table IV constitutes a proof of Theorem A.

B. Deriving the Main Result from Section A by Preliminary Discharging

A more elaborate discharging procedure is required to replace the not geographically good configuration C in $\mathscr{S}^*$ by likely to be reducible configurations.

B.1. *The Preliminary Discharging*

In our more elaborate procedure, prior to applying the fractional discharging of Section A.1, we replace the charge function q_0 on T' by a new charge function q_1, which is obtained by "preliminary discharging." This discharging involves the three configurations C, C_C, and ${}_CC_C$ of Table V, which are called prelimininary discharging situations. The not fully specified vertex (at the bottom) of each is marked to have degree at least 7 and is called the pivot vertex. An image of such a configuration in which a particular

TABLE V

The Preliminary Dischargings

C


C_C


${}_CC_C$


degree, e.g., 7, is specified for the pivot is called a specialization of the configuration. Note that C_C contains a specialization of C as a subconfiguration and ${}_CC_C$ contains two specializations of C. Further note that ${}_CC_C$ contains a geographically good subconfiguration (obtained by deleting the pivot vertex) but this subconfiguration has $n = 20$, $m = 15$, $\varphi = 2$ and would be undesirable to admit to our unavoidable set $\mathscr{S}$. In each of C, C_C, ${}_CC_C$ the pivot has a single V_5-neighbor, the "main V_5", and, as indicated by the arrows in Table 5, the charge of 60 of the main V_5 is transferred to the pivot. This "integral discharging" is carried out simultaneously in all images of C, C_C, and ${}_CC_C$ which are contained in T'. This yields the charge function q_1 on T'. Then the fractional discharging of Section A.1 is applied (with q_1 in place of q_0). This replaces q_1 with a charge function q_3. (The notation is chosen to be consistent with [2]; there a charge function q_2 results from another preliminary discharging, called partial discharging, which is not necessary in this comparatively simple case).

As in Section A, we must consider all possible cases in which the discharging procedure does not completely discharge T', i.e., in which $q_3(V) > 0$ for some vertex V of T'. There are essentially three different ways this can happen.

I. V may be a vertex of degree $k \geqslant 7$ which is the pivot of more than $k - 6$ preliminary discharging situations in T', causing "q_1-overcharging," i.e., $q_1(V) > 0$.

II. V may be a 5-vertex which is not the main V_5 of any preliminary discharging situation in T', i.e., $q_1(V) = q_0(V) = 60$, and have at most one neighbor in T' which has a negative q_1-charge (note that V may have major neighbors with nonnegative charge q_1).

III. V may be a vertex of degree $k \geqslant 7$ such that $q_1(V) < 0$ but $q_3(V) > 0$, i.e. "q_3-overcharging" (note that $q_3(V) > 0$ cannot occur if $q_1(V) = 0$ since the fractional discharging transfers charges only to negative vertices).

In each such case we exhibit 1 of the 47 configurations of Table I which must occur in T'; this completes the proof.

B.2. *Combinations of Preliminary Discharging Situations with the Same Pivot*

In this section we consider all ways in which a vertex V_k of degree $k \geqslant 7$ may have load $L(V_k) \geqslant 1$ and be the pivot of at least one preliminary discharging situation in such a way either that $q_1(V_k) \geqslant -60$ or that $q_1(V_k) = -120$ and $L(V_k) \geqslant 2$. This includes the cases of q_1-overcharging.

The cases in which $q_1(V_k) \leqslant 0$ are important in the next section when we consider all possibilities of q_3-positive V_5's and of q_3-overcharging: The cases of q_3-overcharging can be derived from the cases considered in Table IV by replacing (zero or more) non-5-vertices by vertices of higher degrees which receive preliminary discharging which make their q_1-charges the same as the q_0-charges of the replaced vertices (and so that they also have the same loads, and thus the same fractional discharging takes place). Correspondingly in all cases of q_3-overcharging we obtain likely-to-be-reducible subconfigurations which are "modifications" of configurations in $\mathscr{S}^*$. Note that all non-5-vertices in $\mathscr{S}^*$ (Table III) have positive loads, that all vertices of degree 8 have load $L = 2$, and that there are no vertices of degree greater than 8 (except possibly the pivot of C whose degree is not fully specified). Thus the cases specified in the above paragraph are precisely those we need to consider.

