Homework 8

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1) Correct 4
2) Correct 2,3,4
3) Correct 2,4
4) Correct 4
5) Correct 2,3
6) Correct 1
7) Correct 2,4
8) Correct 3
9) Correct 1,4
10) Correct 1,4
11) Correct 2, 3
12) Correct 2
13) Correct 1
14) Correct 1,2
15) Correct 2
16) Correct 4
17) Correct 2
18) Correct 3
19) Correct 1,3,5
20) Correct 2,4
21) Correct 4
22) Correct 3,4
23) Correct 1,2
24) Correct 1,4
25) Correct 2
26) Correct 1
27) Correct 1, 3
28) Correct 2, 3
29) Correct 1
30) Correct 3
31) Correct 1,3
32) Correct 1,3
33) Correct 2, 3
34) Correct 1,2,4
35) Correct 1,2,3
36) Correct 5,7
37) Correct 1, 3
38) Correct 1,4
39) Correct 3
40) Correct 2,4,6,7
41) Correct 1, 3, 5
42) Correct 3
43) Correct 2.
44) Correct 2,3
45) Correct 1,4
46) Correct 3
47) Correct 2,4

Exercise 48

The set is $[x_1, x_2, x_3, x_4, x_5]$

$P(\omega_1) = \frac{2}{5}$ and $P(\omega_2) = \frac{3}{5}$

Thus, the entropy of the root node is $I = -(P(\omega_1)\log_2 P(\omega_1) + P(\omega_2)\log_2 P(\omega_2))$

$\Rightarrow I = 0.9710$

Computation of the entropy reduction for the second coordinate and value 8 $\quad x_2 \leq 8$

"Yes": It is $[x_2, x_3, x_4] \Rightarrow \frac{N_Y}{N} = \frac{3}{5}$ so,

- $P_Y(\omega_1) = \frac{1}{3}$ and $P_Y(\omega_2) = \frac{2}{3}$

$I_Y = -(P_Y(\omega_1)\log_2(P_Y(\omega_1)) + P_Y(\omega_2)\log_2(P_Y(\omega_2))) = 0.9183$

- "No" It is $[x_1, x_5] \Rightarrow \frac{N_N}{N} = \frac{2}{5}$

$P_N(\omega_1) = \frac{1}{2}$, $P_N(\omega_2) = \frac{1}{2}$

$I_N = -(P_N(\omega_1)\log_2(P_N(\omega_1)) + P_N(\omega_2)\log_2(P_N(\omega_2))) = 1.$

So $\Delta I = I - \frac{N_Y}{N} I_Y - \frac{N_N}{N} I_N$

$= 0.9710 - 0.6 \cdot 0.9183 - 0.4 \cdot 1 = 0.02.$

Exercise 49:

Suppose you are given a data set $Y = \{(y_i, \boldsymbol{x}_i'), i = 1, \dots, N\}$ where $y_i \in \{0,1\}$ is the class label for vector $\boldsymbol{x}_i' \in R^l$. Assume that y and $\boldsymbol{x}'$ are related via the following model: $y = f(\boldsymbol{\theta}^T \boldsymbol{x}' + \theta_0)$, where $\boldsymbol{\theta}$ and θ_0 are the model parameters and $f(z) = 1/(1 + exp(-az))$.

(a) **Plot** the function $f(z)$ for various values of the parameter a.
(b) Propose a gradient descent scheme to **train** this model (that is, to estimate the values of the involved parameters), based on the **minimization** of the sum of error squares criterion, using Y.
(c) Can the model ever respond with a **"clear"** 1 or a **"clear"** 0, for a given $\boldsymbol{x}$?
(d) How can we interpret the response of the model for a given $\boldsymbol{x}$?
(e) Propose a way for leading the model responses very close to 1 (for class 1 vectors) or 0 (for class 0 vectors).

Hints:

(a) Use a more compact notation by setting $\boldsymbol{x}_i = [1\ \boldsymbol{x}_i']^T$, $i = 1, \dots, N$, and $\boldsymbol{\theta} = [\theta_0\ \boldsymbol{\theta}]^T$. The model then becomes $y = f(\boldsymbol{\theta}^T \boldsymbol{x})$.
(b) The sum of error squares criterion in this case is $J(\boldsymbol{\theta}) = \sum_{n=1}^{N} \left(y_n - f(\boldsymbol{\theta}^T \boldsymbol{x}_n)\right)^2$ (Consult also the relative pdf file about optimization theory basics, uploaded in e-class).

