Homework 7
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Answers
1) Correct 1, 4
2) Correct 2, 3
3) Correct 1, 3
4) Correct 3
5) Correct 2
6) Correct 3
7) Correct 2
8) Correct 1
9) Correct 2
10) Correct 4
11) Correct 3
12) Correct 4

Exercise 13: Consider a two-class 1-dim. problem where the classes ω_1 and ω_2 are modeled by the following uniform distributions

$$p(x|\omega_1) = \begin{cases} 1/5, & x \in (0,1) \cup (4,8) \\ 0, & \text{otherwise} \end{cases} \qquad p(x|\omega_2) = \begin{cases} 1/9, & x \in (0,9) \\ 0, & \text{otherwise} \end{cases}$$

(I) Assume that the classes are equiprobable ($P(\omega_1) = P(\omega_2)$).

(i) Depict graphically in the same figure $P(\omega_i)p(x|\omega_i), i = 1,2,$ (as functions of x) and identify the respective decision regions, as they are specified by the Bayes classifier.
(ii) Classify the point $x' = 3.5$ to one of the two classes using the Bayes classifier.

(II) Assume that the classes are **not** equiprobable.

(i) Determine a set of values for the a priori probabilities of the two classes that guarantee that $x' = 5$ is assigned to class ω_2. Justify briefly your choice.

(ii) Is there any combination of the class priori probabilities that guarantees that $x' = 3$ will be assigned to ω_1? Explain.

Hints:

(<u>*H1*</u>) Focus only in the interval [0,10] since all pdfs are zero out of this interval. Note that $P(\omega_1) + P(\omega_2) = 1$.

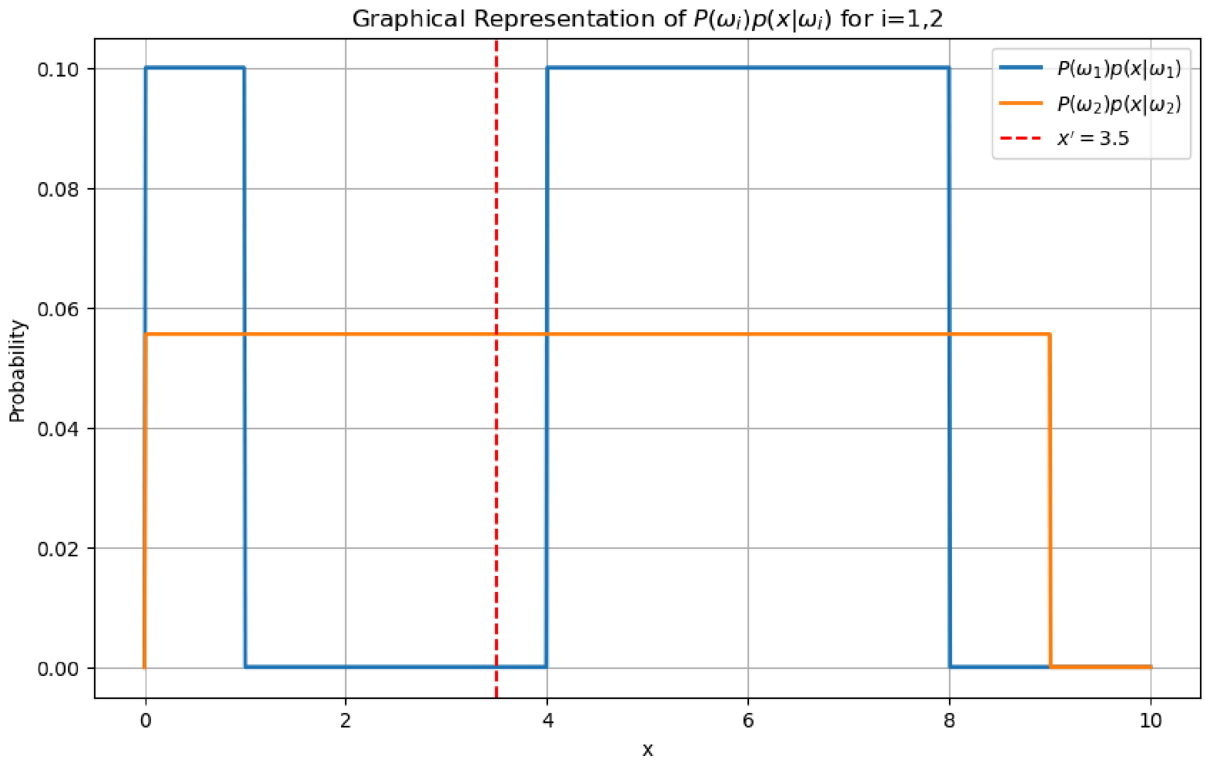
(<u>*H2*</u>) For (II-i): Consider the inequality $P(\omega_1)p(x'|\omega_1) < P(\omega_2)p(x'|\omega_2)$ and the fact that $P(\omega_1) + P(\omega_2) = 1$. Work similarly for (II-ii).

