2.1 Introduction to proofs

A primary endeavor in mathematics is to prove theorems. A ***theorem*** is a statement that can be proven to be true. A ***proof*** consists of a series of steps, each of which follows logically from assumptions, or from previously proven statements, whose final step should result in the statement of the theorem being proven. The proof of a theorem may make use of ***axioms***, which are statements assumed to be true. A proof may also make use of previously proven theorems. Although mathematical proofs are typically expressed in English, the formalism of logic provides a good foundation for mathematical reasoning used in proving theorems. A proof should read like a verbal argument designed to convince a skeptical listener that an assertion is true.

Many of the theorems proven in this material are facts about numbers that can be proven with standard algebra. Theorems and proofs in discrete mathematics presented throughout the rest of this material pertain to a variety of mathematical objects such as graphs, functions, sequences, and sums. The animation below gives the proof of the following theorem:

Theorem 2.1.1: A simple theorem.

Every positive integer is less than or equal to its square.

The statements of the proof itself are shown in black font. Comments which are not part of the proof are shown in red. Every proof should begin with a clear indication that the proof is starting and end with an indication that the proof is complete. In this material, every proof begins with the word **Proof:** and ends with the symbol ▪.

PARTICIPATION ACTIVITY | 2.1.1: A first proof.

Animation captions:

1. The theorem to be proven is stated before the proof. The word "Proof:" indicates that the proof is starting.
2. The first step names a generic object in the domain and the given assumptions about the object.
3. The reasoning in a proof is stated in complete sentences. The proof is followed by an end of proof symbol.

The process of writing proofs

One of the hardest parts of writing proofs is knowing where to start. If a proof can be any sequence of logical steps, how is the prover to know which sequence of steps will lead to a proof of the theorem? Fortunately, many proofs follow one of a relatively small number of patterns. Following one of the common patterns helps give the proof structure and the prover some direction. This material will give a brief overview of some of the common patterns of mathematical proofs. The types of proofs presented are not an exhaustive list, but rather give a first glimpse of the kinds of mathematical arguments used in proving theorems.

Proofs in their final form are expressed in their simplest and most direct way; however, proofs are rarely conceived in their simplest form. Coming up with proofs requires trial and error, even for experienced mathematicians. Often the process includes experimenting with small examples in order to develop intuition about a more general rule. The process almost always entails some dead ends along the way.

In this material, each proof is preceded with an explanation that gives a glimpse of the thought process that went into coming up with the proof. Afterwards, the proof is given in its final, polished state. Going back and forth between the proof and the intuition behind the proof will help further the understanding of the process of proof *creation* as opposed to just proof verification.

Proofs of universal statements

Before proving a theorem, it is essential to understand exactly what the theorem is saying. Rewriting the statement using precise mathematical language is helpful. Most theorems are an assertion about the all elements in a set and are therefore universal statements, even if the statement of the theorem does not explicitly use a universal quantifier. The first step in proving a universal statement is to name a generic object in the domain and prove the statement for that object. "Generic" means we don't assume anything about the object other than the assumptions that are given in the statement of the theorem. Once the generic object is named, write down everything you know about the object and what needs to be proven.

The theorem proven above that states: "Every positive integer is less than or equal to its square". The statement below is equivalent to the theorem and gives a clearer direction for how to proceed with the proof.

If x is an integer and x is positive, then $x \leq x^2$.

Universal statements can also refer to more than one object which may come from different domains. For example, consider the statement:

If x and y are positive real numbers and n is a positive integer, then $(x + y)^n \geq x^n + y^n$.

The statement above concerns three numbers: x, y, and n. x and y are positive real numbers and n is a positive integer.

PARTICIPATION ACTIVITY 2.1.2: Theorem statements.

1) Which of the expressions is equivalent to the following statement:

The average of two real numbers is less than or equal to at least one of the two numbers.

- If x and y are real numbers, then $(x + y)/2 \le x + y$.
- If x and y are real numbers, then $(x + y)/2 \le x$ or $(x + y)/2 \le y$.
- If x and y are real numbers, then $(x + y)/2 \le x$ and $(x + y)/2 \le y$.

2) Which of the expressions is equivalent to the following statement:

The difference of two odd integers is even.

- If x and y are integers and x and y are both odd, then (x - y) is even.
- If x is an odd integer or y is an odd integer, then (x - y) is even.
- If x and y are integers and x and y are both even, then (x - y) is odd.

3) Which of the expressions is equivalent to the following statement:

Among any two consecutive integers, there is an odd number and an even number.

- Let x and y be two integers. Then x is odd and y is even or x is even and y is odd.
- Let x be an integer. Then x is odd and x+1 is even.
- Let x be an integer. Then x is odd and x+1 is even or x is even and x+1 is odd.

Proofs by exhaustion

If the domain of a universal statement is small, it may be easiest to prove the statement by checking each element individually. A proof of this kind is called a ***proof by exhaustion***. For example, consider the statement:

If $n \in \{-1, 0, 1\}$, then $n^2 = |n|$.

