PSY 5939: Longitudinal Data Analysis

Contents

Interpreting fixed and random effects

Review

Linear mixed model

$$Y = \mathbf{X}\beta + \mathbf{Z}\gamma + \epsilon$$

$\mathbf{X}\beta$ are the **fixed effects**

- What all the people have in common

$\mathbf{Z}\gamma$ are the **random effects**

- How the people are different from one another

ϵ is the residual

- Error in estimating each person's trajectory

aka multilevel model

Linear change over time, random intercept, random slope

Level 1 = measurement occasion: Model of observations over time

$$Y_{ij} = \pi_{0i} + \pi_{1i}(Time_{ij}) + e_{ij}$$

Level 2 = participant: Model of participant level differences

$$\pi_{0i} = \beta_{00} + r_{0i}$$
$$\pi_{1i} = \beta_{10} + r_{1i}$$

Combined equation

$$Y_{ij} = \beta_{00} + \beta_{10}(Time_{ij}) + r_{0i} + r_{1i}(Time_{ij}) + e_{ij}$$

Model results - R

```
## Linear mixed model fit by maximum likelihood  ['lmerMod']
## Formula: Y ~ 1 + time1 + (1 + time1 | subject)
##    Data: tall_data
##
##      AIC      BIC   logLik deviance df.resid
##   1345.8   1365.6   -666.9   1333.8      194
##
## Scaled residuals:
##      Min       1Q   Median       3Q      Max
## -2.37379 -0.55722  0.01556  0.49778  2.58483
##
## Random effects:
##  Groups   Name        Variance Std.Dev. Corr
##  subject  (Intercept)  4.94    2.223
##           time1       10.50    3.240    -0.15
##  Residual             27.50    5.244
## Number of obs: 200, groups:  subject, 50
##
## Fixed effects:
##             Estimate Std. Error t value
## (Intercept) 100.5470     0.6955 144.561
## time1         5.3124     0.5657   9.391
##
## Correlation of Fixed Effects:
##       (Intr)
## time1 -0.476
```

Model results - SPSS

Fixed Effects

Type III Tests of Fixed Effects[a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	50.000	20897.726	.000
time1	1	50	88.199	.000

a. Dependent Variable: Y.

Estimates of Fixed Effects[a]

Parameter	Estimate	Std. Error	df	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	100.546956	.695536	50.000	144.560	.000	99.149931	101.943981
time1	5.312409	.565665	50	9.391	.000	4.176238	6.448580

a. Dependent Variable: Y.

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Residual		27.498364	3.888856	7.071	.000	20.841491	36.281475
Intercept + time1 [subject = subject]	UN (1,1)	4.939640	5.551009	.890	.374	.545937	44.693904
	UN (2,1)	-1.115745	3.294692	-.339	.735	-7.573223	5.341733
	UN (2,2)	10.499151	3.292935	3.188	.001	5.677862	19.414381

a. Dependent Variable: Y.

Random Effect Covariance Structure (G)[a]

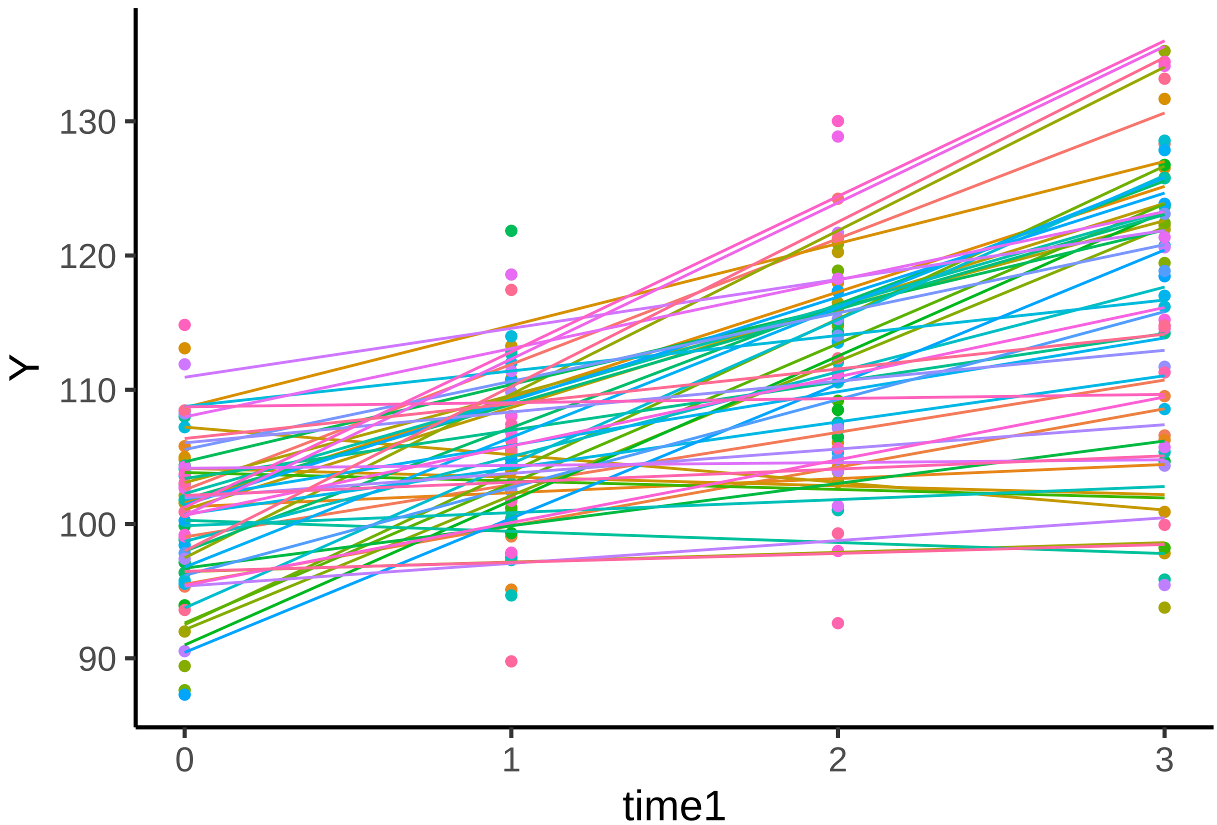
	Intercept | subject	time1 | subject
Intercept | subject	4.939640	-1.115745
time1 | subject	-1.115745	10.499151

