FYS3150 - Project 3

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GIT link:
https://github.com/steffemb/project3/tree/master/project3/solarsystem

Abstract

In this project we are going to model the solar system. The goal is not actually to model the solar system but rather to make an outline of a system which can be improved upon later. And then test the numerical stability.

Much of the work in this project is purely programming, and structuring a class hierarchy so it is as moldable as possible.

We will see that for large systems Runge Kutta methods is the best way to solve the ODE's, everything else gives numerical errors that are unacceptable, unless you use a very small step length, so small that it gets hard to compute. In fact for complicated systems like the whole solar system we should use something better than even Runge Kutta 4, like adaptive methods.

Introduction

Solving ordinary differential equations is not new science. It is something we have done with pen and paper, but also something we have programmed in many courses earlier. So what is new in this project? To be honest not that much, at least not on the ODE solver part. What is new is the object orientation, which is important when modeling such a big system. If I where to model the solar system in a python script it would probably get very messy fast. Not to talk about what a nightmare it would have been to add new objects to the system.

So i guess thats our motivation for this project. We will take a closer look at Runge Kutta 4 also. And of course, we can brag about it too our friends.

Solution

In this project i opted to use the class structure given at the computer lab. Details can be found at the github link given on the front page.

First i want to do a short rundown of how the structure works. On top we have a class "Solarsystem". Inside "Solarsystem" we have objects called "CelestialBody". Each Body in the solar system gets it's own object reference. For example in the Earth-Sun system Earth would be "mySolarsystem.objects[1]". And this object contains all the information about Earth in form of 3 dim.

vectors or double float numbers. The "Solarsystem" only contains information at a specific time t. By using the "Integrate" class we update all the information in the "Solarsystem" from t to (t + dt), where dt is the step length. So sending a solar system to the integrate class will only update the solar system one step. "Solarsystem" also contains a couple of other functions, mostly self explanatory, but the most important is "void Force()" which gives every object in a "Solarsystem" a force vector which is the sum of all acting forces and also updates kinetic and potential energy for the object.

One of the main tasks in this project is to program the "Force()" function. To do this we start with Newtons gravitational force between two objects,

$$F = G\frac{mM}{r^2}, or, \vec{F} = G\frac{mM}{r^3}\vec{r}, \tag{1}$$

Where m and M are the masses, and G is the gravitational constant. First we need to decompose the force to F_x, F_y and F_z. scince $\vec{r} = r_x, r_y, r_z$ we get,

$$\vec{F_x} = G\frac{mM}{r^3}\vec{r_x}, \vec{F_y} = G\frac{mM}{r^3}\vec{r_y}, \vec{F_z} = G\frac{mM}{r^3}\vec{r_z}, \tag{2}$$

If we now have an n-body system we will need to sum n-1 of these forces for each body to find the total force acting on each body. we will fix the origin of the system in space, and we then get the $\vec{r_i}$ vector easily by taking $r_{\mathrm{Object\vec{O}fInterest}} - r_{\mathrm{Other\vec{O}bject}}$. And the total force on an object is,

$$\sum_{i=1}^{i=n-1} \vec{F_i} = G\frac{M_{ObjectOfInterest}M_{Object_i}}{r^3}\vec{r_i}. \tag{3}$$

Now that we have our forces in x, y, z our last task before we have a working simulation is to solve the equations

$$\frac{d^2x}{dt^2} = \frac{\sum F_x}{M_{Object}}, \tag{4}$$

$$\frac{d^2y}{dt^2} = \frac{\sum F_y}{M_{Object}}, \tag{5}$$

$$\frac{d^2 z}{dt^2} = \frac{\sum F_z}{M_{Object}}, \tag{6}$$

for every object in the system.

Runge Kutta 4 method

The main method for solving the ODE's in this project is the forth order Runge Kutta method. We will prefer using this method since it gives us a truncation error of $o(h^4)$ apposed to $o(h^3)$ for Verlet method or $o(h^2)$ for Euler'r method.

