Lehmann representation of the one- and two-particle Green's function

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1 Lehmann representation of the Green's function

1.1 One-particle Green's function

On the real axis we want to obtain the retarded Green's function

$$G^R_{ab}(t,t') = -i\theta(t-t')\left\langle\{c_a(t)c_b^\dagger(t')\}\right\rangle \tag{1.1}$$

$$= -i\theta(t-t')\Big(\left\langle c_a(t)c_b^\dagger(t')\right\rangle - \left\langle c_b^\dagger(t')c_a(t)\right\rangle.\Big) \tag{1.2}$$

To obtain the Lehman representation we evaluate the first term and evaluate the expectation value using the Eigenbasis $|n\rangle$ of the Hamiltonian H

$$G^>_{ab}(t,t') = -i\left\langle c_a(t)c_b^\dagger(t')\right\rangle \tag{1.3}$$

$$= \frac{-i}{Z}\sum_n \mathrm{e}^{-\beta E_n}\left\langle n\right| \mathrm{e}^{iHt}c_a\mathrm{e}^{-iHt}\mathrm{e}^{iHt'}c_b^\dagger\mathrm{e}^{-iHt'}\left|n\right\rangle \tag{1.4}$$

$$= \frac{-i}{Z}\sum_{n,m} \mathrm{e}^{-\beta E_n}\left\langle n\right| \mathrm{e}^{iHt}c_a\mathrm{e}^{-iHt}\left|m\right\rangle\left\langle m\right|\mathrm{e}^{iHt'}c_b^\dagger\mathrm{e}^{-iHt'}\left|n\right\rangle \tag{1.5}$$

$$= \frac{-i}{Z}\sum_{n,m} \mathrm{e}^{-\beta E_n}\mathrm{e}^{i(E_n-E_m)(t-t')}\left\langle n\right| c_a\left|m\right\rangle\left\langle m\right| c_b^\dagger\left|n\right\rangle. \tag{1.6}$$

In the same way we get for the second term

$$G^{<}_{ab}(t,t') = i\langle c^\dagger_b(t')c_a(t)\rangle \tag{1.7}$$

$$= \frac{-i}{Z}\sum_{n,m} \mathrm{e}^{-\beta E_n}\mathrm{e}^{i(E_n-E_m)(t'-t)}\langle n|\, c^\dagger_b\,|m\rangle\langle m|\, c_a\,|n\rangle \tag{1.8}$$

$$\stackrel{n\leftrightarrow m}{=} \frac{-i}{Z}\sum_{n,m} \mathrm{e}^{-\beta E_m}\mathrm{e}^{i(E_n-E_m)(t-t')}\langle n|\, c_a\,|m\rangle\langle m|\, c^\dagger_b\,|n\rangle\,. \tag{1.9}$$

The full retarded Green's function is then given by

$$G^R_{ab}(t-t') = \theta(t-t')\frac{-i}{Z}\sum_{n,m}\left(\mathrm{e}^{-\beta E_n}+\mathrm{e}^{-\beta E_m}\right)\mathrm{e}^{i(E_n-E_m)(t-t')}\langle n|\, c_a\,|m\rangle\langle m|\, c^\dagger_b\,|n\rangle\,. \tag{1.10}$$

A Fourier transform to frequency space yields

$$G_{ab}(\omega) = \int_{-\infty}^{\infty} G^R_{ab}(t)\mathrm{e}^{i(\omega+i\delta)t}\,\mathrm{d}t \tag{1.11}$$

$$= \frac{-i}{Z}\sum_{n,m}\left(\mathrm{e}^{-\beta E_n}+\mathrm{e}^{-\beta E_m}\right)\langle n|\, c_a\,|m\rangle\langle m|\, c^\dagger_b\,|n\rangle\int_0^{\infty}\mathrm{e}^{i(w+i\delta+E_n-E_m)t}\,\mathrm{d}t \tag{1.12}$$

$$= \frac{1}{Z}\sum_{n,m}\left(\mathrm{e}^{-\beta E_n}+\mathrm{e}^{-\beta E_m}\right)\frac{\langle n|\, c_a\,|m\rangle\langle m|\, c^\dagger_b\,|n\rangle}{w+i\delta+E_n-E_m}. \tag{1.13}$$

The Matsubara Green's function is obtained by substituting $\omega+i\delta\to i\omega_n$.

