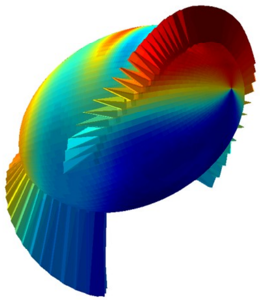
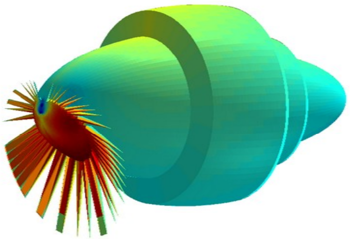
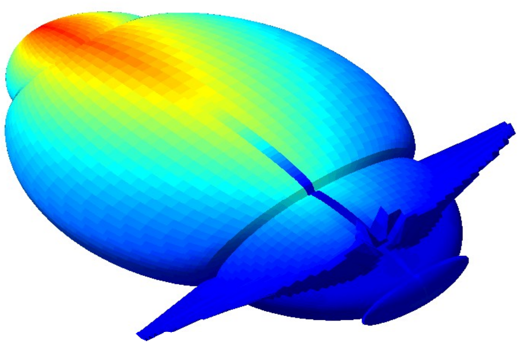
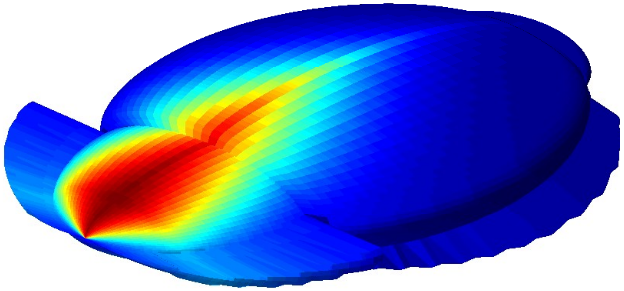




Lecture 12
Associative Memory

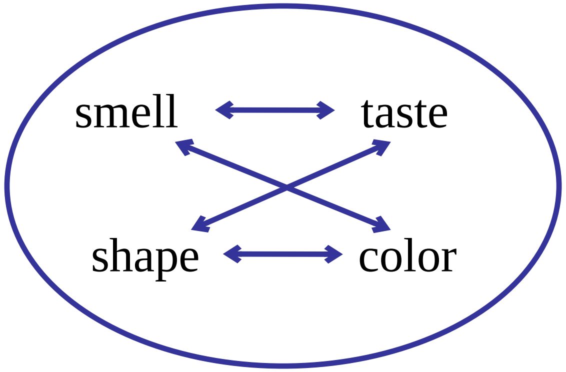




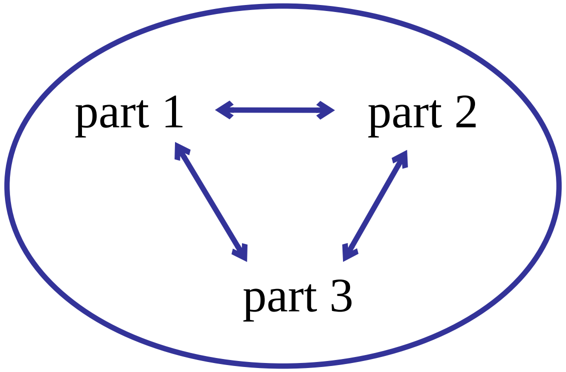
Associative Memory

A memory which stores items by ***association***.

e.g. names ⟷ faces



= apple



= object

The idea is that the entire memory can be recovered given a part of it,

e.g. a name recalls a face

an apple's shape and color recall its taste and smell

We will consider memories to be represented by patterns of activity over a group of neurons.

Associative Memory

- An associative memory neural network stores patterns of activity
(e.g. binary vectors) by association.

  - When part of the stored memory is given as input, the network
recovers (recalls) the rest.

For this reason, associative memories are also called
content-addressable memories

i.e. memories which retrieve stored data based on part of their content
rather than on the address of the storage location.

Indeed, in neural associative memories, there is *no specific storage location*.

↳ "Holographic" memories

Two Types of Associative Memories

Hetero-associative:

pattern 1 → pattern 2 (feed forward)

or

pattern 1 ↔ pattern 2 (bidirectional)

used for recall of information associated with the input pattern, or for recoding.

Auto-associative:

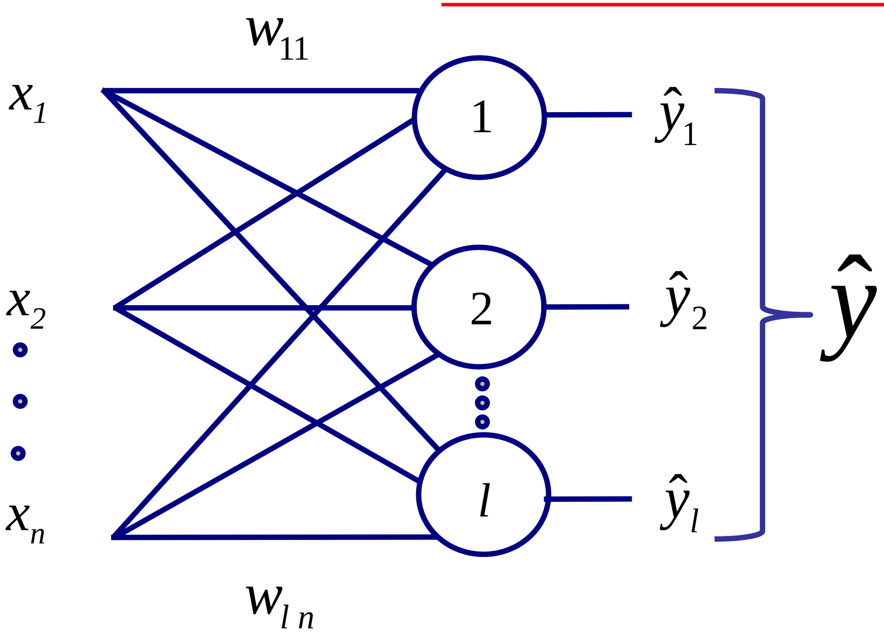
pattern

An incomplete or noisy version of the pattern recalls the correct stored pattern

used for

- pattern completion
- noise-suppression
- optimization

The Correlation Matrix Memory



neuron equation:

