Analyses of a Solar System

Steinn Hauser, steinnhm

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Institute of theoretical astrophysics, Blindern University, Oslo
steinnhauser@mac.com

Abstract

A mathematical model for how to accurately interpret data and produce meaningful insight into how stars behave and whether or not they may have planets in orbit that we can detect. Analyzing data produced from looking at the stars surrounding us to assess which ones may have solar systems of their own is essential to describing our own solar system, as well as how rare or common it really is. Our own solar system has a whopping 8 planets (and no, Pluto is not a planet); four gas giants and four terrestrial planets in a complicated, dynamic orbit around a common center of mass. However, when we look at other stars they often don't have any planets in orbit that we can detect, so it seems as though our solar system is quite a rare one in more ways than one.

Introduction

Let us first assume, for simplicity's sake, that a star has a planet orbiting it in a circular orbit. The star is asserting a gravitational force on the planet which generates a centripetal acceleration that pulls the planet inwards towards the star, keeping it in its orbit. According to Newton, which we can often assume was correct, there must also be an equal and opposite force from the planet on the star. This force causes the star to also be pulled towards the planet and have its own circular orbit around the systems center of mass. The center of mass of the system depends on the distance from the planet to the star, as well as the star and planets masses. If the star is much more massive than the planet, the center of mass will likely be somewhere inside the star. This means that the star does not have a very large scale orbit, but these small variations in the stars velocity can still be detected by what is referred to as the Doppler Effect.

Doppler Effect and data

The Doppler Effect is a physics phenomenon which causes a waves frequency to change depending on the source the waves are coming from. If the source were to have a velocity in a certain direction while sending waves radially outward, the waves get packed closer together in the direction the source is moving towards. Equivalently, the waves get stretched out in the direction the source is moving away from. An excellent example of this the change of pitch heard when a vehicle sounding a siren approaches. The pitch of the siren is higher when the vehicle is approaching, then drops to a lower pitch when the vehicle moves away.

The same can be applied to light waves where, were we to measure the wavelength of the light a star moving towards us radiates, the wavelength would be shorter than expected. Accordingly, the wavelength of the light would be longer than expected if the star were to be moving away from us. These wavelength variations can be measured by our telescopes on the ground where we record the data for analyses. Our data is as follows:

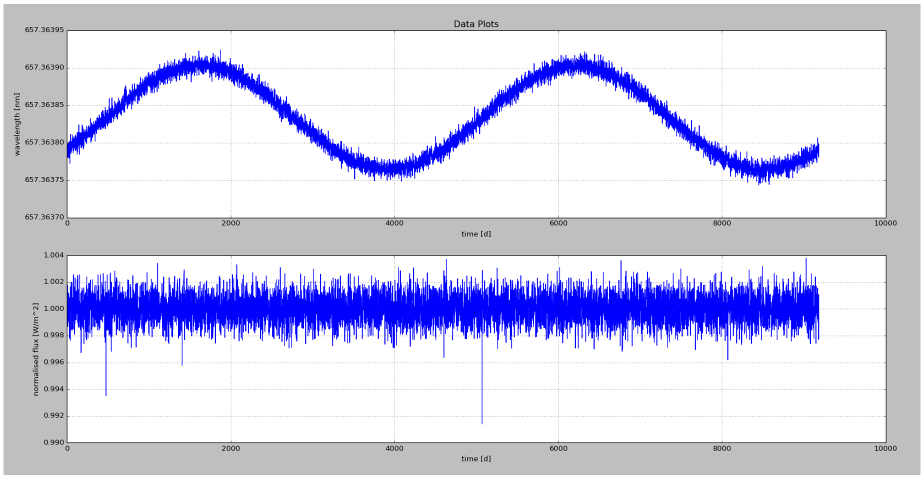

Figure 1: Data received by our telescopes

The upper graph shows the wavelength λ in an interval of $\lambda \approx 657nm$ where the wavelength oscillations were caused by the star orbiting slightly around the systems center of mass, and the lower graph shows the flux of the light received. The wavelength oscillates around a mean value that is not zero, which means the star-planet system could be either moving away or towards us (in addition to orbiting each other), depending on the wavelength of the light it radiates. To know which wavelengths the star will radiate, a spectrograph analyses is needed, but this is irrelevant to the subject at hand.