Unstructured

a. Dependent Variable: Y.

```
exe.
```

Individual (linear) trajectories



Fixed effects

Fixed effects

Using functions built in to `lme4`

```
fixef(model1)
```

```
## (Intercept)       time1
##  100.546956    5.312409
```

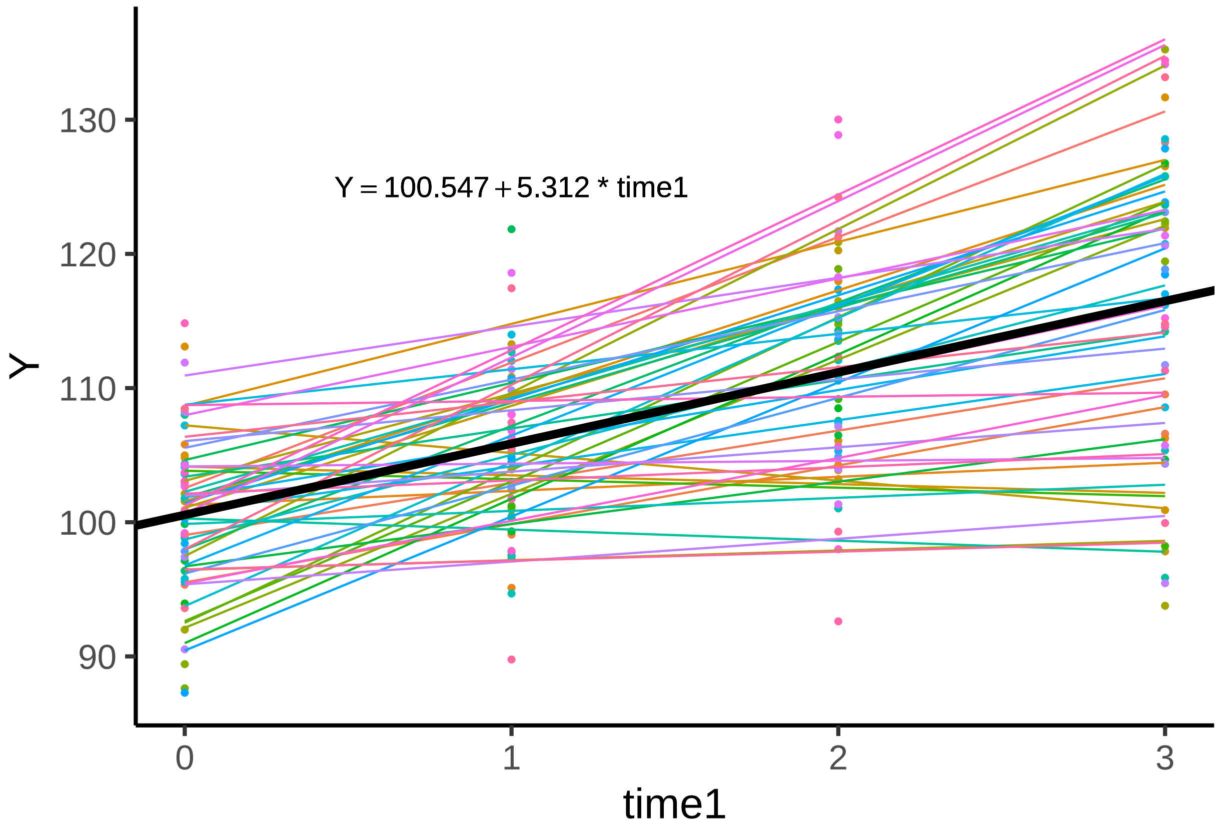
Using the `broom` package

- I don't care for how much it rounds the numbers

```
tidy(model1, effects = "fixed", conf.int = TRUE)
```

```
## # A tibble: 2 x 7
##   effect term        estimate std.error statistic conf.low conf.high
##   <chr>  <chr>          <dbl>     <dbl>     <dbl>    <dbl>     <dbl>
## 1 fixed  (Intercept)   101.       0.696     145.      99.2     102.
## 2 fixed  time1           5.31     0.566       9.39     4.20      6.42
```

Spaghetti plot with fixed effects



Random effects

Random effects

Using functions built in to `lme4`

```
print(VarCorr(model1),comp=c("Variance", "Std.Dev."))
```

```
##  Groups   Name        Variance Std.Dev. Corr
##  subject  (Intercept)  4.9395  2.2225
##           time1       10.4992  3.2402   -0.155
##  Residual             27.4984  5.2439
```

Using the `broom` package

- Here, I don't care for the table format

```
tidy(model1, effects = "ran_pars", scales = "vcov")
```

```
## # A tibble: 4 x 4
##   effect   group    term                       estimate
##   <chr>    <chr>    <chr>                         <dbl>
## 1 ran_pars subject  var__(Intercept)               4.94
## 2 ran_pars subject  cov__(Intercept).time1       -1.12
## 3 ran_pars subject  var__time1                    10.5
## 4 ran_pars Residual var__Observation              27.5
```

Covariance as correlation

Covariance between intercept and slope as a *correlation*

- SAS: "gcorr" option
- R: `VarCorr()` function (and now default in `lmer()`?)
- SPSS: I don't think you can get it
- Calculate by hand:

$$\frac{\sigma_{r_{0i}r_{1i}}}{\sqrt{\sigma^2_{r_{0i}}}\sqrt{\sigma^2_{r_{1i}}}}$$

Also remember that a standard deviation is the square root of a variance

Prediction interval for individuals

Interval for likely values of **individual** intercepts and slopes

- Remember last week, I said that γ is normally distributed with mean 0 and variance $\mathbf{G}$: $\gamma \sim N(0, \mathbf{G})$
- In a normal distribution, 95% of values will be ± 1.96 standard deviation from the mean

95% of **individual intercepts** are in

- $[\beta_{00} - 1.96 \times \sqrt{\sigma^2_{r_{0i}}},\ \beta_{00} + 1.96 \times \sqrt{\sigma^2_{r_{0i}}}]$

95% of **individual slopes** are in

- $[\beta_{10} - 1.96 \times \sqrt{\sigma^2_{r_{1i}}},\ \beta_{10} + 1.96 \times \sqrt{\sigma^2_{r_{1i}}}]$

Prediction interval for individuals

$[\beta_{00} - 1.96 \times \sqrt{\sigma^2_{r_{0i}}},\ \beta_{00} + 1.96 \times \sqrt{\sigma^2_{r_{0i}}}]$

$[100.547 - 1.96 \times \sqrt{4.94},\ 100.547 + 1.96 \times \sqrt{4.94}]$

95% of **individual intercepts** are in [96.191, 104.903]

