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Homework 8 - Q5 to Q6

Question 5:

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a. Use mathematical induction to prove that for any positive integer *n*, 3 divide $n^3 + 2n$ leaving no remainder.

Proof:
1. base case

When n is equal to 1, $\frac{3}{n^3 + 2n} = \frac{3}{1 + 2 * 1} = \frac{3}{3} = 1$

So, when n = 1, 3 divide $n^3 + 2n$ leaving no remainder. Therefore the base case is true.

2. Inductive step

Suppose that for positive integer k, $\frac{3}{k^3 + 2k}$ leaving no remainder, which means 3 evenly divide$k^3 + 2k$, and suppose that all positive integer have the same property, we need to prove that

$$\frac{3}{(k+1)^3 + 2(k+1)} \textit{ also leaving no remainder.}$$

$since\ k^3 + 2k\ =\ 3 * m\ (for\ some\ integer\ number\ m)$, we need to prove that
$(k+1)^3 + 2(k+1) = 3 * n\ \ (for\ some\ integer\ number\ n)$

$$\frac{3}{(k+1)^3 + 2(k+1)} = \frac{3}{k^3 + 3k^2 + 3k + 1 + 2k + 2} = \frac{3}{\left(k^3 + 2k\right) + 3k^2 + 3k + 3}$$
$$= \frac{3}{3m + 3k^2 + 3k + 3} = \frac{3}{3 * \left(m + k^2 + k + 1\right)}$$

$since\ m\ and\ k\ are\ all\ integers,\ \left(m + k^2 + k + 1\right)\ is\ also\ an\ integer,\ 3\ can\ evenly\ divide$
$(k+1)^3 + 2(k+1).$

Therefore, the inductive step is also true.

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b. Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.

Proof:

1. base case:

for n ≥ 2, when n = 2, n can be written as 1 * 2, which is a product of prime.
when n = 3, n can be written as 1 * 3, which is also a product of prime.

So, S(2) and S(3) are all true.

2. Inductive steps

For n ≥ 2, suppose that for any integer k in the range, it can be written as a product of prime, we will show that k+1 can be written as a product of prime.

There are two cases:

- (k+1) is a prime number, so it sure can be written as a product of prime since it is a prime number itself.
- (k+1) is not a prime number, then it can be written as (k+1) = x * y, where x and y are all integers, and x and y are at least 2.

$k+1 = x*y \qquad x = \dfrac{k+1}{y}$

since x and y are all greater or equal to 2

$\dfrac{k+1}{y} < k+1,$ *so* $x < k+1,$ *then we can conclude that* $x \leq k$

By the symmetric argument we can also show that $y \leq k,$

Therefore, x *and* y *both fall in the range between* 2 *and* k. *the inductive hypothesis applied, and they can each be expressed as a product of primes*:

$x \; = \; p1*p2*p3*...*pm$

$y \; = \; q1*q2*q3*...*qn$

and $k+1 = \; x*y = (p1*p2*p3*...*pm)*(q1*q2*q3*...*qn)$ *which is a product of prime.*

Question 6:

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a) Exercise 7.4.1, sections a-g

Define P(n) to be the assertion that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

a. Verify that P(3) is true.

$For\ n = 3,\ sum\ of\ j^2\ is\ equal\ to\ 1^2 + 2^2 + 3^3 = 1 + 4 + 9 = 14\ .$

$For\ n = 3, \frac{n(n+1)(2n+1)}{6} = \frac{3*4*7}{6} = 14$

$Therefore,\ when\ n = 3, \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}\ is\ true,\ P(3)\ is\ true.$

b. Express P(k).

$When\ n = k,\ sum\ of\ j^2\ to\ k\ is\ 1^2 + 2^2 + ... + k^2,\ also\ stands\ for\ \sum_{j=1}^{k} j^2$

$When\ n = k,\ \frac{n(n+1)(2n+1)}{6} = \frac{k(k+1)(2k+1)}{6}$

$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

c. Express P(k + 1).