$$\boxed{\hat{y}_i = \operatorname{sgn}\left(\sum_{j=1}^{n} w_{ij} x_j + \theta_i \right)}$$

weight equation:

$$\boxed{w_{ij} = \frac{1}{n} \sum_{q=1}^{N} x_j^q y_i^q}$$

Stimulus patterns

$$\left. \begin{matrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{matrix} \right\} \in \{-1,1\}^n$$

Response patterns

$$\longrightarrow \left. \begin{matrix} y^1 \\ y^2 \\ \vdots \\ y^N \end{matrix} \right\} \in \{-1,1\}^l$$

$$W = [w_{ij}] = \frac{1}{n} \sum_{q=1}^{N} x^q y^{q^T}$$

correlation matrix

sum of outer products

We will take $\theta_i = 0 \quad \forall i$

Suppose pattern x^r is presented to the network

net input to neuron $i \equiv s_i = \sum_{j=1}^{n} w_{ij} x_j^r$

$$\therefore s_i = \sum_{j=1}^{n} \frac{1}{n} \sum_{q=1}^{N} x_j^q y_i^q x_j^r$$

$$= \frac{1}{n} y_i^r \sum_{j=1}^{n} x_j^r x_j^r + \frac{1}{n} \sum_{q \neq r} y_i^q \sum_{j=1}^{n} x_j^q x_j^r$$

signal term noise term = C

since $x_j^r x_j^r = 1$

$$s_i = y_i^r + C$$

Now $\hat{y}_i = \mathrm{sgn}(s_i) = \mathrm{sgn}(y_i^r + C)$

$\therefore$ if $\mathrm{sgn}(y_i^r + C) = y_i^r \rightarrow \hat{y}_i = y_i^r$

i.e. C does not change the sign of s_i

i.e. no error in $\hat{y}_i$

Condition for error

$$C > 1 \quad \text{if} \quad y_i^r = -1$$
$$C < -1 \quad \text{if} \quad y_i^r = 1$$

or

$$-y_i^r C > 1 \quad \text{for} \quad y_i^r = \pm 1$$

i.e., # matched bits = # mismatched bits

If the x^q are orthogonal, $\sum_j x_j^q x_j^r = 0 \quad \forall q \neq r$

i.e., input is actually x^r

and there are no errors:

$$y_i = \text{sgn}\left(y_i^r\right) = y_i^r$$

If x^q are not orthogonal, recall is still correct if $-y_i^r C$ is not too large (>1)

Using only orthogonal x^q would be too restrictive, so it is good that the system allows us some latitude.

What if the input has some errors?

Suppose input $= \hat{x}^r$ which differs from x^r in m bits

i.e. $$\sum_{j=1}^{n} \hat{x}_j^r x_j^r \equiv n' = (n-m) - m = n - 2m$$

(*n*-*m* bits match, giving 1s
m bits don't, giving -1s)

$$-n \le n' \le n$$

Define $\frac{n'}{n} = \phi$ (called the direction cosine)

Repeating the earlier analysis

$$s_i = \frac{1}{n} y_i^r \sum_{j=1}^{n} \hat{x}_j^r x_j^r + \frac{1}{n} \sum_{q \neq r} y_i^q \sum_{j=1}^{n} \hat{x}_j^r x_j^q$$
$$= \frac{1}{n} y_i^r \sum_{j=1}^{n} \hat{x}_j^r x_j^r + C' = \phi\, y_i^r + C'$$

i.e. $$s_i = \left[\frac{\#\text{correct bits} - \#\text{incorrect bits}}{\#\text{total bits}} \right] y_i^r + C'$$

Error condition:

$$\operatorname{sgn}\left(\phi\, y_i^r + C'\right) \neq y_i^r$$

if $\quad C' > \phi$ and $y_i^r = -1$

or $\quad C' < -\phi$ and $y_i^r = 1$

which, as before gives the general error condition

$$-\, y_i^r C' > \phi \quad \Rightarrow \quad \text{error in } y_i$$

Thus, if $-\, y_i^r C' < \phi$, y_i is still recalled correctly

> The network has the ability to correctly recall the response pattern even if the stimulus patter is somewhat corrupted.

As # stimulus error bits increase, ϕ decreases, and eventually recall is lost

What if some of the weights are missing?

Suppose the weight vector for neuron i, $w_i = [w_{i1}\ w_{i2} \ldots w_{in}]$ is corrupted so that n' of the weights become 0

Denote the corrupted weight vector as $\hat{w}_i = [\hat{w}_{i1}\ \hat{w}_{i2} \ldots \hat{w}_{in}]$

Repeating the earlier analysis

$$s_i = \sum_{j=1}^{n} \hat{w}_{ij} x_j^r = \sum_{j=1}^{n} w_{ij} x_j^r - \sum_{\text{bad } k} w_{ik} x_k^r$$

$$= \frac{1}{n} y_i^r \sum_{j=1}^{n} x_j^r x_j^r - \frac{1}{n} y_i^r \sum_{k} x_k^r x_k^r + \underbrace{\frac{1}{n} \sum_{q \neq r} y_i^q \sum_{j=1}^{n} x_j^r x_j^q - \frac{1}{n} \sum_{q \neq r} y_i^q \sum_{k} x_k^r x_k^q}_{C'}$$

$$= y_i^r - \frac{n'}{n} y_i^r + C' = \frac{n - n'}{n} y_i^r + C'$$

i.e. $s_i = \left[\dfrac{\#\,\text{total bits} - \#\,\text{bits with missing weights}}{\#\,\text{total bits}} \right] y_i^r + C'$