It is straightforward to prove the above statement by verifying the equality for all three possible values of n. Here is the proof:

Proof.
Check the equality for each possible value of n:

- $n = -1$: $(-1)^2 = 1 = |-1|$.
- $n = 0$: $(0)^2 = 0 = |0|$.
- $n = 1$: $(1)^2 = 1 = |1|$. ■

PARTICIPATION ACTIVITY 2.1.3: Proof by exhaustion.

Consider the following statement:

For every positive integer n less than 3, $(n + 1)^2 \geq 3^n$.

1) Which facts must be checked in a proof by exhaustion of the statement?

○
- $1^2 \geq 3^0$
- $2^2 \geq 3^1$
- $3^2 \geq 3^2$

○
- $2^2 \geq 3^1$
- $3^2 \geq 3^2$

○
- $2^2 \geq 3^1$
- $3^2 \geq 3^2$
- $4^2 \geq 3^3$

Counterexamples

It may be tempting to prove statements over larger or even infinite domain by example as well. For example, consider the statement:

If n is an integer greater than 1, then $(1.1)^n < n^{10}$.

The statement certainly holds for n = 2 because

$$(1.1)^2 = 1.21 < 1024 = 2^{10}.$$

The inequality also holds for n = 100:

$$(1.1)^{100} \approx 13780.61 < 100000000000000000000 = 100^{10}.$$

In fact, the statement holds for every number all the way up through 685. However, for n = 686, the statement is false because $(1.1)^{686} > (686)^{10}$.

The example n = 686 is a counterexample for the statement that for every integer greater than 2, $(1.1)^n < n^{10}$. A ***counterexample*** is an assignment of values to variables that shows that a universal statement is false. The example illustrates the danger in generalizing from examples because there can always be a counterexample that was not tried. Therefore, except when the domain is very small, a proof of a universal statement requires a more general argument that holds for all objects in the domain.

A mathematician who does not know whether an unproven statement is true or false may divide his or her time between looking for a counterexample showing that the statement is false or a proof showing that the statement is true.

A counterexample for a conditional statement must satisfy all the hypotheses and contradict the conclusion. For example, consider the statement:

If x is a real number and x < 1, then $x^2 < x$.

The assignment x = 2 would not be a counterexample to the statement above. While the assignment x = 2 makes the conclusion ($x^2 < x$) false, the value x = 2 does not satisfy the hypothesis that x < 1. The assignment x = -1, is a counterexample because when x = -1, x is a real number, x < 1, and it is not true that $x^2 < x$.

PARTICIPATION ACTIVITY | 2.1.4: Matching counterexamples to false statements.

Below is a list of false statements. Match each assignment to the statement for which it is a counterexample.

x = -1 **x = 5** **x = 1**

	For every integer x such that x > 1, $2^x < x^2$
	For all real numbers x, $x \neq x^2$

For all real numbers x, $-x \neq x^2$

Reset

Additional exercises

Exercise 2.1.1: Proof by exhaustion.

About

Prove each statement using a proof by exhaustion.

(a) For every integer n such that $0 \leq n < 3$, $(n + 1)^2 > n^3$.

Solution

(b) For every integer n such that $0 \leq n < 4$, $2^{(n+2)} > 3^n$.

Exercise 2.1.2: Find a counterexample.

About

Find a counterexample to show that each of the statements is false.

(a) Every month of the year has 30 or 31 days.

Solution

(b) If n is an integer and n^2 is divisible by 4, then n is divisible by 4.

(c) For every positive integer x, $x^3 < 2^x$.

(d) Every positive integer can be expressed as the sum of the squares of two integers.

(e) The multiplicative inverse of a real number x, is a real number y such that xy = 1. Every real number has a multiplicative inverse.

Exercise 2.1.3: Restate the theorem.

About

Write each of the following statements as a precise mathematical statement. Use variable names to denote arbitrary numbers in the domain. Your statements should avoid mathematical terms (such as "square root" or "multiplicative inverse") and should be expressed using algebra.

(a) The square root of every positive number less than one is greater than the number itself.

Solution

(b) There is no largest integer.

(c) Every real number besides 0 has a multiplicative inverse.

(d) Among every consecutive three integers, there is a multiple of 3.
Note: a number is a multiple of 3 if it can be expressed as 3k for some integer k.
Avoid the use of the word "consecutive" in your statement.

2.2 Direct proofs

Many mathematical theorems take the form of a conditional statement in which a conclusion follows from a set of hypotheses. Theorems of this kind can be expressed as $p \rightarrow c$, where p is a proposition that is a conjunction of all the hypotheses and c is the conclusion. Sometimes p is referred to as "the hypothesis" for simplicity. In a ***direct proof*** of a conditional statement, the hypothesis p is assumed to be true and the conclusion c is proven as a direct result of the assumption.

Many theorems are conditional statements that also have a universal quantifier such as:

For every integer n, if n is odd then n^2 is odd.

The domain of variable n is the set of all integers. If D(n) is the predicate that says that n is odd, then the statement is equivalent to the logical expression: $\forall n\ (D(n) \rightarrow D(n^2))$. A direct proof of the theorem starts with n, an arbitrary integer, assumes that D(n) is true, and then proves that $D(n^2)$ is true. Frequently, the universal quantifier and domain are expressed as part of the hypothesis, as in

If n is an odd integer, then n^2 is an odd integer.