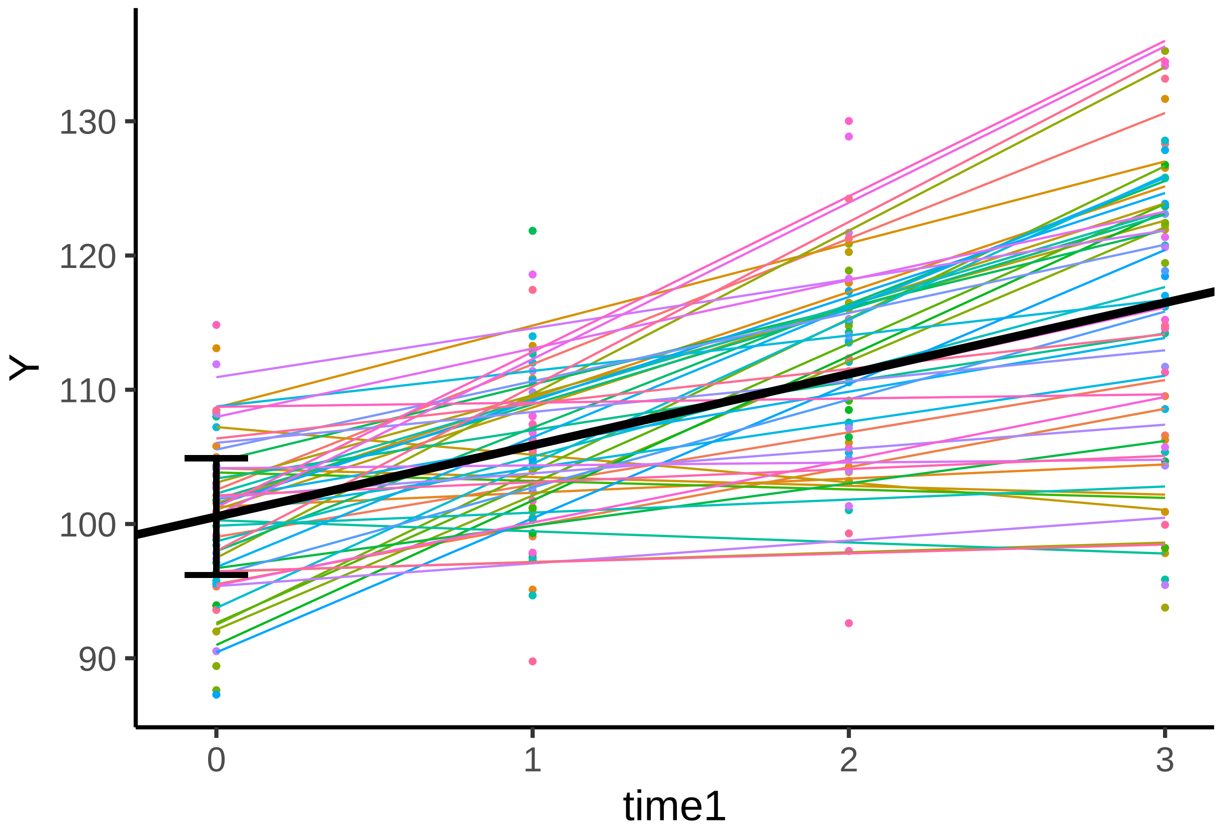
$[\beta_{10} - 1.96 \times \sqrt{\sigma^2_{r_{1i}}},\ \beta_{10} + 1.96 \times \sqrt{\sigma^2_{r_{1i}}}]$

$[5.312 - 1.96 \times \sqrt{10.499},\ 