Mathematically the Runge Kutta method goes as follows:

First we need

$$k_1 = f(t_i, y_i)dt, \tag{7}$$

I have called the step dt here, it is customary to write h instead of dt, but in our case our step is in time, so i called it dt. k_1 is nothing but Eulers method, that is, it is the slope at t_i, y_i. In our case the function f is the force divided by the mass of the object. second we need

$$k_2 = f(t_i + dt/2, y_i + k_1/2)dt. \tag{8}$$

k_2 is a prediction of the slope at the midpoint. In our case we therefore need to calculate the forces on the objects at $t + dt/2$ using k_1 as a prediction. Third we need

$$k_3 = f(t_i + dt/2, y_i + k_2/2)dt. \tag{9}$$

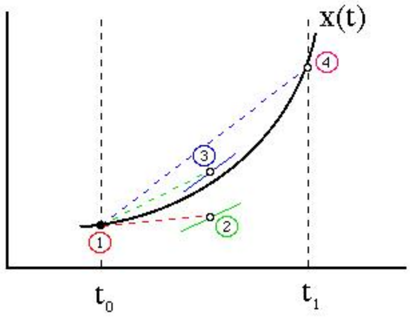
With k_3 we do the same except we say that k_2 is a new and "better" prediction therefore we use that instead of k_1. The forth step is

$$k_4 = f(t_i + dt, y_i + k_3)dt, \tag{10}$$

this is to predict the slope at y_{i+1} at $t_i + dt$. We now have everything we need for our final prediction

$$y_{i+1} = y_i + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]. \tag{11}$$

What we already have is the right hand side of eq 4, 5, 6. Which means we already have the acceleration of the objects at time t. so when we calculate y_{i+1} this will be the velocity in x, y, z. And then we can just run the algorithm again to get the positions.



(a) Runge Kutta 4

Figur 1: *Figure for illustrating rk4 method graphically. 1, 2, 3, 4 stands for k1, k2, k3, k4.*

One problem in the class hierarchy is that we only have one given vector for velocity, position etc and by updating them with, say, k_2 the object "forgets" about k_1. So to solve this i had to make four copy solar systems and compute k_1, k_2, k_3, k_4 in one copy each. This makes for a messy and long code, and is therefore something i would try to improve for the future.

Setting up the solar system

When setting up the solar system we need to give it a center of mass so that everything keeps its symmetry. In an n-body system the center of mass is given by

$$R = \frac{1}{M}\sum_{i=1}^{n} m_i r_i \tag{12}$$

where M is the sum of the masses of all the objects. We will start with the Sun-Earth-Jupiter system.

We also need initial values for all the objects. Since we are in dimensions AU and years we can give the Earth x position $1 - CenterOfMass$, and velocity $2 * \pi$. This is not exactly correct, in reality the Earth has a slightly elliptical orbit. For Jupiter and the Sun it is slightly more difficult. The positions are easy, they are $position - CenterOfMass$, that is $0 - CenterOfMas$ for the Sun, and $5.2 - CenterOfMas$ for Jupiter. For the velocities we can use the formula for centripetal acceleration

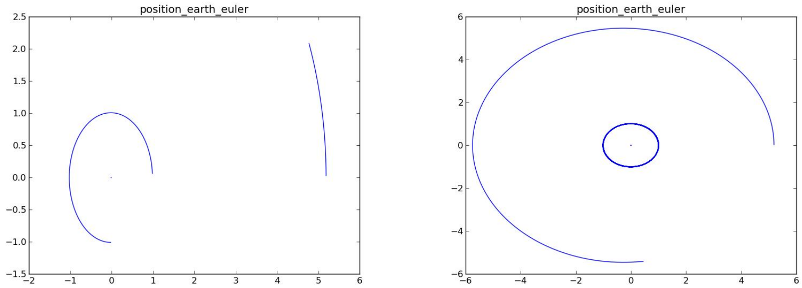
$$a = \frac{v^2}{r} = \frac{F}{m} \Rightarrow v = \sqrt{r\frac{F}{m}} \tag{13}$$

So with the positions given over we now know the magnitude of v. So to make this simple we place all objects along the x-axis and give them velocities in the y-axis. We give the sun negative values and everything else positive. This is because the center of mass will be between the Sun and everything else. If we did not give the Sun and everything else opposite velocity directions they would not orbit the origin where we wish the center of mass to be.