The animation below gives a proof of a theorem that states that the sum of two rational numbers is also rational. A ***rational number***, is defined to be a number that can be expressed as the ratio of two integers in which the denominator is non-zero.

Figure 2.2.1: Overview of a direct proof of a theorem about rational numbers.

- Formally state the theorem: for any two numbers r and s, if r and s are rational, then r + s is also rational. Assigning variable names to the two generic rational numbers provides a more concrete representation of the numbers and makes them easier to work with.
- A direct proof *starts* with the assumption that r and s are rational and proves directly that r + s is also rational.
- The only information given about r and s is that they are rational, so the first step is to state this fact mathematically by assigning variable names to the integers that form the ratios, e.g., r = a/b and s = c/d, where b and d are both non-zero. Giving the four integers variable names, enables the prover to express the sum r + s concretely.
- After plugging in the two ratios for r and s, the rest of the proof is just algebra. The goal is to express r + s as a single ratio of two integers.

PARTICIPATION ACTIVITY 2.2.1: Direct proof example.

Animation captions:

1. A direct proof assumes the hypotheses and then derives the conclusion.
2. The assumption is that r and s are rational numbers: r = a/b and s = c/d, for integers a, b, c, d, such that b and d are non-zero.
3. The next step is to plug the expressions for r and s into r + s and show that the result is equal to (ad +cd)/bd.
4. (ad +cd)/bd is the ratio of two integers and bd is non-zero.
5. Therefore, r+s is rational.

The next example gives a direct proof for the theorem given below. The video and final proof illustrate that the process of coming up with the steps for a proof can look very different than the final proof.

Theorem 2.2.1: An algebraic theorem to be proven by a direct proof.

If x and y are positive real numbers, then

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

Video 2.2.1: Scratch work for the proof.

Example 2.2.1: A direct proof of the algebraic theorem.

Theorem: If x and y are positive real numbers, then

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

Proof.
Since x and y are real numbers, x - y is also a real number. Therefore $(x - y)^2 \geq 0$, because the square of any real number is greater than or equal to 0. Multiplying out the left hand side of the inequality gives

$$x^2 - 2xy + y^2 \geq 0.$$

Since x and y are both greater than 0, we can divide both sides of the inequality by xy to get

$$\frac{x}{y} - 2 + \frac{y}{x} \geq 0.$$

Adding 2 to both sides, gives the conclusion of the theorem:

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

■

PARTICIPATION | 2.2.2: Order the steps of a direct proof.

ACTIVITY

Below are the steps of a direct proof of the following theorem:

Theorem: If x - 3 = 0, then $x^2 - 2x - 3 = 0$.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

Add 3 to both sides of the equation: x = 3.

Assume x - 3 = 0.

Plug in x = 3 into the expression $x^2 - 2x - 3$.

$x^2 - 2x - 3 = 3^2 - 2 \cdot 3 - 3 = 9 - 6 - 3 = 0$.

Step 1.

Step 2.

Step 3.

Step 4.

Reset

Additional exercises

Exercise 2.2.1: Proving conditional statements with direct proofs.

About

Prove each of the following statements using a direct proof.

(a) If n is an odd integer, then n^2 is an odd integer.
(Note: the definition of an odd integer is an integer that can be expressed as 2k + 1, where k is an integer.)

Solution

(b) For any positive real numbers, x and y, $x + y \geq \sqrt{xy}$.

(c) If x is a real number and x ≤ 3, then $12 - 7x + x^2 \geq 0$.

(d)

The product of two odd integers is an odd integer.

(e) If r and s are rational numbers, then the product of r and s is a rational number.

Exercise 2.2.2: Showing a statement is true or false by direct proof or counterexample.

About

Determine whether the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.

(Note: the definition of an even integer is an integer that can be expressed as 2k, where k is an integer.)

(a) If x and y are both even integers, then x+y is an even integer.

Solution

(b) If x+y is an even integer, then x and y are both even integers.

2.3 Proof by contrapositive

A ***proof by contrapositive*** proves a conditional theorem of the form p → c by showing that the contrapositive ¬c → ¬p is true. In other words, ¬c is assumed to be true and ¬p is proven as a result of ¬c.

Many theorems are conditional statements that also have a universal quantifier such as:

For every integer n, if n^2 is odd then n is odd.

The domain of variable n is the set of all integers. If D(n) is the predicate that says that n is odd, then the statement is equivalent to the logical expression: $\forall n\ (D(n^2) \rightarrow D(n))$. A proof by contrapositive of the theorem starts with n, an arbitrary integer, assumes that D(n) is false, and then proves that $D(n^2)$ is false.

The animation below gives a proof by contrapositive that for every integer n, if 3n + 7 is odd then n is even. The theorem is a conditional statement in which the hypothesis is that 3n + 7 is odd and the conclusion is that n is even. The proof implicitly uses the fact that every integer is even or odd, so if an integer is not even, then it is odd. A contrapositive proof assumes the negation of the conclusion (n is odd) and uses the assumption to prove the negation of the hypothesis (3n + 7 is even). Why use a contrapositive proof for this proof? One reason, is that the negation of the assumption (n is odd) is a little simpler and therefore easier to work with than the hypothesis (3n + 7 is odd).

