MEDICAL IMAGING

Skriptum

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Contents

Inverse Problems 1
- What is an Inverse Problem? 1
  - Example: Computed Tomography 1
  - Example: Deconvolution 1
- What is an Inverse Problem? (formal) 2
- Vector Space 2
- Vector Space Examples 2
- Definition: Inverse Problem 2
- Definition: Well-Posedness (Hadamard) 3
- Well-Posedness Example 3
- Definition: Inner Product 3
- Definition: Vector Norm 3
- Definition: Matrix Norm 3
- Injection, Surjection, Bijection 3
- Null Space and Range 4
- Connection to Hadamard 4
- Singular Value Decomposition 4
- Least Squares (m > n) 4
- Minimum Norm Solution (n > m) 4
- Generalized Inverse 4
- Regularization 4
- Typical Regularization Terms 4
- Tikhonov Optimality Condition 5
- Probabilistic Interpretation 5

X-rays and CT 5
- Discovery of X-rays 5
- Nature and Properties of X-rays 5
- Ionizing Radiation 5
- Interaction of Energetic Electrons with Matter 5
- Interaction of X-rays with Matter 5
- Generation of X-rays 6
- Attenuation of Electromagnetic Radiation 6
- Narrow Beam vs Broad Beam 6
- Linear Attenuation Coefficient 6
- Projection Radiography 6
- Blurring in Projection Imaging 7
- Noise in Projection Imaging 7
- Tomography 7

Computed Tomography (CT) . . . 7
CT Generations . . . 7
Image Formation in CT . . . 7
Parallel-Ray Geometry . . . 8
Sinogram . . . 8
Backprojection . . . 8
Fourier Slice Theorem . . . 8
Filtered Backprojection (FBP) . . . 8
Iterative Reconstruction . . . 9
CT Artifacts . . . 9
Hounsfield Units . . . 9
Summary . . . 9
Learned Reconstruction . . . 9
MRI . . . 9
Image Registration . . . 9
Image Segmentation . . . 10
Federated Learning . . . 10
Microscopy . . . 10
A Subsection . . . 10
Tables and Figures . . . 10

Inverse Problems

What is an Inverse Problem?

There exit a “Forward Problem” which estimate the effect from the cause and then there is inverse Problem which estimates the cause from the effect. In the medical context that would be finding the cause illness given from a certain syntom/effect. Typically, the forward problem is ”easy” and well described. The challenge here is: We need to solve the inverse problem given only the observed effect of the forward problem.

As an Example from the real world: forward problem: The street becomes wet when it rains. backward problem would be: We observe that the street is wet. Why?

There are multiple different causes: • Rain • Fog • Cleaning

And this can be already problematic as we have multiple different options for what the cause could be.

Example: Computed Tomography

Forward Problem X-ray emitter and detector rotating around the body. Detectors measure the number of photons passing through the body and hitting the detector

Inverse Problem Reconstruct the interior of the body from the measured detector signals.

Note that a CT Scan can be very large in file size. A scan from shoulder to belt line is already 18GB of data for just a single scan. So we bascially have y and we want to get to x

Example: Deconvolution

Forward Problem Observe a blurred image

$$f = k * u$$

on a domain

$$\Omega \subset \mathbb{R}^2$$

.

Inverse Problem Estimate the sharp image

$$u : \Omega \to \mathbb{R}$$

given the blur kernel

$$k : \Omega \times \Omega \to \mathbb{R}_+$$

On of the oldest clasical methods to do that is the Wiener Filter. Deconvolution is linked to Fourier F:

$$f = k * u$$

$$F(f) = F(k) \odot F(u)$$

If we wanna do the inverse:

$$F^{-1}\{F(f)\} = F^{-1}\{F(k) \odot F(u)\} = f$$

where $\odot$ is a pointwise multiplication

So a estimate $\hat{u}$ would be

$$\hat{u} = F^{-1}\left(\frac{F(f)}{F(k)}\right)$$

The only problem here is when we have 0 frequencies in the kernel. The Wiener Filtering introduces

$$\hat{u} = F^{-1}\left(\frac{F(f)}{I\sigma^2 F(k)}\right)$$

What is an Inverse Problem? (formal)

> **Definition 1** (Inverse Problem) . Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $x \in \mathbb{R}^n$ the forward problem is $y = Ax \in \mathbb{R}^m$
>
> The inverse problem is: Given A and y, estimate x.