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
from sympy import symbols, solve, Eq

# Define the probability density functions
def p_x_given_w1(x):
    return (1/5) * ((0 < x < 1) or (4 < x < 8))

def p_x_given_w2(x):
    return (1/9) * (0 < x < 9)

# (I.i) Plot scaled PDFs
P_w1 = 0.5
P_w2 = 0.5
x_values = np.linspace(0, 10, 1000)
P_w1_p_x_given_w1 = P_w1 * np.array([p_x_given_w1(x) for x in x_values])
P_w2_p_x_given_w2 = P_w2 * np.array([p_x_given_w2(x) for x in x_values])

plt.figure(figsize=(10, 6))
plt.plot(x_values, P_w1_p_x_given_w1, label=r"$P(\omega_1)p(x|\omega_1)$", linewidt
plt.plot(x_values, P_w2_p_x_given_w2, label=r"$P(\omega_2)p(x|\omega_2)$", linewidt
plt.axvline(x=3.5, color='red', linestyle='--', label=r"$x'=3.5$")
plt.title("Graphical Representation of $P(\omega_i)p(x|\omega_i)$ for i=1,2")
plt.xlabel("x")
```

```python
plt.ylabel("Probability")
plt.legend()
plt.grid()
plt.savefig("decision_regions.png")
plt.show()

# (I.ii) Classify x' = 3.5
x_prime = 3.5
p_x_w1 = p_x_given_w1(x_prime)
p_x_w2 = p_x_given_w2(x_prime)
decision_class = "ω1" if P_w1 * p_x_w1 > P_w2 * p_x_w2 else "ω2"
print(f"(I.ii) Point x' = {x_prime} is classified as class {decision_class}.")

# (II.i) Determine range for x' = 5 favoring ω2
P_w1_sym, P_w2_sym = symbols('P_w1 P_w2')
P_w2_eq = Eq(P_w2_sym, 1 - P_w1_sym)
x_prime_5 = 5
p_x_5_w1 = p_x_given_w1(x_prime_5)
p_x_5_w2 = p_x_given_w2(x_prime_5)
inequality_5 = P_w1_sym * p_x_5_w1 < P_w2_sym * p_x_5_w2
inequality_5 = inequality_5.subs(P_w2_eq.lhs, P_w2_eq.rhs)
P_w1_range_5 = solve(inequality_5, P_w1_sym)
print(f"(II.i) For x' = 5 to favor ω2, P(ω1) must satisfy: {P_w1_range_5}")

# (II.ii) Determine range for x' = 3 favoring ω1
x_prime_3 = 3
p_x_3_w1 = p_x_given_w1(x_prime_3)
p_x_3_w2 = p_x_given_w2(x_prime_3)
inequality_3 = P_w1_sym * p_x_3_w1 > P_w2_sym * p_x_3_w2
inequality_3 = inequality_3.subs(P_w2_eq.lhs, P_w2_eq.rhs)
P_w1_range_3 = solve(inequality_3, P_w1_sym)
print(f"(II.ii) For x' = 3 to favor ω1, P(ω1) must satisfy: {P_w1_range_3}")
```