We see that for a given $|n\rangle$ with N_n electrons, the summation over m can be restricted to the subspace with $N_m = N_n+1$ electrons, Or, respectively, for a given $|m\rangle$ with N_m electrons, the summation over n can be restricted to the subspace with $N_n = N_m - 1$ electrons.

1.2 Two-particle Green's function

We consider the two-particle Green's function $G^{(2)}_{\uparrow\downarrow}$ as defined in the PhD Thesis of G.Rohringer

$$G^{(2)}_{\uparrow\downarrow}(\tau_1,\tau_2,\tau_3,\tau_4) = \langle Tc^\dagger_\uparrow(\tau_1)c_\uparrow(\tau_2)c^\dagger_\downarrow(\tau_3)c_\downarrow(\tau_4)\rangle\,. \tag{1.14}$$

The other spin combinations like $\uparrow\uparrow$ can by obtained from $G^{(2)}_{\uparrow\downarrow}$ by using symmetry relations.

Due to time-translational invariance we can set $\tau_4 = 0$, which leaves us with 6 distinct time orderings for τ_1, τ_2, τ_3

$$\begin{aligned}
G^{(2)}_{\uparrow\downarrow}(\tau_1,\tau_2,\tau_3,\tau_4) = \theta_{\tau_1>\tau_2>\tau_3} & \langle c^\dagger_\uparrow(\tau_1) c_\uparrow(\tau_2) c^\dagger_\downarrow(\tau_3) c_\downarrow(0)\rangle \\
- \theta_{\tau_1>\tau_3>\tau_2} & \langle c^\dagger_\uparrow(\tau_1) c^\dagger_\downarrow(\tau_3) c_\uparrow(\tau_2) c_\downarrow(0)\rangle \\
- \theta_{\tau_2>\tau_1>\tau_3} & \langle c_\uparrow(\tau_2) c^\dagger_\uparrow(\tau_1) c^\dagger_\downarrow(\tau_3) c_\downarrow(0)\rangle \\
+ \theta_{\tau_2>\tau_3>\tau_1} & \langle c_\uparrow(\tau_2) c^\dagger_\downarrow(\tau_3) c^\dagger_\uparrow(\tau_1) c_\downarrow(0)\rangle \\
+ \theta_{\tau_3>\tau_1>\tau_2} & \langle c^\dagger_\downarrow(\tau_3) c^\dagger_\uparrow(\tau_1) c_\uparrow(\tau_2) c_\downarrow(0)\rangle \\
- \theta_{\tau_3>\tau_2>\tau_1} & \langle c^\dagger_\downarrow(\tau_3) c_\uparrow(\tau_2) c^\dagger_\uparrow(\tau_1) c_\downarrow(0)\rangle
\end{aligned} \tag{1.15}$$