5.312 + 1.96 \times \sqrt{10.499}]$

95% of **individual slopes** are in $[-1.038,\ 11.663]$

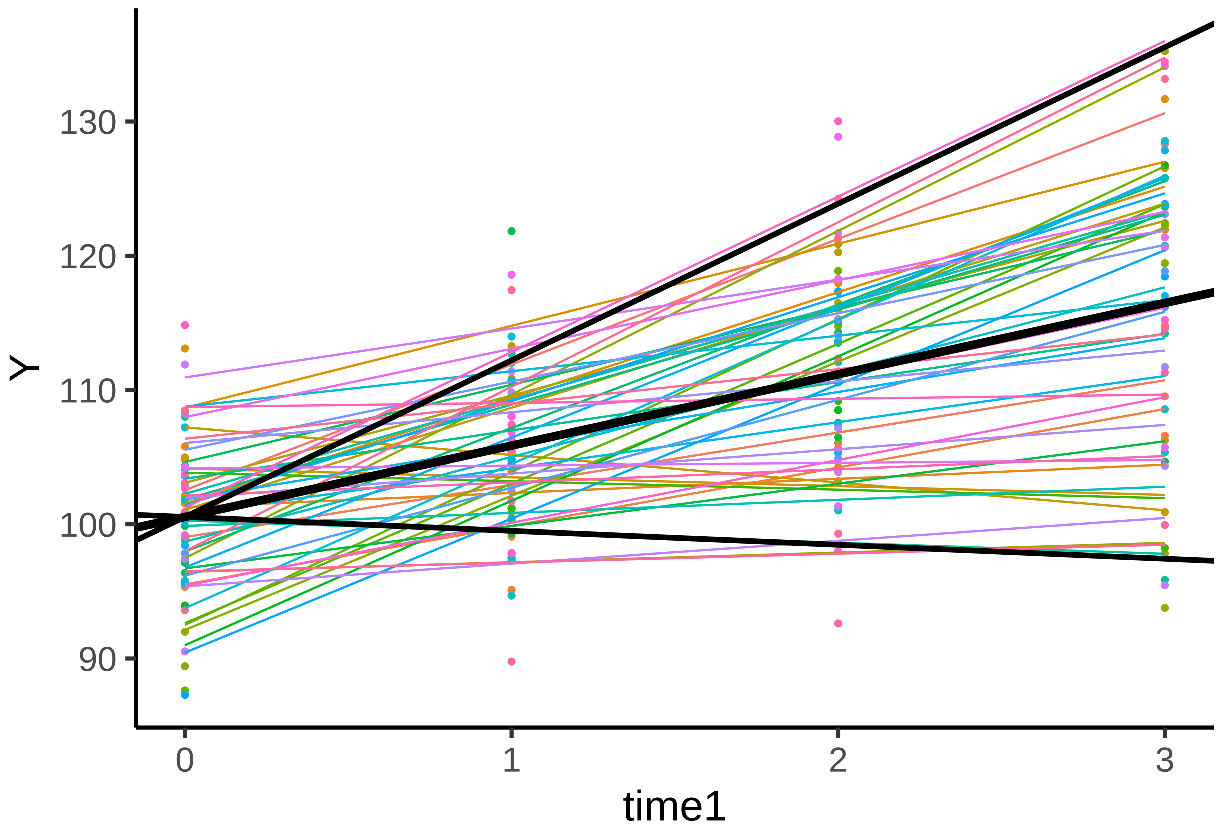
Spaghetti plot with fixed and random intercepts