Error condition:

$$\mathrm{sgn}\left(\frac{n-n'}{n}y_i^r + C'\right) \neq y_i^r$$

$$\text{if} \quad C' > \frac{n-n'}{n} \text{ and } y_i^r = -1$$

$$\text{or} \quad C' < -\frac{n-n'}{n} \text{ and } y_i^r = 1$$

which, as before gives the general error condition

$$-y_i^r C' > \frac{n-n'}{n} \quad \Rightarrow \quad \text{error in } y_i$$

Thus, if $-y_i^r C' < \dfrac{n-n'}{n}$, y_i is still recalled correctly

The network has the ability to correctly recall the response pattern even if some of the weights are missing

As # missing weights increase, $n - n' \rightarrow 0$ and eventually recall is lost

A similar argument can be made if weights are corrupted by noise, some inputs are set to 0, etc.

Conclusion:

This associative memory is ***very robust***, and can produce correct recall even even when some inputs or weights are incorrect or missing.

Some comments on the weight equation

it is not strictly necessary to use $\frac{1}{n}$ in the learning equation

The necessary condition is: $w_{ij} \propto \sum_{q=1}^{N} y_i^q x_j^q$

Thus, the weights can be determined by learning as follows:

$\boxed{1}$ set $w_{ij} = 0 \quad \forall\ i, j$

$\boxed{2}$ For $q = 1$ to N

$$w_{ij} = w_{ij} + \eta y_i^q x_j^q$$

The rule

$$\Delta w_{ij} \propto y_i^q x_j^q$$

is the Hebb Rule

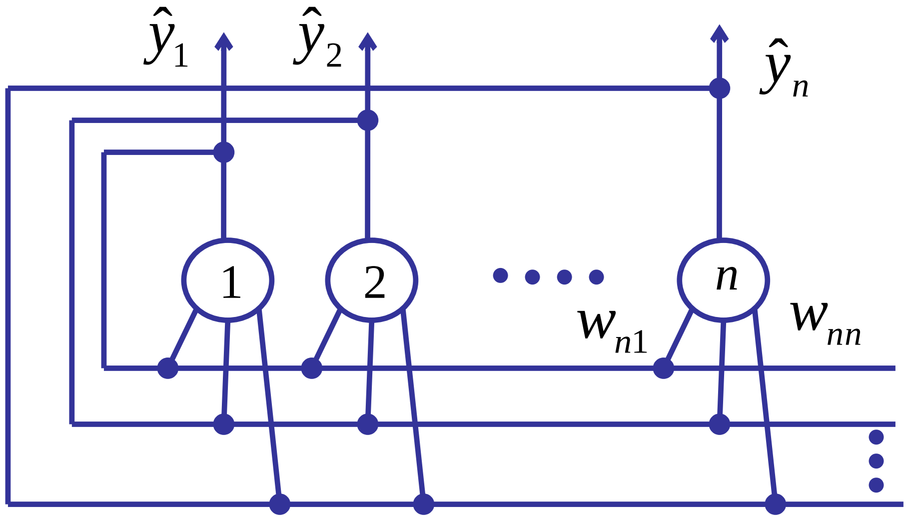
if $\eta = \frac{1}{n}$, we get the weight equation given earlier

In general, the Hebb Rule for associative learning states:

- if the activity of i and j is correlated $\rightarrow$ increase w_{ij}
- if the activity of i and j is anti-correlated $\rightarrow$ decrease w_{ij}
- if i and j are uncorrelated $\rightarrow$ $w_{ij} = 0$

Many variants of Hebbian Learning exist in the literature, especially for the case of 0/1 neurons, which are biologically more plausible.

Recurrent Autoassociative Memory



stored patterns : $\left\{ y^q = \left[y_1^q \ y_2^q \ \cdots y_n^q \right]^T \right\}_{q=1}^{N}$

$$\hat{y}_i(t+1) = \operatorname{sgn}\Bigg(\underbrace{\sum_{j=1}^{n} w_{ij} \hat{y}_j(t) + \theta_i}_{s_i(t+1)} \Bigg)$$

same as the
correlation matrix memory

$$w_{ij} = \frac{1}{n} \sum_{q=1}^{N} y_i^q y_j^q \qquad \text{(Hebb Rule)}$$

The weights associate each y^q with itself.

If the system is started with $\hat{y}(0) \approx y^q$ it recovers y^q over a few iterations

