Exercise 10.8

K - class problem, current model $f_k(x)$, $k=1,\dots,K$ s.t. $\sum_{k=1}^{K} f_k(x_i)=0$

$$y_{ik} = \begin{cases} 1 & \text{if obs } i \text{ is in class } k \\ 0 & \text{otherwise} \end{cases}$$

GOAL: update the estimate in a region R by a constant γ_k, $f_k(x)+\gamma_k$, $\gamma_K=0$

a)
$$p_k(x_i) = \frac{e^{f_k(x_i)}}{\sum_{m=1}^{K} e^{f_m(x_i)}} \qquad (10.21) \qquad \sum_{k=1}^{K} p_k(x_i)=1, \quad p_k(x_i)\geq 0 \ \forall k$$

$$L_i(p(x)) = \frac{N!}{y_1!\times\dots\times y_K!}\, p_1(x_i)^{y_{i1}} \times \dots \times p_K(x_i)^{y_{iK}}$$

$\mathbb{1}(y_i = \hat{g}_i)$

$$\ell_i(p(x)) = y_{i1}\log p_1(x_i) + \dots + y_{iK}\log p_K(x_i) = \sum_{k=1}^{K} y_{ik}\log p_k(x_i)$$

$$\ell_i(f(x)) = \sum_{k=1}^{K} y_{ik}\log\frac{e^{f_k(x_i)}}{\sum_{m=1}^{K} e^{f_m(x_i)}}$$

$$= \sum_{k=1}^{K} y_{ik} f_k(x_i) - \underbrace{\sum_{k=1}^{K} y_{ik}}_{\text{one } 1\text{, the rest } 0} \overbrace{\log\sum_{m=1}^{K} e^{f_m(x_i)}}^{\text{does not depend on } k}$$

$$= \sum_{k=1}^{K} y_{ik} f_k(x_i) - \log\sum_{m=1}^{K} e^{f_m(x_i)}$$

$$L(\theta) = \prod_{i=1}^{n} L_i(\theta) \qquad \ell(\theta) = \sum_{i=1}^{n} \ell_i(\theta)$$

$$\ell(f(x)) = \sum_{x_i\in R}\sum_{k=1}^{K} y_{ik} f_k(x_i) - \sum_{x_i\in R}\log\sum_{m=1}^{K} e^{f_m(x_i)}$$

$$\ell(\gamma) = \sum_{x_i\in R}\sum_{k=1}^{K} y_{ik}\left(f_k(x_i)+\gamma_k\right) - \sum_{x_i\in R}\log\sum_{m=1}^{K} e^{f_m(x_i)+\gamma_m}$$

$$\ell_{\gamma_k}(\gamma) = \frac{\partial \ell(\gamma)}{\partial \gamma_k} = \sum_{x_i\in R} y_{ik} - \sum_{x_i\in R}\frac{e^{f_k(x_i)+\gamma_k}}{\sum_{m=1}^{K} e^{f_m(x_i)+\gamma_m}}$$

$$\ell_{\gamma_k\gamma_k}(\gamma) = \frac{\partial^2 \ell(\gamma)}{\partial \gamma_k^2} = -\sum_{x_i\in R}\frac{e^{f_k(x_i)+\gamma_k}\left(\sum_{m=1}^{K} e^{f_m(x_i)+\gamma_m}\right) - \left(e^{f_k(x_i)+\gamma_k}\right)^2}{\left(\sum_{m=1}^{K} e^{f_m(x_i)+\gamma_m}\right)^2}$$

$$\ell_{\gamma_k\gamma_q}(\gamma) = \frac{\partial^2 \ell(\gamma)}{\partial \gamma_k\,\partial \gamma_q} = +\sum_{x_i\in R}\frac{+e^{f_k(x_i)+\gamma_k}\, e^{f_q(x_i)+\gamma_q}}{\left(\sum_{m=1}^{K} e^{f_m(x_i)+\gamma_m}\right)^2}$$