Assignment 1

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Exercise 1.

$P(A) = 1/2$

$P(B'|A) = 3/4$

$P(A \cup B) = 3/4$

A) $P(A \cap B)$: We have that:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A) \cdot P(A) \quad ①$$

$P(B'|A) = 1 - P(B|A) = 3/4 \Rightarrow P(B|A) = 1 - 3/4 = 1/4$

$① \rightarrow P(A \cap B) = 1/4 \cdot 1/2 = 1/8$

B) $P(A \cap B) = P(A) + P(B) - P(A \cap B) \Rightarrow$

$P(B) = P(A \cap B) - P(A) + P(A \cap B) \Rightarrow$

$P(B) = 1/8 - 1/2 + 3/4 = 3/8$

Γ) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{3/8} = \frac{1}{3}$

Δ) For A, B to be independent:

$P(A \cap B) = P(A) \cdot P(B)$

$P(A) = 1/2$, $P(B) = 3/8$, $P(A \cap B) = 1/8$

$P(A) \cdot P(B) = 3/16 \neq 1/8 \Rightarrow$

$P(A) \cdot P(B) \neq P(A \cap B)$ so A, B are not independent.

Exercise 2.

For the specific problem we get the geometric random variable $p = 1/6$ where each trial is independent.

a) The probability mass function is (PMF):

$$P(X=n) = (1-p)^{n-1}\, p.$$

$p = 1/6 \Rightarrow p(X=n) = (5/6)^{n-1} \cdot 1/6;\ n \in \mathbb{Z}_{>0}$.

b) For 6 to appear for the first time in the 10th roll we have: $X = 10$.

$$P(X=10) = (5/6)^{10-1} \cdot 1/6 = \left(\frac{5}{6}\right)^{9} \cdot \frac{1}{6} = \frac{5^9}{6^{10}}.$$

c) $E(X) = \frac{1}{p}$, $p = 1/6 \Rightarrow E(X) = \frac{1}{1/6} = 6.$

Exercise 3

- Transmission Probabilities: $P(0) = 0.6$, $P(1) = 0.4$.
- Error Probabilities: $P(1|0) = 0.1$, $P(0|1) = 0.2$
- Correct Transmission Probabilities
  $P(0|0) = 1 - P(1|0) = 0.9$ and $P(1|1) = 1 - P(0|1) = 0.8$

a) if "1" is received, what is the probability that "1" was actually sent?

From Bayes' Theorem:

$$P(1 \mid \text{received } 1) = \frac{P(\text{received } 1 \mid 1)\, P(1)}{P(\text{received } 1)}$$

$$P(\text{received } 1) = P(\text{received } 1 \mid 1)\, P(1) + P(\text{received } 1 \mid 0)\, P(0)$$

We have that: $P(1 \mid \text{received } 1) = \dfrac{P(1|1) \cdot P(1)}{P(\text{received } 1)} = \dfrac{0.8 \cdot 0.4}{0.38} = \dfrac{16}{19}$

b) If two "0" are received consecutively what is the probability that two "0" symbols were sent.

- Let the events: A: two "0" are sent, B: two "0" are received.

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}.$$

$P(B|A) = P(0|0) \cdot P(0|0) = 0.9^2 = 0.81$. in case two "0" sent and two "0" received.

- Two "0" are sent: $P(A) = P(0) \cdot P(0) = 0.6^2 = 0.36$
- Total probability of receiving two "0" symbols.
  Two cases: i) two "0" sent and received correctly
  ii) two "1" sent but both receives as "0"

$$P(B) = P(B|A)\,P(A) + P(B \mid \text{two 1's sent})\, P(\text{two 1's sent})$$

$P(B \mid \text{two 1's sent}) = P(0|1) \cdot P(0|1) = 0.2^2 = 0.04$

$P(\text{two 1's sent}) = P(1) \cdot P(1) = 0.4^2 = 0.16$.

So we have that.

$$P(B) = 0.81 \cdot 0.36 + 0.04(0.16)$$
$$= 0.2916 + 0.0064 = 0.298.$$

From Baye's theorem.