```
<>:23: SyntaxWarning: invalid escape sequence '\o'
<>:23: SyntaxWarning: invalid escape sequence '\o'
C:\Users\steli\AppData\Local\Temp\ipykernel_22304\883146398.py:23: SyntaxWarning: in
valid escape sequence '\o'
  plt.title("Graphical Representation of $P(\omega_i)p(x|\omega_i)$ for i=1,2")
```




```
(I.ii) Point x' = 3.5 is classified as class ω2.
(II.i) For x' = 5 to favor ω2, P(ω1) must satisfy: (-oo < P_w1) & (P_w1 < 0.35714285
7142857)
(II.ii) For x' = 3 to favor ω1, P(ω1) must satisfy: (1.0 < P_w1) & (P_w1 < oo)
```

In [ ]:

Exercise 14

To depict $P(\omega_1)p(x|\omega_1)$ and $P(\omega_2)p(x|\omega_2)$ we will find the decision regions

$$R_1: P(\omega_1)p(x|\omega_1) > P(\omega_2)P(x|\omega_2) \overset{P(\omega_1)=P(\omega_2)=1/2,\ \text{equib}}{\Longleftrightarrow}$$

$$P(x|\omega_1) > P(x|\omega_2)$$

$$\frac{1}{\cancel{\sqrt{2\pi}}} \exp\left(-\frac{(x+1)^2}{2}\right) > \frac{1}{2\cancel{\sqrt{2\pi}}} \exp\left(-\frac{(x-1)^2}{8}\right) \Leftrightarrow$$

$$\exp\left(-\frac{(x+1)^2}{2}\right) > \frac{1}{2} \exp\left(-\frac{(x-1)^2}{8}\right) \Leftrightarrow$$

$$2\exp\left(-\frac{(x+1)^2}{2}\right) > \exp\left(-\frac{(x-1)^2}{8}\right) \Leftrightarrow$$

$$\ln 2 - \frac{(x+1)^2}{2} > -\frac{(x-1)^2}{8} \Leftrightarrow$$

$$8\ln 2 - 4(x+1)^2 > -(x-1)^2 \Leftrightarrow$$

$$4x^2 + 8x + 4 - 8\ln 2 < x^2 - 2x + 1 \Leftrightarrow$$

$$3x^2 + 10x + 3 - 8\ln 2 < 0$$

$$\Delta = 100 - 36 + 96\ln 2 = 64 + 96\ln 2$$

$$x_{1,2} = \frac{-10 \pm \sqrt{64 + 96\ln 2}}{6} = \frac{-10 \pm 11.4}{6} \to x_1 = -3.5,\quad x_2 = 0.24$$

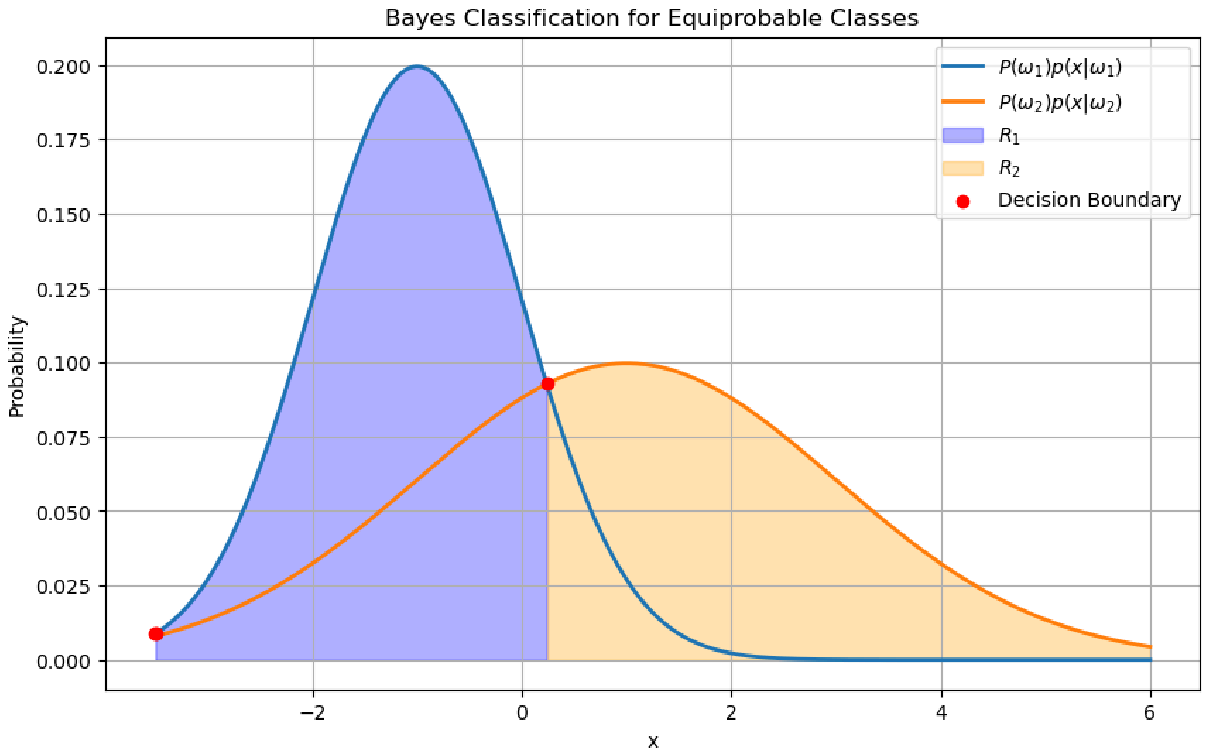
x		-3.5		0.24	
	$+$	0	$-$	0	$+$

Thus $R_1: x \in (-3.5, 0.24)$, R_2 $x \in (-\infty, -3.5) \cup (0.24, +\infty)$

Exercise 14: Consider a two-class 1-dim. classification problem of two equiprobable classes ω_1 and ω_2 ($P(\omega_1) = P(\omega_2)$) that are modeled by the normal distributions $N(-1,1)$ and $N(1,4)$, respectively. Depict the quantities $P(\omega_j)p(x|\omega_j)$ for $j = 1,2$, in the same graph and determine the decision regions R_1 and R_2 corresponding to the two classes, according to the Bayes classification rule.