In Table VI we discuss all configurations which can be obtained by merging a k-wheel with $k \geqslant 7$ and $L \geqslant 1$ (see Table II) with one or more preliminary discharging situations whose pivots are identified to the central vertex V_k of the wheel so that $q_1(V_k)$ and $L(V_k)$ meet the specifications given in the first paragraph of this section.

We note from Table II that the case $q_1(V_k) = -120$, $L(V_k) \geqslant 2$ cannot occur and that the possible values for k are 7, 8, 9, 10. Table VI contains 41 configurations denoted $M1, \ldots, M41$. All but 7 of these configurations contain subconfigurations which appear in $\mathscr{S}$ (Table 1); these are circled and marked with their numbers from Table 1. The remaining seven "critical combinations" are marked "critical".

Note that the critical combinations $M15$, $M17$, and $M18$ contain likely-to-be-reducible subconfigurations with n-values of 16; but we chose not to include these configurations in $\mathscr{S}$ since they are relatively large and (as will be seen in the next section) those modifications of configurations of $\mathscr{S}^*$ in which a vertex is replaced by one of $M15$, $M17$, $M18$ contain smaller, likely-to-be-reducible subconfigurations.

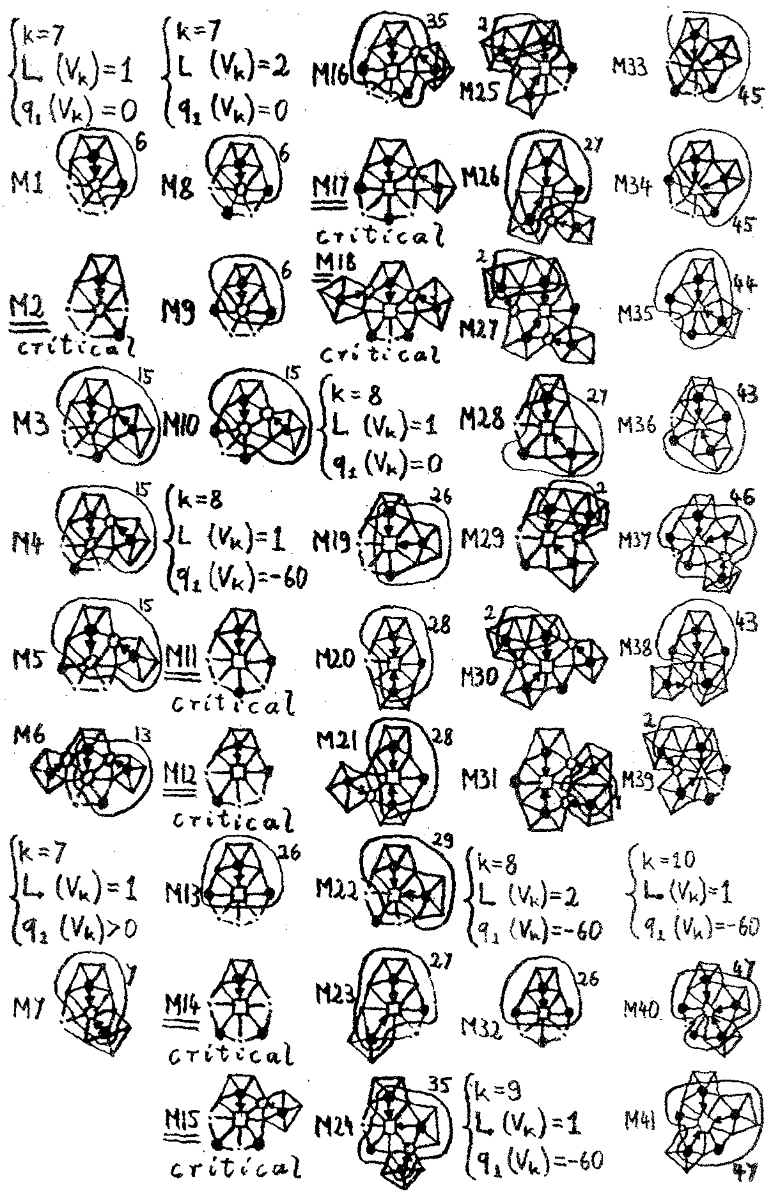
All those configurations which contain noncritical combinations as proper subconfigurations are omitted from Table VI. For example, in the case $k = 7$, $L = 1$, $q_1 > 0$, all those combinations which can be obtained by merging one of $M1$, $M3, \ldots, M6$ (as listed in the case $k = 7$, $L = 1$, $q_1 = 0$) with one of C, C_C, ${}_CC_C$ are omitted.