Interpreting this data involves looking past the static produced by the Earth's atmosphere and seeing the harmonic oscillation curve of the star in the first graph. The second graph informs us about the light intensity received and whether or not the planet may have eclipsed our view of the star. Two noticeable drops in the light intensity occur at $t \approx 500d$ and $t \approx 5000d$ and indicate that some celestial object is obscuring the telescope's view of the star. The two eclipses have a noticeably similar time period to that of the star oscillations, since, were we to calculate the two bodies oscillation periods, we would expect to get the same time result. Seeing as we can estimate from the graph that the time it took between eclipses is about the same as the period of the star movements really furthers certainty of it being a planet in orbit around the star that is the cause of the brightness fluctuations. Furthermore, if the system only has one star and one planet, we can expect the planet to eclipse when the star only has a velocity component that we can't see (because the star is not moving towards or away from us in relation to its reference system) and the fluctuations would occur when the star only has an angular velocity and zero radial velocity. This is precisely what we can tell from our data graphs; the planet eclipses each time the stars radial velocity is on its way from minimum to maximum just as expected of a star-planet system.

Method

To accurately compute the characteristics of the system, the data measurements will need to have a much smaller margin of error than that of the telescope data. The static produced by the Earth's atmosphere can easily compromise the data measurements, such as the stars radial velocity v_r, the period of the system P or the time t_0 when the velocity v_r is at a maximum. In order to accurately measure these values, the static must be removed or minimized. One way to do this is to utilize the Least-Squares method from regression analyses. The method of least squares involves adjusting the appropriate parameters of a function model to best fit the data. In this case the function model can be described by a general harmonic oscillator function:

$$v_r^{model}(t) = v_r cos(2\pi \frac{t - t_0}{P})$$

Where v_r, P and t_0 are adjustable parameters. The Least-Squares model is based on finding the smallest absolute value when subtracting our function from the data. The more accurate our function parameters, the smaller the subtraction result becomes. We must therefore program an algorithm which will take in a minimum and a maximum value for each parameter (with the least square solution somewhere between them) and cycle through them, saving the results as it goes. The algorithm will save the values in a large array of squares, which we can later find the lowest value for, and what parameters produced that value. The algorithm is as follows:

$$\Delta(v_r, P, t_0) = \sum_t (v_r^{data}(t) - v_r^{model}(t, v_r, P, t_0))^2$$

Choosing the parameter bounds (what values each parameter will cycle through) is not easy. On one hand it is reasonable to choose parameter intervals that are quite big, so as to be sure that the best value is not excluded. However, the computation needed for computing say, 100 values for each parameter, becomes 100^3 calculations. This would be very inefficient in terms of computation time. So in order to get as accurate a measurement as possible, a close parameter interval must be chosen by eye, making sure that the true value is somewhere in between. Only then can the program compute, say 40 values per parameter with a good computation time and a reasonably good accuracy.

These parameters, once accurately computed, can be used to calculate a few different things such as the mass of the planet, the planets mean distance from the star as well as the planets radius, should the data measurements be accurate enough. Calculating a planets radius involves having a lot of data points of the time interval Δt where the planet goes from not obscuring the star at all, to its full eclipse state. Calculating the radius involves having a estimable time interval Δt which can be read off of the light intensity (flux) plot. This time interval, paired with the velocity of the planet (calculable using other formulas) gives us the radius of the planet, as the planet will travel a distance equal to its diameter in the time interval Δt, mathematically written as:

$$v_p = \frac{2R}{\Delta t} \Rightarrow R = \frac{v_p \Delta t}{2}$$

Where v_p is the planets orbital velocity and R is the planets radius. The radius (in addition to the planets mass) can be used to calculate whether or not it is a gas or rock planet, but it seems as though the data received by the telescopes is not accurate enough to calculate such a Δt. However, the mass of the planet is calculable through manipulation of simple velocity equations and Kepler's third law:

$$v_* = \frac{2\pi a_*}{P}, v_p = \frac{2\pi a_p}{P}$$

Seeing as the period P is equal for both objects, and the force upon one is equal to the force upon the other, we get:

$$\Rightarrow m_p a_p = m_* a_*$$

Where a_p and a_* are the distances from the systems center of mass to the planet and the star. Let us implement the angle of inclination to get an expression for the radial velocity of the planet:

$$v_* = \frac{2\pi a_*}{P}, v_p = \frac{2\pi a_p}{P} \Rightarrow \frac{a_*}{a_p} = \frac{v_*}{v_p} = \frac{v_{*r}/\sin i}{v_{pr}/\sin i} = \frac{v_{*r}}{v_{pr}}$$

