Name: Minjie Shen
NetID: ms10733
Email: ms10733@nyu.edu

Homework 5 - Q7 to Q11

## Question 7:

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a) 8.2.2 section b
f(n) = $n^3 + 3n^2 + 4$. Prove that f = $\Theta(n^3)$.

If this function f(n) is $\Theta(n^3)$, which means that there exist positive constants c1 and c2 that
$c1 * g(n) < f(n) < c2 * g(n)$

select c1 = 1, c2 = 12, n0 =1, we will show that for any n $\geqslant 1$, $1 * g(n) < f(n) < 12 * g(n)$

$for\ n \geqslant 1,\ f(n) = n^3 + 3n^2 + 4 < n^3 + 3n^2 + 8, \quad \because 8 > 4$

for $n \geqslant 1,\ n^3 > n^2,\ so\ \ n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 8$

$since\ n \geqslant 1,\ so\ n^3 \geqslant 0,\ n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 8n^3$

$n^3 + 3n^3 + 8n^3 = 12n^3,\ \therefore\ f(n) = n^3 + 3n^2 + 4 \leq 12n^3$

$Since\ n \geqslant 1,\ n^2 \geqslant 0,\ so\ 3n^2 + 4 \geqslant 0$

$Also\ since\ n \geqslant 1,\ n^3 \geqslant 0,\ we\ add\ n^3\ to\ both\ side\ and\ get: 3n^2 + 4 + n^3 \geqslant n^3$

$which\ means\ \ f(n) = 3n^2 + 4 + n^3 \geqslant 1 * n^3$

$Therefore,\ for\ n \geqslant 1,\ 1*g(n) \leq f(n) \leq 12*g(n),\ so\ f = \Theta(n^3).$

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b) Use the definition of Θ to show that $\sqrt{7n^2 + 2n - 8} = \Theta(n)$

To show that $\sqrt{7n^2 + 2n - 8} = \Theta(n)$, we need to show that there exist two positive constants c1 and c2 which makes $c1*n < \sqrt{7n^2 + 2n - 8} < c2*n,$ for any n $\geqslant 1$

we take c1 = 1, c2 = 4,n0 = 1, we will show that for any n $\geqslant 1, 1 * n < \sqrt{7n^2 + 2n - 8} < 4*n$

for c1 = 1, n0 = 1, $\sqrt{7n^2 + 2n - 8}$ = $\sqrt{7 + 2 - 8}$ = $\sqrt{1} = 1$, this is equal to c1 * n0 = 1;
for c1 = 1, n0 = 2, $\sqrt{7n^2 + 2n - 8}$ = $\sqrt{28 + 4 - 8}$ = $\sqrt{24} = 4.89,$ this is greater than c1 * n0 = 2;
$\therefore 1 * n < \sqrt{7n^2 + 2n - 8}$ $for\ c1 = 1\ and\ n0 \geqslant 1.$

for c2 = 4, n0 = 1, $\sqrt{7n^2 + 2n - 8}$ = $\sqrt{7 + 2 - 8}$ = $\sqrt{1} = 1$, this is less than c2 * n0 = 4;
for c2 = 4, n0 = 2, $\sqrt{7n^2 + 2n - 8}$ = $\sqrt{28 + 4 - 8}$ = $\sqrt{24} = 4.89,$ this is less than c2 * n0 = 8;
$\therefore \sqrt{7n^2 + 2n - 8} < 4*n\ for\ c2 = 4\ and\ n0 \geqslant 1$

Therefore, $for\ n \geqslant 1, 1 * n < \sqrt{7n^2 + 2n - 8} < 4*n$, based on the definition,
$\sqrt{7n^2 + 2n - 8} = \Theta(n)$

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c) Solve Exercise 8.3.5, sections a-e

a.
this algorithm makes the sequence into two part, one part is less than p, one part is greater than p.

b.
it depends on the value p compares, counter increases on the basis of value and p. when all other integers are negative and p is 0, it will iterate till the end, which is n times; when p is 0 and when all other integers are positive, it will iterate 0 times.

c.
the answer depends on the value. if the list is all negetive numbers and p = 0, swap will not be executed; swap will be maximime n/2 numbers.

d.
the lower bound is O(n)

e.
the upper bound is O(n)

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## Question 8:

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a) Exercise 5.1.1, sections b, c

**b.**

We assume the sets of special characters, digits, or letters be A, B, C, and these sets are mutually exclusive.

$|A \cup B \cup C| \;=\; |A| + |B| + |C| = 4 + 10 + 26 = 40$

Therefor, for string of length 7, each space will have 40 different selections, total the string of length 7 have $40^7$ different combinations.