In [4]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

# Define the parameters for the normal distributions
mu_w1, sigma_w1 = -1, 1  # N(-1, 1)
mu_w2, sigma_w2 = 1, 2   # N(1, 4) (std deviation is sqrt(4) = 2)

# Define the PDFs for each class
def p_x_given_w1(x):
    return norm.pdf(x, mu_w1, sigma_w1)

def p_x_given_w2(x):
    return norm.pdf(x, mu_w2, sigma_w2)

# Define the priors (equiprobable case)
P_w1 = 0.5
P_w2 = 0.5

# Generate x values
x_values = np.linspace(-3.5, 6, 1000)

# Calculate P(w_i)p(x|w_i)
P_w1_p_x_given_w1 = P_w1 * p_x_given_w1(x_values)
P_w2_p_x_given_w2 = P_w2 * p_x_given_w2(x_values)

# Find decision boundary (where P(w1)p(x|w1) = P(w2)p(x|w2))
decision_boundary = np.abs(P_w1_p_x_given_w1 - P_w2_p_x_given_w2) < 1e-3

# Plot the results
plt.figure(figsize=(10, 6))
plt.plot(x_values, P_w1_p_x_given_w1, label=r"$P(\omega_1)p(x|\omega_1)$", linewidt
plt.plot(x_values, P_w2_p_x_given_w2, label=r"$P(\omega_2)p(x|\omega_2)$", linewidt
plt.fill_between(x_values, 0, P_w1_p_x_given_w1, where=P_w1_p_x_given_w1 > P_w2_p_x
                 color='blue', alpha=0.3, label=r"$R_1$")
plt.fill_between(x_values, 0, P_w2_p_x_given_w2, where=P_w2_p_x_given_w2 > P_w1_p_x
                 color='orange', alpha=0.3, label=r"$R_2$")
plt.scatter(x_values[decision_boundary], P_w1_p_x_given_w1[decision_boundary], colo
            label="Decision Boundary", zorder=5)
plt.title("Bayes Classification for Equiprobable Classes")
plt.xlabel("x")
plt.ylabel("Probability")
plt.legend()
plt.grid()
plt.savefig("decision_regions_ex14_corrected.png")
plt.show()
```

```
# Print decision regions
decision_points = x_values[decision_boundary]
if len(decision_points) >= 2:
    print(f"Decision boundary points: {decision_points[0]:.2f}, {decision_points[-1
    print(f"Region R1: x < {decision_points[0]:.2f} or x > {decision_points[-1]:.2f
    print(f"Region R2: {decision_points[0]:.2f} <= x <= {decision_points[-1]:.2f}")
else:
    print("No clear decision boundaries found in this range.")
```




```
Decision boundary points: -3.50, 0.24
Region R1: x < -3.50 or x > 0.24
Region R2: -3.50 <= x <= 0.24
```