B.3. *Critical Modifications of the Members of $\mathscr{S}^*$*

We still must consider all possible failures of the fractional discharging procedure, i.e., all cases in which II or III of Section B.1 applies. We have dealt with a similar task in Section A.2; so we need only consider those complications which arise because of the fact that the q_1-charge of a vertex V may differ (by an integral multiple of 60) from its q_0-charge. Because of the

TABLE VI

The Critical and Noncritical Combinations



discussion in Section B.2, we may ignore cases in which a vertex V_k, $k \geqslant 7$, appears which is the pivot of a noncritical combination. Thus we need just consider those cases of vertices V_k, $k \geqslant 7$ with $q_1(V_k) \neq q_0(V_k)$, where V_k is the pivot of one of the seven critical combinations in Table VI; in particular, we must have $q_1(V_k) = q_0(V_k) + 60$, $L(V_k) = 1$. This implies that such a vertex V_k cannot be q_3-overcharged (since either its q_1-charge is zero or its capacity is at least 30). Hence we come to the following conclusions.

(a) Suppose V is a 5-vertex in T' with $q_3(V) > 0$ (Case II in Section B.1). If T' contains no noncritical combinations then T' contains a "critical modification" of either Configuration 3* or C (Table III), i.e., a configuration obtained from 3* or C by (i) raising the degrees of zero or more 6-vertices to 7 and (ii) attaching copies of $M2$ so that their pivots are identified to the 7-vertices created in (i). (Of course, this must be done in such a way that the degree specifications of the merged configurations are compatible.) Note that the configuration so obtained cannot be a preliminary discharging situation (Table V) with main V_5 at V, since otherwise $q_3(V) = q_1(V) = 0$.

(b) Suppose that V is a major vertex in T' with $q_1(V) < 0$ and $q_3(V) > 0$ (Case III of Section B.1). If T' contains no noncritical combinations then T' contains a "critical modification" of one of the configurations 4*,..., 34* in Table III which is obtained by

(i) raising the degrees of zero or more vertices of load $L = 1$ from 6 to 7 or from 7 to 8 and

(ii) attaching copies of critical combinations so that each vertex with raised degree 7 is identified to the pivot of a copy of $M2$ and each vertex with raised degree 8 is identified to the pivot of a copy of one of $M11$, $M12$, $M14$, $M15$, $M17$, $M18$.

Note that we need not consider the case that a 6-vertex is replaced by a 5-vertex V_5 with $q_1(V_5) = 0$ since this would introduce a pair of adjacent 5-vertices which, by hypothesis, cannot occur in T'. It should be noted that the hypothesis forbidding adjacent V_5's eliminates practically all of the difficulties which arise in connection with the modification concept in the general case as treated in [2]. In the case treated in this paper we may conclude the following useful rule on constructing critical modifications step by step (which is a simplified version of in [2, Theorem 10]).

