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SANTA CRUZ

**EMERGENT PHENOMENA IN ACTIVE POLYMER ARRAYS**

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requirements for the degree of

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in

PHYSICS

by

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## **Abstract**

Emergent Phenomena in Active Polymer Arrays

by

Stephen E. Martin

Theses have elements. Isn't that nice?

To myself,

Perry H. Disdainful,

the only person worthy of my company.



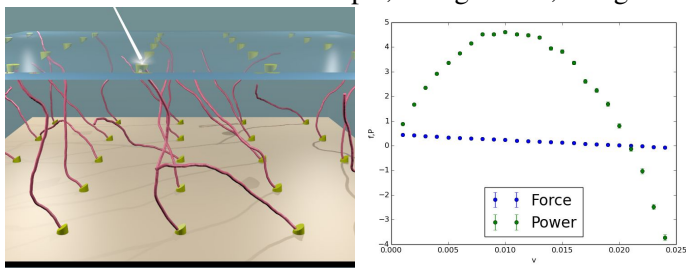
## **Acknowledgments**

I want to “thank” my committee, without whose ridiculous demands, I would have graduated so, so, very much faster.

# Chapter 1

## Introduction

Every dissertation should have an introduction. You might not realize it, but the introduction should introduce the concepts, background, and goals of the dissertation.



A device is investigated that continuously and directly converts light into mechanical energy, using polymers and photodissociation. A polymer brush tethered to a surface, is brought into contact with a parallel plate a small distance above it that contains reaction sites where photodissociation of bound polymer and light can occur. Under the appropriate conditions, the collective effect of these polymers is predicted to apply a force parallel to the plates, converting incoming light into mechanical work. Numerical work is carried out to understand this effect, a three dimensional Langevin simulation, solution to the Fokker Planck equation, and a one

dimensional Monte Carlo simulation. Theoretical analysis of the Fokker Planck equation is used to study a model where equilibration of the unbound state occurs and equilibration to a metastable equilibrium is achieved in the bound state. It is shown that the work per cycle can be made much larger than the thermal energy but at the expense of requiring a greatly diminished photodissociation rate. Parameters are discussed in order to optimize mechanical energy conversion.

The physical mechanism behind the spontaneous formation of metachronal waves in microtubule arrays in a low Reynolds number fluid has been of interest for the past several years, yet is still not well understood. We present a model implementing the hydrodynamic coupling hypothesis from first principles, and use this model to simulate kinesin-driven microtubule arrays and observe their emergent behavior. The results of simulations are compared to known experimental observations by Sanchez et al.[?, ?]. By varying parameters, we determine regimes in which the metachronal wave phenomenon emerges, and categorize other types of possible microtubule motion outside these regimes.

## Chapter 2

# Photomechanical Energy Conversion Using Polymer Brush Dissociation

### 2.1 Introduction

There are many proposals to convert light into mechanical energy using smart polymeric photo-responsive materials [?, ?, ?, ?, ?] or the synthesis of individual molecules that lead to rotary or linear motion [?].

Photoresponsive materials use a conformational change of a macroscopic collection of polymers in response to light or an external change in temperature and pH. This will cause a change in volume. If this process is reversible, which it is in many instances, this device can be cycled to derive mechanical energy [?, ?, ?].

In general, there are many scenarios leading to a change in configuration of a molecule in response to light [?]: *cis-trans* isomerization, zwitter ion formation, radical formation, ionic

dissociation, and ring formation/cleavage. If such light-sensitive elements are incorporated into a macromolecular system, this can in principle, through thermodynamic cycles, to conversion of light to mechanical power.

Remarkable advances in the design of molecules have led to prototypes for light-driven synthetic molecular motors [?]. Many of these are based on photoisomerization reactions that enable rotary motion of molecules when combined with thermal rotation steps [?, ?, ?, ?]. It has also been possible to make threading-dethreading systems [?, ?], and shuttle molecular rings [?, ?]. There has even been progress on a macroscopic level, such as creating droplet motion by modifying surface properties [?] or inducing mechanical deformation [?] similar to what is achieved with photo-responsive materials [?, ?, ?, ?, ?].

Much of the research in this area is inspired by biological machines such as myosin II that work by quite a different principle than man-made macroscopic motors [?]. Biological motors use chemical energy rather than photons to produce mechanical power however this difference does not affect the basic mechanism of operation. A myosin head can be thought of as being in two states, bound or unbound. Thermal noise in the unbound state can cause the head to bind to actin, producing a force. The hydrolysis of ATP releases energy causing the head to return to the unbound state. The biochemistry of a real motor protein is considerably more complicated, but by simplifying this description to one involving only these two states, the motion can be analyzed, and it is easily seen that electromagnetic energy can be used instead of chemical energy [?].

In this paper, we investigate the use of photons in powering a two state motor system similar to biological motors. We consider creating motion between two surfaces that are very

close together by placing an *asymmetric* polymer brush between them. This could potentially have advantages. A continuous source of light would create a motion that is constant on a macroscopic scale, that could for example, rotate the surfaces relative to each other. It would not require any additional mechanisms to keep it moving, other than the microscopic motion of molecules between the plates.

The device proposed here falls into a distinct category different than the experimental approaches above. It is not a macroscopic smart material that changes properties in response to external stimuli. The device described in the next section works on the scale of individual polymers, and the force generated on the surfaces is the sum of these molecules acting independently of each other. However it is unlike the chemical synthesis approach that requires different states of isomerization. The principles that it relies on are robust and just require photodissociation of polymer to a binding site, and, as in the case of biological motors, some degree of asymmetry. This asymmetry, plus a disruption of thermal equilibrium due to energy input are the two factors needed to convert energy from chemical or electromagnetic energy, to mechanical [?].

The paper is organized as follows: The photo-mechanical system is described (The System) and rough estimates of its operation are given (Estimate of system parameters), including a discussion of its potential efficiency. In order to illustrate the characteristics that need to be understood, a three dimensional simulation of this device is carried out (Three dimensional model). The physics of this system is then investigated in more analytical detail, starting with general considerations of its steady state behavior using Fokker Planck equations. The results will be used in subsequent sections. An exact description of this system is possible in steady state (Steady State Probability Distribution) and the resulting differential equation is solved nu-

merically in one dimension (One dimensional solution). This allows to understand better how effective the asymmetry in the force produces power, and influences the direction that is taken in the rest of this work. Next, we study the useful large power stroke limit that is hard to probe numerically, but allows us to understand how the efficiency of this system is related to spring stiffness, metastability, relaxation times, and the photo-dissociation rate (Unbound Equilibration Model). We then use this model to understand the efficiency, using a one dimensional Monte Carlo model that is in this regime (Efficiency in large power stroke limit). Finally, we conclude on how the physics that has been learned might be useful in optimizing experimental parameters for such a system (Conclusions).

## 2.2 The System

The components needed to do constant photomechanical energy conversion are illustrated in Fig. 2.1.

- **A flat plate of semi-flexible polymer brushes.** Polymer brushes are polymers, each with one end tethered to a wall in a suitable solvent.
- **A parallel plate right above the brush.** These polymers are put in contact with a parallel plate close to the surface so that their ends are able to interact with it.
- **The parallel plate contains an array of photoreactive binding sites.** It is crucial that these polymers bind with the surface in a sterically asymmetric way, so that the average binding orientation of each (called  $\hat{\mathbf{n}}_a$  in Fig. 2.1) is the same.