We now introduce the Eigenbasis $|m\rangle, |n\rangle, |o\rangle, |p\rangle$

$$\begin{aligned}
\sum_{mnop} \mathrm{e}^{-\beta E_m} \Big(& + \theta_{\tau_1>\tau_2>\tau_3} \langle m|c^\dagger_\uparrow(\tau_1)|n\rangle \langle n|c_\uparrow(\tau_2)|o\rangle \langle o|c^\dagger_\downarrow(\tau_3)|p\rangle \langle p|c_\downarrow(0)|m\rangle \\
& - \theta_{\tau_1>\tau_3>\tau_2} \langle m|c^\dagger_\uparrow(\tau_1)|n\rangle \langle n|c^\dagger_\downarrow(\tau_3)|o\rangle \langle o|c_\uparrow(\tau_2)|p\rangle \langle p|c_\downarrow(0)|m\rangle \\
& - \theta_{\tau_2>\tau_1>\tau_3} \langle m|c_\uparrow(\tau_2)|n\rangle \langle n|c^\dagger_\uparrow(\tau_1)|o\rangle \langle o|c^\dagger_\downarrow(\tau_3)|p\rangle \langle p|c_\downarrow(0)|m\rangle \\
& + \theta_{\tau_2>\tau_3>\tau_1} \langle m|c_\uparrow(\tau_2)|n\rangle \langle n|c^\dagger_\downarrow(\tau_3)|o\rangle \langle o|c^\dagger_\uparrow(\tau_1)|p\rangle \langle p|c_\downarrow(0)|m\rangle \\
& + \theta_{\tau_3>\tau_1>\tau_2} \langle m|c^\dagger_\downarrow(\tau_3)|n\rangle \langle n|c^\dagger_\uparrow(\tau_1)|o\rangle \langle o|c_\uparrow(\tau_2)|p\rangle \langle p|c_\downarrow(0)|m\rangle \\
& - \theta_{\tau_3>\tau_2>\tau_1} \langle m|c^\dagger_\downarrow(\tau_3)|n\rangle \langle n|c_\uparrow(\tau_2)|o\rangle \langle o|c^\dagger_\uparrow(\tau_1)|p\rangle \langle p|c_\downarrow(0)|m\rangle \Big)
\end{aligned} \tag{1.16}$$

$$\begin{aligned}
= \sum_{mnop} \mathrm{e}^{-\beta E_m} \Big(& + \theta_{\tau_1>\tau_2>\tau_3} \mathrm{e}^{\tau_1(E_m-E_n)} \mathrm{e}^{\tau_2(E_n-E_o)} \mathrm{e}^{\tau_3(E_o-E_p)} \langle m|c^\dagger_\uparrow|n\rangle \langle n|c_\uparrow|o\rangle \langle o|c^\dagger_\downarrow|p\rangle \langle p|c_\downarrow|m\rangle \\
& - \theta_{\tau_1>\tau_3>\tau_2} \mathrm{e}^{\tau_1(E_m-E_n)} \mathrm{e}^{\tau_3(E_n-E_o)} \mathrm{e}^{\tau_2(E_o-E_p)} \langle m|c^\dagger_\uparrow|n\rangle \langle n|c^\dagger_\downarrow|o\rangle \langle o|c_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle \\
& - \theta_{\tau_2>\tau_1>\tau_3} \mathrm{e}^{\tau_2(E_m-E_n)} \mathrm{e}^{\tau_1(E_n-E_o)} \mathrm{e}^{\tau_3(E_o-E_p)} \langle m|c_\uparrow|n\rangle \langle n|c^\dagger_\uparrow|o\rangle \langle o|c^\dagger_\downarrow|p\rangle \langle p|c_\downarrow|m\rangle \\
& + \theta_{\tau_2>\tau_3>\tau_1} \mathrm{e}^{\tau_2(E_m-E_n)} \mathrm{e}^{\tau_3(E_n-E_o)} \mathrm{e}^{\tau_1(E_o-E_p)} \langle m|c_\uparrow|n\rangle \langle n|c^\dagger_\downarrow|o\rangle \langle o|c^\dagger_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle \\
& + \theta_{\tau_3>\tau_1>\tau_2} \mathrm{e}^{\tau_3(E_m-E_n)} \mathrm{e}^{\tau_1(E_n-E_o)} \mathrm{e}^{\tau_2(E_o-E_p)} \langle m|c^\dagger_\downarrow|n\rangle \langle n|c^\dagger_\uparrow|o\rangle \langle o|c_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle \\
& - \theta_{\tau_3>\tau_2>\tau_1} \mathrm{e}^{\tau_3(E_m-E_n)} \mathrm{e}^{\tau_2(E_n-E_o)} \mathrm{e}^{\tau_1(E_o-E_p)} \langle m|c^\dagger_\downarrow|n\rangle \langle n|c_\uparrow|o\rangle \langle o|c^\dagger_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle \Big).
\end{aligned} \tag{1.17}$$

