Exercise 15.1 Derive $\rho\sigma^2 + \frac{1-\rho}{B}\sigma^2$

variance of the average of B i.d. r.v. with variance σ^2

$$E[A^2] = Var[A] + (E[A])^2 \qquad E[X_b] = \mu$$

$$E\left[\left(\frac{1}{B}\sum_{b=1}^{B} X_b\right)^2\right] = \frac{1}{B^2} E\left[\sum_{b=1}^{B} X_b^2 + 2\sum_{b=1}^{B}\sum_{c>b} X_b X_c\right]$$

$$= \frac{1}{B^2}\left(\sum_{b=1}^{B} E[X_b^2] + 2\sum_{b=1}^{B}\sum_{c>b} E[X_b X_c]\right)$$

$$= \frac{1}{B^{\cancel{2}}}\cancel{B}(\sigma^2+\mu^2) + \frac{2}{B^2}\binom{B}{2} E[X_b X_c]$$

$$= \frac{1}{B}(\sigma^2+\mu^2) + \frac{\cancel{2}}{B^2}\frac{B(B-1)}{\cancel{2}}(\rho\sigma^2+\mu^2)$$

$$\rho = \frac{E[(X_b-\mu)(X_c-\mu)]}{\sigma^2} \to \rho = \frac{E[X_b X_c] - \mu E[X_b] - \mu E[X_c] + \mu^2}{\sigma^2}$$

$$\rho = \frac{E[X_b X_c] - \mu^2}{\sigma^2} \to E[X_b X_c] = \sigma^2\rho + \mu^2$$

$$Var\left[\frac{1}{B}\sum_{b=1}^{B} X_b\right] = \frac{1}{B}(\sigma^2+\mu^2) + \frac{B-1}{B}(\rho\sigma^2+\mu^2) - \mu^2$$

$$= \frac{\sigma^2}{B} + \cancel{\frac{\mu^2}{B}} + \frac{B-1}{B}\rho\sigma^2 + \cancel{\frac{B-1}{B}\mu^2} \cancel{-\mu^2}$$

$$= \frac{\sigma^2}{B} + \rho\sigma^2 - \frac{\rho\sigma^2}{B}$$

$$= \rho\sigma^2 + \frac{(1-\rho)}{B}\sigma^2$$