- **At least one of the plates is transparent.** Light causes the unbinding of polymer ends from the photoreactive binding sites.
- **Binding catalyst.** To control the rate at which binding occurs, the binding of the end of the polymer to a binding site can be facilitated by the use of a catalyst. The concentration of the catalyst can be used to control the rate of binding.

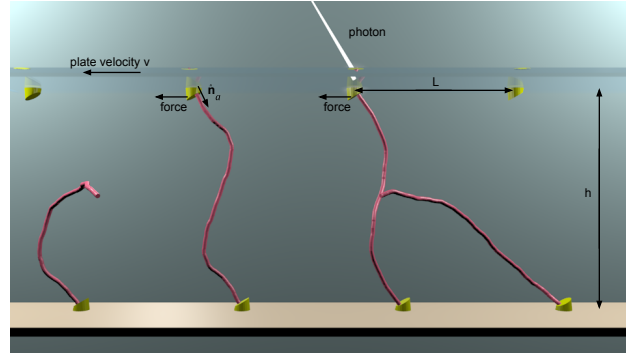


Figure 2.1: (a) A schematic of the light energy to mechanical converter. A polymer brush of semiflexible chains is put in contact with a plate a height  $h$  above the base, containing an array of anisotropic binding sites with preferred binding direction  $\hat{n}_a$ . Polymer ends are trapped by unlikely thermal fluctuations, applying a non-zero average force parallel to the plate. Photodissociation of the polymer ends with the binding sites releases the chain end that will move until it finds another site. The upper plate moves at a velocity  $v$ , thereby generating power. A single track is shown, and the separation between binding sites is  $L$ . The shape of the binding sites is meant only as a guide to the eye concerning the direction of  $\hat{n}_a$ , and is not necessarily an indicator of the binding sites' physical structure.

In most of the discussion below, the polymers can be considered to be separated from each other by a sufficient distance so that we can ignore inter-chain interactions. The absorption of light causing unbinding will happen asynchronously. The ends tethered to the lower plates are at positions that are either random or incommensurate with the binding sites of the upper



plate. Altogether this means that the motion of each polymer is uncorrelated with the others in the system.

The total effect of the forces acting on the plates can be used to perform work against a force acting on the upper plate. Because we assume that the number of polymers contributing is very large and are asynchronous, the net velocity of the upper plate  $v$  will be constant, as will be the net force acting on the plate.

In the following we will analyze the system and model it in different ways to better understand the power generation.

## 2.3 Estimate of system parameters

We will now make a crude estimate with realistic parameters of how well this device should work. We envisage separate one dimensional tracks of the kind shown in Fig. 2.1. We do not enforce that the arrangement of binding sites need be a square two-dimensional array—the spacing between the tracks could indeed be much greater than the distances between polymers on a single track. This would have the effect of limiting polymers to a particular track of binding sites. In reality, it might prove more efficacious to manufacture a square array of polymers and binding sites but we will allow two separate length scales in our analysis below.

We will assume a solar intensity of  $I = 600 \text{ W}/\text{m}^2$  [?], and an average photon energy of  $e = 2 \text{ eV}$  [?]. We will take the relaxation time of a polymer to be  $\tau = 10^{-7} \text{ s}$ , which corresponds roughly to a polymer size of  $3 \text{ nm}$  [?]. The number of unbinding events, if every photon was absorbed at a binding site, is  $I/e \approx 2 \times 10^{21}/(\text{m}^2 \text{ s})$ . If the density of polymers is  $\sigma$ , then for

all of these events to be utilized requires  $I/e = \sigma/\tau$ , or  $\sigma = 2 \times 10^{14}/m^2$ , which is a separation of  $1/\sqrt{\sigma} \approx 70nm$ . The amount of energy per step that is gained by a photo-dissociation event is of order  $k_B T$ . With such parameters, this is expected to yield an efficiency of order 1%. However, as we will show below, longer relaxation times allow for more energy per step, and this requires a higher polymer density. However the two dimensional density is limited by inter-chain interactions. In the case considered here, this density could be increased roughly by three orders of magnitude. However the relaxation time for this system depends exponentially on the force generated, so a three order of magnitude increase in relaxation time would only increase the efficiency by a factor of about 7. Indeed, the detailed analysis presented below suggests that with optimal conditions, the efficiency of conversion is about 7%. To increase the efficiency of solar conversion further might require stacking devices.

There are other means of increasing the solar conversion efficiency by first converting solar radiation to lower energy photons. For example, two methods for doing this are fluorescent down-conversion of solar photons [?], or thermophotovoltaic down-conversion [?].

Further speculation on device efficiency is premature as there will undoubtedly be many unforeseen technical problems that will likely provide other obstacles to increasing device efficiency. However an efficiency for direct photo-mechanical conversion of about 7% is still a useful amount of power comparable to photovoltaic conversion with amorphous solar cells, and also because power is lost in electro-mechanical conversion, which is not a problem with direct mechanical energy conversion.

## 2.4 Three dimensional model

We start by simulating a three dimensional model of this system. There are two components to the system, the polymer and the binding sites. The polymer chain is modeled as having  $N$  links of fixed length. Denote the coordinates of the  $i$ th bead as  $\mathbf{r}_i$ . The elastic potential for the middle of the chain for the  $i$ th bead is

$$U_E(i) \equiv -\frac{C}{2}(|\mathbf{r}_{i+2} - \mathbf{r}_i|^2 + |\mathbf{r}_{i-2} - \mathbf{r}_i|^2) \quad (2.1)$$

where  $C$  is the stiffness constant.

The binding potential of the polymer has two components, an isotropic component  $U_i$  and directional component  $U_d$ .  $U_i(r)$  is short range with a length scale  $r_s$  and scale  $V_a$ :

$$U_i(r) \equiv -V_a \exp\left(-\frac{Ar^2}{r_s^2 - r^2}\right), \quad (2.2)$$

where  $A > 0$  is a parameter to adjust the shape of the potential. This binding potential is chosen for two reasons. First,  $U_i(r)$  and all of its derivatives vanish as  $r \rightarrow r_s$ , thus all derivatives of these forces are continuous. Second, for small  $A$ ,  $U_i(r)$  rises quickly as  $r \rightarrow r_s$ , resulting in a large binding cross section. A plot of  $U_i(r)$  for  $A = 0.2$ ,  $V_a = 60$ , and  $r_s = 1$  is shown in Fig. 2.2.

$U_d$  uses a direction  $\hat{\mathbf{n}}_a$  so that the difference between the last two end beads  $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$  will give a minimum in  $U_d$  along that direction:

$$U_d(\mathbf{r}_0, \mathbf{r}_1) \equiv \left( \left| \frac{\Delta\mathbf{r}}{|\Delta\mathbf{r}|} - \hat{\mathbf{n}}_a \right|^2 + 1 \right) U_i(r). \quad (2.3)$$

The end attached to the lower surface is always bound. The other end can bind to a periodic linear array of binding sites equally spaced at a distance  $L$ .