For the same reason, string of length 8 and 9 will have $40^8\ and\ 40^9$ different combinations.

In total, the number of password of string of length 7 or 8 or 9 will have $40^7 + 40^8 + 40^9$ choices.

**c.**

Based on the rule that $|A| \;=\; |U| - |\overline{A}|$, so in order to find strings of length 7, 8, or 9, Characters can be special characters, digits, or letters, and first character cannot be a letter. We can only find the total strings of length 7 or 8, or 9 and the set of the first character being a letter, L.

String of length 7, first character being a letter: 26 * $40^6$
String of length 8, first character being a letter: 26 * $40^7$
String of length 9, first character being a letter: 26 * $40^8$

so, L = 26 * $40^6$ + 26 * $40^7$ + 26 * $40^8$ = 26 * ($40^6$ + $40^7$ + $40^8$)

Therefore, the strings of length 7, 8, or 9, and first character cannot be a letter:
A = $40^7 + 40^8 + 40^9$ - 26 * ($40^6$ + $40^7$ + $40^8$)
= 14 * $40^6$ + 14 * $40^7$ + 14 * $40^8$

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b) Exercise 5.3.2, section a

**a.**

We can choose the first character of length 10 from 3 options, a, b, and c;
for the second character of length 10, we can only choose from 2 options, if the first character is a, we can only choose b and c for the second character, since no two consecutive characters can be the same.

based on this idea, the total number of strings without repetition will be 3 * $2^9$

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c) Exercise 5.3.3, sections b, c

**b.**

Since no digits can apprears more than once, we will have total number of different plate:

$$P = 10 * 26^4 * 9 * 8 = 720 * 26^4$$

**c.**

Since no digits and letters can appears more than once, we will have total number of different plate:

$$P = 10 * 26 * 25 * 24 * 23 * 9 * 8$$

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d) Exercise 5.2.3, sections a, b

**a.**

For a function to be bijection, it has to be both one-to-one and onto.

We create a function: f(x) = $\begin{cases} x+1 \; for \; x \notin E_9 \\ x+0 \; for \; x \in E_9 \end{cases}$ if x is in $E_9$ the binary string with 9 bits that have even number of 1's, if it add 0, it still have even number of 1's; if x is not in $E_9$, then it means that x has odd number of 1's, if it add 1 to the end, it will have even number of 1's.

let f(a) = f(b) = c, there are two cases:

if the last digit for c is 0, which means a, b $\in E_9$, a + 0 = b + 0 = c, which means a = b.
if the last digit for c is 1, which means a, b $\notin E_9$, a + 1 = b + 1 = c, which also means a = b.
Thus f(x) is one-to-one.

let k $\in E_{10}$, then there are two cases,:

the last bit is 1, let A be the string with the last bit removed, k = A + 1, since k has even 1's, A
must be have odd 1's. k = f(A)

the last bit is 0, let A be the string with the last bit removed, k = A + 0, since k has even 1's, A must be have even 1's too. k = f(A).
Thus, f(x) is onto.

Since f(x) is both one-to-one and onto, it is bijection.

**b.**

Since there is a bijection between $B^9$ $and$ $E_{10}$, $|E_{10}| = |B^9| = 2^9$

## Question 9:

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a) Exercise 5.4.2, sections a, b

**a.**

Since the telephone number starts with either824 or 825 with 7 strings, the total number of different telephone numbers are:

$$10^4 + 10^4 = 2 * 10^4$$

**b.**

Since the last four digits cannot have dupulicate numbers, the total number of different telephone numbers are:

$$10*9*8*7 + 10*9*8*7 = 2 * 10 * 9 * 8 * 7$$

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b) Exercise 5.5.3, sections a-g

**a.**

when there are no restrictions, there are $2^{10}$ 10-bit strings

**b.**

The string starts with 001. there are $2^7$ different 10-bit strings.

**c.**

The string starts with 001 or 10. there are $2^7$ + $2^8$ different 10-bit strings.

**d.**

The first two bits are the same as the last two bits. there are 4 * $2^6$ different 10-bit strings.

**e.**

The string has exactly six 0's. There are 10*9*8*7*6 = 10! / 4! different 10-bit strings.

**f.**

The string has exactly six 0's and the rst bit is 1. there are $\binom{9}{6} = 9!/3!$

**g.**

exactly one 1's in the first half and three 1's in the second half.