95% of **individual intercepts** range from 96.191 to 104.903



Spaghetti plot with fixed and random slopes

95% of **individual slopes** range from -1.038 to 11.663



ICC and model comparison

Intra-class correlation (ICC)

Quantifying similarity

Previously:

Non-independence due to repeated measuring the same individual

- Standard errors are **underestimated**
- **How much** the standard errors are underestimated depends on how much the observations are related to one another

The **intraclass correlation (ICC)** quantifies this "more alike" ness

Intraclass correlation

- Calculated from the results of an "unconditional mixed model" that has no predictors, **not even time**
- Number between 0 and 1
- Proportion of total variability that is at level 2
  - Since level 2 is the person, ICC tells you what proportion of the variability is **between people**, not **between occasions**

Unconditional model

Also called "random effects ANOVA"

Level 1 model:

$$Y_{ij} = \pi_{0i} + e_{ij}$$

Level 2 model:

$$\pi_{0i} = \beta_{00} + r_{0i}$$

Combined model:

$$Y_{ij} = \beta_{00} + r_{0i} + e_{ij}$$

Overall mean (β_{00}) + variability between people (var(r_{0i}) + variability in observations over time (from the same person) (var(e_{ij}))

- Note that all variance is either *between people* or *within people*

Calculating ICC

Use the level 2 intercept variance: $\sigma^2_{r_{0i}}$

And the level 1 residual variance: σ^2_e

$$ICC = \frac{\sigma^2_{r_{0i}}}{\sigma^2_{r_{0i}} + \sigma^2_e}$$

ICC is the proportion of variance that is due to differences between people

- Level 2 variation

$1 - ICC$ is the proportion of variance due to errors in predicting individuals over time

- Level 1 variation

Unconditional model

```r
uncond <- lmer(Y ~ 1 + (1|subject), tall_data, REML = "false")
summary(uncond)
```

```
## Linear mixed model fit by maximum likelihood  ['lmerMod']
## Formula: Y ~ 1 + (1 | subject)
##    Data: tall_data
##
##      AIC      BIC   logLik deviance df.resid
##   1494.6   1504.5   -744.3   1488.6      197
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -2.2096 -0.6350 -0.1259  0.6475  2.5714
##
## Random effects:
##  Groups   Name        Variance Std.Dev.
##  subject  (Intercept)  9.082   3.014
##  Residual             92.033   9.593
## Number of obs: 200, groups:  subject, 50
##
## Fixed effects:
##             Estimate Std. Error t value
## (Intercept) 108.5156     0.8011   135.5
```

Calculating ICC

$$ICC = \frac{\sigma^2_{r_{0i}}}{\sigma^2_{r_{0i}} + \sigma^2_e} = \frac{9.0818}{9.0818 + 92.0331} = \frac{9.0818}{101.1149} = 0.0898$$

9% of the variance in the outcome is **between people**

- Differences between people

91% of the variance is **within people**

- People varying over time

Design effect

The design effect is how much standard errors still be underestimated

$$D_{eff} = 1 + (n - 1) * ICC$$

where n is the cluster size

- Here, the (average) number of repeated observations per person

In this example, $D_{eff} = 1 + (n - 1) * ICC = 1 + (4 - 1) * 0.0898 = 1.269$

- If we ignored the repeated measures, our standard errors would be need to be multiplied by 1.269 to account for non-independence
- (This is also ignoring a bunch of other reasons we might need to do a mixed model that we haven't yet talked about)
- In general, design effect goes up with ICC and with number of repeated measures

Likelihood ratio (LR) test

Deviance

Mixed models are estimated with **maximum likelihood (ML)**

Unlike linear regression, no sums of squares (SS)

- No $SS_{explained}$ or $SS_{residual}$
- No R^2, which is $\frac{SS_{explained}}{SS_{explained}+SS_{residual}}$
- No F-tests of the overall model

Maximum likelihood instead provides **deviance** (also called -2 log likelihood)