(c) Suppose that a configuration B is a critical modification of a configuration A which is obtained from A by degree-raising at more then one vertex (and attaching critical combinations). Let $V^{(1)},..., V^{(\alpha)}$ be those vertices of A whose degrees have been raised, indexed in an arbitrary order. Then there is a (uniquely defined) sequence of configurations $B_0 = A, B_1,...,$ $B_\alpha = B$ such that B_{i+1}, $i = 0,..., \alpha - 1$, is obtained from B_i by raising the degree of $V^{(i+1)}$ (by one) and attaching one critical combination. Then all of $B_1, \ldots, B_\alpha$ are critical modifications of A, and B_{i+1} is called an extension of B_i (at $V^{(i+1)}$).

Now we must consider all possible critical modifications of the configurations in $\mathscr{S}^*$ and, in each case, exhibit a member of $\mathscr{S}$ as a subconfiguration, in the case that the preliminary discharging situation C is modified it is sufficient to exhibit either a member of $\mathscr{S}$ or another preliminary discharging situation with the same main V^j as a subconfiguration (by the observation in (a)). This finishes the proof.

B.3.1. *Modifications at One Vertex.* The cases in which no degree-raising takes place are trivial (and have been taken care of in Section A). Thus we consider first all critical modifications "at precisely one vertex" (i.e., where the degree of precisely one vertex has been raised). Such a modification, say B, is completely determined by the following data:

(1) The member, say A, of $\mathscr{S}^*$ which is modified;

(2) the vertex V of A whose degree is raised;

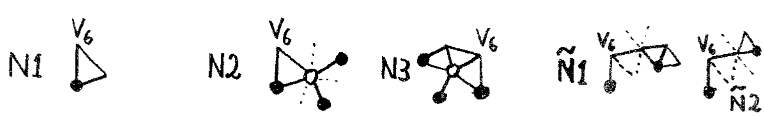
(3) the critical combination M which is attached to A at V;

(4) the orientation of the attachment (i.e., whether or not the copy of M which is attached to A is a mirror image of M as drawn in Table VI).

To handle the orientation of the attachment conveniently, henceforth we regard the members of $\mathscr{S}^*$ as oriented configurations (i.e., as essentially different from their mirror images). We indicate a reflected configuration by attaching the letter "r" to its number, e.g., 10* stands for Configuration 10* in Table III, 10*r for its mirror image. We also regard the critical combinations as oriented and (by symmetry) we may assume (for all modifications at

TABLE VII

The Common Neighborhoods of Vertices V_0 with Load 1



precisely one vertex) that these combinations occur in the modifications B only with the orientation shown in Table VI (while the members of $\mathscr{S}^*$ may occur reflected or nonreflected).

In order to reduce the number of cases to be considered we distinguish two types of vertices, "common" and "noncommon" in configurations as follows. All vertices of load 2 or of degree 5 are common, all 7-vertices of load 1 are noncommon; a 6-vertex is common if it has a "common neighborhood $N1, N2, N3, \tilde{N}_1$, or $\tilde{N}_2$, as drawn in Table VII," i.e., if the given configuration contains a subconfiguration $N1$, $N2$, $N3$, $\tilde{N}1$, or $\tilde{N}2$ with the orientation as given in Table VII so that the 6-vertex in question is contained in that subconfiguration at the place marked 'V_6" in Table VII; otherwise the 6-vertex is noncommon. In Table III, the noncommon vertices of the nonreflected configurations are marked "n" while the noncommon vertices of the reflected configurations are marked "nr"; (the 7-vertices with load 1 are both "n" and "nr"). It may be noted that the common neighborhoods $\tilde{N}1$ and $\tilde{N}2$ do not occur in any member of $\mathscr{S}^*$, but they occur at a later stage of the procedure (see (ii) below).