Vector Space

> **Definition 2** (Vector Space) . A non-empty set V is a vector space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ if there are operations of vector addition: $V \times V \to V$ and scalar multiplication: $\mathbb{F} \times V \to V$ satisfying the following axioms. **Vector addition**
> 1. $u + v \in V \quad \forall u, v \in V$
> 2. $u + v = v + u$
> 3. $(u + v) + w = u + (v + w) \quad \forall u, v, w \in V$
> 4. $\exists 0 \in V : u + 0 = u$
> 5. $\forall u \in V : \exists -u : u + (-u) = 0$
>
> **Scalar multiplication**
> 1. $au \in V$
> 2. $a(u + v) = au + av$
> 3. $(a + b)u = au + bu$
> 4. $a(bu) = (ab)u$
> 5. $1u = u$

Vector Space Examples

- $\mathbb{R}^n = (x_1, ..., x_n)^T$
- Continuous functions: $C(\mathbb{R}^n, \mathbb{R})$
- Once continuously differentiable functions: $C^1(\mathbb{R}^n, \mathbb{R})$
- Lebesgue space: $L^2(\mathbb{R}^n) = \left\{f \mid \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty\right\}$
- Sobolev space: $H^1(\mathbb{R}^n) = \left\{f \in L^2 \mid \int_{\mathbb{R}^n} |\nabla f(x)|^2 dx < \infty\right\}$

Definition: Inverse Problem

> **Definition 3** (Inverse Problem) . Let X, Y be vector spaces and $A : X \to Y$. The forward problem is $y = Ax$. The inverse problem is to find $x \in X$ such that
>
> $$Ax = y$$

.

Definition: Well-Posedness (Hadamard)

Definition 4 (Well-Posedness) . We can now start to categorize inverse problems: The inverse problem $Ax = y$ is well-posed if:

1. **Existence:** a solution exists (EXISTENCE)
2. **Uniqueness:** the solution is unique (UNIQUENESS)
3. **Stability:** the solution depends continuously on the data (STABILITY)

If one condition fails, the problem is ill-posed.

Well-Posedness Example

Let $X = Y = \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}, x \to x^2$ Is this example well posed?

- Existence: for $y = -1$ no solution exists (if it would be $\mathbb{R}^+$ it would be okay)
- Uniqueness: for $y = 1, x = \pm 1$
- Stability: yes, since A is continuous

Definition: Inner Product

Definition 5 (Inner Product) . An inner product is a mapping

$$\langle ., . \rangle : Y \times Y \to \mathbb{F}$$

with properties:

1. Symmetry: $\langle x, y \rangle = \langle y, x \rangle \quad x, y \in Y$
2. Additivity: $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle \quad x, y, z \in Y$
3. Homogeneity: $\langle ax, y \rangle = a \langle x, y \rangle \quad x, y \in Y \quad a \in \mathbb{R}$
4. Positivity: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$

Definition: Vector Norm

Definition 6 (Inner Product) . A vector norm is a vector space Y over a field F is a map $\|.\| : Y \to \mathbb{R}$ with:

1. Non-negativity:$\|x\| \geq 0$
2. Definiteness:$\|x\| = 0 \iff x = 0$
3. Homogeneity: $\|ax\| = |a| \|x\|$
4. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

Definition: Matrix Norm

Definition 7 (Inner Product) . Let $\|.\|_a$ on $\mathbb{R}^n$ and $\|.\|_b$ on $\mathbb{R}^m$.

For $A \in \mathbb{R}^{m \times n}$ the induced matrix norm is $\|A\|_{a,b} = \sup_{x \neq 0} \left(\frac{\|Ax\|_b}{\|x\|_a} \right)$

Injection, Surjection, Bijection

Let $A : X \to Y$

- Injective:

$$Ax_1 = Ax_2 \Rightarrow x_1 = x_2$$

- Surjective: $\forall y \in Y, \exists x \in X : Ax = y$
- Bijective: injective and surjective

Null Space and Range

- Null space: $N(A) = \{x \in X \mid Ax = 0\}$
- Range: $R(A) = \{Ax \mid x \in X\}$

Connection to Hadamard

- Existence $\Longleftrightarrow R(A) = Y$
- Uniqueness $\Longleftrightarrow N(A) = \{0\}$
- Well-posed $\Longleftrightarrow A$ bijective (and stable)

Singular Value Decomposition

Let $A \in \mathbb{R}^{m \times n}$ then $A = U \Sigma V^T$ where $\Sigma = \left(\sigma_1, ..., \sigma_p\right)$ with $\sigma_i > 0$ and $p = (A)$.