We now perform a Fourier transform $\tau_i \to i\omega_i$. The **first** term evaluates to

$$\int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{i\omega_3\tau_3} \int_0^\beta \mathrm{d}\tau_2 \mathrm{e}^{i\omega_2\tau_2} \int_0^\beta \mathrm{d}\tau_1 \mathrm{e}^{i\omega_1\tau_1} \Big(\mathrm{e}^{-\beta E_m} \theta_{\tau_1>\tau_2>\tau_3} \mathrm{e}^{\tau_1(E_m-E_n)} \mathrm{e}^{\tau_2(E_n-E_o)} \mathrm{e}^{\tau_3(E_o-E_p)} \Big) \tag{1.18}$$

$$= \mathrm{e}^{-\beta E_m} \int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_o-E_p)} \int_{\tau_3}^\beta \mathrm{d}\tau_2 \mathrm{e}^{\tau_2(i\omega_2+E_n-E_o)} \int_{\tau_2}^\beta \mathrm{d}\tau_1 \mathrm{e}^{\tau_1(i\omega_1+E_m-E_n)} \tag{1.19}$$

$$= \mathrm{e}^{-\beta E_m} \int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_o-E_p)} \int_{\tau_3}^\beta \mathrm{d}\tau_2 \mathrm{e}^{\tau_2(i\omega_2+E_n-E_o)} \frac{-\mathrm{e}^{\beta(E_m-E_n)} - \mathrm{e}^{\tau_2(i\omega_1+E_m-E_n)}}{i\omega_1+E_m-E_n} \tag{1.20}$$

$$= \frac{-1}{i\omega_1+E_m-E_n} \int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_o-E_p)} \int_{\tau_3}^\beta \mathrm{d}\tau_2 \mathrm{e}^{\tau_2(i\omega_2+E_n-E_o)} \Big(\mathrm{e}^{-\beta E_n} + \mathrm{e}^{\tau_2(i\omega_1+E_m-E_n)} \mathrm{e}^{-\beta E_m} \Big) \tag{1.21}$$

$$\begin{aligned} = \frac{-1}{i\omega_1+E_m-E_n} \int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_o-E_p)} \Big(&\mathrm{e}^{-\beta E_n} \frac{-\mathrm{e}^{\beta(E_n-E_o)} - \mathrm{e}^{\tau_3(i\omega_2+E_n-E_o)}}{i\omega_2+E_n-E_o} \\ &+ \mathrm{e}^{-\beta E_m} \frac{+\mathrm{e}^{\beta(E_m-E_o)} - \mathrm{e}^{\tau_3(i\omega_1+i\omega_2+E_m-E_o)}}{i\omega_1+i\omega_2+E_m-E_o} \Big) \end{aligned} \tag{1.22}$$

$$\begin{aligned} = \frac{1}{i\omega_1+E_m-E_n} \int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_o-E_p)} \Big(&\frac{\mathrm{e}^{-\beta E_o} + \mathrm{e}^{-\beta E_n} \mathrm{e}^{\tau_3(i\omega_2+E_n-E_o)}}{i\omega_2+E_n-E_o} \\ &- \frac{\mathrm{e}^{-\beta E_o} - \mathrm{e}^{-\beta E_m} \mathrm{e}^{\tau_3(i\omega_1+i\omega_2+E_m-E_o)}}{i\omega_1+i\omega_2+E_m-E_o} \Big) \end{aligned} \tag{1.23}$$

$$\begin{aligned} = \frac{1}{i\omega_1+E_m-E_n} \Big(&\frac{\int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_o-E_p)}}{i\omega_2+E_n-E_o} \Big[\mathrm{e}^{-\beta E_o} + \mathrm{e}^{-\beta E_n} \mathrm{e}^{\tau_3(i\omega_2+E_n-E_o)} \Big] \\ &- \frac{\int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_o-E_p)}}{i\omega_1+i\omega_2+E_m-E_o} \Big[\mathrm{e}^{-\beta E_o} - \mathrm{e}^{-\beta E_m} \mathrm{e}^{\tau_3(i\omega_1+i\omega_2+E_m-E_o)} \Big] \Big) \end{aligned} \tag{1.24}$$