→ Attractor Network

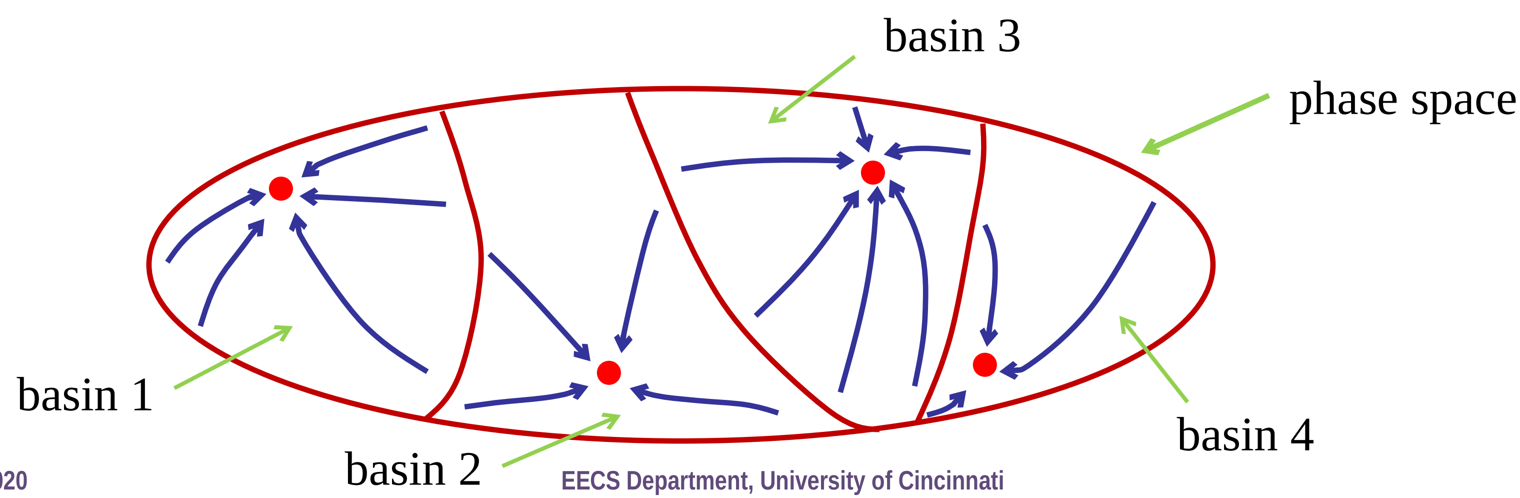
Invariant set: A set ϑ in phase space such that if the system is in ϑ at time t_0 , it remains in ϑ for all time $t > t_0$.

e.g. an equilibrium point.

Attractor: An invariant set in phase space towards which a dynamical system is attracted from certain initial conditions.

e.g. a stable fixed point or stable limit cycle.

Basin of attraction: The basin of attraction for an attractor, *a*, is the set of all initial conditions from which a system is attracted to *a*.



Recurrent associative memories learn by embedding the desired memory patterns as attractors of the network dynamics.

Given an initial pattern, $\hat{y}(0)$, the network relaxes to the attractor whose basin of attraction includes $\hat{y}(0)$.

→ pattern completion
pattern recall

Hazards

- $\hat{y}(0)$ may not be in the desired BOA.

- The system may have unwanted attractors (spurious memories)

Network Dynamics

Synchronous Updating

$$\hat{y} = \left[\hat{y}_1 \; \hat{y}_2 \; \cdots \hat{y}_n\right]^T$$

$$\hat{y}(t+1) = \text{sgn}\left(W\hat{y}(t)\right)$$

i.e. all $\hat{y}_i$ are updated in parallel at the same time

$$\hat{y}_i(t+1) = \text{sgn}\left(w_i^T \hat{y}(t)\right)$$

Asynchronous updating

Bits are updated one at a time, in random order or sequentially.

→ we no longer have strict synchronous time indexed by *t*.

In a recurrent network with $w_{ij} = w_{ji} \quad \forall i, j$ (symmetric network)

- Synchronous update → fixed point or 2-cycle
- Asynchronous update → fixed point

Discrete-Time Hopfield Network

$$w_{ij} = \begin{cases} \dfrac{1}{n}\sum_{q=1}^{N} y_i^q y_j^q & i \neq j \\ 0 & i = j \end{cases} \longrightarrow w_{ij} = w_{ji} \quad \forall i, j$$

or, defining

$$w = \begin{bmatrix} w_{11} & w_{12} & \cdot & \cdot & \cdot & w_{1n} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ w_{n1} & \cdot & \cdot & \cdot & \cdot & w_{nn} \end{bmatrix} = \begin{bmatrix} w_1^T \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_n^T \end{bmatrix}$$

$$w = \frac{1}{n}\left[\sum_{q=1}^{N} y^q y^{q^T}\right] - \frac{N}{n} I \longleftarrow$$

n x n
Identity matrix

$$\hat{y}(t+1) = \operatorname{sgn}\left(w_i^T \hat{y}(t)\right)$$

Are the stored memories attractors?

Case I: Only one stored pattern y^1

$$w_{ij} = \frac{1}{n} y_i^1 y_j^1$$

$$s_i(t+1) = \sum_j w_{ij} \hat{y}_j(t)$$

$$= \frac{1}{n} \sum_j y_i^1 y_j^1 \hat{y}_j(t) = \frac{1}{n} y_i^1 \sum_j y_j^1 \hat{y}_j(t)$$

$$= y_i^1 \, \frac{1}{n} \left[\begin{array}{c} \#\, \hat{y}(t) \\ \text{bits matching } y^1 \end{array} - \begin{array}{c} \#\, \hat{y}(t) \\ \text{bits not matching } y^1 \end{array} \right]$$

$\rightarrow$ if the majority of $\hat{y}(t)$ bits match y^1 bits, $\operatorname{sgn}(s_i) = y_i^1$ $\rightarrow$ recall of y^1

$\therefore$ if $\hat{y}(0)$ matches y^1 in > $\frac{n}{2}$ bits then $\hat{y}(1) = y^1$ $\longrightarrow$ recall