Least Squares (m > n)

Solve $Ax = y$ by minimizing $\arg\min_x \|Ax - y\|^2$ Normal equations: $A^T Ax = A^T y$

Minimum Norm Solution (n > m)

Underdetermined system $Ax = y$

Choose the minimum norm solution:

$\arg\min_x \|x\| \text{ subject to } Ax = y$

Solution: $x = A^T \left(AA^T\right)^{-1} y$

Generalized Inverse

Using the SVD:

$A^{\dagger} = V \Sigma^{-1} U^T$

This interpolates between least squares and minimum norm solutions.

Regularization

Instead of solving $Ax = y$, solve

$\arg\min_x \|Ax - y\|^2 + \lambda R(x)$

Typical Regularization Terms

- Tikhonov: $R(x) = \|x\|^2$
- L2: $R(x) = \|x\|^2$
- H1: $R(x) = \|\nabla x\|^2$
- L1: $R(x) = \|x\|_1$

- Total Variation: $R(x) = \|\nabla x\|_1$

Tikhonov Optimality Condition

$(A^T A + \lambda I)x = A^T y$

Probabilistic Interpretation

Assume noisy measurements: $y = Ax + \varepsilon \; \varepsilon \sim (0, \sigma^2 I)$

Bayes' rule yields: $\arg\max_x \log(p(x \mid y))$ which is equivalent to $\arg\min_x \|Ax - y\|^2 - \log(p(x))$

Hence, regularization corresponds to MAP estimation.

X-rays and CT

Discovery of X-rays

- In 1895 Wilhelm Röntgen discovered "rays of mysterious origin", later called X-rays.
- On 22.12.1895 the first radiograph of the hand of Röntgen's wife was produced.
- This immediate medical application marks the birth of medical imaging.

Nature and Properties of X-rays

- X-rays are electromagnetic waves.
- They are a form of ionizing radiation, i.e. radiation with enough energy to eject electrons from atoms.

Ionizing Radiation

Two main forms:

1. **Particulate radiation** Subatomic particles (electrons, protons, neutrons) with sufficient kinetic energy.
2. **Electromagnetic radiation** Acts as wave or particle (photon).

EM radiation is ionizing if photon energy exceeds the hydrogen binding energy: $E > 13.6eV$

Relations: $E = h\nu \; \lambda = \frac{c}{\nu}$

Interaction of Energetic Electrons with Matter

When electrons hit matter:

- **Collision transfer** (99%): Energy transferred to other electrons $\longrightarrow$ heat.
- **Radiative transfer** (1%): a) Inner-shell ionization $\longrightarrow$ characteristic X-rays b) Braking near nucleus $\longrightarrow$ bremsstrahlung radiation

Interaction of X-rays with Matter

Photoelectric effect Photon ejects an inner-shell electron:

$E_e = h\nu - E_B$

- Filling the vacancy emits characteristic X-rays.

- Alternatively produces Auger electrons.

Compton scattering Photon interacts with outer-shell electrons, losing energy and changing direction.

Generation of X-rays

X-rays are generated using an X-ray tube:

- Heated cathode emits electrons
- High voltage accelerates electrons
- Electrons hit anode $\longrightarrow$ X-rays produced

Attenuation of Electromagnetic Radiation

Consider a narrow monoenergetic X-ray beam.

Let:
- $N(x)$ = number of photons
- $\mu(x)$ = linear attenuation coefficient

Photon loss:

$dN = -\mu(x) N dx$

Divide and integrate:

$d\frac{N}{N} = -\mu(x) dx$
$\ln\left(\frac{N}{N_0}\right) = -\int \mu(x) dx$

Resulting intensity:

$N = N_0 \exp(-\int \mu(x) dx)$

Narrow Beam vs Broad Beam

- Broad beam: scattering contributes to detector signal.
- Monoenergetic assumption fails due to energy loss.

Solution:
- Collimation
- Narrow-beam geometry

Then attenuation law holds approximately.

Linear Attenuation Coefficient

$\mu(x)$ depends on:
- material
- photon energy

Higher $\mu \longrightarrow$ stronger attenuation.