The statements "n is odd" and "3n + 7 is even" need to be translated from English into a mathematical expression to which the prover can apply the standard rules of algebra. An ***even integer*** can be expressed as 2k for some integer k. An ***odd integer*** can be expressed as 2k + 1 for some integer k. The (2k + 1) expression for n can be plugged into 3n + 7. The goal of the algebra is then to show the resulting expression is 2 times some integer.

PARTICIPATION ACTIVITY 2.3.1: Proof by contrapositive example.

Animation captions:

1. A proof by contrapositive starts with the negation of the conclusion and derives the negation of the hypothesis.
2. The negation of the conclusion is that n is an odd integer. Therefore, $n = 2k + 1$ for some integer k.
3. The next step is to plug the expression for n into $3n + 7$ and show that the result is equal to $2(3k + 5)$, which is 2 times an integer.
4. Therefore, $3n + 7$ is even, which is the negation of the hypothesis.

Deciding whether to prove a conditional statement using a direct proof or a proof by contrapositive often involves some trial and error. The decision should be based on whether the hypothesis or the negation of the conclusion provides a more useful assumption to work with. Consider the statement:

For every integer x, if x^2 is even, then x is even.

A direct proof assumes that x^2 is even, which in mathematical terms means that $x^2 = 2k$, for some integer k. Deriving an expression for x requires taking the square root of both sides, and it is not clear how to reason that $\sqrt{2k}$ is an even integer. Alternatively, a proof by contrapositive assumes that x is odd, which in mathematical terms means that x = 2k + 1, for some integer k. The expression for x can then be plugged into x^2 resulting in an expression that is much easier to reason about.

Example 2.3.1: A proof by contrapositive of a theorem about even integers.

Theorem: For every integer x, if x^2 is even, then x is even.

Proof.
Proof by contrapositive. Let x be an integer. We assume that x is odd and prove that x^2 is odd. If x is odd, it can be expressed as 2k + 1, for some integer k. Plug in x = 2k + 1 into x^2 to get

$$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since k is an integer, $2k^2 + 2k$ is also an integer. Therefore x^2 can be expressed as 2k' + 1, where k' = $2k^2 + 2k$ is an integer. We can conclude that x^2 is odd. ■

The theorem proven below states that for every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational. An ***irrational number*** is a real number that is not rational.

A direct proof would assume that r is irrational which means that r can not be expressed as a ratio of two integers. The assumption that r can not be expressed in a certain way does not provide a concrete expression for r to work with. Alternatively, a proof by contrapositive uses the assumption that $\sqrt{r}$ is rational which provides a useful expression for r (the ratio of two integers) that can be plugged into other expressions.

Example 2.3.2: Another proof by contrapositive.

Theorem: For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.
Proof by contrapositive. Let r be a positive real number. We assume that $\sqrt{r}$ is not irrational and prove that r is not irrational.

Since r is a positive real number, then $\sqrt{r}$ is a positive real number. Since $\sqrt{r}$ is not irrational and is real, then $\sqrt{r}$ must be a rational number. Therefore $\sqrt{r} = x/y$ for some two integers, x and y, where y ≠ 0. Squaring both sides of the equation gives:

$$(\sqrt{r})^2 = r = \frac{x^2}{y^2}.$$

Since x and y are integers, x^2 and y^2 are both integers. Since y ≠ 0, y^2 is also non-zero. The number r is equal to the ratio of two integers in which the denominator is non-zero, so r is a rational number. Therefore r is not irrational. ■

PARTICIPATION ACTIVITY | 2.3.2: Proof by contrapositive - order the steps.

Below are the steps of a proof by contrapositive of the following theorem:

Theorem: For every real number x, if $x^3 + 2x + 1 \le 0$, then $x \le 0$.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

Since x^3 , 2x, and 1 are all greater than 0, their sum is also greater than zero.

$x^3 + 2x + 1 > 0$. Therefore the theorem is true.

Assume $x > 0$, where x is a real number.

If $x > 0$, then $2x > 0$ and $x^3 > 0$.

Step 1.

Step 2.

Step 3.

Step 4.

Reset

PARTICIPATION ACTIVITY

2.3.3: Proof by contrapositive - fill in the blank.

Below is a statement of a theorem and a proof by contrapositive with some parts of the argument replaced by capital letters in red font.

Theorem: For every positive integer x, if x^3 is even, then x is even.

Proof.
Proof by contrapositive.

Let x be a positive integer. Assume A. We will prove that B.

If x is odd, then it can be written as C for some integer k. Plug in the expression for x into x^3 to get D. The expression for x^3 can be written as E. Since $(4k^3 + 6k^2 + 3k)$ is an integer, we can conclude that x^3 is odd. ■

1) What is the correct expression for A?

- x^3 is even
- x^3 is odd
- x is even
- x is odd

2) What is the correct expression for B?