All critical modifications of the (reflected and nonreflected) members of $\mathscr{S}^*$ at precisely one vertex are discussed in Table VIII. The columns correspond to the critical combinations which are attached. In those cases where the degree of a common V_6 has been raised we have drawn only the subconfiguration of the modification which consists of $M2$ and the common neighborhood $N1$, $N2$, $N3$, $\tilde{N}1$, or $\tilde{N}2$ of V_6 (in which the degree of V_6 has been raised to 7); in this way a majority of all modifications can be discussed by only five drawings. When the degree of a noncommon vertex is raised the complete member of $\mathscr{S}^*$ is drawn. The only member of $\mathscr{S}^*$ that in one given orientation contains more than one noncommon vertex of the same degree is 12*; here the noncommon 6-vertices are distinguished by subscripts (1), (2). The subconfigurations S which are (reflected or nonreflected) copies of members of $\mathscr{S}$ are circled and marked with the numbers used in Table I. The only configuration in Table VIII which does not contain a member of $\mathscr{S}$ is the modification of C at the noncommon V_6. (The modification of the reflection of C at its noncommon V_6 yields the same configuration due to the symmetry of C and thus is not drawn in Table VIII.) However, this modification is the preliminary discharging situation C_C with main V_5 at the main V^l of C which is sufficient to dispose of this case.

Next we note that the discussion displayed in Table VIII not only covers all critical modifications at precisely one vertex, but also all those critical modifications B at two or more vertices $V^{(1)},\ldots, V^{(\alpha)}$ for which one of the following cases applies.

(i) The order of the vertices $V^{(1)},\ldots, V^{(\alpha)}$ can be so arranged that (with the notation as in Rule (c) above) none of $V^{(2)},\ldots, V^{(\alpha)}$ belongs to the subconfiguration S (as circled in Table VIII) of the modification B_1 at $V^{(1)}$. (Then, trivially, the extension B of B_1 contains S also.)

(ii) The order of $V^{(1)},\ldots, V^{(\alpha)}$ can be so arranged that $V^{(\alpha)}$ is a common vertex of the modification $B_{\alpha-1}$. Note that $V^{(\alpha)}$ need not be a common vertex of the original configuration A; for example, the modification of 12*** by $M12$ in Table VIII contains a common 6-vertex (which has a common neighborhood $\tilde{N}1$) which was not a common vertex of 12***.

B.3.2. *Modifications at two vertices.* Next we consider all those critical modifications "at precisely two vertices" which have not been covered according to (i) or (ii) in B.3.1. Again we taken advantage of Rule (c) and use the notation introduced there. In order to avoid duplications, the indexing of $V^{(1)}$, $V^{(2)}$ is arranged to satisfy the following conditions. (1) If one of the two vertices is common and the other is not, then the common vertex is $V^{(1)}$, (2) if both vertices are noncommon but one of them is a V_6 and the other is a V_7, then the V_7 is $V^{(1)}$; (3) if A is 12* and the two vertices are noncommon V_6's, then $V^{(1)}$ is the vertex marked (1) in Table III. Since A contains at most one 7-vertex with load 1 (see Table III) it follows that $V^{(2)}$ is always a 6-vertex. Moreover, $V^{(2)}$ belongs to the circled subconfiguration S of B_1 (as displayed in Table VIII).