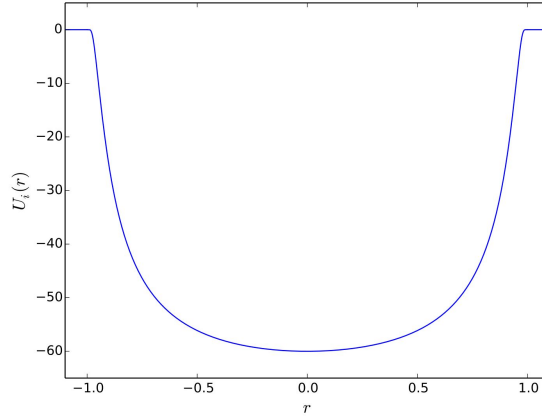


Figure 2.2: A plot of the isotropic binding potential  $U_i(r)$  vs.  $r$ , for  $A = 0.2$ , potential well depth  $V_a = 60$ , and interaction length scale  $r_s = 1$ .

There are two states that the system can be in, unbound, 0, and bound, 1. There are two parameters that control binding: the rate at which dissociation occurs  $c_1$ , and the rate that dissociated ends can be rebound  $c_0$ .

The distance between the lower and upper surface is  $h$  and their relative velocity is  $v$ .

The model was simulated with a Langevin equation, at finite temperature  $T$ . Although inertial effects are typically small at these microscopic scales, it was included for completeness. An algorithm was used that efficiently updates this system with fixed link lengths [?].

In order to increase the rate of binding, we also restrict the vertical motion of the polymer to be between the two plates (i.e.  $0 \leq z \leq 1$  for  $h = 1$ ) and the lateral motion (perpendicular to  $\mathbf{v}$ ) to a maximum distance of  $h/2$  from the binding sites. These restrictions are not unrealistic—we expect motion of polymers in such a device to lie between the two solid plates, and lateral motion may be restricted by the polymer stiffness and/or interactions with other polymers.

The simulation was run to determine how the force and power generated are influenced by the plate velocity. Fig. 2.3 shows the results of two simulations with different values of  $c_0$  and  $c_1$ . The parameters used were: 4 links,  $h = 1$ , spacing between binding sites  $L = 2$ , the direction of  $\hat{\mathbf{n}}_a$  is  $\pi/4$ ,  $V_a = 60$ , link length of 1, particle mass of 1,  $C = 0.5$ ,  $r_s = 1.0$ ,  $A = 0.2$ , and a coefficient of damping of 10.

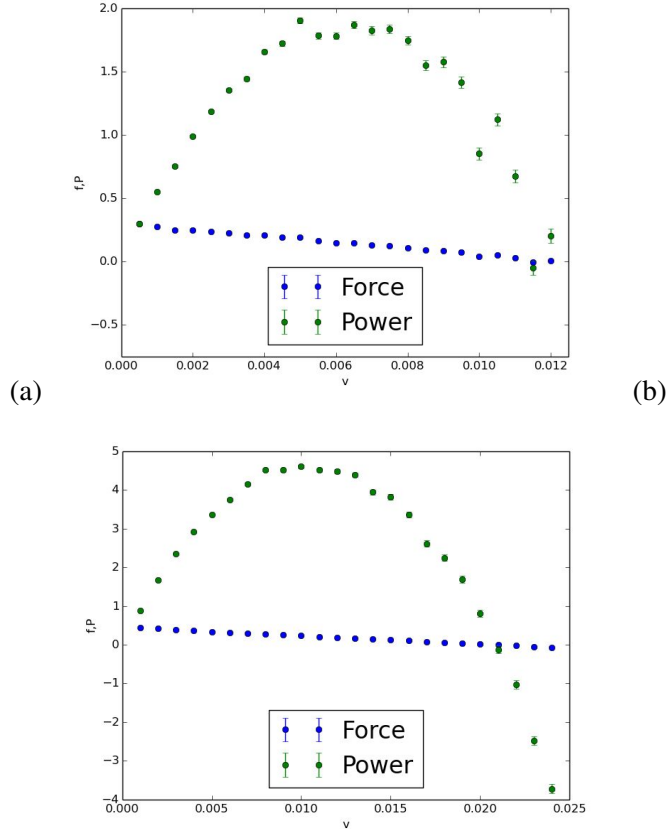


Figure 2.3: The average force  $f$  and the power  $P = fv$  measured as a function of the relative velocity between the plates  $v$ . For clarity, the power is multiplied by 2000. There were 4 links, with a plate separation of  $h = 1$ , spacing between binding sites  $L = 2$ , the direction of  $\hat{\mathbf{n}}_a$  (that is, the binding angle) is  $\pi/4$ ,  $V_a = 60$ ,  $A = 0.2$ , link length of 1, stiffness constant  $C = 0.5$ , potential range  $r_s = 1.0$ , and a coefficient of damping of 10. The coefficients of binding and unbinding, respectively, are (a)  $c_0 = c_1 = 0.05$ , and (b)  $c_0 = c_1 = 0.4$ .

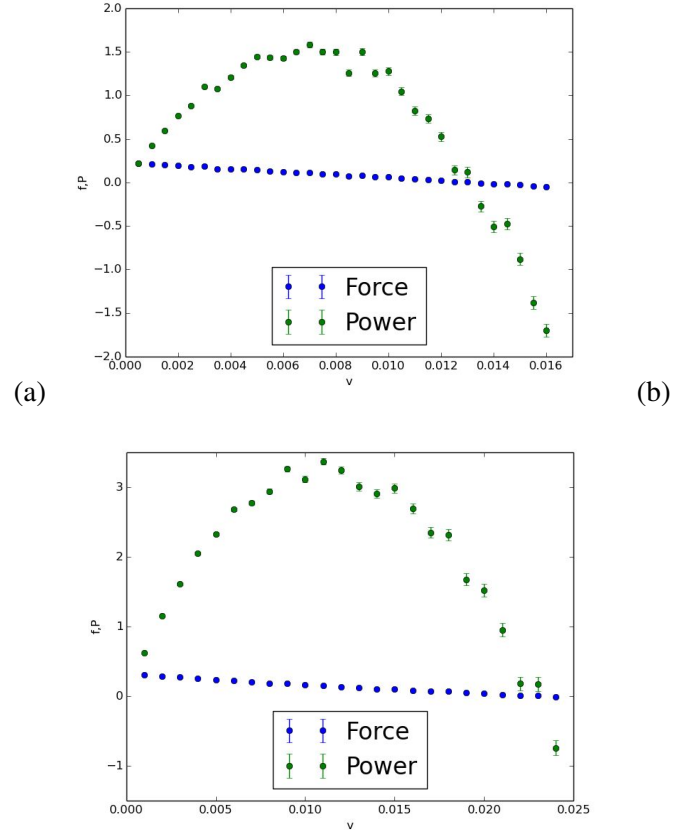


Figure 2.4: The average force  $f$  and the power  $P = fv$  measured as a function of the relative velocity between the plates  $v$ . The binding sites are placed randomly, with average binding site separation  $\bar{d} = 2$  and minimum binding site separation  $d_{min} = 0.1$ . Other than this, the same parameters and scaling are used as in Fig. 2.3: (a)  $c_0 = c_1 = 0.05$ , and (b)  $c_0 = c_1 = 0.4$ .

The force generated is highest approaching zero velocity and decreases until it becomes negative, at which point it is taking mechanical energy to move it at such high speeds. Qualitatively, this is the point where frictional drag dominates over photo-energy conversion.