- x^3 is even
- x^3 is odd
- x is even
- x is odd

3) What is the correct expression for C?

- 2k
- 2k+1
- k^3

4) What is the correct expression for D?

- $8k^3+12k^2+6k+1$
- $8k^3$

5) What is the best expression for E? Both choices are equal to x^3, but one does a better job showing that x^3 is odd.

- $2(4k^3 + 6k^2 + 3k) + 1$
- $8k^3+12k^2+6k+1$

The next questions compare direct proofs to proofs by contrapositive.

PARTICIPATION ACTIVITY 2.3.4: Direct proofs and proofs by contrapositive.

Consider the following statement:

For every real number x, if $0 \leq x \leq 3$, then $15 - 8x + x^2 > 0$

1) What would be the starting assumption in a direct proof of the statement above?

- $0 \leq x \leq 3$
- $15\text{-}8x+x^2 > 0$
- $15\text{-}8x+x^2 \leq 0$

2) What would be proven in a direct proof of the statement, given the assumption?

- $0 \leq x \leq 3$
- $15\text{-}8x+x^2 > 0$
- $15\text{-}8x+x^2 \leq 0$

3) What would be the starting assumption in a proof by contrapositive of the statement above?

- $0 \leq x \leq 3$

- $15\text{-}8x+x^2 > 0$
- $15\text{-}8x+x^2 \leq 0$

4) What would be proven in a proof by contrapositive of the statement, given the assumption?
   - $0 \leq x \leq 3$
   - $x < 0$ or $x > 3$.
   - $x < 0$ and $x > 3$.

Additional exercises

Exercise 2.3.1: Proving conditional statements by contrapositive.

About

Prove each statement by contrapositive

(a) For every integer n, if n^2 is an odd, then n is odd.

Solution

(b) For every integer n, if n^3 is even, then n is even.

(c) For every integer n, if $5n + 3$ is even, then n is odd.

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

(e) For every real number x, if x is irrational, then $-x$ is also irrational.

Solution

(f) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

(g) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

(h) For every integer n, if n^2 is not divisible by 4, then n is odd.

(i)

For every pair of real numbers x and y, if $x + y$ is irrational, then x is irrational or y is irrational.

(j)

For every pair of integers x and y, if xy is even, then x is even or y is even.

(k) For every pair of integers x and y, if $x - y$ is odd, then x is odd or y is odd.

Solution

(l)

For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

(m) For every pair of positive real numbers x and y, if $xy > 400$, then $x > 20$ or $y > 20$.

2.4 Proof by contradiction

A ***proof by contradiction*** starts by assuming that the theorem is false and then shows that some logical inconsistency arises as a result of this assumption. If t is the statement of the theorem, the proof begins with the assumption ¬t and leads to a conclusion r ∧ ¬r, for some proposition r. If the theorem being proven has the form p → q, then the beginning assumption is p ∧ ¬q which is logically equivalent to ¬(p → q). Unlike direct proofs and proofs by contrapositive, a proof by contradiction can be used to prove theorems that are not conditional statements. A proof by contradiction is sometimes called an ***indirect proof***.

The animation below gives a proof that if a and b are positive real numbers, then

$$\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}.$$

A proof by contradiction assumes that the entire statement is false. In this case, the assumption is that there are positive integers a and b such that

$$\sqrt{a} + \sqrt{b} = \sqrt{a+b}.$$

The algebra in the proof is guided by the goal of simplifying the equation. Square roots are complicated to handle, so each side of the equation is squared to help eliminate (or reduce) terms with square roots. The next steps are to distill the equation into its simplest form which will hopefully enable the prover to see a contradiction.

PARTICIPATION ACTIVITY | 2.4.1: Proof by contradiction example.

Animation captions:

1. A proof by contradiction shows that assuming that the theorem is false leads to an inconsistency.
2. The negation of the theorem is the same as saying that there are positive real numbers a and b such that $\sqrt{a} + \sqrt{b} = \sqrt{(a+b)}$.
3. The next step is to square both sides of the equation and derive that ab = 0 which implies that either a = 0 or b = 0.
4. The fact that a = 0 or b = 0 contradicts the assumption that a and b are both positive, which is an inconsistency.

The proof above of the fact that for positive real numbers, a and b, $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$, resembles a proof by contrapositive. In fact, a proof by contrapositive is actually a special case of a proof by contradiction. In proving a theorem of the form p → q by contrapositive, we assume ¬q and prove ¬p. A proof by contrapositive can be recast as a proof by contradiction by assuming p ∧ ¬q and arriving at the contradiction p ∧ ¬p. Proofs by contradiction are more general than proofs by contrapositive because a proof by contradiction of the theorem p → q could start with the assumption ¬q ∧ p and conclude with some contradiction other than p ∧ ¬p. Also, proof by contrapositive is a proof technique that is specific to proofs of conditional statements, whereas a proof by contradiction can be used to prove theorems that are not of the form p → q.

PARTICIPATION ACTIVITY 2.4.2: Matching proof types to logical arguments.

Match the type of the proof to its logical format. In each case the theorem being proved is p→ q.