In [ ]:

Exercise 15

$p(x|\omega_1) = N(\mu_1, \Sigma_1)$, $\mu_1 = [6,0]^T$, $\Sigma_1 = 2I$

$p(x|\omega_2) = N(\mu_2, \Sigma_2)$, $\mu_2 = [0,6]^T$, $\Sigma_2 = 2I$

(a) $p(x|\omega_j) = \frac{1}{2\pi|\Sigma_j|^{1/2}} \exp\left(-\frac{([x_1,x_2]-[\mu_{11},\mu_{12}])\begin{pmatrix}2&0\\0&2\end{pmatrix}^{-1}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}\mu_{11}\\\mu_{12}\end{bmatrix}\right)}{2}\right)$

$= \frac{1}{4\pi} \exp\left(-\frac{([x_1,x_2]-[\mu_{11},\mu_{12}])\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}\mu_{11}\\\mu_{12}\end{bmatrix}\right)}{2}\right)$

$x_1 = [2,4]^T$

- $p(\omega_1)p(x_1|\omega_1) = \frac{1}{2}\cdot\frac{1}{4\pi} \exp\left(-\frac{([2,4]-[6,0])\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\left(\begin{bmatrix}2\\4\end{bmatrix}-\begin{bmatrix}6\\0\end{bmatrix}\right)}{2}\right)$

$= \frac{1}{8\pi} \exp\left(-\frac{[-4,4]\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\begin{bmatrix}-4\\-4\end{bmatrix}}{2}\right)$

$= \frac{1}{8\pi} \exp\left(-\frac{[-2\ 2]\begin{bmatrix}-4\\4\end{bmatrix}}{2}\right) = \frac{1}{8\pi}\exp(-8)$

$\simeq 0.04 e^{-8}$

- $p(\omega_2)p(x_1|\omega_2) = \frac{1}{8\pi} \exp\left(-\frac{([2,4]-[0,6])\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\left(\begin{bmatrix}2\\4\end{bmatrix}-\begin{bmatrix}0\\6\end{bmatrix}\right)}{2}\right) =$

$= \frac{1}{8\pi} \exp\left(-\frac{[2,-2]\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\begin{bmatrix}2\\-2\end{bmatrix}}{2}\right) = \frac{1}{8}\exp(-2)$

$\simeq 0.04 e^{-2}$, $\Rightarrow x_1 \in (\omega_2) \Leftrightarrow p(\omega_2)p(x_1|\omega_2) > p(\omega_1)p(x_1|\omega_1)$