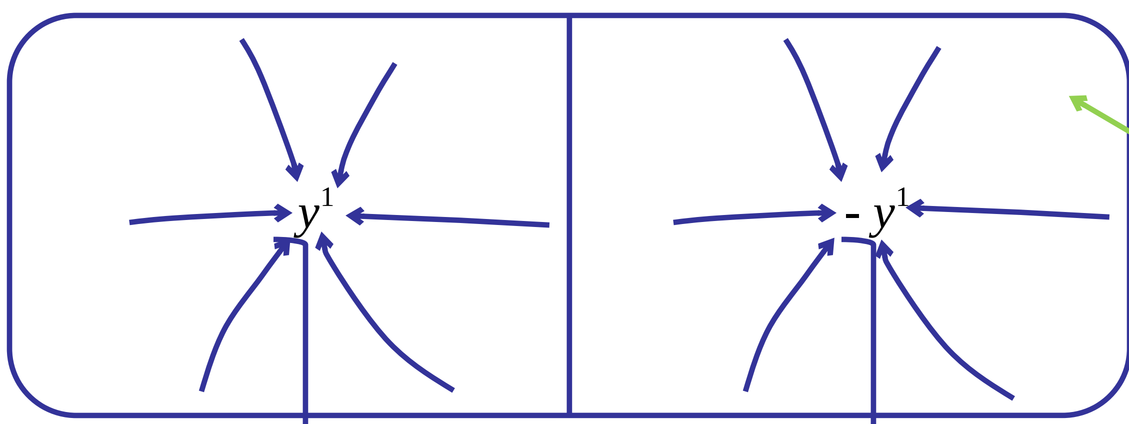
Are the stored memories the *only* attractors?

What happens if $\hat{y}(0)$ matches y^1 in $< \dfrac{n}{2}$ bits?

In that case $\hat{y}_i(1) = - y_i^1 \quad \forall i$

and $- y_i^1$ is also stable

spurious memory



phase space

region of correct recall

region of incorrect recall

Case II: Multiple Patterns

$$w_{ij} = \frac{1}{n} \sum_{q=1}^{N} y_i^q y_j^q$$

Let us examine the recall of a particular pattern, y^r

Let $$\frac{1}{n} \sum_{i=1}^{n} \hat{y}_i(t) y_i^r = \phi^r(t)$$

$$s_i(t+1) = \sum_j w_{ij} \hat{y}_j(t)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \sum_{q=1}^{N} y_i^q y_j^q \hat{y}_j(t)$$

$$= \frac{1}{n} y_i^r \sum_j y_j^r \hat{y}_j(t) + \frac{1}{n} \sum_{q \neq r} y_i^q \sum_j y_j^q \hat{y}_j(t)$$

$$= \phi^r(t) y_i^r + C_i(t)$$

$$\text{if } -y_i^r C_i(t) < \phi^r(t) \qquad \text{(i.e. if } -y_i^r C_i(t) \text{ is "small enough")}$$

$$\text{sgn}(s_i(t+1)) = y_i^r \quad \Rightarrow \hat{y}_i(t+1) = y_i^r$$

if $-y_i^r C_i(t)$ is "small enough" for all i

$$\equiv \text{if } \phi^r(t) \text{ is "sufficiently large" to overcome } -y_i^r C_i(t) \quad \forall i$$

y_i^r is recalled in one step from $\hat{y}(t)$

$$\phi^r(t) \text{ "sufficiently large"} \equiv \hat{y}(t) \text{ overlaps sufficiently with } y^r$$

If this occurs for all i, then starting with a state sufficiently similar to y^r, the network will recall y^r

→ y^r has a finite basin of attraction in phase space

≡ <u>y^r is an attractor</u>

Note that the basin size will be much smaller than in the single memory case.

Stability of Hopfield Networks

We have shown that the embedded memories are fixed points, but not that the system always converges from any initial condition

To prove this, we can use a Lyapunov Function (or an energy function)

A Lyapunov function is a function of the system state that is:

- Constant at system equilibrium points
- A decreasing function of time when the system is not at equilibrium

Existence of a Lyapunov Function → stable system

Lyapunov function for Hopfield network:

$$J = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\hat{y}_i\hat{y}_j$$

J.J. Hopfield (1982) Neural networks and physical systems with emergent collective computational abilities, *Proceedings of the National Academy of Sciences, USA*, 79: 2554-2558.

Each *ij* combination occurs twice

$$J = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\hat{y}_i\hat{y}_j$$

For symmetric networks with $w_{ij} = w_{ji}$ (e.g., Hopfield networks):

$$J = -\sum_{i=1}^{n}\sum_{j>i}^{n} w_{ij}\hat{y}_i\hat{y}_j - \frac{1}{2}\sum_{i=1}^{n} w_{ii}$$

Each *ij* combination occurs once

$$= -\sum_{i=1}^{n}\sum_{j>i}^{n} w_{ij}\hat{y}_i\hat{y}_j - K$$

- Hopfield: $K = 0$
- Hebb: $K = N/2n$

½ goes away because we are counting half the weights

Consider the case with asynchronous update – only bit k updated at time t (i.e., $j = k$)

$$J(t) = -\sum_{i=1}^{n}\sum_{j>i}^{n} w_{ij}\hat{y}_i(t)\hat{y}_j(t) - K$$

$$J(t+1) = J(t) - \left(-\sum_{i\neq k} w_{ik}\hat{y}_i(t)\hat{y}_k(t) \right) + \left(-\sum_{i\neq k} w_{ik}\overbrace{\hat{y}_i(t)}^{\text{This remained unchanged}}\hat{y}_k(t+1) \right)$$

Remove the old terms involving neuron k

Add the new terms involving neuron k

Terms affected by the update

$$\begin{bmatrix} w_{11} & \cdots & w_{1k} & \cdots & w_{1n} \\ \vdots & & \vdots & & \vdots \\ & & w_{(k-1)k} & & \\ w_{k1} & \cdots & w_{kk} & w_{k(k+1)} \cdots & w_{kn} \\ \vdots & & \vdots & & \vdots \\ w_{n1} & \cdots & w_{nk} & \cdots & w_{nn} \end{bmatrix}$$

Note that the w_{kk} term is excluded because it is covered in the constant K which remains unchanged.