Proof by contradiction **Direct proof** **Proof by contrapositive**

	Assume p. Follow a series of steps to conclude q.
	Assume ¬q. Follow a series of steps to conclude ¬p.
	Assume p ∧ ¬q. Follow a series of logical steps to conclude r ∧ ¬r for some proposition.

Reset

Below is a proof of the fact that $\sqrt{2}$ is irrational. The proof starts out with the assumption that $\sqrt{2}$ is rational and, by a series of logical steps, ends up with a statement known to be false. The proof makes repeated use of the fact that if x is an integer and x^2 is even, then x must be even, which can be proven by contrapositive by showing that if x is an odd integer, then x^2 is also odd.

Theorem 2.4.1: Theorem to be proven by contradiction.

$\sqrt{2}$ is an irrational number.

Proof.
We assume that $\sqrt{2}$ is a rational number and therefore can be expressed as the ratio of two integers n/d, where d ≠ 0. Any ratio can be simplified into lowest terms. Therefore, we can assume that there is no integer greater than 1 that evenly divides both n and d.

Squaring both sides of the equation $\sqrt{2} = n/d$ gives $2 = n^2/d^2$. Multiplying both sides by d^2 gives

$$2d^2 = n^2.$$

Since n^2 is an integer multiple of 2, n^2 is even. If the square of an integer is even, then the integer itself must be even. Therefore, n is even which means that n can be expressed as 2k for some integer k. Plugging in the expression 2k for n into n^2:

$$n^2 = (2k)^2 = 4k^2.$$

Putting the two equations above together yields that $2d^2 = 4k^2$. Dividing both sides by 2 results in the equation $d^2 = 2k^2$. Therefore d^2 is even. Now we use the fact again that if the square of an integer is even, then that integer must also be even. Therefore d is even.

We have proven that n and d are both even. This means that n and d are both divisible by 2, which contradicts the assumption that the ratio n/d is expressed in its lowest terms. Thus, we have established a contradiction and we must conclude that the assumption that $\sqrt{2}$ is a rational number is a false assumption. ■

The proof above may be difficult to follow for the reader who is not used to reading proofs. The next question gives an outline of the steps of the proof and asks you to put the parts of the outline in the correct order. When you are done with the question, see if you can prove each line in the outline on your own.

PARTICIPATION ACTIVITY | 2.4.3: Outline of the proof that root-2 is irrational.

Below is an outline of the steps of the proof that $\sqrt{2}$ is irrational. Put the steps in the outline in the correct order.

This contradicts the assumption that there is no integer greater than 1 that divides both n an

Assume $\sqrt{2}$ is rational.

Since d^2 is even, d is even.

Express $\sqrt{2}$ as n/d, where n and d are integers, d ≠ 0, and there is no integer greater than 1 th

Since n is even, and $n^2 = 2d^2$, d^2 is also even.

Since n and d are both even, 2 evenly divides n and d.

$n^2 = 2d^2$.

n^2 is even and therefore n is even.

	Step 1.
	Step 2.
	Step 3.
	Step 4.
	Step 5.
	Step 6.
	Step 7.
	Step 8.

Reset

The next example proves a theorem unrelated to the properties of numbers.

Example 2.4.1: Proof by contradiction - birthday months.

Theorem: Among any group of 25 people, there must be at least three who are all born in the same month.

Proof.
Proof by contradiction: assume that there is a group of 25 people such that no three people have their birthdays in the same month. We introduce twelve variables: $x_1,...,x_{12}$. The variable x_i is the number of people in the group whose birthday falls in the i^{th} month. x_1 is the number of people in the group with January birthdays, x_2 the number of people with February birthdays, etc.

Since everyone is born in some month, the number of people in the group must be $x_1 + x_2 + x_3 + \cdots x_{12}$. By our assumption, there are no three people born in the same month, so each x_i is at most 2. Therefore the number of people in the group is at most 24:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}$$

$$\leq\ 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$$

$$=\ 2 * 12 = 24.$$

The fact that the number of people is at most 24 contradicts the fact that there are 25 people in the group. Therefore, there must be at least three people in the group who are all born in the same month. ■

If you had trouble following the equations in this proof, here is a video explanation.

PARTICIPATION ACTIVITY | 2.4.4: Proof by contradiction from geometry - fill in the blank.

Below is a statement of a theorem and a proof by contradiction with some parts of the argument replaced by capital letters in red font.

Theorem: Every triangle has at least one acute angle.

Proof.
Assume that the theorem is false which means that A. If an angle is not acute, it is right or obtuse, meaning it has degree 90 or greater. Let the degree of the three angles be x, y, and z. We will show that if B, then x + y + z > 180. The conclusion that x + y + z > 180 is a contradiction because it violates the known theorem that the sum of the degrees of the angles of a triangle equals 180.

Since B, x + y + z ≥ C. Since C > 180, then

$$x + y + z \geq C > 180$$

■

1) What is the correct expression for A?

- ○ Every triangle has at least one angle that is not acute.
- ○ There is a triangle with at least one angle that is not acute.
- ○ There is a triangle which does not have an acute angle.

2) What is the correct expression for B?