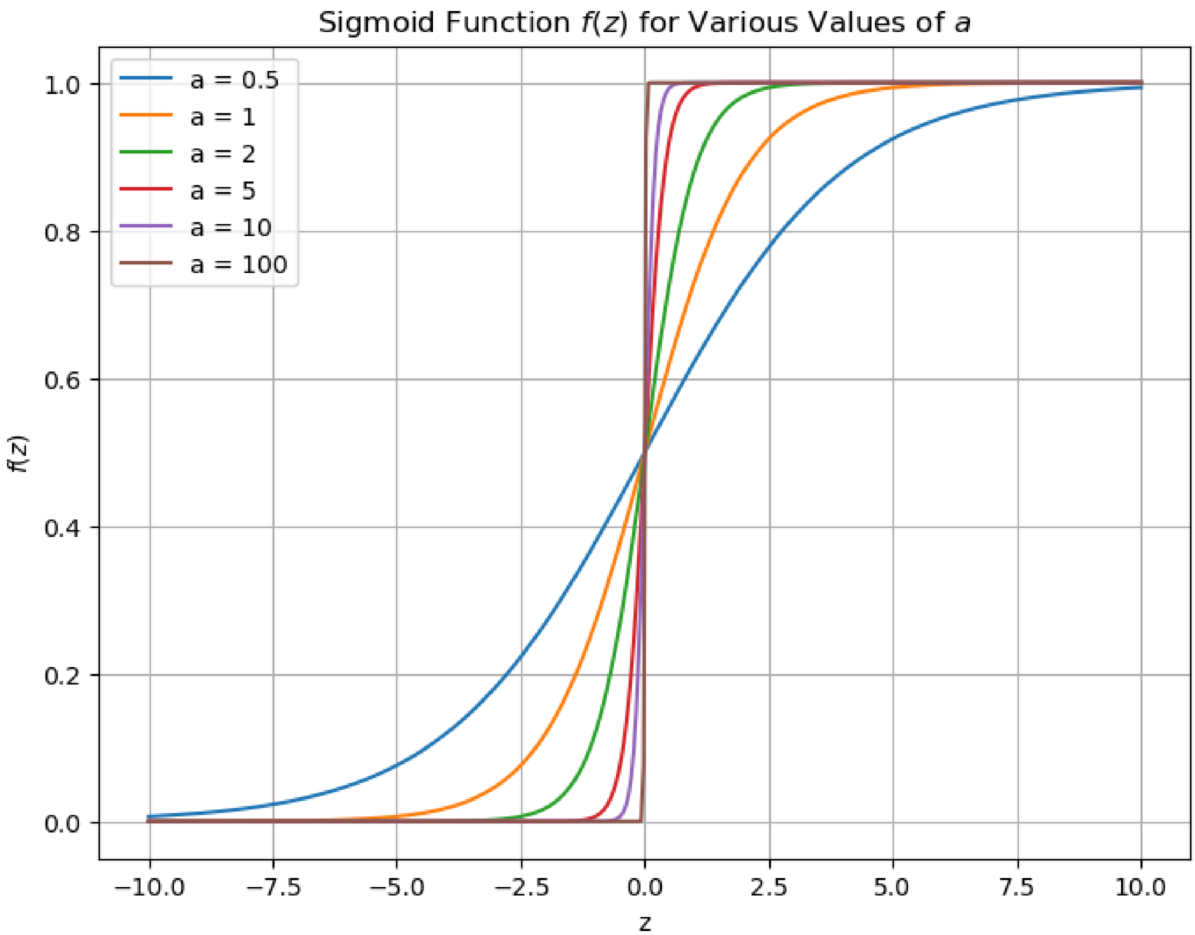
It is $f'(z) = \frac{df(z)}{dz} = af(z)\left(1 - f(z)\right)$.