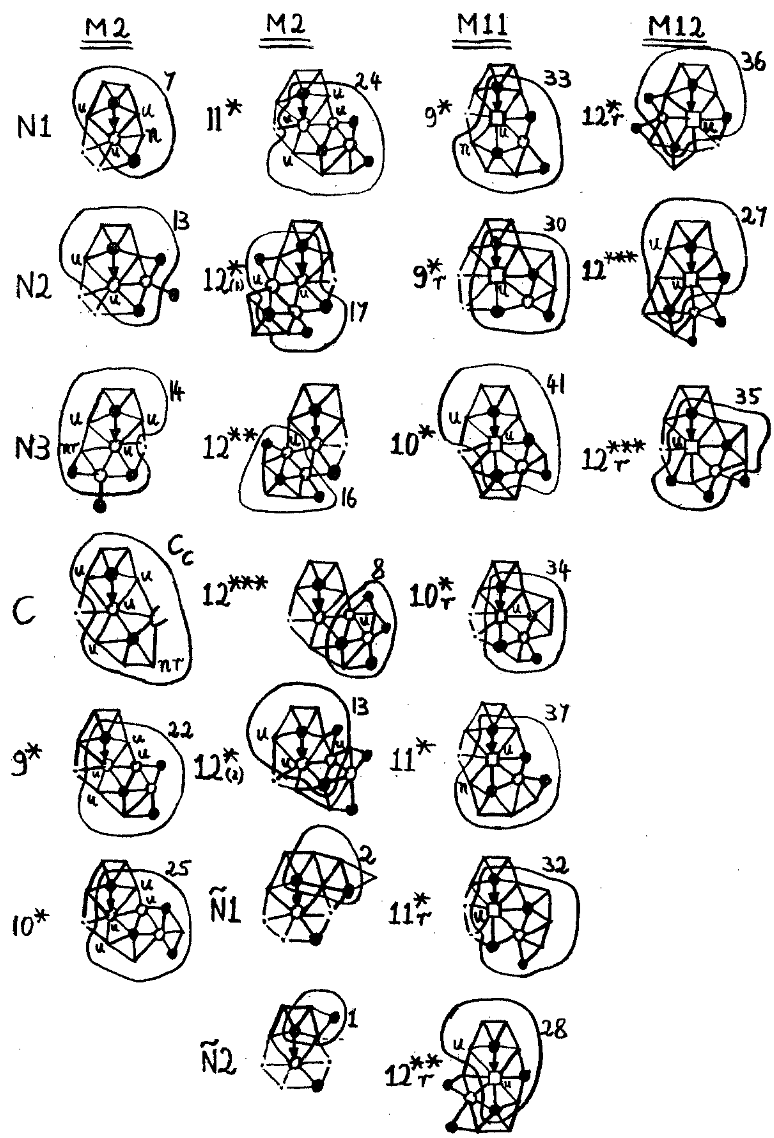
Thus we need only consider those cases in which $V^{(2)}$ is a noncommon 6-vertex of one of the configurations in Table VIII and belongs to the circled subconfiguration of that configuration. Moreover, if $V^{(1)}$ is a noncommon vertex of A then $V^{(2)}$ must not be a common vertex of A; for example, if A is 9* and B_1 is the modification of 9* at its noncommon V_6 by $M2$ (as drawn in the first column of Table VIII) then B_1 contains a 6-vertex (the lower 6-vertex marked "u" in Table VIII) which is noncommon in B_1 but was common in A (before the degree of $V^{(1)}$ was raised from 6 to 7). Further we observe that the critical combination M in B_1 (the pivot of which is identified with $V^{(1)}$) must not be altered by the degree-raising at $V^{(2)}$, i.e., $V^{(2)}$ must not belong to the preliminary discharging situation with pivot at $V^{(1)}$. With these restrictions, most vertices in Table VIII do not qualify as possibilities for $V^{(2)}$. In Table VIII we have marked with a "u" all those vertices which are noncommon in the drawn configurations and belong to the circled subconfigurations but cannot be chosen for $V^{(2)}$ for one of the above reasons; we have marked with "n" or "nr" those vertices which can be chosen for $V^{(2)}$ in the non-reflected or reflected configurations.