- ○ x ≥ 90, y ≥ 90, and z ≥ 90.
- ○ x ≥ 90, y ≥ 90, or z ≥ 90.
- ○

x < 90, y < 90, or z < 90.

3) What is the correct expression for C?

- 180
- 360
- 270

PARTICIPATION ACTIVITY | 2.4.5: Proof by contradiction - no smallest positive real number.

Below are the steps for a proof by contradiction of the following theorem:

Theorem: There is no smallest positive real number.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

Assume there is a smallest positive real number called r.

r/2 is a positive real number that is smaller than r.

This contradicts the assumption that r is the smallest positive real number.

Since r is positive, r/2 is also positive. Since r is positive, r > r/2.

Step 1.

Step 2.

Step 3.

Step 4.

Reset

Additional exercises

Exercise 2.4.1: Proofs by contradiction. About

Give a proof for each statement.

(a) If a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won at least 12 trophies.

Solution

(b) If a person buys at least 400 cups of coffee in a year, then there is at least one day in which the person has bought at least two cups of coffee.

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

(d) $\sqrt[3]{2}$ is irrational.
You can use the following fact in your proof:
If n is an integer and n^3 is even, then n is even.

(e) There is no smallest integer.

(f) There is no largest rational negative number.

2.5 Proof by cases

Many theorems can be phrased as ∀x P(x), where P is a predicate with variable x whose value can be any element from some domain. Sometimes proving such a theorem simultaneously for all elements in the domain is difficult. However, the theorem is more approachable if the domain is broken down into different classes where each class can be addressed separately. A ***proof by cases*** of a universal statement such as ∀x P(x) breaks the domain for the variable x into different classes and gives a different proof for each class. Every value in the domain must be included in at least one class.

Take as an example, the theorem:

Theorem 2.5.1: Theorem to be proven by cases.

For every integer x, x^2 - x is an even integer.

The domain is the set of all integers. This problem would be easier to approach if there were a mathematical expression to substitute for x in x^2 - x in order to determine its parity. (The ***parity*** of a number is whether that number is odd or even.) An even number can be expressed as 2k for some integer k, and an odd number can be expressed as 2k + 1 for some integer k. Even though we don't know whether x is odd or even, we can prove the theorem by treating each case separately. The key point is that the set of odd integers together with the set of even integers account for all possible integers, ensuring that all integers are addressed.

PARTICIPATION ACTIVITY 2.5.1: Proof by cases.

Animation content:

undefined

Animation captions:

1. The two cases (that x is odd and x is even) cover all possible integers.
2. If x is even, then $x = 2k$ for some integer k. The next step plugs $2k$ in for x into $x^2 - x$ and shows that the result is 2 times an integer, which is even.
3. If x is odd, then $x = 2k + 1$ for some integer k. The next step plugs $2k + 1$ in for x into $x^2 - x$ and shows that the result is 2 times an integer, which is even.

The next example gives a proof by cases for a theorem that is not about the properties of numbers:

Example 2.5.1: Proof by cases - mutual friends and enemies.

Theorem: Consider a group of six people. Each pair of people are either friends or enemies with each other. Then there are three people in the group who are all mutual friends or all mutual enemies.

This video gives a pictorial overview of the proof which may make the formal proof given below easier to understand.

Proof.
Select a particular individual from the group and call that person x. Person x is either friends or enemies with each of the five other people in the group. Since five is an odd number, person x can not have the same number of friends as enemies in the group. We consider the case in which person x has more friends than enemies and the case where person x has more enemies than friends.

Case 1: person x has more friends than enemies in the group. Then it must be the case that person x has at least three friends. Select three friends of person x. If the three selected friends of x are all mutual enemies, then the theorem holds. If they are not all mutual enemies, then at least two of them must be friends with each other. Thus, there are two people who are friends with each other who are also both friends of person x. They form a set of three people in the group who are all mutual friends.

Case 2: person x has more enemies than friends in the group. Then it must be the case that person x has at least three enemies. Select three enemies of person x. If the three selected enemies of x are all mutual friends, then the theorem holds. If they are not all mutual friends, then at least two of them must be enemies of each other. Thus, there are two people who are enemies of each other who are also both enemies of person x. They form a set of three people in the group who are all mutual enemies. ■

The next example uses the expression max{x, y} which is equal to the larger of x and y. If x and y are equal, then max{x, y} is equal to both x and y. It would be convenient in a proof to have an algebraic expression to use for max{x, y}. The fact that the value of max{x, y} depends on whether x or y is larger suggests two natural cases for the proof. In the first case, y ≥ x and y = max{x, y}. In the second case, x > y and x = max{x, y}. The situation in which x = y can be included in either case.

Example 2.5.2: Proof by cases - the maximum and average of two numbers.

Theorem: For any two real numbers, x and y, max{x, y} is greater than or equal to the average of x and y.

Proof.
Case 1: y ≥ x. Since y ≥ x, y = max{x, y}, and we need to show that y ≥ (x+y)/2. Start with the inequality y ≥ x and divide both sides by 2 to get y/2 ≥ x/2. Now add y/2 to both sides of the inequality:

$$\frac{y}{2} + \frac{y}{2} \geq \frac{x}{2} + \frac{y}{2} = \frac{x+y}{2}.$$

The left side of the inequality is equal to y which in this case is the same as max{x, y}. The right side of the inequality is equal to the average of x and y. Therefore max{x, y} is greater than or equal to the average of x and y.