(a) Plot the function (f(z)) for various values of the parameter (a)

In [9]:
```
import scipy.io as sio
import numpy as np
import matplotlib.pyplot as plt

# Load the dataset
Dataset = sio.loadmat('HW8.mat')
train_x = Dataset['train_x']
train_y = Dataset['train_y']

test_x = Dataset['test_x']
test_y = Dataset['test_y']

# Values for z from -10 to 10
z = np.linspace(-10, 10, 400)

# Different values for a
a_values = [0.5, 1, 2, 5,10,100]

# Plot f(z) for different values of a
plt.figure(figsize=(8, 6))
```

```
for a in a_values:
    f_z = 1 / (1 + np.exp(-a * z))
    plt.plot(z, f_z, label=f'a = {a}')

plt.title('Sigmoid Function $f(z)$ for Various Values of $a$')
plt.xlabel('z')
plt.ylabel('$f(z)$')
plt.legend()
plt.grid(True)
plt.show()
```

```
C:\Users\steli\AppData\Local\Temp\ipykernel_7808\2985167747.py:22: RuntimeWarning: o
verflow encountered in exp
  f_z = 1 / (1 + np.exp(-a * z))
```



(b) Propose a gradient descent scheme to train this model

(b) Gradient Descent Scheme for Estimating Parameters

To estimate the parameters θ, we aim to minimize the cost function $J(\theta)$. The gradient descent update rule to achieve this is as follows:

$$\theta_i = \theta_{i-1} - \mu \nabla_\theta J(\theta) \quad (16)$$

where the cost function $J(\theta)$ is defined by the sum of squared errors:

$$J(\theta) = \sum_{n=1}^{N} \left(y_n - f(\theta^T x_n)\right)^2$$

In this formulation, we have incorporated the bias term θ_0 into the parameter vector θ by augmenting the feature vector x_n to include a 1 as its first component. This allows us to treat θ_0 as part of θ.

The gradient of the cost function, $\nabla_\theta J(\theta)$, with respect to the parameter vector is given by:

$$\nabla_\theta J(\theta) = -2 \sum_{n=1}^{N} \left(y_n - f(\theta^T x_n)\right) \left(a f(\theta^T x_n)(1 - f(\theta^T x_n))\right) x_n \quad (17)$$

Where $f(z) = \frac{1}{1+\exp(-z)}$ is the sigmoid function, and a is a scaling factor. The term $f(\theta^T x_n)(1 - f(\theta^T x_n))$ corresponds to the derivative of the sigmoid function.

Using this gradient, we update the parameters θ at each iteration of the gradient descent process:

$$\theta_i = \theta_{i-1} + 2\mu \sum_{n=1}^{N} \left(y_n - f(\theta^T x_n)\right) \left(a f(\theta^T x_n)(1 - f(\theta^T x_n))\right) x_n \quad (18)$$

Where:

- μ represents the learning rate, controlling the size of each update.
- θ_i is the parameter vector at iteration i, and θ_{i-1} is the parameter vector at the previous iteration.
- x_n is the augmented feature vector of the n-th sample.

Through this iterative process, the parameters θ are updated in a way that minimizes the cost function $J(\theta)$, gradually improving the model's predictions as the iterations progress.

(c) Can the model ever respond with a “clear” 1 or a “clear” 0, for a given (\mathbf{x})?

No, the logistic regression model (sigmoid function) cannot give an absolute 1 or 0. It outputs a probability between 0 and 1. The closer the value is to 1 or 0, the more confident the model is in its prediction, but it will never give a "hard" 1 or 0 unless we apply a threshold, such as 0.5, to make a decision.

(d) How can we interpret the response of the model for a given (\mathbf{x})?

The response of the model can be interpreted as the probability that the input vector $\mathbf{x}$ belongs to class 1. For example, if $f(z) = 0.8$, it means the model predicts an 80% probability that the sample belongs to class 1, and 20% probability for class 0.

(e) Propose a way for leading the model responses very close to 1 (for class 1 vectors) or 0 (for class 0 vectors)

To make the model's responses closer to 1 or 0 for class 1 or class 0 vectors, we can:

1. **Regularization**: Apply a regularization term (e.g., L2 regularization) to penalize large parameter values and encourage sharper predictions.
2. **Increase the steepness of the sigmoid function**: By increasing the value of (a) in the sigmoid function, the model will become more confident in its predictions. However, this can also make the model less generalizable. This is also observed in question (a).
3. **Thresholding**: After training, apply a thresholding mechanism to convert the probabilities into clear class labels. For example, classify anything above 0.9 as class 1, and anything below 0.1 as class 0.