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$
$$= \frac{0.81 \cdot 0.36}{0.298} = \frac{0.2916}{0.298} \simeq 0.978$$

So, i) P(1 | received 1) = 0.849.

ii) P(two "0" sent | two "0" received) = 0.978.

Exercise 4.

To verify that $g(x)$ is a valid pdf two conditions must be satisfied:

1. $g(x) \geq 0 \ \forall x$
2. $\int_{-\infty}^{+\infty} g(x)dx = 1.$

- 1^{st} condition:
  - For the first condition we have that:
    for $x \geq x_0$ $f(x)$ is the pdf of X so $f(x) \geq 0$.
    Moreover $1 - F(x_0) > 0 \Rightarrow F(x_0) < 1$
    So $g(x) \geq 0$ for $x \geq x_0$.
  - Also for $x < x_0$ $g(x) = 0$ so $g(x) \geq 0$ here as well.
    So $g(x) \geq 0 \ \forall x$.

- 2^{nd} condition:

$$\int_{-\infty}^{+\infty} g(x)dx = \int_{x_0}^{+\infty} g(x)dx + \int_{-\infty}^{x_0} g(x)dx.$$

We have $g(x) = 0$ for $x < x_0$ so

$$\int_{-\infty}^{+\infty} g(x)dx = \int_{x_0}^{+\infty} \frac{f(x)}{1-F(x_0)} dx \Leftrightarrow$$

$$\int_{x_0}^{+\infty} \frac{f(x)}{1-F(x_0)} dx = \frac{1}{1-F(x_0)} \int_{x_0}^{+\infty} f(x)dx$$

$\int_{x_0}^{+\infty} f(x)dx = 1 - F(x_0)$ so we get:

$\frac{1}{1-F(x_0)}(1 - F(x_0)) = 1$. which means that

$$\int_{-\infty}^{+\infty} g(x)dx = 1.$$

So the function $g(x)$ is a valid pdf.

Exercise 5

"Poisson" problem: with a rate $\lambda = 3$ arrivals/minute.
Generally, the probability for k arrivals in t period is

$$P(N(t)=k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k=0,1,2,\dots$$

λt is avg number of arrivals.

a) The average number of arrivals is $\lambda t = 3$.

$$P(N(1)=0) = \frac{e^{-3}(3)^0}{0!} = e^{-3} \simeq 0.05$$

b) $P(N(1) \geq 3) = 1 - P(N(1) < 3)$

$= 1 - [P(N(1)=0) + P(N(1)=1) + P(N(1)=2)]$

$$= 1 - \left[e^{-3} + \frac{e^{-3} \cdot 3}{1!} + \frac{e^{-3}(3)^2}{2!}\right]$$

$= 1 - [e^{-3} + 3e^{-3} + 4.5e^{-3}] = 1 - 8.5e^{-3} = 0.575$

c) $\lambda t = 3 \cdot 3 = 9$.

We want $P(N(3) \leq 5) = \sum_{k=0}^{5} \frac{e^{-9}(9)^k}{k!}$.

We have.

$$P(N(3)=0) = \frac{e^{-9} \cdot 9^{(0)}}{0!} = e^{-9}$$

$P(N(3)=1) = e^{-9} \cdot 9/1! = 9e^{-9}$

$P(N(3)=2) = 81e^{-9}/2! = 40.5e^{-9}$

$P(N(3)=3) = 729 \cdot e^{-9}/6 = 121.5e^{-9}$

$P(N(3)=4) = 6561e^{-9}/24 = 273.4e^{-9}$

$P(N(3)=5) = 59049e^{-9}/120 = 492e^{-9}$

So $P(N(3) \leq 5) = 937.45e^{-9} \simeq 0.115$

Exercise 6.