Case 2: x > y. Since x > y, x = max{x, y}, and we need to show that x ≥ (x+y)/2. Start with the inequality x > y and divide both sides by 2 to get x/2 > y/2. Now add x/2 to both sides of the inequality:

$$\frac{x}{2} + \frac{x}{2} > \frac{x}{2} + \frac{y}{2} = \frac{x+y}{2}.$$

The left side of the inequality is equal to x which in this case is the same as max{x, y}. The right side of the inequality is equal to the average of x and y. Therefore max{x, y} is greater than the average of x and y which in turn implies that it is greater than or equal to the average of x and y. ■

The following question is a statement about the properties of a perfect square. A perfect square is a number that is equal to k^2 for some integer k. An integer is a multiple of 4 if it can be expressed as 4c for some integer c.

PARTICIPATION ACTIVITY 2.5.2: Proof by cases - multiples of 4.

Below is a statement of a theorem and a proof by cases with some parts of the argument replaced by capital letters in red font.

Theorem: Every perfect square is either a multiple of 4 or a multiple of 4 plus 1.

Proof.
Every perfect square can be expressed as n^2 for some integer n. We consider two cases:

Case 1: n is even. If n is even, then it can be expressed as A, for some integer k. Plug in the expression for n into n^2:

$$n^2 = (A)^2 = B$$

Since k^2 is an integer, if n is even, then n^2 is a multiple of 4.

Case 2: n is C. If n is C, then n can be expressed as D, for some integer k. Plug in the expression for n into n^2:

$$n^2 = (D)^2 = E$$

Since (k^2+k) is an integer, if n is C, then n^2 is one plus a multiple of 4. ■

1) What is the correct expression for A?
- 2k
- 2k+1
- even

2) What is the correct expression for B?
- 2k+1
- $4k^2$
- multiple of 4

3) What is the correct expression for C?
- even
- odd
- a multiple of 4

4) What is the correct expression for D?
- 2k
- 2k+1
- odd

5) What is the best expression for E? Note that more than one of the choices may make the equality true. However, the goal is to express n^2 as a multiple of 4 or one plus a multiple of 4.

- $4k^2 + 2k + 1$
- $(2k)^2 + 4k + 1$.
- $4(k^2+k) + 1$

The following question explores a proof by cases involving the absolute value function. The ***absolute value*** of a real number x is defined to be |x| = -x if x ≤ 0, and |x| = x if x ≥ 0. For example, if x = -3, then

$$|x| = |-3| = 3 = -(-3) = -x.$$

If x = 3, then

$$|x| = |3| = 3 = x.$$

In a proof by cases, it is acceptable for a particular value for x or y to be included in more than one case. For example, the situation in which x = 0 is covered in all the cases because when x = 0, then x ≥ 0 and x ≤ 0 are both true.

PARTICIPATION ACTIVITY 2.5.3: Proof by cases - absolute value.

Below are the arguments for a proof by cases.

Theorem: For every pair of real numbers, x and y, |x · y| = |x| · |y|.

Match the assumptions of each case with the argument that matches the case.

x ≤ 0 and y ≤ 0. | **x ≤ 0 and y ≥ 0.** | **x ≥ 0 and y ≥ 0.** | **x ≥ 0 and y ≤ 0.**

	xy ≥ 0 and therefore |x · y| = xy. Meanwhile |x| = -x and |y| = -y, so |x| · |y| = (-x)(-y) = xy.
	xy ≤ 0 and therefore |x · y| = -xy. Meanwhile |x| = -x and |y| = y, so |x| · |y| = (-x)(y) = -xy.
	xy ≤ 0 and therefore |x · y| = -xy. Meanwhile |x| = x and |y| = -y, so |x| · |y| = (x)(-y)= -xy.
	xy ≥ 0 and therefore |x · y| = xy. Meanwhile |x| = x and |y| = y, so |x| · |y| = (x)(y)= xy.

Reset

Additional exercises

Exercise 2.5.1: Proofs by cases.

About

Prove each statement.

(a) If x is an integer, then $x^2 + 5x - 1$ is odd.

Solution

(b) If x and y are real numbers, then max(x, y) + min(x, y) = x + y.

(c) If integers x and y have the same parity, then x + y is even.
The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

(d) For any real number x, $|x| \geq 0$.

(e) For any real number x, $|x| \geq x$ and $|x| \geq -x$.
You can use the fact proven in the previous problem that for any real number x, $|x| \geq 0$.

(f) For real numbers x and y, $|x + y| \leq |x| + |y|$.
You can use the fact proven in the previous problem that for any real number z, $z \leq |z|$ and $-z \leq |z|$.

(g) For integers x and y, if xy is odd, then x is odd and y is odd.

(h) Let x and y be two integers. If xy is not an integer multiple of 5, then neither x nor y is an integer multiple of 5.

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