The weights involved are highlighted.

Consider the case with asynchronous update – only bit k updated at time t (i.e., $j = k$)

$$J(t) = -\sum_{i=1}^{n}\sum_{j>i}^{n} w_{ij}\hat{y}_i(t)\hat{y}_j(t) - K$$

This remained unchanged

$$J(t+1) = J(t) - \left(-\sum_{i\neq k} w_{ik}\hat{y}_i(t)\hat{y}_k(t)\right) + \left(-\sum_{i\neq k} w_{ik}\overbrace{\hat{y}_i(t)}\hat{y}_k(t+1)\right)$$

Remove the old terms involving neuron k

Add the new terms involving neuron k

Terms affected by the update

$$\begin{bmatrix} w_{11} & \cdots & w_{1k} & \cdots & w_{1n} \\ \vdots & & \vdots & & \vdots \\ & & w_{(k-1)k} & & \\ w_{k1} & \cdots & w_{kk} & w_{k(k+1)} \cdots & w_{kn} \\ \vdots & & \vdots & & \vdots \\ w_{n1} & \cdots & w_{nk} & \cdots & w_{nn} \end{bmatrix}$$

Note that the w_{kk} term is excluded because it is covered in the constant K which remains unchanged.

The weights involved are highlighted.

Also, in the equation, we have chosen to count the $w_{(k+1)k} \ldots w_{nk}$ terms instead of the $w_{k(k+1)} \ldots w_{kn}$ terms, which are the same

$$J(t+1) - J(t) = -\left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right) + \left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t+1) \right)$$

Case I: $\hat{y}_k(t+1) = \hat{y}_k(t)$ (bit *k* remains unchanged by the update)

$$J(t+1) - J(t) = -\left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right) + \left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right) = 0$$

J remains unchanged

Case II: $\hat{y}_k(t+1) = -\hat{y}_k(t)$ (bit *k* is flipped by the update)

$$J(t+1) - J(t) = -\left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right) + \left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t+1) \right)$$

$$= -\left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right) + \left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t)(-\hat{y}_k(t)) \right)$$

$$J(t+1) - J(t) = -\left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right) + \left(-\sum_{i \neq k} w_{ik} \hat{y}_i(t)(-\hat{y}_k(t)) \right)$$

$$= \left(\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right) + \left(\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \right)$$

$$= 2\sum_{i \neq k} w_{ik} \hat{y}_i(t) \hat{y}_k(t) \;=\; 2\sum_{i \neq k} w_{ki} \hat{y}_i(t) \hat{y}_k(t)$$

Replace w_{ik} with w_{ki}

$$= 2\hat{y}_k(t) \sum_{i \neq k} w_{ki} \hat{y}_i(t)$$

$$= 2\hat{y}_k(t) \left(\sum_{i=1}^{n} w_{ki} \hat{y}_i(t) - w_{kk} \hat{y}_k(t) \right)$$

Summing over all i including k

$$= 2\hat{y}_k(t) \sum_{i=1}^{n} w_{ki} \hat{y}_i(t) - 2w_{kk}$$

$$J(t+1) - J(t) = 2\hat{y}_k(t)\sum_{i=1}^{n} w_{ki}\hat{y}_i(t) - 2w_{kk}$$

But $\hat{y}_k(t+1) = \text{sgn}\left(\sum_{i=1}^{n} w_{ki}\hat{y}_i(t)\right)$

and $\hat{y}_k(t)$ and $\hat{y}_k(t+1)$ have opposite signs (Case II)

$$\Rightarrow 2\hat{y}_k(t)\sum_{i\neq k}^{n} w_{ki}\hat{y}_i(t) < 0$$

$\boxed{\text{A}}$

Also, $w_{kk} = N/n > 0 \quad \Rightarrow \quad -2w_{kk} < 0$ (Hebb)

or $w_{kk} = 0 \quad \Rightarrow \quad -2w_{kk} = 0$ (Hopfield)

$\boxed{\text{B}}$

$\boxed{\text{A}}$ $\boxed{\text{B}}$ $\Rightarrow J(t+1) < J(t)$

Thus, *J* either remains unchanged or decreases until there are no bit flips → **Equilibrium**

As we know, for a stored memory y^q

$$\hat{y}(t) = y^q \quad \Rightarrow \quad \hat{y}(t+1) = y^q$$

$$\Rightarrow \quad J(t+1) = J(t) \quad \text{at} \quad \hat{y} = y^q$$

When the system is not at equilibrium, i.e., bits are still flipping
$\rightarrow$ J is still decreasing

J will keep decreasing until the system reaches equilibrium.

$\Rightarrow$ J is a Lyapunov function $\Rightarrow$ System is stable

Conclusions about Hopfield network dynamics:

- Symmetric networks with asynchronous dynamics converge to fixed points
- Stored memories are not the only equilibria.
- Setting $w_{ii} = 0$ can reduce spurious equilibria.

Capacity of Hopfield Networks

How many memories can a Hopfield network store?

Suppose the network is already in state $y(t) = y^r$

The condition for an erroneous flip in bit i is:

$$Q_i^r = -y_i^r C_i(t) \geq 1$$

Since we are starting in y^r, $\phi^r = 1$

$$Q_i^r = -y_i^r \left(\frac{1}{n} \sum_j \sum_{q \neq r} y_i^q y_j^q \hat{y}_j(t) \right) \geq 1$$

$$Q_i^r \approx \frac{1}{n} \left[\text{sum of } n(N\text{-}1) \text{ random } \pm 1 \text{ terms} \right]$$

$$\Rightarrow \quad \text{mean of } Q_i^r \approx 0$$

This assumes that +1 and -1 are equally probable, which may not always be the case.