- Conceptually, kind of similar to $SS_{residual}$
- How far you are from a "perfect" model
- Relative, not absolute

Likelihood ratio (LR) test

Deviance = -2 × log-likelihood

Difference in deviance between two models is $\sim \chi^2$ with degrees of freedom equal to the difference in number of parameters estimated

- Significant likelihood ratio test:

    - Models are different, use the more complex one

- NS likelihood ratio test:
    - Models are equivalent, so use the simpler one (parsimony)

Unconditional versus linear time model

Linear model parameters = 6: β_{00}, β_{10}, $\sigma^2_{r_{0i}}$, $\sigma^2_{r_{1i}}$, $\sigma_{r_{0i}r_{1i}}$, σ^2_e

Unconditional model parameters = 3: β_{00}, $\sigma^2_{r_{0i}}$, σ^2_e

Model	# parameters	deviance
Linear time	6	1333.8436
Unconditional	3	1488.6397
(Absolute) difference	3	154.7962

Critical value for $\chi^2(3) = 7.815$

- $154.796 > 7.815$, so the test is significant

The more complex model (linear time) is significantly better than the simpler (unconditional) model

- Adding linear time to the model is better than not having linear time

Variance explained

R^2 analogues

Mixed models produce **deviance**

- Not $SS_{explained}$ or $SS_{residual}$
- No R^2, which is $\frac{SS_{explained}}{SS_{explained}+SS_{residual}}$

Pseudo-R^2 values for mixed models

- Not as good as R^2
    - R^2 can never be negative but pseudo-R^2 can (report as 0)

Variance explained and variance reduction

When you compare your model to the **unconditional model**

- Variance *explained*
- How much variance does my model explain?
- Like R^2

When you compare your model to **some other (simpler) model**

- Variance *reduction*
- How much is (error) variance reduced by adding whatever you added?
- Like R^2_{change}

Variance explained and variance reduction

Model 1 is simpler, Model 2 is more complex

- The model you "care about" is Model 2

Reduction in variance =

$$\frac{\sigma_e^2(Model1) - \sigma_e^2(Model2)}{\sigma_e^2(Model1)}$$

Variance explained example

Unconditional model: $\sigma_e^2 = 92.0331$

Linear model with time centered at first wave: $\sigma_e^2 = 27.4984$

Variance explained = $\frac{\sigma_e^2(uncond) - \sigma_e^2(linear)}{\sigma_e^2(uncond)} = (92.0331 - 27.4984)/(92.0331) = 0.7012$

Interpret similar to R^2

- 70.1% of the *residual variance* is **explained** by adding the linear trend
- **Reduced** 70.1% of the *variance in the unconditional model* by adding the linear trend

Adding predictors

Predictors of growth

Growth parameters

The model so far...

- β_{00} and β_{10} are mean intercept and slope
- $\sigma^2_{r_{0i}}$ is the intercept variance
- $\sigma^2_{r_{1i}}$ is the slope variance
- $\sigma_{r_{0i}r_{1i}}$ is the covariance between intercept and slope

Describe growth (and individual variability in growth)

- But we're not yet trying to **explain** growth

Predictors of growth

Mixed models can have predictors at different levels

- Predictors at level 1: **Time varying** predictors = change over time
  - At the level of the **measurement occasion**
  - Have a (potentially) different value at each measurement occasion
- Predictors at level 2: **Time invariant** predictors = don't change
  - At the level of the **participant**
  - Same value at every measurement occasion

Level 2 predictors are easy, but level 1 predictors are a bit more complex

Level 2 predictors

Level 2 predictors

Time invariant predictors **don't** change with time

- Biological sex, experimental group, etc.
- Something that can change but is unlikely to change during the study: e.g., SES
- Something that changes but is perfectly correlated with time: e.g., age

Level 2 predictors

Level 2 predictors go in the level 2 equation(s)

$$\pi_{0i} = \beta_{00} + \beta_{01}(L2PRED) + r_{0i}$$

$$\pi_{1i} = \beta_{10} + \beta_{11}(L2PRED) + r_{1i}$$

β_{01} = the effect of L2PRED on average *intercept* value

β_{11} = the effect of L2PRED on average *slope* value

You can add a predictor of the intercept, the slope, or both

- **You decide**, as a researcher, what the predictor does

Level 2 predictors

Level 1:

$$Y_{ij} = \pi_{0i} + \pi_{1i}(time_{ij}) + e_{ij}$$

Level 2:

$$\pi_{0i} = \beta_{00} + \beta_{01}(L2PRED) + r_{0i}$$
$$\pi_{1i} = \beta_{10} + \beta_{11}(L2PRED) + r_{1i}$$

Combined model:

$$Y_{ij} = \beta_{00} + \beta_{01}(L2PRED) + \beta_{10}(time_{ij}) + \underline{\beta_{11}(L2PRED)(time_{ij})} + r_{0i} + r_{1i}(time_{ij}) + e_{ij}$$

Centering level 2 predictors

You don't need to center level 2 predictors if they are

- dummy coded (0, 1)
- effects coded (-1,1)

If your level 2 predictor is coded another way, **center it**

- You should "grand mean center" the level 2 predictor
- Subtract the mean of **all** observations from the predictor

Interpreting fixed effects

I'm going to work with a model with **only** a random intercept here.