$$\begin{aligned} = \frac{1}{i\omega_1+E_m-E_n} \Big(&\frac{1}{i\omega_2+E_n-E_o} \Big[\mathrm{e}^{-\beta E_o} \frac{-\mathrm{e}^{\beta(E_o-E_p)} - 1}{i\omega_3+E_o-E_p} + \mathrm{e}^{-\beta E_n} \frac{+\mathrm{e}^{\beta(E_n-E_p)} - 1}{i\omega_2+i\omega_3+E_n-E_p} \Big] \\ &- \frac{1}{i\omega_1+i\omega_2+E_m-E_o} \Big[\mathrm{e}^{-\beta E_o} \frac{-\mathrm{e}^{\beta(E_o-E_p)} - 1}{i\omega_3+E_o-E_p} - \mathrm{e}^{-\beta E_m} \frac{-\mathrm{e}^{\beta(E_m-E_p)} - 1}{i\omega_1+i\omega_2+i\omega_3+E_m-E_p} \Big] \Big) \end{aligned} \tag{1.25}$$

$$\begin{aligned} = \frac{1}{i\omega_1+E_m-E_n} \Big(&\frac{1}{i\omega_2+E_n-E_o} \Big[\frac{-\mathrm{e}^{-\beta E_p} - \mathrm{e}^{-\beta E_o}}{i\omega_3+E_o-E_p} + \frac{\mathrm{e}^{-\beta E_p} - \mathrm{e}^{-\beta E_n}}{i\omega_2+i\omega_3+E_n-E_p} \Big] \\ &- \frac{1}{i\omega_1+i\omega_2+E_m-E_o} \Big[\frac{-\mathrm{e}^{-\beta E_p} - \mathrm{e}^{-\beta E_o}}{i\omega_3+E_o-E_p} - \frac{-\mathrm{e}^{-\beta E_p} - \mathrm{e}^{-\beta E_m}}{i\omega_1+i\omega_2+i\omega_3+E_m-E_p} \Big] \Big) \end{aligned} \tag{1.26}$$

$$\begin{aligned} = \frac{-1}{i\omega_1+E_m-E_n} \Big(&\frac{1}{i\omega_2+E_n-E_o} \Big[\frac{\mathrm{e}^{-\beta E_o} + \mathrm{e}^{-\beta E_p}}{i\omega_3+E_o-E_p} + \frac{\mathrm{e}^{-\beta E_n} - \mathrm{e}^{-\beta E_p}}{i\omega_2+i\omega_3+E_n-E_p} \Big] \\ &- \frac{1}{i\omega_1+i\omega_2+E_m-E_o} \Big[\frac{\mathrm{e}^{-\beta E_o} + \mathrm{e}^{-\beta E_p}}{i\omega_3+E_o-E_p} - \frac{\mathrm{e}^{-\beta E_m} + \mathrm{e}^{-\beta E_p}}{i\omega_1+i\omega_2+i\omega_3+E_m-E_p} \Big] \Big). \end{aligned} \tag{1.27}$$

The **second** term evaluates to

$$-\int_0^\beta \mathrm{d}\tau_2 \mathrm{e}^{i\omega_2\tau_2}\int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{i\omega_3\tau_3}\int_0^\beta \mathrm{d}\tau_1 \mathrm{e}^{i\omega_1\tau_1}\Big(\mathrm{e}^{-\beta E_m}\theta_{\tau_1>\tau_3>\tau_2}\mathrm{e}^{\tau_1(E_m-E_n)}\mathrm{e}^{\tau_3(E_n-E_o)}\mathrm{e}^{\tau_2(E_o-E_p)}\Big) \tag{1.28}$$