Note that the highest force  $f$ , peak operating velocity, and power  $P = fv$  are all seen for  $c_0 = c_1 = 0.4$ . Higher maxima in the power are generally seen with increasing  $c_0$  and  $c_1$ . However, this higher power is at the expense of efficiency, as higher dissociation rates imply a higher photon flux.

It is also of interest to determine if this photomechanical effect persists if the binding sites are no longer periodic but instead are arranged at random. To investigate this situation, we place the binding sites so that the probability distribution of spacings,  $d$ , between binding sites followed Poisson statistics, that is  $p_{spacing}(d) \propto \exp(-d/\bar{d})$ . Here  $\bar{d}$  is the average spacing between sites. However we also must introduce a lower cutoff,  $d_{min}$ , to prevent binding sites from overlapping so that when generating the next site's position, we reject spacings less than this cutoff. For our simulations, we set  $\bar{d} = 2$  and  $d_{min} = 0.1$ . The purpose of this is to simulate possible experiments in which a rigidly periodic array of binding sites is impractical to construct.

Fig. 2.4 shows the results of these simulations. The results are strikingly similar to those found with periodic sites. This shows that, at least for this parameter regime, the motor function of the device does not depend strongly on the regular placement of binding sites—an optimistic result from an experimental perspective.

## 2.5 Steady State Probability Distribution

To understand the above simulation in more detail requires a better understanding of the mechanisms involved. There have been a very large number of works on the theory of motor proteins [?, ?], and we will follow the two state approach mentioned above of Prost *et. al.* [?, ?]. The major difference is that they considered the two states to be periodic potentials, whereas here we model the system to more closely mimic the particular device we are investigating allowing the exploration of force versus velocity. Hence instead of considering the position of a particle moving between two periodic potentials, we consider the unbound state to have a free end described as moving in a bound potential around the tether point. Likewise, because of this tethering, the bound state potential is not periodic but has a periodic component as we will describe in detail below. Here we consider the probability distribution for this system in steady state, which is described by a Fokker Planck equation.

Initially to calculate the power that is produced, we will concentrate on the low velocity limit, with the two plates moving so slowly that this motion does not affect chain conformations appreciably. In this way, we can consider the average force exerted between the plates by a polymer in steady state in the limit  $v = 0$  so that the tether point is not moving. It is most convenient to let the point at which the polymer is tethered to the bottom plate be a variable parameter  $\mathbf{r}' = x'\hat{\mathbf{x}}$ . With this point fixed, we consider the distribution of the other end of the polymer  $\mathbf{r}$ . We will assume that the internal dynamics of the chain are much faster than the binding and unbinding rates, so that the only degrees of freedom are  $\mathbf{r}$ , and if the polymer is bound,  $i = 0$  is unbound and  $i = 1$  is bound. Therefore the probability distribution of the system



can be described by a function  $P_i(\mathbf{r}; \mathbf{r}')$ . The equations describing this are [?]

$$\partial_t P_0(\mathbf{r}) = \nabla \cdot (\nabla - \mathbf{f}_0) P_0(\mathbf{r}) - c_0 P_0(\mathbf{r}) + c_1 P_1(\mathbf{r}) \quad (2.4a)$$

$$\partial_t P_1(\mathbf{r}) = \nabla \cdot (\nabla - \mathbf{f}_1) P_1(\mathbf{r}) + c_0 P_0(\mathbf{r}) - c_1 P_1(\mathbf{r}) \quad (2.4b)$$

where the last  $\mathbf{r}'$  argument of  $P$  has been left out for notational simplicity. The units here absorb the diffusion coefficient  $D$  together with the time, that is  $t$  as used here and below is really  $D$  times the time. The force terms have absorbed a temperature factor  $T$ , that is  $\mathbf{f}_i$  is really the force times  $1/k_B T$ . Those forces  $\mathbf{f}_0$  and  $\mathbf{f}_1$  are the total forces acting on the upper end of the chain. When the system is unbound, it is the force of a (possibly) nonlinear spring

$$\mathbf{f}_0 = \mathbf{f}_s(\mathbf{r} - \mathbf{r}'). \quad (2.5)$$

When the system is bound we have total force is the sum of the spring force and a periodic force representing the binding potential

$$\mathbf{f}_1 = \mathbf{f}_s(\mathbf{r} - \mathbf{r}') + \mathbf{f}_p(\mathbf{r}). \quad (2.6)$$

These forces are assumed to be conservative,  $\mathbf{f}_s$  and  $\mathbf{f}_p$  are derived from potentials that respectively are  $V_s$  and  $V_p$ .

In steady state the left hand sides of Eqs. 2.4a-b are zero. We can eliminate the last two terms on the right hand side to obtain

$$\hat{O}_0 P_0 + \hat{O}_1 P_1 = 0, \quad (2.7)$$

where

$$\hat{O}_i \equiv \nabla \cdot (\nabla - \mathbf{f}_i), \quad i = 0, 1. \quad (2.8)$$

We can now calculate the average force exerted by the two potentials, is zero. In steady state, we multiply Eq. 2.7 by  $\mathbf{r}$  and integrate with respect to  $x$ ,  $y$  and  $z$ . Then using integration by parts, we have boundary terms at infinity. Because we are assuming that the spring potential grows without bounds, this confines  $P_0$ , and  $P_1$ , to a neighborhood around  $\mathbf{r}'$ , so that the boundary terms vanish. We are then left with

$$\langle \mathbf{f} \rangle = \int (\mathbf{f}_0 P_0 + \mathbf{f}_1 P_1) d^3 \mathbf{r} = 0 \quad (2.9)$$

as is expected because  $r$  is confined to a region of space so that the average velocity, and hence average force, will be zero.

We can also determine the total fraction of time spent in the bound or unbound states in steady state. First, because the probability of being in any state is unity,

$$\int (P_0(\mathbf{r}) + P_1(\mathbf{r})) d^3 \mathbf{r} = 1. \quad (2.10)$$

Then by integrating Eq. 2.4a over all space, the derivative term integrates to 0, giving

$$\int (-c_0 P_0(\mathbf{r}) + c_1 P_1(\mathbf{r})) d^3 \mathbf{r} = 0 \quad (2.11)$$

hence

$$\int P_0(\mathbf{r}) d^3 \mathbf{r} = \frac{c_1}{c_0 + c_1}, \quad \int P_1(\mathbf{r}) d^3 \mathbf{r} = \frac{c_0}{c_0 + c_1}. \quad (2.12)$$

To obtain the power produced by this device, we are not interested in the total force acting on the upper chain end because this includes the binding potential, but the average force due to the spring acting on the lower plate  $\langle f_s \rangle$ . We would like to calculate the work done in moving the lower point  $\mathbf{r}'$  by one period of  $\mathbf{f}_p$ . After moving one period the system is statistically

identical to its starting point, and this method can therefore give the work done in moving  $n$  such periods. Denoting the period of  $\mathbf{f}_p$  by  $L$ , we would like to calculate

$$\begin{aligned} W_l &= \int_0^L \hat{\mathbf{x}} \cdot \langle \mathbf{f}_s \rangle dx' \\ &= \int_0^L \int \hat{\mathbf{x}} \cdot \mathbf{f}_s(\mathbf{r}, x' \hat{\mathbf{x}}) (P_0(\mathbf{r}, x' \hat{\mathbf{x}}) + P_1(\mathbf{r}, x' \hat{\mathbf{x}})) d^3 \mathbf{r} dx' \end{aligned} \quad (2.13)$$

Using Eq. 2.5 and 2.6, this may be rewritten as

$$W_l = \int_0^L \int \hat{\mathbf{x}} \cdot (\mathbf{f}_0 P_0(\mathbf{r}, x' \hat{\mathbf{x}}) + (\mathbf{f}_1 - \mathbf{f}_p) P_1(\mathbf{r}, x' \hat{\mathbf{x}})) d^3 \mathbf{r} dx' \quad (2.14)$$

Eq. 2.9 then yields

$$W_l = - \int_0^L \int \hat{\mathbf{x}} \cdot \mathbf{f}_p(\mathbf{r}) P_1(\mathbf{r}, x' \hat{\mathbf{x}}) d^3 \mathbf{r} dx'. \quad (2.15)$$

In thermal equilibrium, where the transition rates  $c_0$  and  $c_1$  are both zero, we recover the Gibbs distribution. Let us assume, that the system starts, and therefore remains in state  $i = 1$ .