Combined model:

$$Y_{ij} = \beta_{00} + \beta_{01}(L2PRED) + \beta_{10}(time_{ij}) + \beta_{11}(L2PRED)(time_{ij}) + r_{0i} + e_{ij}$$

Just the fixed effects:

$$Y_{ij} = \beta_{00} + \beta_{01}(L2PRED) + \beta_{10}(time_{ij}) + \beta_{11}(L2PRED)(time_{ij})$$

This should look familiar

- Just like a two predictor regression with an **interaction**

Example level 2 predictor

Remember that there is a level 2 variable in the dataset: `group`

```
head(tall_data, n = 12)
```

```
## # A tibble: 12 x 5
##    subject group  time     Y time1
##      <dbl> <dbl> <dbl> <dbl> <dbl>
##  1       1     1     0 101.      0
##  2       1     1     1 113.      1
##  3       1     1     2 124.      2
##  4       1     1     3 128.      3
##  5       2     0     0  95.3     0
##  6       2     0     1 106.      1
##  7       2     0     2 111.      2
##  8       2     0     3 107.      3
##  9       3     1     0  96.4     0
## 10       3     1     1  99.1     1
## 11       3     1     2 103.      2
## 12       3     1     3 110.      3
```

Combined model:

$$Y_{ij} = \beta_{00} + \beta_{01}(group) + \beta_{10}(time_{ij}) + \beta_{11}(group)(time_{ij}) + r_{0i} + e_{ij}$$

Model with level 2 predictor

```
model2 <- lmer(Y ~ 1 + group + time1 + time1*group + (1|subject), tall_data, REML = "false")
summary(model2)
```

```
## Linear mixed model fit by maximum likelihood  ['lmerMod']
## Formula: Y ~ 1 + group + time1 + time1 * group + (1 | subject)
##    Data: tall_data
##
##      AIC      BIC   logLik deviance df.resid
##   1295.2   1315.0   -641.6   1283.2      194
##
## Scaled residuals:
##      Min       1Q   Median       3Q      Max
```

```
## -2.20536 -0.58245  0.01897  0.55369  2.42595
##
## Random effects:
##  Groups   Name        Variance Std.Dev.
##  subject  (Intercept)  8.874   2.979
##  Residual             29.368   5.419
## Number of obs: 200, groups:  subject, 50
##
## Fixed effects:
##             Estimate Std. Error t value
## (Intercept) 101.2629     1.1839  85.537
## group        -1.2344     1.5545  -0.794
## time1         1.7138     0.5289   3.241
## group:time1   6.2045     0.6944   8.935
##
## Correlation of Fixed Effects:
##             (Intr) group  time1
## group       -0.762
## time1       -0.670  0.510
## group:time1  0.510 -0.670 -0.762
```

Interpreting fixed effects

$$Y_{ij} = \beta_{00} + \beta_{01}(group) + \beta_{10}(time_{ij}) + \beta_{11}(group)(time_{ij}) =$$

$$101.263 + -1.234(group) + 1.714(time_{ij}) + 6.205(group)(time_{ij})$$

Interpretation of the fixed effects plays out the same as a **2 predictor regression with continuous (time) and categorical (group) predictors and an interaction**

β_{00}: Mean Y value for `group` $= 0$ when `time1` $= 0$

- Mean value of Y for control group at first time point
- 101.263

β_{01}: Difference between `groups` when `time1` $= 0$

- Difference between control (`group` $= 0$) and treatment (`group` $= 1$) at first time point (baseline differences)
- -1.234

Interpreting fixed effects

$$Y_{ij} = \beta_{00} + \beta_{01}(group) + \beta_{10}(time_{ij}) + \beta_{11}(group)(time_{ij}) =$$

$$101.263 + -1.234(group) + 1.714(time_{ij}) + 6.205(group)(time_{ij})$$

Interpretation of the fixed effects plays out the same as a **2 predictor regression with continuous (time) and categorical (group) predictors and an interaction**

β_{10}: Linear change over time for `group` $= 0$

- Increase in Y for a 1 unit change in `time1` for the control group
- 1.714

β_{11}: Difference between groups in linear change over time

- Difference between control and treatment in the increase in Y for a 1 unit change in `time1`
- 6.205

Simple slopes

When `group` $= 0$ (control):

$$101.263 + -1.234(group) + 1.714(time_{ij}) + 6.205(group)(time_{ij})$$

$$101.263 + -1.234(0) + 1.714(time_{ij}) + 6.205(0)(time_{ij})$$

$$101.263 + 1.714(time_{ij})$$

When `group` $= 1$ (treatment):