Projection Radiography

Basic imaging equation:

$I = \int S(E) \exp(-\int \mu(x, E) dx) dE$

Assuming effective monoenergetic spectrum:

$I = I_0 \exp(-\int \mu(x) dx)$

Taking logarithm:

$-\ln\left(\frac{I}{I_0}\right) = \int \mu(x) dx$

Blurring in Projection Imaging

Sources of blur:
- Finite focal spot (penumbra)
- Detector blur
- Compton scattering outside field of view

Noise in Projection Imaging

Photon detection is a counting process:

$N \sim (N)$

Variance:

$\mathrm{Var}(N) = N$

Signal-to-noise ratio:

$\mathrm{SNR} = \frac{N}{\sqrt{N}} = \sqrt{N}$

To increase SNR:
- Increase photon count
- Use contrast agents

Tomography

Tomography = imaging by sectioning a volume.

From Greek:
- **tomos** = slice
- **grapho** = to write

Computed Tomography (CT)

Basic principle:
- Acquire many projections
- Different orientations around object
- Reconstruct cross-sectional image

CT Generations

- 1st generation: translate–rotate, pencil beam
- 2nd generation: fan beam, detector array
- 3rd generation: rotating source and detectors
- 4th generation: stationary detector ring

Image Formation in CT

Using attenuation model:

$I = I_0 \exp(- \int \mu(x) dx)$

Define projection value:

$p = -\ln\left(\frac{I}{I_0}\right) = \int \mu(x) dx$

Thus each projection is a line integral of μ.

Parallel-Ray Geometry

Parameterization:

$x(s) = s\cos(\theta) - t\sin(\theta)\ y(s) = s\sin(\theta) + t\cos(\theta)$

Projection:

$g(t, \theta) = \int \mu(x(s), y(s)) ds$

This is the **Radon transform**.

Sinogram

- $g(t, \theta)$ plotted over t and θ
- Each object point traces a sinusoid
- Sinogram contains all projection data

Backprojection

Idea:
- Smear each projection back over the image

Backprojection operator:

$f_{\mathrm{BP}(x,y)} = \int g(x\cos(\theta) + y\sin(\theta), \theta) d\theta$

Produces blurred image.

Fourier Slice Theorem

1D Fourier transform of projection:

$G(\omega, \theta) = F_1[g(t, \theta)]$

Equals slice of 2D Fourier transform of image:

[

$$F_2(\mu)(u, v)$$

]

with:

$u = \omega\cos(\theta)\ v = \omega\sin(\theta)$

Filtered Backprojection (FBP)

Steps:
1. Filter projections with high-pass filter

2. Backproject filtered projections

Reconstruction: $\mu(x, y) = \int (g * h)(x\cos(\theta) + y\sin(\theta), \theta) d\theta$

where h is the reconstruction filter.

Iterative Reconstruction

- Start with initial guess
- Forward project
- Compare to measured data
- Update estimate

(Details skipped in lecture)

CT Artifacts

- **Aliasing**: insufficient number of projections
- **Beam hardening**: low-energy photons absorbed more strongly

Results in streaks and cupping artifacts.

Hounsfield Units

To standardize CT values:

[

$$HU = 1000\frac{\mu - \mu_{\text{water}}}{\mu_{\text{water}} - \mu_{\text{air}}}$$

]

Reference values:

- Air: −1000
- Water: 0
- Fat: −120 to −90
- Muscle: +35 to +55
- Bone: +300 to +1900
- White matter: +20 to +30
- Grey matter: +37 to +45

Summary

- CT reconstructs attenuation coefficients from projections
- Based on Radon transform and Fourier theory
- Filtered backprojection is classical reconstruction method
- Regularization and learning-based methods improve reconstruction

Learned Reconstruction

MRI

Image Registration

Image Segmentation

Federated Learning

Microscopy

Here's some example text. Notice how the section heading uses elegant spaced small caps.

A Subsection

Subsections use italic text for a subtle hierarchy.

Definition 8 (Important Concept) . A definition block with a distinctive left border. Use this to define key terms in your work.

Theorem 1 (Main Result) . A theorem block for stating important results. The numbering is automatic.

Example 1 — Practical Application . An example block with a subtle gray background. Use this to illustrate concepts with concrete examples.

Remark. A remark block for additional observations or notes that don't fit the formal structure of theorems and definitions.

Inline code looks like `this`, and code blocks are formatted cleanly:

```
def hello_world():
    """A simple function."""
    print("Hello, ClassicThesis!")
```

Tables and Figures

Item	Description	Value
Alpha	First item	100
Beta	Second item	200
Gamma	Third item	300

Table 1 : A sample table with clean styling.