Exercise 50 (python code + text):

Consider a two-class, two-dimensional classification problem for which you can find attached two sets: one for training and one for testing (file *HW8.mat*). Each of these sets consists of pairs of the form $(y_i, \boldsymbol{x}_i)$, where y_i is the class label for vector $\boldsymbol{x}_i$. Let N_{train} and N_{test} denote the number of training and test sets, respectively. The data are given via the following arrays/matrices:

- ***train_x*** (a $N_{train} \times 2$ **matrix** that contains in its rows the training vectors $\boldsymbol{x}_i$)
- ***train_y*** (a N_{train}–dim. column **vector** containing the **class labels** (1 or 2) of the corresponding training vectors $\boldsymbol{x}_i$ included in ***train_x***).
- ***test_x*** (a $N_{test} \times 2$ **matrix** that contains in its rows the test vectors $\boldsymbol{x}_i$)
- ***test_y*** (a N_{test}–dim. column **vector** containing the **class labels** (1 or 2) of the corresponding test vectors $\boldsymbol{x}_i$ included in ***test_x***).

Assume that the two classes, ω_1 and ω_2 are modeled by normal distributions.

(a) Adopt the **k-nearest neighbor classifier**, for $k = 5$ and estimate the classification error probability.

(b) Depict graphically the training set, using different colors for points from different classes.

(c) Report the classification results obtained by the k-NN classifier and compare them with the results obtained by the Bayes and the naïve Bayes classifier (see relevant exercises in HW7 and HW7a).

Hint: Use the attached Python code in file *HW8.ipynb* (also given in Homework 7).

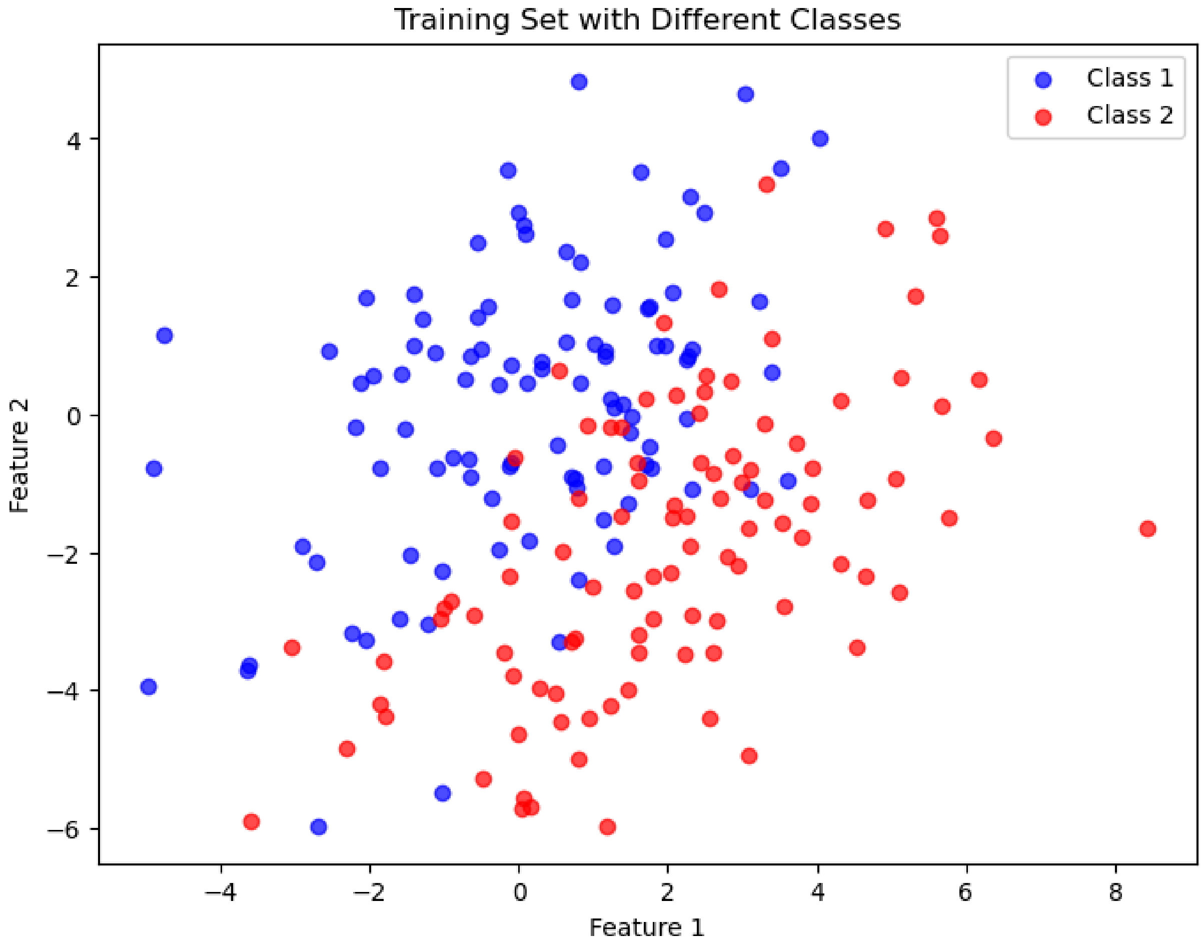
(a) Adopt the k-nearest neighbor classifier, for $k = 5$ and estimate the classification error probability.