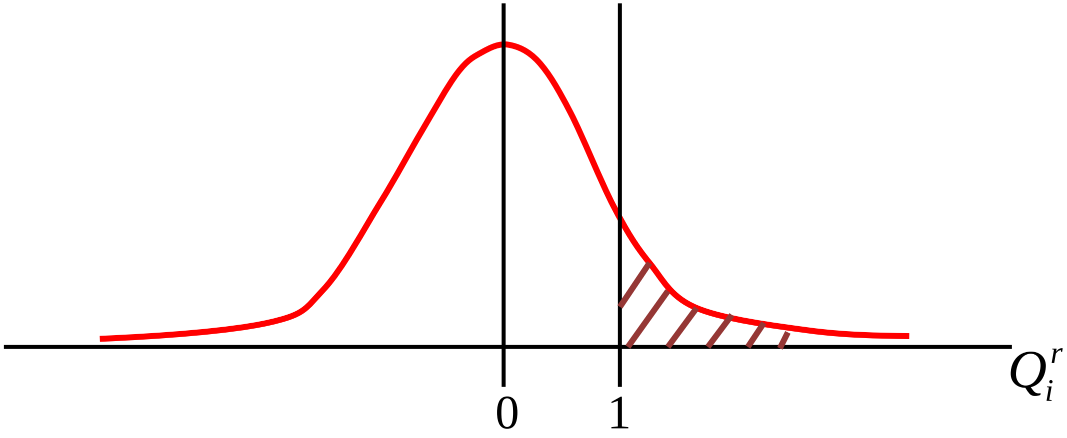
$$\text{variance of } Q_i^r \approx \frac{n(N-1)}{n^2} \approx \frac{N}{n}$$

Since each sample has a variance of 1 and there are $N(n\text{-}1)$ such samples

$$\text{distribution of } Q_i^r \approx \text{Gaussian (Central Limit Theorem)}$$

$$Q_i^r \sim G(0, N/n)$$

$$P(\text{error in bit } i) = P(Q_i^r \geq 1) \approx \frac{1}{\sqrt{2\pi N/n}} \int_1^{\infty} e^{-\frac{x^2}{2N/n}} dx$$