There are $\binom{5}{1}*\binom{5}{3} = 5*5*4*3$

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c) Exercise 5.5.5, section a

**a.**

*since there are* $30$ *boys and* $35$ *girls, and we will choose* $10$ *boys and* $10$ *girls from them.* *there are* $\binom{35}{10}*\binom{30}{10}$ *ways to make the selection.*

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d) Exercise 5.5.8, sections c-f

**c.**

How many five-card hands are made entirely of hearts and diamonds?
within the 52 cards in the deck, there are 13 hearts and 13 diamonds, which in total are 26 cards of hearts and diamonds. we choose 5 from 26 cards, so the possibilities are $\binom{26}{5}$

**d.**

How many five-card hands have four cards of the same rank?
Since there are 13 ranks in a deck and each rank have 4 cards, we choose 4 cards from the same rank, which is 13 ways of selection, then we choose another card from the remaining card, which is 52 - 13 = 48, so the total number of possibilities are 13 * 48

**e.**

How many five-card hands contain a full house?
we choose 2 out ot the 4 same rank card, total possibilities are $\binom{4}{2}*13 = 12*13$;
then we choose 3 out of 4 same rank card from the left 12 ranks, total possibilities are 12 * 4 * 3 * 2 * 1= $12^2*2$,
total the number of possibilities of full house are $12^3*26$

**f.**

How many five-card hands do not have any two cards of the same rank?
Since there are 13 different ranks in a deck, we set the first card have 13*4 possibilities, since no any two cards have the same rank, the second card will have 12*4 possibilites.
Total number of possibilities are 13 * 12 * 11 * 10 * 9 * $4^5$

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e) Exercise 5.6.6, sections a, b

**a.**

Since in the 10 senate members, 5 of them are from Demon and 5 of them are from Repub, the total number of selection are $\binom{44}{5}*\binom{56}{5}$

b.

selection of a speaker are
$\binom{44}{1}*\binom{43}{1}$ *for Demon, since the speaker and vice speaker cannot be the same person.*
selection of a speaker are $\binom{56}{1}*\binom{55}{1}$ *for Repub.*
total number of selections are $\binom{44}{1}*\binom{43}{1}*\binom{56}{1}*\binom{55}{1}$

## Question 10:

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a) Exercise 5.7.2, sections a, b

**a.**

How many 5-card hands have at least one club?

We can count this problem by counting the number of 5-card hand have no club, since

$|A| = |U| - |\overline{A}|$

$|\overline{A}|$ = $\binom{39}{5}$

$|U|$ = $\binom{52}{5}$

so, $|A| = \binom{52}{5} - \binom{39}{5}$

**b.**

How many 5-card hands have at least two cards with the same rank?

$|A| = |U| - |\overline{A}|$

$|\overline{A}|$ = $\binom{13}{5} * 4^5$ which is 5-card hands with no two cards with the same rank.

$|U|$ = $\binom{52}{5}$

so, $|A| = \binom{52}{5} - \binom{13}{5} * 4^5$

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b) Exercise 5.8.4, sections a, b

a.

the total number of possibilities are $5^{20}$

b.

the total number of possibilities are $\binom{20}{4} * \binom{16}{4} * \binom{12}{4} * \binom{8}{4} * \binom{4}{4} = \frac{20!}{4!4!4!4!4!}$

## Question 11:

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a)
There will not be any function from a set of five elements to set with 4 elements that have one-to-one relationship. Based on the definition of one-to-one relation, if x1 is not equal to x2, f(x1) should not be equal to f(x2). But in this case, there is one f(x) value in the set with 4 elements that will necessarily have two different x value, which contradicts the definition of one-to-one function.

b)
Since both domain and target sets have five elements, each one element in the domain will have a different element in the target set.
for the first element, it has 5 ways to choose from the target set; for the second element, it has 4 ways to choose from the target set, and so on. We get that there are 5 * 4 * 3 * 2 * 1 = 120 different one-to-one functions.

c)
Since domain have 5 elements and target sets have 6 elements, each one element in the domain will have a different element in the target set. and there will be one element in the target set that is not pointed.
for the first element, it has 6 ways to choose from the target set; for the second element, it has 5 ways to choose from the target set, and so on. We get that there are 6 * 5 * 4 * 3 * 2 = 720 different one-to-one functions.

d)
Since domain have 5 elements and target sets have 7 elements, each one element in the domain will have a different element in the target set. and there will be two elements in the target set that are not pointed.
for the first element, it has 7 ways to choose from the target set; for the second element, it has 6 ways to choose from the target set, and so on. We get that there are 7 * 6 * 5 * 4 * 3 = 2520 different one-to-one functions.