In [35]:
```python
import scipy.io as sio
import numpy as np
from sklearn.neighbors import KNeighborsClassifier
from sklearn.metrics import accuracy_score

# Load the dataset from HW8.mat
Dataset = sio.loadmat('HW8.mat')
train_x = Dataset['train_x']
train_y = Dataset['train_y'].flatten()  # Convert to 1D array if it's not already
test_x = Dataset['test_x']
test_y = Dataset['test_y'].flatten()  # Convert to 1D array if it's not already

# Initialize the k-NN classifier with k=5
knn = KNeighborsClassifier(n_neighbors=5)

# Fit the classifier on the training data
knn.fit(train_x, train_y)

# Predict on the test set
predictions = knn.predict(test_x)

# Calculate the classification error
error = 1 - accuracy_score(test_y, predictions)

print(f"Classification Error Probability: {error:.4f}")
```

```
Classification Error Probability: 0.1700
```

(b) Depict graphically the training set, using different colors for points from different classes.

In [39]:
```python
import matplotlib.pyplot as plt

# Plot the training data
plt.figure(figsize=(8, 6))

# Class 1 points (train_y == 1)
class_1 = train_x[train_y == 1]
# Class 2 points (train_y == 2)
class_2 = train_x[train_y == 2]

# Scatter plot for class 1 and class 2
plt.scatter(class_1[:, 0], class_1[:, 1], color='blue', label='Class 1', alpha=0.7)
plt.scatter(class_2[:, 0], class_2[:, 1], color='red', label='Class 2', alpha=0.7)

# Adding labels and title
plt.title('Training Set with Different Classes')
plt.xlabel('Feature 1')
plt.ylabel('Feature 2')
plt.legend()

# Show the plot
plt.show()
```



(c) Report the classification results obtained by the k-NN classifier and compare them with the results obtained by the Bayes and the naïve Bayes classifier (see relevant exercises in HW7 and HW7a).

The classification error rates for each classifier on the test set are as follows:

k-NN Classifier:

- **Error Rate**: 0.1700

Bayes Classifier:

- **Error Rate**: 0.150

Naïve Bayes Classifier:

- **Error Rate**: 0.165

Comparison:

- The **Bayes classifier** outperforms both the **Naïve Bayes classifier** and the **k-NN classifier** by achieving the lowest error rate (0.150).

- The **Naïve Bayes classifier** performs slightly worse than the **Bayes classifier** due to its assumption of feature independence, resulting in a higher error rate of 0.165.
- The **k-NN classifier** has an error rate of 0.1700, which is slightly worse than both the **Naïve Bayes** and **Bayes classifiers**, showing that while **k-NN** is a strong classifier, it does not perform as well as the probabilistic models on this dataset.

```
In [ ]:
```