The Fourier transform and the FFT algorithm

Steffen Haug

The Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} dt \tag{1}$$

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Analytic evaluation generally not feasible.

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Most engineering applications use its discrete counterpart – the DFT:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i k n}{N}} \tag{2}$$

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We want a way to compute this *efficiently*.

Preliminaries

Eulers formula and the roots of unity

Eulers identity

$$e^{i\pi} = -1 \tag{3}$$

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Just a special case of $e^{i\vartheta}$ when $\vartheta = \pi$.

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta \tag{4}$$

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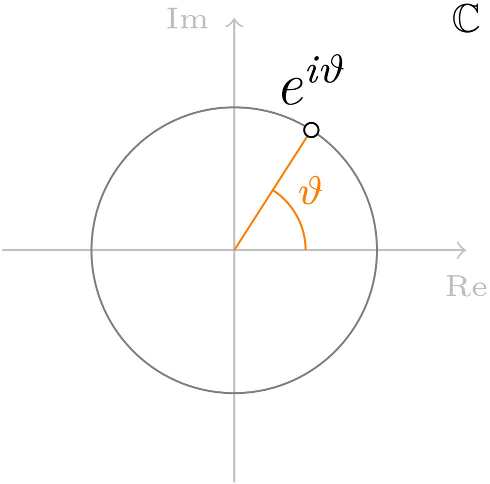

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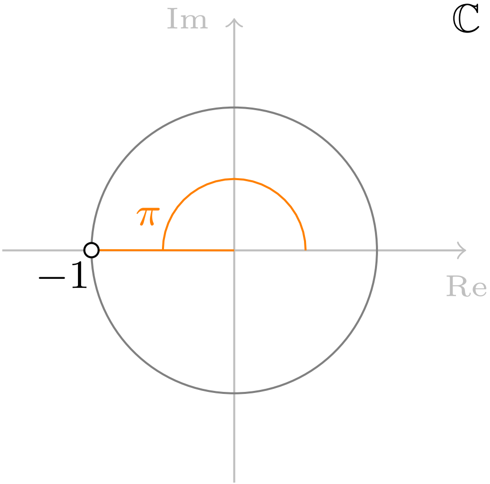

The roots of unity

$$z^N = 1$$

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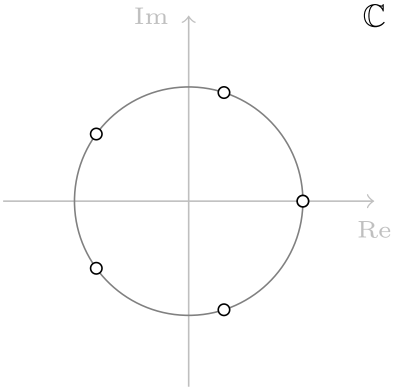
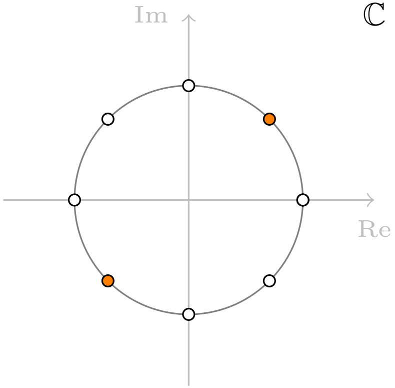


SOLUTIONS: $e^{\frac{2\pi i n}{N}}$ for n in $0, \ldots, N-1$.

Important point



When N is even, every root has a “friend” with the opposite sign.

The key idea

Decomposing the DFT

Simplifying assumption

N is a power of two

Let $W = e^{\frac{-2\pi i}{N}}$ (notation from Cooley-Tukey).
We can restate (4) as

$$X_k = \sum_n x_n W^{nk} \tag{5}$$

Gaining speed

Reusing intermediate results

References

- **Cooley & Tukey**: .....