We have that the probability density function PDF for a single observation x_i is.

$$f(x_i;k,\lambda) = \frac{k}{\lambda}\left(\frac{x_i}{\lambda}\right)^{k-1} e^{-(x_i/\lambda)^k}$$

So for n independent observations $X_1, X_2, \dots X_n$ the joint PDF is: $f(X_1, X_2, \dots, X_n; \lambda) = \prod_{i=1}^{n} f(x_i; k, \lambda)$

Substituting the expression $f(x_i;k,\lambda)$ we have.

$$f(X_1, X_2, \dots X_n) = \prod_{i=1}^{n} \frac{k}{\lambda}\left(\frac{x_i}{\lambda}\right)^{k-1} e^{-(x_i/\lambda)^k}$$

$$\Leftrightarrow f(X_1, X_2, \dots X_n; \lambda) = \left(\frac{k}{\lambda}\right)^n \prod_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{k-1} e^{-\sum_{i=1}^{n}(x_i/\lambda)^k}.$$

Using the factorization theorem.

$$f(X_1, X_2, \dots X_n; \lambda) = g(T(x);\lambda) \cdot h(X_1, X_2, \dots, X_n)$$

where $T(x) = \sum_{i=1}^{n} X_i^k$.

Factorizing PDF:

$$g(T(x);\lambda) = \frac{k^n}{\lambda^{nk}} e^{-1/\lambda^k T(x)}$$

$$h(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} X_i^{k-1}$$

So the sufficient statistic is $T(x) = \sum_{i=1}^{n} X_i^k$

Exercise 7

a) The CI for the mean is $\bar{x} = \pm t_{u-1,\alpha/2} \frac{S}{\sqrt{u}}$

- Sample 1.

$\bar{x}_1 = 45.3,\ S_1^2 = 4.1,\ u_1 = 15,\ S_1 = \sqrt{4.1} = 2.02.$

Degrees of freedom: $u_1 - 1 = 14$

$t_{14,0.025}$ for 95% $t \simeq 2.145$

Margin of error $= 2.145 \cdot \frac{2.02}{\sqrt{15}} = 1,116$

CI for μ_1: $45 \pm 1.116 = (44,18, 46.42)$

- Sample 2.

$\bar{x}_2 = 47.8,\ S_2^2 = 39,\ u_2 = 18,\ S_2 = \sqrt{3.9} = 1.97$

Degrees of freedom $= u_2 - 1 = 17.$

$t_{17,0.025}$: $t \simeq 2.110.$

Margin of error $= t \cdot \frac{S_2}{\sqrt{u_2}} = 2.110 \cdot \frac{1.97}{\sqrt{18}} \simeq 0.980$

CI for μ_2: $= 47.8 \pm 0.980 = (46.82, 48.78)$

b) $H_0: \mu_1 = \mu_2,\ H_a: \mu_1 \neq \mu_2.$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_1^2}{u_1}} + \sqrt{\frac{S_2^2}{u_2}}} \simeq \frac{45.3 - 47.8}{\sqrt{\frac{4.1}{15}} + \sqrt{\frac{3.9}{18}}} = \frac{-2.5}{\sqrt{0.49}} \simeq -3.57$$

$$v = \frac{\left(\frac{S_1^2}{u_1} + \frac{S_2^2}{u_2}\right)^2}{\frac{(S_1^2/u_1)^2}{Df_1} + \frac{(S_2^2/u_2)^2}{Df_2}} = \frac{0.49^2}{0.005 + 0.003} \simeq 29.64 \simeq 30.$$

Critical value $t \simeq 2,042 \Rightarrow |t| > 2.042$ H_0 rejected at (5%)

c) $H_0: \mu_1 = 45$, $H_a: \mu_1 > 45$.

$$t = \frac{\bar{x}_1 - \mu_0}{s_1/\sqrt{n_1}}$$

$$t = \frac{45.3 - 45}{2.02/\sqrt{15}} = \frac{0.3}{0.5215} \simeq 0.575.$$

Degrees of freedom $n_1 - 1 = 14$
Critical value $t_{14,0,01}$ for 1% $t \simeq 2.62$

$t = 0.575 < 2.62$ we cannot reject H_0.

So the mean concentration of the first sample is not significantly greater 45 mg at 1% significance level.

Transmission Probabilit...

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So the mean concentration of the first sample is not significantly greater 45 mg at 1% significance level.