Kepler's third law states:

$$m_* + m_p = \frac{4\pi^2 a^3}{P^2 G}$$

Using the correlation $a = a_* + a_p$ and $a = \frac{P}{2\pi}(v_* + v_p)$ in Kepler's third law returns:

$$m_* + m_p = \frac{P}{2\pi G}(v_* + v_p)^3$$

Knowing that the radial velocity is normally the only measurable component, it is useful to substitute:

$$m_* + m_p = \frac{P}{2\pi G}\frac{(v_{*r} + v_{pr})^3}{\sin^3 i}$$

Using the correlation $v_{pr} = v{*}r\frac{m_*}{m_p}$ returns:

$$m_* + m_p = \frac{P v_{*r}^3}{2\pi G \sin^3 i}(1 + \frac{m_*}{m_p})^3$$

If the star is far more massive than the planet, which tends to be the case, the expression can be simplified to:

$$m_* = \frac{Pv_{*r}^3}{2\pi G \sin^3 i} \frac{m_*^3}{m_p^3} \Rightarrow m_p \sin i = \frac{m_*^{2/3} v_{*r} P^{1/3}}{(2\pi G)^{1/3}}$$

Where m_p is the planets mass, i is the inclination angle of the system, m_* is the mass of the star (here 1.57 solar masses), v_{*r} is the star's maximum radial velocity, P is the system's oscillation period, and G is Newton's gravitational constant.

Results

Following is a table of results, where N is the number of calculations for the parameters v_r, P and t_0, and m_p is the lower bound mass of the planet (assuming $i = 90°$) calculated with said parameters.

Least-Square Calculation Results			
x	$N = 20$	$N = 50$	$N = 80$
Radial Velocity v_r	$31.6316\frac{m}{s}$	$31.4898\frac{m}{s}$	$31.5949\frac{m}{s}$
Orbital Period P	$4580.16d$	$4587.14d$	$4600.58d$
Peak time t_0	$1635.16d$	$1633.31d$	$1621.95d$
Least mass m_p	$1.505 \cdot 10^{26}kg$	$1.499 \cdot 10^{26}kg$	$1.502 \cdot 10^{26}kg$
Program run-time	$3.83s$	$30.43s$	$122.40s$

Comments

The results produced by the program varied slightly but it is safe to say that they become more accurate when the number of calculations N increase. The values calculated with the Least-Square method are verifiable (at least somewhat) by taking by-eye measurements of the parameters on the top graph in figure 1. The parameters calculated seem to agree very well with by-eye measurements. The mass of the planet turned out to be quite massive although this is the minimal mass (assuming $i = 90°$), so it is likely more massive than this. It is clear that the inclination angle is not too far off 90° since the planet eclipsed the star in the first place. However, since the flux drops (from graph two) had such a short time interval Δt, it is likely that the planet eclipsed the stars edge and not at the equator. Other inaccuracies in mass could come from assuming the star mass is much larger than the planets mass, but this is also a very small factor in most cases. It may therefore be that the calculations of the planet mass are a few decimals off, but no more than that.

Conclusion

The data received by our telescopes will often be plagued by inaccuracies produced in our line of sight of the object measured. Large inaccuracies in the data are caused by atmospheric discrepancies and instabilities. There are many ways to minimize these data inaccuracies such as the Least-Square method and other more complicated methods such as the Laser guide star technique, which are designed to do just that. These inaccuracies caused by atmospheric distortion of the light received will always be a factor when measuring from the Earth's surface. This is why the best measurements and images of the cosmos are taken by telescopes such as the Hubble Space Telescope in orbit around the earth, in the vacuum of space.

Summary

We want our data to be as accurate as possible in order to perfect our experimental and observational physics models. Getting our theoretical calculations to match our astronomical observations accurately includes having both methods include as little inaccuracy as possible. The more accurate the data the more accurate our testing of our models becomes and we can fine-tune some theoretical calculations to match with our astronomical observations.

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