To check if we are doing things right we can compute the potential energy, the kinetic energy, and the angular momentum. in the case that we have circular orbits these should all be constants, on each object, but also the total. This is because it is always a balance between kinetic and potential energy. When the orbit is circular the magnitude of $\vec{r}$ is a constant, which makes all the latter constants. If we want to play with elliptical orbits, as a test, we can check the total energy, which should always be conserved.

Results

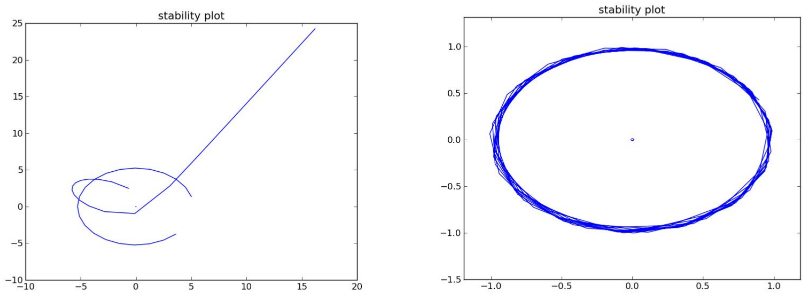
The first result we get is that for the case where we make circular orbits the energy is indeed conserved, we can also see from a plot that the system is doing something that is intuitive pleasing.



(a) Sun Earth Jupiter after 0,75y (b) Sun Earth Jupiter after 10y

Figur 2:

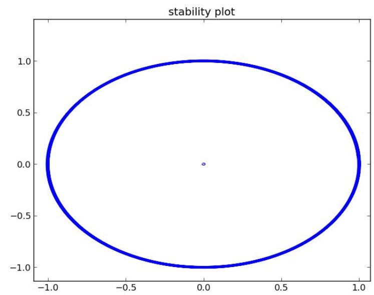
If we test for different values of dt and plot we see that not all values gives us something that works.



(a) stability at dt = 0,5 (b) stability at dt = 0,7

Figur 3:

For dt = 0,5 we see that something really strange is happening. Although the total energy is conserved this is not physically pleasing. When testing for dt = 0,7 we see that we have a stable orbit. But i would not trust the results, because we see that the orbit is choppy.

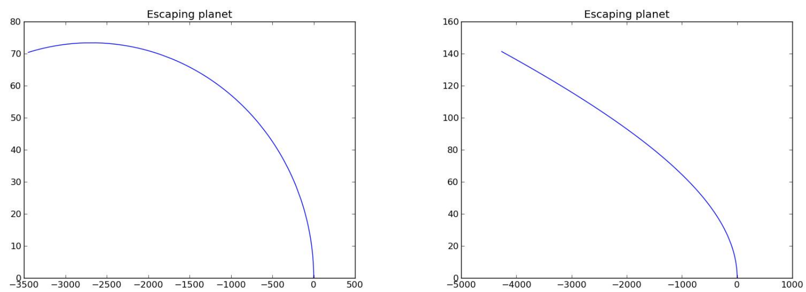


(a) stability at dt = 0,01

Figur 4:

I found that for most cases dt = 0,01 is fine. you could use bigger dt, possibly up to dt = 0,05. but i used dt = 0,01 to be sure. One thing to note is that the system gets more unstable the lower the orbit. So if we were to out in mercury, for example, in the system, we would probably need a smaller dt.

We can now test the system a little. By trial and error i found that by giving the Earth (or any object placed at 1 Au from the Sun) that by giving it an initial velocity of $8.884AUy^{-1}$ it still has an orbit but at $8.885AUy^{-1}$ i could no longer get an orbit. So the object escaped the solar system.