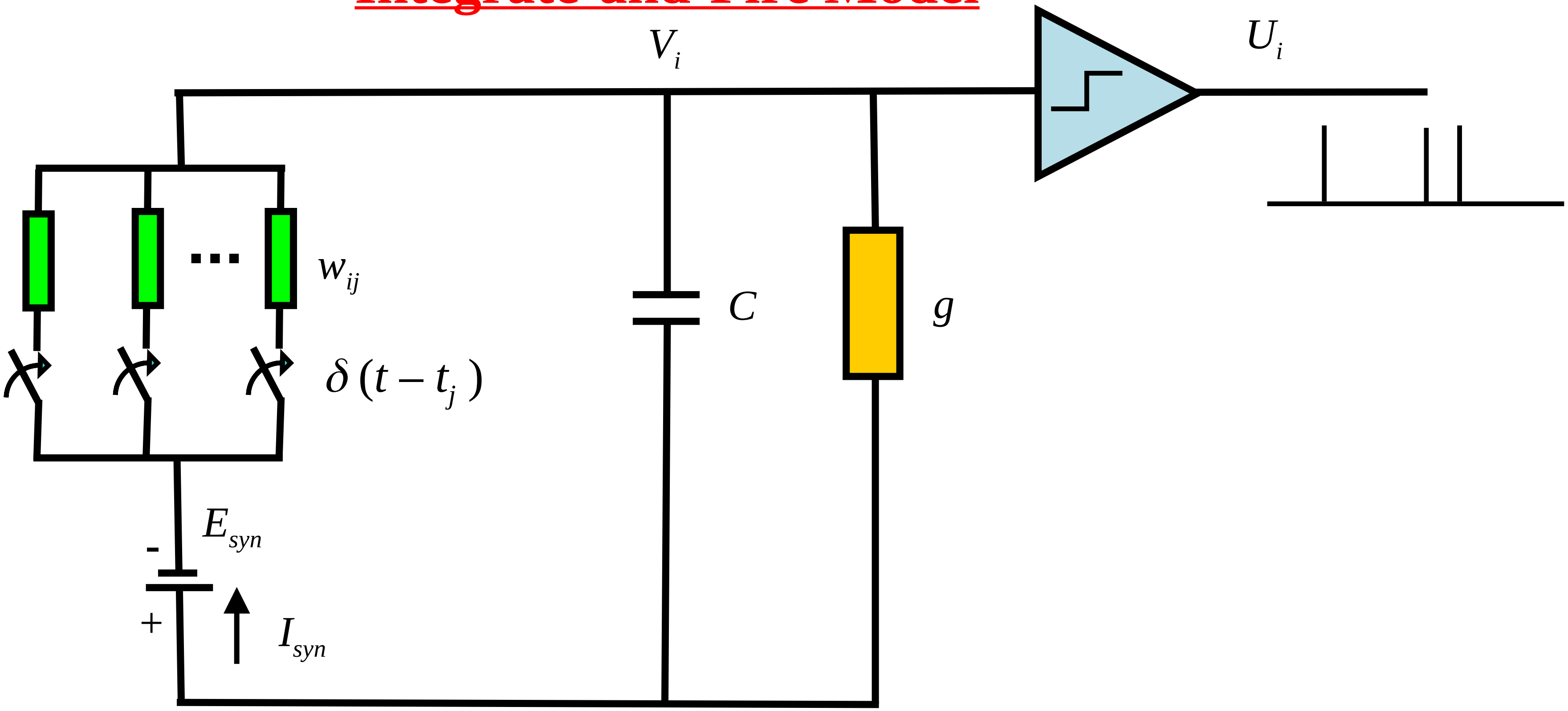


Solving for P(error) < 0.01, this gives $\boxed{N_{\max} \approx 0.15n}$

In fact, $0.138n$ is a better estimate

$\Rightarrow$ a 1000 neuron Hopfield network can store only about 130 memories

Integrate-and-Fire Model

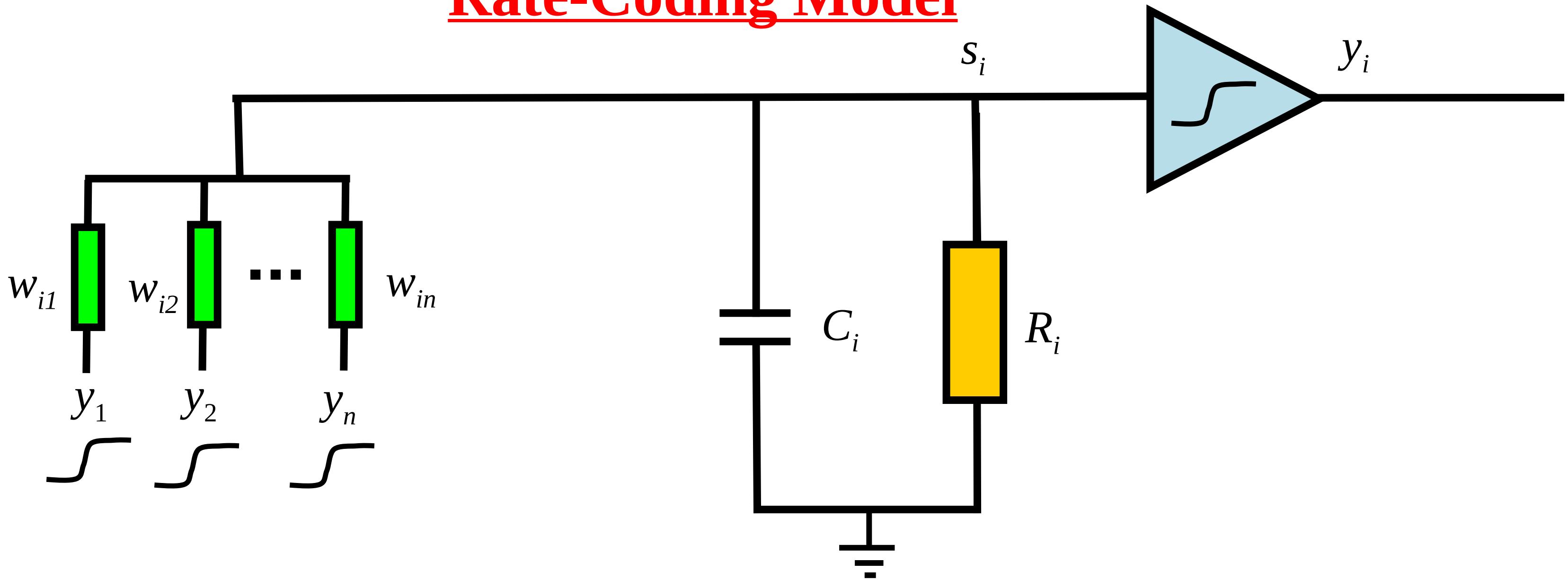


$$C\frac{dV_i}{dt} = -g\ V_i + \sum_j w_{ij} \sum_k \delta(t - t_{jk})\,(E_{syn} - V_i)$$

$$U_i(t) = \begin{cases} 1 & if\ V_i(t) > \theta \\ 0 & else \end{cases}$$

V_i is reset after each spike

Rate-Coding Model



$$C_i \frac{ds_i}{dt} = -\frac{1}{R_i} s_i(t) + \sum_j w_{ij} y_j(t)$$

Note that this is a ***nonlinear*** differential equation

Defining $\tau_i = R_i C_i$ and measuring time in units of τ_i

State: $$\frac{ds_i}{dt} = -s_i(t) + \sum_j w_{ij} y_j(t) \quad = -s_i(t) + \sum_j w_{ij} f\left(s_j(t)\right)$$

Output: $$y_i(t) = f\left(s_i(t)\right)$$ where $f(x)$ is a sigmoid function

Stability and Attractors

It can be shown that if

$$f(x) = \tanh(\alpha x / 2)$$

The continuous-time Hopfield network has a Lyapunov function

$$J = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\hat{y}_i\hat{y}_j + \sum_{j=1}^{n}\frac{1}{\alpha R_i}\int_0^{y_j}\tanh^{-1}(y/2)\,dy$$

As $\alpha \to \infty$, $\quad J \to -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\hat{y}_i\hat{y}_j$

which is the Lyapunov function for the discrete-time network with the same weights

$\Rightarrow$ The continuous-time network with extremely steep sigmoids has the same attractors as the equivalent discrete-time network.

Stochastic Model (Boltzmann Machine)

$$C_i \frac{ds_i}{dt} = -\frac{1}{R_i} s_i(t) + \sum_j w_{ij} y_j(t)$$

Note that this is a ***nonlinear*** differential equation

Defining $\tau_i = R_i C_i$ and measuring time in units of τ_i

State: $$\frac{ds_i}{dt} = -s_i(t) + \sum_j w_{ij} y_j(t)$$

Output: $$P(y_i(t) = 1) = \frac{1}{1 + \exp\left(-s_i / T\right)}$$

where T is a *temperature* parameter.

When T is high, $P(y_i(t) = 1) \approx 0.5$ Random update

As $T \to 0$ $P(y_i(t) = 1) \to \begin{cases} 1 & \text{if } s_i > 0 \\ 0 & \text{if } s_i < 0 \end{cases}$ Threshold function