$$= -\mathrm{e}^{-\beta E_m}\int_0^\beta \mathrm{d}\tau_2 \mathrm{e}^{\tau_2(i\omega_2+E_o-E_p)}\int_0^\beta \mathrm{d}\tau_3 \mathrm{e}^{\tau_3(i\omega_3+E_n-E_o)}\int_0^\beta \mathrm{d}\tau_1 \mathrm{e}^{\tau_1(i\omega_1+E_m-E_n)}, \tag{1.29}$$

which we see is precisely the same expression as the first term, except we have to swap $i\omega_2 \leftrightarrow i\omega_3$, and flip the overall sign. Therefore, we get $f(\omega_1,\omega_3,\omega_2)$, where we have defined

$$f(\omega_1,\omega_2,\omega_3) = \frac{1}{i\omega_1+E_m-E_n}\Big(\frac{1}{i\omega_2+E_n-E_o}\Big[\frac{\mathrm{e}^{-\beta E_o}+\mathrm{e}^{-\beta E_p}}{i\omega_3+E_o-E_p}+\frac{\mathrm{e}^{-\beta E_n}-\mathrm{e}^{-\beta E_p}}{i\omega_2+i\omega_3+E_n-E_p}\Big] \\ -\frac{1}{i\omega_1+i\omega_2+E_m-E_o}\Big[\frac{\mathrm{e}^{-\beta E_o}+\mathrm{e}^{-\beta E_p}}{i\omega_3+E_o-E_p}-\frac{\mathrm{e}^{-\beta E_m}+\mathrm{e}^{-\beta E_p}}{i\omega_1+i\omega_2+i\omega_3+E_m-E_p}\Big]\Big). \tag{1.30}$$

For the **third** term we have to swap $i\omega_1 \leftrightarrow i\omega_2$ compared to the first term and obtain $f(\omega_2,\omega_1,\omega_3)$

For the **fourth** term we have to swap $i\omega_1 \leftrightarrow i\omega_3$ compared to the third term and obtain $-f(\omega_2,\omega_3,\omega_1)$

For the **fifth** term we have to swap $i\omega_2 \leftrightarrow i\omega_3$ compared to the third term and obtain $-f(\omega_3,\omega_1,\omega_2)$

For the **sixth** term we have to swap $i\omega_1 \leftrightarrow i\omega_2$ compared to the fifth term and obtain $f(\omega_3,\omega_2,\omega_1)$

Collecting all terms we obtain the final expression for the two-particle Green's function

$$\begin{aligned} G^{(2)}_{\uparrow\downarrow}(\omega_1,\omega_2,\omega_3) = \sum_{mnop}\Big(&-\langle m|c^\dagger_\uparrow|n\rangle\langle n|c_\uparrow|o\rangle\langle o|c^\dagger_\downarrow|p\rangle\langle p|c_\downarrow|m\rangle f(\omega_1,\omega_2,\omega_3) \\ &+\langle m|c^\dagger_\uparrow|n\rangle\langle n|c^\dagger_\downarrow|o\rangle\langle o|c_\uparrow|p\rangle\langle p|c_\downarrow|m\rangle f(\omega_1,\omega_3,\omega_2) \\ &+\langle m|c_\uparrow|n\rangle\langle n|c^\dagger_\uparrow|o\rangle\langle o|c^\dagger_\downarrow|p\rangle\langle p|c_\downarrow|m\rangle f(\omega_2,\omega_1,\omega_3) \\ &-\langle m|c_\uparrow|n\rangle\langle n|c^\dagger_\downarrow|o\rangle\langle o|c^\dagger_\uparrow|p\rangle\langle p|c_\downarrow|m\rangle f(\omega_2,\omega_3,\omega_1) \\ &-\langle m|c^\dagger_\downarrow|n\rangle\langle n|c^\dagger_\uparrow|o\rangle\langle o|c_\uparrow|p\rangle\langle p|c_\downarrow|m\rangle f(\omega_3,\omega_1,\omega_2) \\ &+\langle m|c^\dagger_\downarrow|n\rangle\langle n|c_\uparrow|o\rangle\langle o|c^\dagger_\uparrow|p\rangle\langle p|c_\downarrow|m\rangle f(\omega_3,\omega_2,\omega_1) \end{aligned} \tag{1.31}$$

