

Natural cubic splines have the additional constraints of being linear beyond the boundary knots. Start from the truncated power series:

$$f(X) = \sum_{j=0}^3 \beta_j X^j + \sum_{k=1}^K \theta_k (X - \xi_k)_+^3$$

For the left boundary knot

$$f(X) = \sum_{j=0}^3 \beta_j X^j, \quad X < \xi_1$$

and we need the constraints $\beta_2 = 0$ and $\beta_3 = 0$ for the function to be linear.

The truncated power series representation

$$f(X) = \sum_{j=0}^3 \beta_j X^j + \sum_{k=1}^K \theta_k (X - \xi_k)_+^3$$

with the constraints on the coefficients

$$\beta_2 = 0, \quad \beta_3 = 0, \quad \sum_{k=1}^K \theta_k = 0, \quad \sum_{k=1}^K \xi_k \theta_k = 0$$

Taking into account first the β restrictions, we can construct a new basis with the first two basis functions as

$$f(X) = \beta_0 \cdot \underbrace{1}_{N_1(x)} + \beta_1 \underbrace{X}_{N_2(x)} + 0 \cdot X^2 + 0 \cdot X^3 + \dots$$

For the right boundary knot

$$\begin{aligned} f(X) &= \sum_{j=0}^3 \beta_j X^j + \sum_{k=1}^K \theta_k (X - \xi_k)_+^3, \quad \xi_K \leq X \\ &= \sum_{j=0}^3 \beta_j X^j + \sum_{k=1}^K \theta_k X^3 - \sum_{k=1}^K \theta_k \xi_k 3X^2 + \sum_{k=1}^K \theta_k \xi_k^2 3X - \sum_{k=1}^K \theta_k \xi_k^3 \end{aligned}$$

and we need the additional constraints $\sum_{k=1}^K \theta_k = 0$ and $\sum_{k=1}^K \xi_k \theta_k = 0$ for the function to be linear.

For the θ constraints, we utilize that

$$\sum_{k=1}^{K-2} \theta_k = -\theta_{K-1} - \theta_K, \quad \sum_{k=1}^{K-2} \xi_k \theta_k = -\xi_{K-1} \theta_{K-1} - \xi_K \theta_K$$

Take out the last two terms of the truncated basis functions:

$$\sum_{k=1}^K \theta_k (X - \xi_k)_+^3 = \sum_{k=1}^{K-2} \theta_k (X - \xi_k)_+^3 + \theta_{K-1} (X - \xi_{K-1})_+^3 + \theta_K (X - \xi_K)_+^3,$$

and use the θ constraints to show that the two last terms can be rewritten as sums over the $N - 2$ first terms.