Then

$$P_1(\mathbf{r}, \mathbf{r}') = \frac{e^{-V_1}}{Z} \quad (2.16)$$

where the partition function

$$Z(\mathbf{r}') = \int e^{-V_1} d^3 \mathbf{r} \quad (2.17)$$

which will also have periodicity of  $L$ . In this case  $W_l$  can be easily calculated because  $P_0 = 0$  and  $\mathbf{f}_s = -\nabla V_s(\mathbf{r} - \mathbf{r}') = \nabla' V_s(\mathbf{r} - \mathbf{r}')$ . So

$$\begin{aligned} W_l &= \int_0^L \int (\partial_{x'} V_s(\mathbf{r}, x' \hat{\mathbf{x}})) \frac{e^{-(V_s(\mathbf{r} - \hat{\mathbf{x}} x') + V_p(\mathbf{r}))}}{Z(x')} d^3 \mathbf{r} dx' \\ &= -T(\log Z(L) - \log Z(0)) = 0 \end{aligned} \quad (2.18)$$

as it must be by the second law of thermodynamics.

## 2.6 One dimensional solution

In order to investigate how the power conversion depends on the forces acting on this system, it is important to simplify the three dimensional model to obtain a minimal model that depends on far fewer parameters. Therefore we investigate this model in one dimension.

A key point to understand is how asymmetry in the form of the force produces power. With symmetric spring and binding potentials, it is easily seen by symmetry that no net power can be produced from this system. We now ask how asymmetry affects the results. We will see that even with a large asymmetry in the spring potential, the work defined by Eq. 2.13 is very small. Simulations using Monte Carlo or Langevin equations are too noisy to provide good estimates. We therefore instead use a more analytical approach.

The coupled Focker Planck Eqs. 2.4 (a) and (b) can be solved to produce an equation only involving one distribution function,  $P_1$  in steady state. Using the definitions in Eq. 2.8, we can eliminate  $P_0$ .

$$(\hat{O}_0 \hat{O}_1 - c_1 \hat{O}_0 - c_0 \hat{O}_1)P_1 = 0 \quad (2.19)$$

which is a fourth order equation in spatial variables.

Now we restrict the analysis to one dimension. In this case,  $\hat{O}_i = \partial_x(\partial_x - f_i)$  so we can integrate with respect to  $x$ . We note that because the spring confines  $P_1$  to a localized region, it will go to zero as  $x \rightarrow \pm\infty$ . Therefore the integration constant must also be zero

$$((\partial_x - f_0)\partial_x(\partial_x - f_1) - c_1(\partial_x - f_0) - c_0(\partial_x - f_1))P_1 = 0 \quad (2.20)$$

which is a third order linear differential equation.

Eq. 2.20 was solved by the shooting method [?]. The boundary conditions were obtained by considering the system far from  $x'$  where  $P_0$  is very small. In that domain,  $f_p$  was artificially cutoff so that  $f_0 = f_1 = f_s$ . Because the potential there is no longer changing between the two states, the solution is that of a system in thermal equilibrium. The solutions were required to match to these thermal solutions in this regime, far from  $x'$ . The equation was solved with three different initial conditions. An appropriate linear combination of these were constructed to match the boundary conditions as just described.

The periodic binding potential that was used is

$$V_p = A_1 \cos(x) - A_2 \sin(2x) . \quad (2.21)$$

We first consider asymmetry in  $V_s$  but with symmetric functions  $V_p$ , that is,  $A_2 = 0$  in Eq. 2.21.

$$V_s(x) \equiv \frac{1}{2}kx^2 - \frac{a}{1 + (b(x-d))^2} . \quad (2.22)$$

The first term describes a linear spring with spring constant  $k$ , the second adds an asymmetric dip. The parameters are chosen so that this dip is close in potential to the one created predominantly by the linear term,  $k = 4$ ,  $a = 8$ ,  $b = 2$ , and  $d = 2$ . The function is plotted in Fig. 2.5.

Plots are shown in Fig. 2.6 of  $P_1(x)$  within a period for four values of  $x'$ :  $-\pi$ ,  $-\pi/2$ ,  $0$ ,  $\pi/2$ .

By integrating using these distributions, Eq. 2.15, the work can be obtained. With  $c_0 = 0.025$  and  $c_1 = 0.05$  the work  $W_L = 0.00080954$ . With the  $c_i$ 's 10 times those values,  $c_0 = 0.25$ , and  $c_1 = 0.5$ ,  $W_L = 0.0025$ .

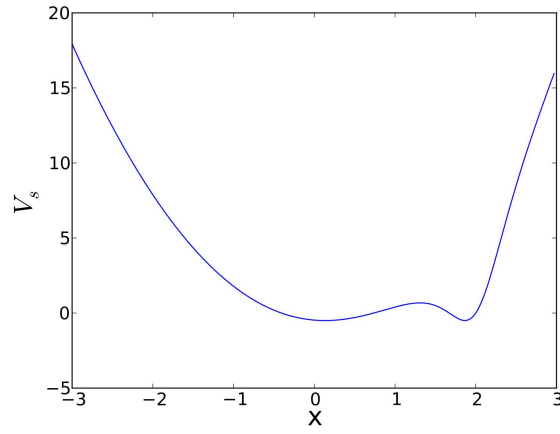


Figure 2.5: An asymmetric spring potential used to tether the chain. It has a second dip at approximately  $x = 2$

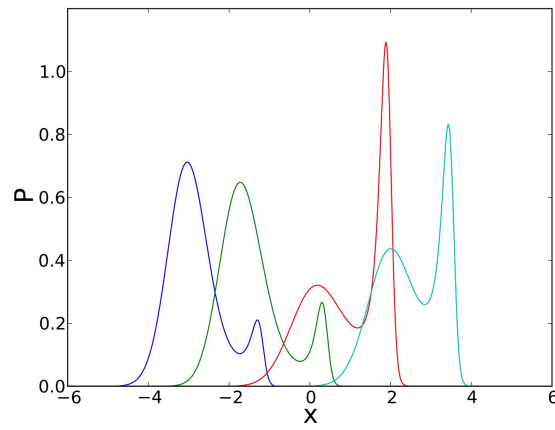


Figure 2.6: Plots of the probability distribution of  $P_1$  as a function of position  $x$  for  $x' = -\pi, -\pi/2, 0$ , and  $\pi/2$  for the asymmetric spring model Eq. 2.22.