When analyzing the overlap elements we see that each term corresponds to the following transitions: Starting with $|m\rangle$ being a state $m(N,S)$ with electron number

N and total spin S we get by going from right to left

$$
\begin{aligned}
&(1): \ m(N,S) \to p(N-1,S+1) \to o(N,S) &&\to n(N-1,S-1) \to m(N,S) \\
&(2): \ m(N,S) \to p(N-1,S+1) \to o(N-2,S) &&\to n(N-1,S-1) \to m(N,S) \\
&(3): \ m(N,S) \to p(N-1,S+1) \to o(N,S) &&\to n(N+1,S+1) \to m(N,S) \\
&(4): \ m(N,S) \to p(N-1,S+1) \to o(N,S+2) &&\to n(N+1,S+1) \to m(N,S) \\
&(5): \ m(N,S) \to p(N-1,S+1) \to o(N-2,S) &&\to n(N-1,S+1) \to m(N,S) \\
&(6): \ m(N,S) \to p(N-1,S+1) \to o(N,S+2) &&\to n(N-1,S+1) \to m(N,S)
\end{aligned}
\tag{1.32}
$$

So we rearrange the terms grouping them by similar transitions in order to reuse results as much as possible in the code

$$
\begin{aligned}
&(1): \ m(N,S) \to p(N-1,S+1) \to o(N,S) &&\to n(N-1,S-1) \to m(N,S) \\
&(3): \ m(N,S) \to p(N-1,S+1) \to o(N,S) &&\to n(N+1,S+1) \to m(N,S) \\
&(4): \ m(N,S) \to p(N-1,S+1) \to o(N,S+2) &&\to n(N+1,S+1) \to m(N,S) \\
&(6): \ m(N,S) \to p(N-1,S+1) \to o(N,S+2) &&\to n(N-1,S+1) \to m(N,S) \\
&(2): \ m(N,S) \to p(N-1,S+1) \to o(N-2,S) &&\to n(N-1,S-1) \to m(N,S) \\
&(5): \ m(N,S) \to p(N-1,S+1) \to o(N-2,S) &&\to n(N-1,S+1) \to m(N,S)
\end{aligned}
\tag{1.33}
$$

which results in the final expression to be evaluated

$$
\begin{aligned}
G^{(2)}_{\uparrow\downarrow}(\omega_1,\omega_2,\omega_3) = \sum_{mnop} \Big(&- \langle m|c^\dagger_\uparrow|n\rangle \langle n|c_\uparrow|o\rangle \langle o|c^\dagger_\downarrow|p\rangle \langle p|c_\downarrow|m\rangle f(\omega_1,\omega_2,\omega_3) \\
&+ \langle m|c_\uparrow|n\rangle \langle n|c^\dagger_\uparrow|o\rangle \langle o|c^\dagger_\downarrow|p\rangle \langle p|c_\downarrow|m\rangle f(\omega_2,\omega_1,\omega_3) \\
&- \langle m|c_\uparrow|n\rangle \langle n|c^\dagger_\downarrow|o\rangle \langle o|c^\dagger_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle f(\omega_2,\omega_3,\omega_1) \\
&+ \langle m|c^\dagger_\downarrow|n\rangle \langle n|c_\uparrow|o\rangle \langle o|c^\dagger_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle f(\omega_3,\omega_2,\omega_1) \\
&+ \langle m|c^\dagger_\uparrow|n\rangle \langle n|c^\dagger_\downarrow|o\rangle \langle o|c_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle f(\omega_1,\omega_3,\omega_2) \\
&- \langle m|c^\dagger_\downarrow|n\rangle \langle n|c^\dagger_\uparrow|o\rangle \langle o|c_\uparrow|p\rangle \langle p|c_\downarrow|m\rangle f(\omega_3,\omega_1,\omega_2) \Big).
\end{aligned}
\tag{1.34}
$$