$When\ n = k+1,\ sum\ of\ j^2\ to\ k\ is\ 1^2 + 2^2 + ... + (k+1)^2,\ also\ stands\ for\ \sum_{j=1}^{k+1} j^2$

$When\ n = k+1,\ \frac{n(n+1)(2n+1)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

d. In an inductive proof that for every positive integer n, what must be proven in the base case?

$In\ the\ base\ case,\ we\ need\ to\ prove\ P(1)\ is\ true,\ which\ is \sum_{j=1}^{1} j^2 = \frac{1(1+1)(2+1)}{6}$

$1^2 = 1$

$\frac{1(1+1)(2+1)}{6} = \frac{2*3}{6} = 1$

$Therefore,\ P(1)\ is\ true,\ the\ base\ case\ is\ true.$

e.In an inductive proof that for every positive integer n, what must be proven in the inductive step?

In the inductive step, we need to prove if $P(k)$ is true, $P(k+1)$ is also true.

$$P(k) \ = 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$P(k+1) = 1^2 + 2^2 + \dots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

f. What would be the inductive hypothesis in the inductive step from your previous answer?

The hypothesis is that $P(k)$ is true.

g.Prove by induction that for any positive integer n, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

Base case: $P(1)$ is true

$1^2 = 1$

$$\frac{1(1+1)(2+1)}{6} = \frac{2*3}{6} = 1$$

Therefore, $P(1)$ is true, the base case is true.

Inductive steps: for all positive integer $k \geq 1$, $P(k)$ is true, then $P(k+1)$ is true

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}$$

Therefore, $P(k)$ and $P(k+1)$ are all true for all positive integer $k \ \geq 1$.

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b) Exercise 7.4.3, section c

Prove that for n ≥ 1, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

$Base\ case:\ when\ n\ =\ 1,\ P(1)\ is\ true.$

$\frac{1}{1^2} = 1$

$2 - \frac{1}{1} = 1$

$so,\ \sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}\ when\ n\ =\ 1,\ P(1)\ is\ true.$

$Inductive\ steps:\ for\ all\ k\ \geq 1,\ if\ P(k)\ is\ true,\ P(k+1)\ is\ also\ true.$

$P(k):\ \sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$

$since\ k \geq 1,\ then\ k+1 > k,\ then \frac{1}{k+1} < \frac{1}{k},\ then\ -\frac{1}{k+1} > -\frac{1}{k}$

$2 - \frac{1}{k+1} > 2 - \frac{1}{k}$

$P(k+1): \sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)}\ \ based\ on \frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)} \\
&= 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)} \\
&= 2 - \frac{1}{k+1}
\end{aligned}$$

$Therefore,\ \ for\ all\ k\ \geq 1,\ if\ P(k)\ is\ true,\ P(k+1)\ is\ also\ true.$

c. Exercise 7.5.1, section a

Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$

$Base\ case:\ prove\ P(1)\ is\ true$

$when\ n\ =\ 1,\ 3^{2n} - 1 = 3^{2*1} - 1 = 9 - 1 = 8$
$4\ can\ evenly\ divides\ 8$

$Inductive\ step:\ for\ all\ positive\ integer,\ P(k)\ is\ true,\ then\ P(k+1)\ is\ true.$

$P(k)\ :\ 3^{2*k} - 1\ is\ evenly\ divides\ by\ 4,\ so\ 3^{2k} - 1 = 4 * m$
$P(k+1):\ \ prove\ 3^{2*(k+1)} - 1 = 4 * n\ (n\ is\ an\ positive\ integer)$

$$\begin{aligned} 3^{2k} + 3^{2} - 1 &= 3^{2k} * 9 - 1 \\ &= 8 * 3^{2k}\ +\ 3^{2k}\ -\ 1 \\ &= 8 * 3^{2k} + 4m \\ &= 4 * \left(2 * 3^{2k} + m\right) \end{aligned}$$

$since\ m\ and\ k\ are\ all\ integers,\ 3^{2*(k+1)} - 1\ can\ be\ evenly\ divided\ by\ 4$