In discussing the remaining cases we regard B_1 as oriented (according to the drawing in Table VIII) and assume that the critical combination which is attached at $V^{(2)}$ is not reflected while B_1 may or may not be reflected. There are only five occurrences of "n" or "nr" in Table VIII. The corresponding modifications are drawn in Table IX. Each of these modifications contains a member of $\mathscr{S}$ (circled and marked in Table 9), except for the modification of C which extends the modification $(M2 \| C)_4$; but this contains the preliminary discharging situation ${}_C C_C$ with main V_5 at the main V_5 of C which suffices in this case.

Now we note that the discussion in Table IX not only covers all remaining

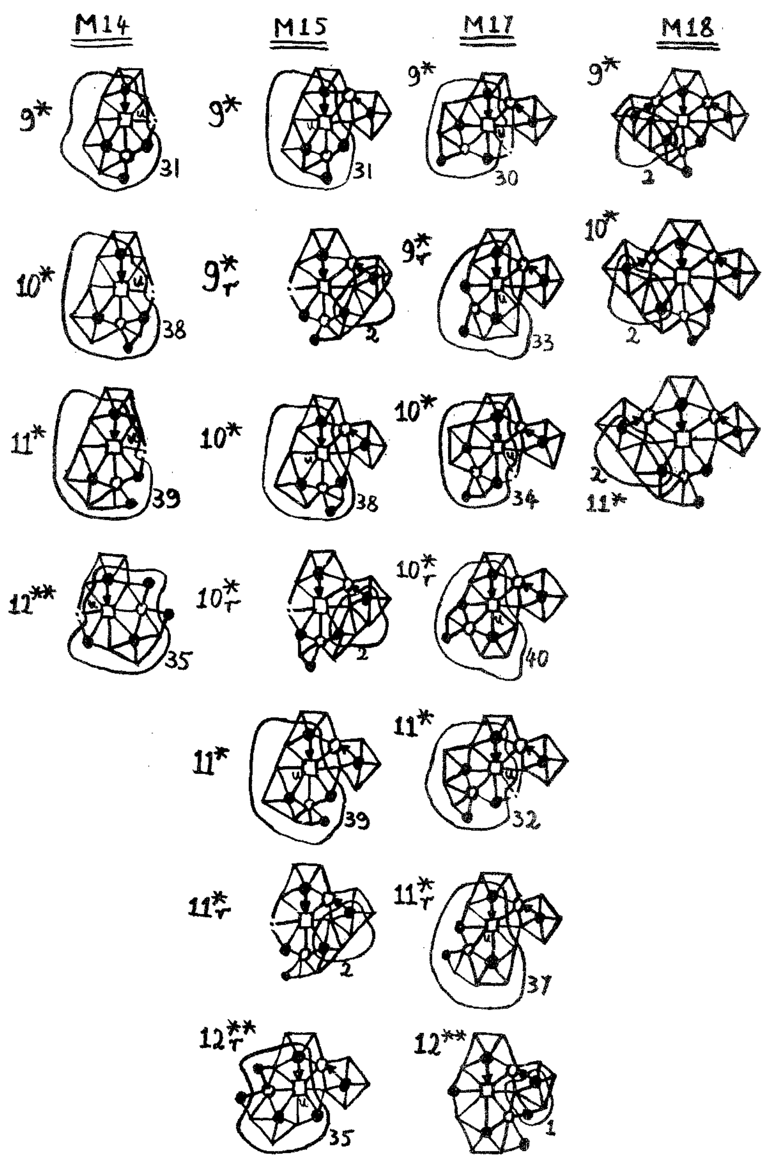
TABLE VIII

The Critical Modifications at Precisely One Vertex



(table continued)

TABLE VIII—*Continued*



critical modifications at precisely two vertices, but also those critical modifications B at three or more vertices $V^{(1)}$, $V^{(2)}$,..., $V^{(\alpha)}$ for which one of the following cases applies (analogous to (i) and (ii) in B.3.1).