(a) Sun Earth initial v = 8.884 AU/y (b) Sun Earth initial v = 8.885 AU/y

Figur 5:

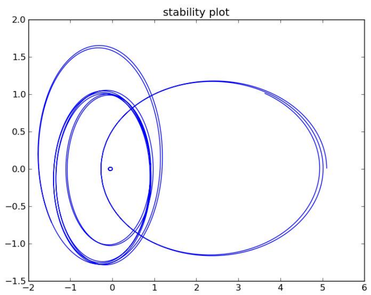
This can be explained analytically. From astronomy we know that the total energy in a two body system is given by

$$E = \frac{1}{2}\mu v^2 - \frac{\mu m}{r}, \text{ where } \mu = \frac{m_1 m_2}{m_1 + m_2}, \text{ and } m = G(m_1 + m_2) \tag{14}$$

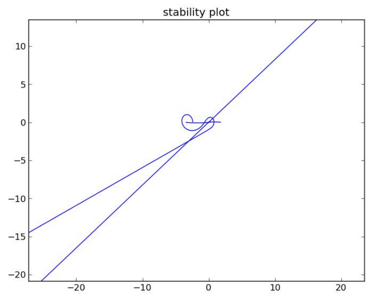
In this equation the energy is negative for bound systems, in the case $E = 0$ we get a parabolic trajectory, which means that the object will escape. For $E = 0$ we then get

$$v = \sqrt{\frac{2m}{r}} = 8,8848Auy^{-1}. \tag{15}$$

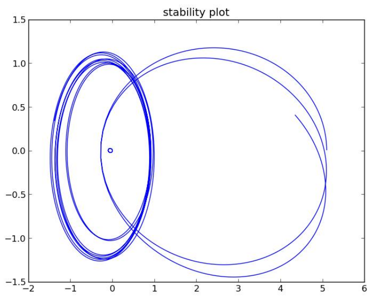
This result is very close to what i found by trial and error. We could in fact use this result to find out if any object is bound to the system or not, and how much increase in velocity it would need to escape.



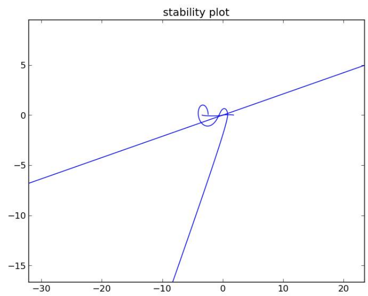
(a) rk4 solver for Jupiter mass with a factor 10



(b) rk4 solver for Jupiter mass with a factor 1000



(c) verlet solver for Jupiter mass with a factor 10



(d) verlet solver for Jupiter mass with a factor 1000

Figur 6: *test of solvers with dt = 0,01*

In this next plot we test the the different solvers with Jupiter's mass 10 times bigger and 1000 times bigger. We see that the solvers seem to be doing almost the same thing. Though, especially for mass = 10 times, we can see that the Verlet solver seems to be tilting the orbit of Jupiter allot. When i test if the total energy is conserved the Runge Kutta solver keeps the total energy conserved whilst the Verlet solver seems to be adding energy to the system for each step. I am not sure if this is because the solver is programmed wrong. But clearly it is not as good as Runge Kutta in my integrator.

Conclusion

I have not added all the planets in the system, but in theory this should be easy and it should not alter the results in this paper much since the Sun and Jupiter are the dominating objects in our solar system. What could be fun to do would be to add all the objects in the system, also the moons, and see what happens with the moons of Saturn. It was told in one lecture that it is impossible to predict the motion of the moons of Saturn, if this is true then a model should also yield different results every time. If not that means that it is not impossible, we just cant accurately determine the initial values, jet.

For future development of the system i would like to implement adaptive Runge Kutta methods. Especially for modeling moons around planets. In that case you would want a much smaller dt for the moons who have a wery small orbit in contrast to the planets. If we where to fully model the solar system with this static Runge Kutta algorithm, we would probably lose precision or have a really slow code.

Attachments

rk4 figure lent from:

$http://www.physics.drexel.edu/students/courses/Comp_Phys/Integrators/rk4.html$