Exercise 18.4 Show that for ridge regression $\hat{\beta} = V\left(R^TR + \lambda I\right)^{-1}R^Ty$ where $X = RV^T$

$$\hat{\beta} = (X^TX + \lambda I)^{-1}X^Ty$$

$$\overset{!}{=} (VR^TRV^T + \lambda I)^{-1}VR^Ty$$

matrix inverse lemma: $(A + UCB)^{-1} = \underset{a\times a}{A^{-1}} - \underset{a\times a}{A^{-1}}\underset{a\times b}{U}(\underset{b\times b}{C^{-1}} + \underset{b\times a}{B}\underset{a\times a}{A^{-1}}\underset{a\times b}{U})^{-1}\underset{b\times a}{B}\underset{a\times a}{A^{-1}}$

$$A = \lambda I$$
$$U = V$$
$$C = R^TR$$
$$B = V^T$$

$$\hat{\beta} = \left[\lambda^{-1}I - \lambda^{-1}IV\left((R^TR)^{-1} + V^T\lambda^{-1}IV\right)^{-1}V^T\lambda^{-1}I\right]VR^Ty$$

$$\overset{!}{=} \left[\lambda^{-1}VV^T - \lambda^{-1}V\left((R^TR)^{-1} + \lambda^{-1}I\right)^{-1}\lambda^{-1}V^T\right]VR^Ty$$

$$\overset{!}{=} \lambda^{-1}V\left[I - \lambda^{-1}\left((R^TR)^{-1} + \lambda^{-1}I\right)^{-1}\right]\underbrace{V^TV}_{I}R^Ty$$

matrix inverse lemma special case $\quad (A + B)^{-1} = A^{-1} - A^{-1}(B^{-1} + A^{-1})^{-1}A^{-1}$

here $A = \lambda^{-1}I$

$B = (R^TR)^{-1}$

$$\hat{\beta} = \lambda^{-1}V\left[I - \lambda^{-1}\left(\lambda I - \lambda\left(R^TR + \lambda I\right)^{-1}\lambda\right)\right]R^Ty$$

$$\overset{!}{=} \cancel{\lambda^{-1}}V\left[\cancel{I} - \cancel{I} + \cancel{\lambda}\left(R^TR + \lambda I\right)^{-1}\right)R^Ty$$

$$\overset{!}{=} V\left(R^TR + \lambda I\right)^{-1}R^Ty$$