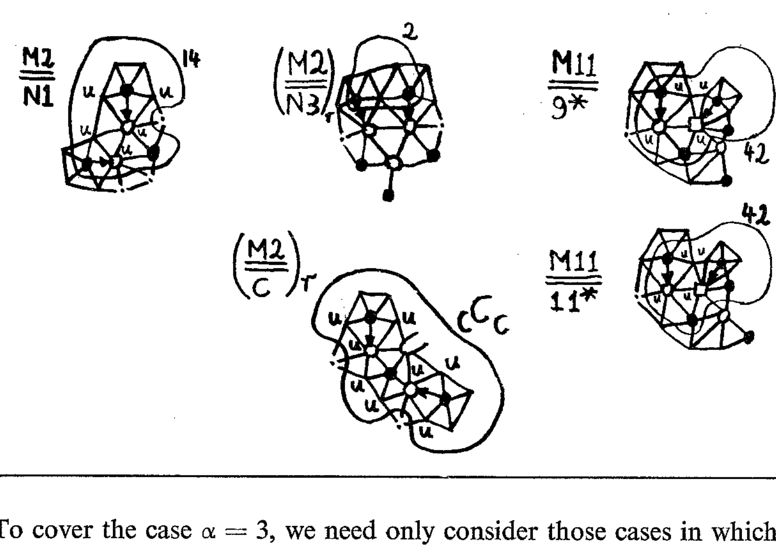
(i*) The order of $V^{(1)},..., V^{(\alpha)}$ can be so arranged that none of $V^{(3)},...,$ $V^{(\alpha)}$ belongs to the subconfiguration S_2 of B_2 (as circled in Table IX).

(ii*) The order of $V^{(1)},..., V^{(\alpha)}$ can be so arranged that $V^{(\alpha-1)}$ is a common vertex of $B_{\alpha-2}$.

B.3.3. *Modifications at Three or More Vertices.* Finally we must discuss all those critical modifications B of members A of $\mathscr{S}^*$ at more than two vertices which are not already covered in B.3.1 or B.3.2. Using Rule (c), the indexing of the vertices $V^{(1)},..., V^{(3)}$ is arranged (in accordance with the arrangement for the case $\alpha = 2$ in B.3.2) so that the following hold. (1*) Common vertices of A have smaller indices than noncommon ones; (2*) noncommon 7-vertices of A have smaller indices than noncommon 6-vertices; (3*) the noncommon V_6 marked (1) in 12* has a smaller index than the V_6 marked (2) (in case that both of them are among the $V^{(i)}$).

TABLE IX

Critical Modifications at Precisely Two Vertices



To cover the case $\alpha = 3$, we need only consider those cases in which $V^{(3)}$ is a noncommon vertex of one of the configurations in Table IX and belongs to the circled subconfiguration of that configuration. Moreover, if $V^{(1)}$ is a noncommon vertex of A then $V^{(3)}$ must not be a common vertex of A. In Table IX we have marked "u" all those vertices which are noncommon in the configurations drawn and belong to the circled subconfigurations but cannt be chosen for $V^{(3)}$ (under the above restrictions). It turns out that there are no possible choices left for $V^{(3)}$ in Table IX. This means that all critical modifications with $\alpha = 3$ have been covered already in B.3.1 and B.3.2. Now the same conclusion follows easily for all $\alpha > 3$ by induction on α. (It may be remarked that α cannot be greater than five since no configuration in $\mathscr{S}^*$ has more than five vertices of load 1.)

This finishes the discussion of the critical modifications of the members of $\mathscr{S}^*$ and completes the proof of the theorem.

Note added in proof. This paper, submitted in 1975, describes the application, to a restricted problem, of an approach which was modified to yield a proof of the Four Color Theorem in 1976. (See Every planar map is four colorable, Part I: Discharging, by K. Appel and W. Haken and Part II: Reducibility, by K. Appel, W. Haken, and J. Koch, *Illinois J. Math.* **21** (1977), 429–567.)

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