$x_2 = [4,2]^T$

- $p(\omega_1)p(x_2|\omega_1) = \frac{1}{8\pi} \exp\left(\frac{([4,2]-[6,0])\begin{bmatrix}1/2&0\\0&1/2\end{bmatrix}\left(\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}6\\0\end{bmatrix}\right)}{2}\right)$

$= \frac{1}{8\pi}\exp(-2) \simeq 0{,}04e^{-2}$

- $p(\omega_2)p(x_2|\omega_2) = \frac{1}{8\pi} \exp\left(-\frac{([4,2]-[0,6])\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}\left(\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}0\\6\end{bmatrix}\right)}{2}\right)$

$= \frac{1}{8\pi}\exp(-8) \simeq 0{,}04e^{-8}$

$x_2 \in \omega_1 \Leftrightarrow p(\omega_1)p(x_2|\omega_1) > p(\omega_2)p(x_2|\omega_2)$

$x_3 = [2,2]^T$

- $p(\omega_1)p(x_3|\omega_1) = \frac{1}{8\pi} \exp\left(-\frac{[4,2]\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\begin{bmatrix}-4\\ 2\end{bmatrix}}{2}\right)$

$= \frac{1}{8\pi} \exp\left(-\frac{[-2,1]\begin{bmatrix}-4\\ 2\end{bmatrix}}{2}\right) \simeq 0{,}04\exp(-5)$

- $p(\omega_2)p(x_3|\omega_2) = \frac{1}{8\pi} \exp\left(-\frac{[2,-4]\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\begin{bmatrix}2\\ -4\end{bmatrix}}{2}\right) \simeq 0{,}04e^{-5}$

So x_3 lies on the decision boundary of ω_1, ω_2

b) $p(x|\omega_1)p(\omega_1) = p(x|\omega_2)p(\omega_2)$

$p(x|\omega_1) = p(x|\omega_2)$

$\exp\left(-\frac{(x-\mu_1)^T\Sigma^{1/2}(x-\mu_1)}{2}\right) = \exp\left(-\frac{(x-\mu_2)^T\Sigma^{1/2}(x-\mu_2)}{2}\right)$

$(x-\mu_1)^T\Sigma^{1/2}(x-\mu_1) = (x-\mu_2)^T\Sigma^{1/2}(x-\mu_2)$

$([x_1,x_2]-[6,0])\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\begin{bmatrix}6\\ 0\end{bmatrix}\right)$

$= ([x_1,x_2]-[0,6])\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\begin{bmatrix}0\\ 6\end{bmatrix}\right)$

$= \left[\frac{x_1-6}{2}, \frac{x_2}{2}\right]\begin{bmatrix}x_1-6\\ x_2\end{bmatrix} = \left[\frac{x_1}{2}, \frac{x_2-6}{2}\right]\begin{bmatrix}x_1\\ x_2-6\end{bmatrix}$

$= \frac{(x_1-6)^2}{2} + \frac{x_2^2}{2} = \frac{x_1^2}{2} + \frac{(x_2-6)^2}{2}$

$(x_1-6)^2 + x_2^2 = x_1^2 + (x_2-6)^2$

$\cancel{x_1^2} - 12x_1 + \cancel{36} + \cancel{x_2^2} = \cancel{x_1^2} + \cancel{x_2^2} - 12x_2 + \cancel{36}$

$\varepsilon: -x_1 + x_2 = 0$ is the line that separates the 2 classes

Exercise 16

$p(x|\omega_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$p(x|\omega_2) = \begin{cases} 1/2\sqrt{2\pi} & , x \in [-\sqrt{2\pi}, \sqrt{2\pi}] \\ 0 & , \text{otherwise} \end{cases}$

$x \in [-\sqrt{2\pi}, \sqrt{2\pi}]$, x belongs to ω_1 if:

$P(\omega_1|x_1) > P(\omega_2|x_2) \iff$

$p(x_1|\omega_1)\, p(\omega_1) > p(x_2|\omega_2)\, p(\omega_2) \iff$

$1/\sqrt{2\pi} \exp(-x^2/2) > 1/2\sqrt{2\pi} \iff$

$\exp(-x^2/2) > 1/2$

$x^2/2 < \ln 2 \iff x^2 < 2\ln 2 \iff |x| < \sqrt{2\ln 2}$

$-\sqrt{2\ln 2} < x < \sqrt{2\ln 2}$

Thus

Classifier: 1. Assign x to ω_1 if

$x \in [-\sqrt{2\pi}, \sqrt{2\pi}]$

$x \in (-\sqrt{2\ln 2}, \sqrt{2\ln 2})$

2. Assign x to ω_2 if

$x \in [-\sqrt{2\pi}, \sqrt{2\pi}$ and $|x| > \sqrt{2\ln 2}$

So Decision Regions are:

$R_1 = (-\infty, -\sqrt{2\pi}) \cup (-\sqrt{2\ln 2}, \sqrt{2\ln 2}) \cup (\sqrt{2\pi}, +\infty)$

$R_2 = (-\sqrt{2\pi}, -\sqrt{2\ln 2}) \cup (\sqrt{2\ln 2}, \sqrt{2\pi})$

Exercise 17

$x = [4, 1.5]^T$, $x' = [8, 1.5]^T$, using Parzen with $h=1$

classes are

Class ω_1: Points $x_1, \dots, x_8$

Class ω_2: Points $x_9, \dots x_{12}$

Priors $= \frac{\text{points}(\omega_i)}{\text{Total points}} \Rightarrow P(\omega_1) = \frac{8}{12} = \frac{2}{3}$, $P(\omega_2) = \frac{4}{12} = \frac{1}{3}$

$$P(x|\omega_1) = \frac{1}{8} \frac{1}{2\pi} \sum_{i=1}^{8} \exp\left(-\frac{(x-x_i)^T(x-x_i)}{2}\right)$$

$$P(x|\omega_2) = \frac{1}{4} \frac{1}{2\pi} \sum_{i=9}^{8} \exp\left(-\frac{(x-x_i)^T(x-x_i)}{2}\right)$$

$x = [4, 1.5]^T$

$\|x-x_1\| = \sqrt{18.25}$, $\|x-x_2\| = \sqrt{3.25}$, $\|x-x_3\| = \sqrt{18.25}$

$\|x-x_4\| = \sqrt{3.25}$, $\|x-x_5\| = \sqrt{81.25}$, $\|x-x_6\| = \sqrt{120.25}$

$\|x-x_7\| = \sqrt{135.25}$, $\|x-x_8\| = \sqrt{174.25}$, $\|x-x_9\| = \sqrt{27.25}$

$\|x-x_{10}\| = \sqrt{66.25}$, $\|x-x_{11}\| = \sqrt{27.25}$, $\|x-x_{12}\| = \sqrt{66.25}$

$$p(\omega_1)p(x|\omega_1) = \frac{2}{3} \frac{1}{16\pi} \sum_{i=1}^{8} \exp\left(-\frac{\|x-x_i\|^2}{2}\right) = 0.0523$$

$$p(\omega_2)p(x|\omega_2) = \frac{1}{3} \cdot \frac{1}{8\pi} \sum_{i=1}^{12} \exp\left(-\frac{\|x-x_i\|^2}{2}\right) = 0.00321 \cdot 10^{-5}$$

$\boxed{x \in \omega_1}$

$x' = [8, 1.5]$

$p(\omega_1)p(x'|\omega_1) \simeq 0.00321 \cdot 10^{-5}$

$p(\omega_2)p(x'|\omega_2) = 0.0523$

$\boxed{x \in \omega_2}$

b) Class ω_1: PDF could be GMM, points look like 2 clusters, bimodal

Class ω_2: Points look like from same distribution, unimodal.