Now we consider the case of a linear spring so that the nonlinear parameter  $a = 0$  in Eq. 2.22. Instead we make the periodic potential asymmetric by setting  $A_1 = A_2 = 2$  in Eq. 2.21, still with  $c_0 = 0.25$ , and  $c_1 = 0.5$ . Fig. 2.7 plots  $P_1(x)$  within a period for the same four values of  $x'$  in Fig. 2.6. In this case, the work  $W_L = -0.1596$ .

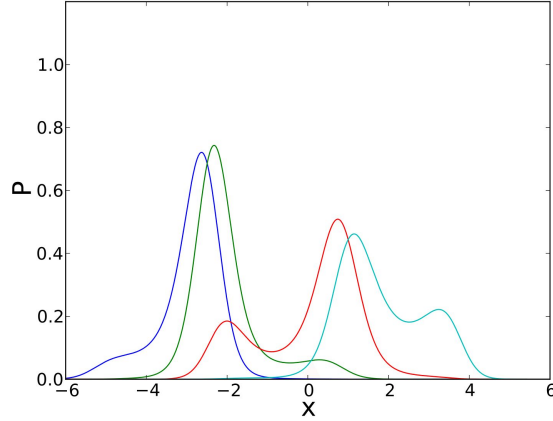


Figure 2.7: Plots of the probability distribution of  $P_1$  as a function of position  $x$  for the values of  $x'$  shown in Fig. 2.6, but for an asymmetric periodic potential and a linear spring.

What the numerical results have shown is that asymmetry in the spring potential is quite ineffective at producing work, whereas asymmetry in the periodic binding potential is much more effective. We shall use this result below and concentrate on systems with symmetric spring potentials, but asymmetric binding potentials.

## 2.7 Unbound Equilibration Model

It is worthwhile to understand in more detail what constrains the maximum force of photo-mechanical conversion by tuning the potentials employed and the binding rate, and to ask

how the design will depend on the flux of photons. We are limited in our choice of the potentials  $V_0$  and  $V_1$  that both must be bounded. In this system, the unbound potential is not periodic which limits the amount of power that can be generated. In cases considered earlier [?], where both potentials are periodic, it is possible to get much more efficient motion by alternating between states with different potential maxima. However this is not relevant to our system.

To understand this better, it is useful to examine a limit where we can treat the system analytically. We therefore examine the case where the binding rate of an unbound chain is sufficiently small, so that we can regard it in thermal equilibrium.

The model is illustrated in Fig. 2.8. We take the unbound potential to be that of a linear spring below a cutoff

$$V_s = V_0 = \begin{cases} k(x - x')^2/2, & \text{if } -l_c < (x - x') < l_c. \\ \infty, & \text{otherwise.} \end{cases} \quad (2.23)$$

we choose the spring coefficient  $k$  such that  $l_c \exp(-kl_c^2/2) \ll 1$ . Below, we will take  $l_c = L/2$  or  $\infty$ .

The periodic potential  $V_p$  is taken to be localized at periodic points (with a separation of  $L$ ) that rapidly vary from a large maximum  $V_{max}$  to a minimum  $V_{min}$ , as shown. We take the region over which this happens to be negligibly small compared to other length scales in the problem.

In the following, we present a new analytical method that allows us to systematically solve for the behavior of this kind of nonequilibrium system in the limit of large activation barriers. To understand how equilibration can occur in this model, we can rewrite Eqs. 2.4 in



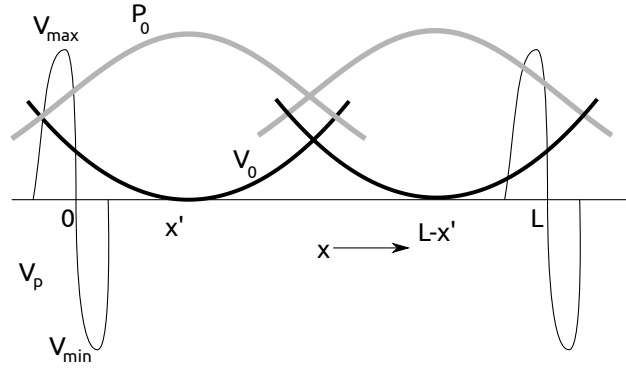


Figure 2.8: Illustration of the kind of potentials employed in the Unbound Equilibration Model. The periodic potential  $V_p$  is nonzero only on small regions on the  $x$ -axis. It shows a large peak of height  $V_{max}$  and large negative dip  $V_{min}$ . The spring potential  $V_0$  is parabolic, except that it has a cutoff where it becomes infinite when stretched by more than  $l_c$ . The corresponding probability distribution  $P_0$  (grey curve) is assumed to have relaxed to equilibrium. The spring potential is shown at two different position, when it is centered at  $x'$  and  $L-x'$ .

the steady state limit

$$\nabla \cdot (\nabla - \mathbf{f}_i) P_i(\mathbf{r}) - c_i P_i(\mathbf{r}) = -c_{1-i} P_{1-i}(\mathbf{r}), \quad i = 0, 1 \quad (2.24)$$

Although this is time independent, we consider the related time dependent equations for variables  $\tilde{P}_i(\mathbf{r}, t)$

$$\partial_t \tilde{P}_i(\mathbf{r}) - \nabla \cdot (\nabla - \mathbf{f}_i) \tilde{P}_i(\mathbf{r}) = 0, \quad i = 0, 1 \quad (2.25)$$

where  $P_i$ ,  $i = 0, 1$ , is the Laplace transform of  $\tilde{P}_i$ ,  $P_i(s) = \mathcal{L}\{\tilde{P}_i\}$ , where the conjugate variable for  $i = 0, 1$  are  $s = c_0$  and  $s = c_1$ , respectively. Taking the Laplace transform of both sides of Eqns. 2.25 yields Eqns. 2.24, with the initial conditions

$$\tilde{P}_i(\mathbf{r}, t = 0) = c_{1-i} P_{1-i}(\mathbf{r}, s = c_{1-i}), \quad i = 0, 1 \quad (2.26)$$

These equations have a direct physical interpretation. The left hand sides in Eqs. 2.25

are those of particles diffusing in potentials, but with conservation of particles. For long enough times, independent of initial conditions, the solution to these equations will go to thermal equilibrium given by the Gibbs distribution. Eq. 2.25,  $i = 0$ , describes diffusion in a quadratic potential. We require that we are probing this at long enough times  $t$ , so that it will have nearly reached this equilibrium state. Call this longest relaxation time  $\tau_0$ . The solution to the diffusion equation  $\tilde{P}_0(\mathbf{r}, t)$  for either  $i = 0, 1$  in Eq. 2.25, can be written as a sum over spatial eigenfunctions,  $\phi_n(\mathbf{r})$  that decay at different rates  $\lambda_n$ , (arranged to be monotonically increasing):

$$\tilde{P}_0(\mathbf{r}, t) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}) \exp(-\lambda_n t), \quad (2.27)$$

of which the smallest  $\lambda$ ,  $\lambda_0 = 0$ , corresponds to the equilibrium state. The relaxation time is  $\tau_0 = 1/\lambda_1$ , thus the Laplace transformed variable  $P_0$  can be written

$$P_0(\mathbf{r}, s) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}) \frac{1}{s + \lambda_n}. \quad (2.28)$$

If the  $n = 0$  term is to dominate, we therefore require  $s = c_0 \ll 1/\tau_0$ . The physical interpretation of this condition is that binding typically occurs only after many relaxation times of the unbound end.

