Exercise 3.3 (a) prove Gauss-Markov theorem

$$\hat{\theta}_{LS} = a^T \hat{\beta} = \underbrace{a^T (X^T X)^{-1} X^T}\, y \qquad E[a^T \hat{\beta}] = a^T \beta$$

$$\tilde{\theta} = c^T y \qquad c^T = a^T (X^T X)^{-1} X^T + \delta^T$$

$$E[y] = X\beta$$

$$\begin{aligned}
E_{\text{nk}}[\tilde{\theta}] &= E_{\text{nk}}[c^T y] \\
&\overset{!}{=} E_{\text{nk}}[a^T (X^T X)^{-1} X^T y + \delta^T y] \\
&\overset{!}{=} a^T \cancel{(X^T X)^{-1}} \cancel{X^T X} \beta + \delta^T X \beta \\
&\overset{!}{=} a^T \beta + \delta^T X \beta \qquad \Rightarrow \underbrace{\delta^T X = 0}
\end{aligned}$$

$$\begin{aligned}
Var(\tilde{\theta}) &= Var[c^T y] \\
&\overset{!}{=} c^T \sigma^2 c \\
&\overset{!}{=} \sigma^2 (a^T (X^T X)^{-1} X^T + \delta^T)(a^T (X^T X)^{-1} X^T + \delta^T)^T \\
&\overset{!}{=} \sigma^2 (a^T (X^T X)^{-1} X^T + \delta^T)(X (X^T X)^{-1} a + \delta) \qquad 0 = (\delta^T X)^T = X^T \delta = 0 \\
&\overset{!}{=} \sigma^2 (a^T \cancel{(X^T X)^{-1}} \cancel{X^T X} (X^T X)^{-1} a + a^T (X^T X)^{-1} \underbrace{X^T \delta}_{=0} + \\
&\quad + \underbrace{\delta^T X}_{0} (X^T X)^{-1} a + \delta^T \delta) \\
&= \underbrace{\sigma^2 a^T (X^T X)^{-1} a}_{Var(\hat{\theta}_{LS})} + \underbrace{\delta^T \delta \sigma^2}_{\geq 0} \qquad \Rightarrow Var(\tilde{\theta}) \geq Var(\hat{\theta}_{LS})
\end{aligned}$$