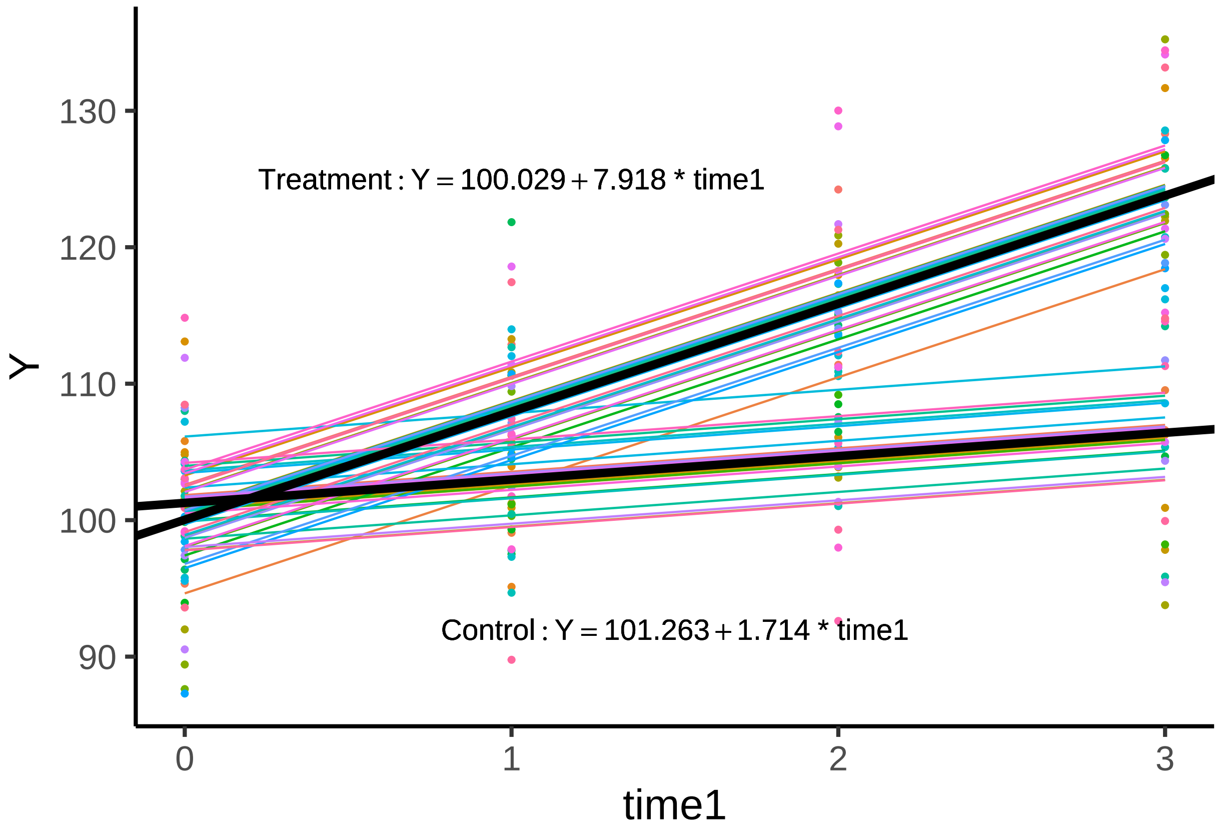
$$101.263 + -1.234(group) + 1.714(time_{ij}) + 6.205(group)(time_{ij})$$

$$101.263 + -1.234(1) + 1.714(time_{ij}) + 6.205(1)(time_{ij})$$

$$101.263 + -1.234 + 1.714(time_{ij}) + 6.205(time_{ij})$$

$$100.029 + 7.919(time_{ij})$$

Spaghetti plot with fixed effects



Random effects

```
print(VarCorr(model2), comp = c("Variance", "Std.Dev"))
```

```
##  Groups   Name        Variance Std.Dev.
##  subject  (Intercept)  8.8743  2.9790
##  Residual             29.3675  5.4192
```

$[\beta_{00} - 1.96 \times \sqrt{\sigma^2_{r_{0i}}}, \beta_{00} + 1.96 \times \sqrt{\sigma^2_{r_{0i}}}]$

95% of **individual intercepts** in the **control group** are in:

$$[101.263 - 1.96 \times \sqrt{8.874}, 101.263 + 1.96 \times \sqrt{8.874}]$$

$$[95.424, 107.102]$$

95% of **individual intercepts** in the **treatment group** are in:

$$[100.029 - 1.96 \times \sqrt{8.874}, 100.029 + 1.96 \times \sqrt{8.874}]$$

$$[94.19, 105.867]$$

Spaghetti plot with fixed and random effects