$P_0(\mathbf{r}, s = c_0)$  will be dominated by the first term in Eq. 2.28 and therefore proportional to the eigenfunction  $\phi_0(\mathbf{r})$  which is the equilibrium distribution and is  $\propto \exp(-V_0)$ . An important simplifying point in the above approach is that the precise form of the initial condition Eq. 2.26,  $i = 0$ , is not important, but because of particle conservation, only the total area under  $P_1$  affects the result.

Now we consider the solution for  $P_1$ . If we consider unbinding times much longer than the relaxation time in the bound state, we also arrive at thermal equilibrium, which was

shown by Eq. 2.18 to lead to no work being performed. Instead we will consider situations where there are very long lived metastable states. In Fig. 2.8, the potential seen by a particle in Eq. 2.25,  $i = 1$ , is  $V_1 = V_0 + V_p$ . If a particle starts between 0 and  $L$ , it will remain trapped in that region for a Kramer's time which (ignoring algebraic pre-factors) depends on  $x'$ , but has a minimum value of  $\tau_m \propto \exp(V_{max})$ . By choosing large enough  $V_{max}$  this can be made arbitrarily long.

A particle in such a metastable state will relax to a metastable equilibrium, obeying Eq. 2.27 that will eventually fail for times  $t > \tau_m$ . In this expansion, the longest relaxation time to this metastable state  $\tau_1 = 1/\lambda_1$ , will be taken to be much smaller than  $\tau_m$ . Thus the above argument on the range of  $c_0$  can be used *mutatis mutandis* to restrict the unbinding rate to  $1/\tau_m \ll c_1 \ll 1/\tau_1$ .

In this regime we can understand the solution to Eqs. 2.25 and 2.26,  $i = 1$  by considering the corresponding Green's function  $\tilde{G}(\mathbf{r}, t; \mathbf{r}_0)$ . We replace the initial condition Eq. 2.26,  $i = 1$ , by

$$\tilde{G}_1(\mathbf{r}, t = 0; \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0). \quad (2.29)$$

We can then obtain  $\tilde{P}_1$  from the Green's function from

$$\tilde{P}_1(\mathbf{r}, t) = \int \tilde{G}_1(\mathbf{r}, t; \mathbf{r}_0) c_0 P_0(\mathbf{r}_0, s = c_0) d^3 \mathbf{r}_0. \quad (2.30)$$

For a wide range of times, and regions of  $\mathbf{r}'$ ,  $\tilde{G}_1$  will approach the same metastable state. A particle starting at any point within a certain region will end up stuck in the same state and hence approach the same metastable equilibrium. Confining our attention to the one dimension model of Fig. 2.8, for  $0 < x_0 < L/2$  the solution will be strongly localized at the

minimum  $x = 0$ . For  $L/2 < x_0 < L$  the effects of  $V_p$  are negligible and  $\tilde{P}_1(x) \propto \exp(-V_0)$ . This is because the potential  $V_0$  in Eq. 2.23 is cutoff which does not allow the particle to visit the  $x = 0$  region. Hence the effect of  $V_p$  is only seen close to  $x = L$  where it provides a strong repulsion, but over a negligibly small region of  $x$ . For  $-L/2 < x_0 < 0$ ,  $V_p$  also does not contribute. For  $L < x_0 < 3L/2$  the solution will be strongly localized at the minimum at  $x = L$ . Because of particle conservation, the area under  $\tilde{G}_1$  is always unity.

A physical interpretation of the above equations in terms of a one dimensional one particle system can now be made using the Laplace transformed variables and the metastable limit considered above is now apparent. In the unbound state, the particle reaches thermal equilibrium relaxing to the Gibbs distribution,  $P_0 \propto \exp(-V_0)$ . Then the periodic potential  $V_p$  is suddenly added in. The position at the time of switching is labeled  $x_0$ . Depending on which interval  $x_0$  is in,  $x$  will equilibrate to the corresponding metastable equilibrium with  $P_1 \propto \exp(-V_1)$  and is completely confined to that interval. The relative probabilities being in one of the three above regions is obtained by the area under  $P_0$  for that interval.

Now that we know how to determine  $P_0$  and  $P_1$ , we would like to calculate the work  $W_l$  given by Eq. 2.13. Because of the symmetric form assumed for  $f_s$ , the  $f_0 P_0$  term in the integrand gives zero contribution and we are left with

$$W_l = \int_0^L \int_{-\infty}^{\infty} f_0(x - x') P_1(x, x') dx dx' \quad (2.31)$$

and consider first how to calculate the inner integral

$$f_l(x') \equiv \int_{-\infty}^{\infty} f_0(x - x') P_1(x, x') dx \quad (2.32)$$

where  $x'$  is the position of the tethered end. Eq. 2.23 gives  $f_0(x) = -kx$  for  $|x| < L/2$ .

Using the prescription we have found for  $P_1(x, x')$ , which is simplified by the above physical interpretation, we can partition this integral into the different  $x$ -intervals of metastability:  $I_- \equiv [-L/2, 0]$ ,  $I_0 \equiv [0, L]$ , and  $I_+ \equiv [L, 3L/2]$ . The value of  $P_1(x, x')$  depends on the probability of initially being trapped in one of those three intervals. Because of the cutoff we have imposed on  $V_0$ , only two intervals need be considered for a given value of  $x'$ . For  $0 < x' < L/2$ , only intervals  $I_-$  and  $I_0$  occur. The probability that  $x \in I_-$  given  $x'$  is

$$E(x') \equiv P(x \in I_- | x') = \int_{-\infty}^0 p_0(x - x') dx = \int_{x'}^{\infty} p_0(x) dx \quad (2.33)$$

and the probability that  $x \in I_0$  is  $P(x \in I_0 | x') = 1 - E(x')$ . Here  $p_0(x)$  is proportional to  $P_0(x)$  but normalized to unity, to simplify the presentation. We can obtain the values for  $L/2 < x' < L$  by symmetry, so that  $P(x \in I_+ | x') = E(L - x')$  and  $P(x \in I_0 | x') = 1 - E(L - x')$ . In the limit considered here, the effects of the cutoff in the potential will have a negligible effect. For this reason,  $p_0(x)$  will be a normal distribution and thus  $E(x')$  is simply related to the complementary error function,  $E(x') = \frac{1}{2} \text{erfc}(\sqrt{k}x')$ .

At this point, it is possible to evaluate Eq. 2.31 directly. However, for simplicity, we take a more intuitive approach: there is a symmetry in many of the quantities considered, as shown in Fig. 2.8 where the potential  $V_0$  and corresponding probability distribution  $p_0$  is for the tethering point at  $x'$  and at  $L - x'$ . Therefore it is convenient to consider the quantity  $f_{l/2}(x') \equiv f_l(x') + f_l(L - x')$ .

Additionally, by equation 2.32,  $f_l$  can be thought of as an expectation value, or weighted average. Any expectation value can be expressed as a weighted sum of expectation

values broken down by region. This yields

$$W_l = \frac{kc_0}{c_0 + c_1} \int_0^{L/2} 2E(x')(\langle x \rangle' - x') + x' dx'. \quad (2.34)$$

In the limit of large  $L$ , which we are considering by virtue of the condition on the cutoff in  $V_0$  imposed by Eq. 2.23, the integrand simplifies because  $\langle x \rangle'$  becomes exponentially close to  $x'$ , and the only term remaining is  $x'$ . Thus for large  $L$ ,  $W_l = (c_0/(c_0 + c_1))kL^2/8$ . The factors involving the  $c$ 's represent the fraction of time spent in the bound configuration. The last term increases quadratically with  $L$ . This result is misleading if not taken with the appropriate limits that have been assumed in its derivation. The factor  $(c_0/(c_0 + c_1))$  is very close to unity as we are assuming that the relaxation time in the unbound state is much faster than in the bound state, hence  $c_0 \gg c_1$ . However the work  $W_L$  to move a distance  $L$  was assumed to be in the adiabatic limit, and here the time scales associated with bound state relaxation are exponentially long. This is because to reach this metastable equilibrium the particle has to hop over barriers of size  $V_0(x')$ , see Fig. 2.8. Hence the longest relaxation time for this is at  $x' = L/2$  and is of order  $\exp(kL^2/4)$ . Note that this does not contradict our assumption that we are still in a region of metastability, which requires times much less than  $\exp(V_{max})$ . But for this to work, we require that  $V_{max} \gg V_0(L/2)$ .

Now we consider the case where the spring length cutoff  $l_c \rightarrow \infty$ . The disadvantage of this is that the chain end can, in principle, hop over many barriers ending up arbitrarily far from the tether point, and that these hopping times should be included in the above analysis. However, the probability of such a hop becomes negligible when the probability of finding the chain end there is small. Hence we still have a clear separation of time scales between metastable states as

discussed above, and fully equilibrated system, which requires surmounting the energy barrier  $V_{max}$ . In this case, we can therefore assume that when the chain end  $x'$  is between  $nL$  and  $(n+1)L$ , it will strongly localized at  $x' = nL$ . Therefore

$$f_l(x') = -k \sum_{n=-\infty}^{\infty} \Delta_n(x')(nL - x') \quad (2.35)$$

where  $\Delta_n(x')$  is the probability of initially finding the chain end between  $nL$  and  $(n+1)L$ ,

$$\Delta_n(x') = E(nL - x') - E((n+1)L - x'). \quad (2.36)$$

It is easily seen that  $\Delta_{-n}(L - x') = \Delta_n(x')$ . Using this and Eq. 2.35

$$f_l(L - x') = -f_l(x') + kL \sum_{n=-\infty}^{\infty} \Delta_n(x') = -f_l(x') + kL. \quad (2.37)$$

To obtain the work, we follow the same procedure as above and consider  $f_l(x') + f_l(L - x')$ , which here is just  $kL$ . Therefore in this case,

$$W_l = \frac{c_0}{c_0 + c_1} \frac{kL^2}{2}. \quad (2.38)$$

This is an exact value for this model, for all  $k$  and  $L$ , given that we are in a region of strong metastability.

### 2.7.1 Efficiency in large power stroke limit

We now are in a position to answer a central question about the performance of this kind of device: is the efficiency limited by the small value of  $k_B T$  compared to photon energies? We have seen from the above analysis that it is possible to get arbitrarily large forces developing at the expense of exponentially slow operation. However this is in the limit of infinitesimal

plate velocity  $v$ . In contrast, the power obtained is instead average force times this,  $\langle f \rangle v$  and we would like to operate the device at the velocity of maximum power, which necessitates the operation of it far from equilibrium, because compensating the increase in  $v$  is a decrease in  $\langle f \rangle$  due to dissipation. It could be that the optimum velocity of operation decreases very quickly with  $kL^2/2$  meaning that the device becomes increasingly inefficient as the spring constant  $k$  is increased. This would make it impossible to harvest more than of order  $k_B T$  energy per cycle.

The efficiency is defined as the ratio of the amount of power produced to the amount of power put in. The amount of energy needed to dissociate a polymer end from a binding site is  $V_{min}$ . Binding to  $V_{min}$  must produce an energy less than that of an unbound polymer. The power put in is  $V_{min}c$  where  $1/c = 1/c_0 + 1/c_1$ . Therefore the efficiency is

$$\eta = \frac{\langle f \rangle v}{c V_{min}}. \quad (2.39)$$

To investigate this problem further, the one dimensional model of the last section with  $l_c \rightarrow \infty$  was implemented using Metropolis Monte Carlo. We chose the periodic potential  $V_p$  to vary as

$$V_p(x) = \begin{cases} -4V_{min} \frac{x}{\delta} (1 - \frac{x}{\delta}), & \text{if } 0 < x < \delta. \\ -4V_{max} \frac{x}{\delta} (1 + \frac{x}{\delta}), & \text{if } 0 > x > -\delta. \\ 0, & \text{otherwise.} \end{cases} \quad (2.40)$$

Here we set  $\delta = 0.1$ ,  $V_{max} = V_{min} = 100$ , and  $L = 1$ . In order to preserve diffusional dynamics, steps in  $x$  were attempted uniformly in the range  $[-0.025, 0.025]$  ensuring that the periodic potential cannot be jumped across in one move. One move increased the time by .05, though this number was arbitrary and aside from an obvious rescaling, does not affect the results obtained.



To observe behavior in the limit of metastability as discussed in the previous section, the rate of unbinding must be set to be small compared to the inverse metastable equilibration time in the bound state. Hence we chose the unbinding rate  $c_1 = 10^{-5}$  and  $c_0 = 5 \times 10^{-4}$ .

As the simulation was running, the tether point was moved at velocity  $v$  which was typically small, The average spring force  $\langle f \rangle$  was measured as a function of  $v$  for a given spring constant  $k$ . The results of a run of  $3 \times 10^{10}$  steps are shown in Fig. 2.9(a).

We are interested in the limit of large  $k$ , although this is hard to achieve numerically owing to exponentially long relaxation times. The parameters used allow us to probe up to  $k = 20$ . In the large  $k$  limit, the minimum value of  $V_{min}$  needed to bind is  $kl^2/2$ , where  $l$  is the maximum amount the spring will need to stretch from the tether to the binding site. Because  $\delta = 0.1$  and  $L = 1$ , this implies  $l = 0.9$ . Therefore our formula for the efficiency, given this input energy is  $\eta = P_{max}/(ckl^2/2)$ , where  $P_{max}$  is the maximum of the power versus velocity, as shown in Fig. 2.9(a). Plotting the efficiency for different values of  $k$  ranging from 5 to 20 yields the points in Fig. 2.9(b). Despite the fact that the relaxation time for metastable relaxation of the system varies over more than two orders of magnitude, the overall efficiency is almost constant. We expect at higher value of  $k$ , the efficiency will eventually drop owing to the fact that the average unbinding time becoming smaller than the metastable relaxation time.

What the above analysis shows is that in the limit where the photon cross section can be made arbitrarily low, the efficiency can be adjusted to be constant, independent of the photon energy. This is accomplished by choosing a large spring coefficient. In reality with photon energy of  $2eV$  and  $k_B T \approx 1/40eV$ , photon flux would have to be far too low for this optimal regime to be realizable. However the above analysis also shows that the efficiency can

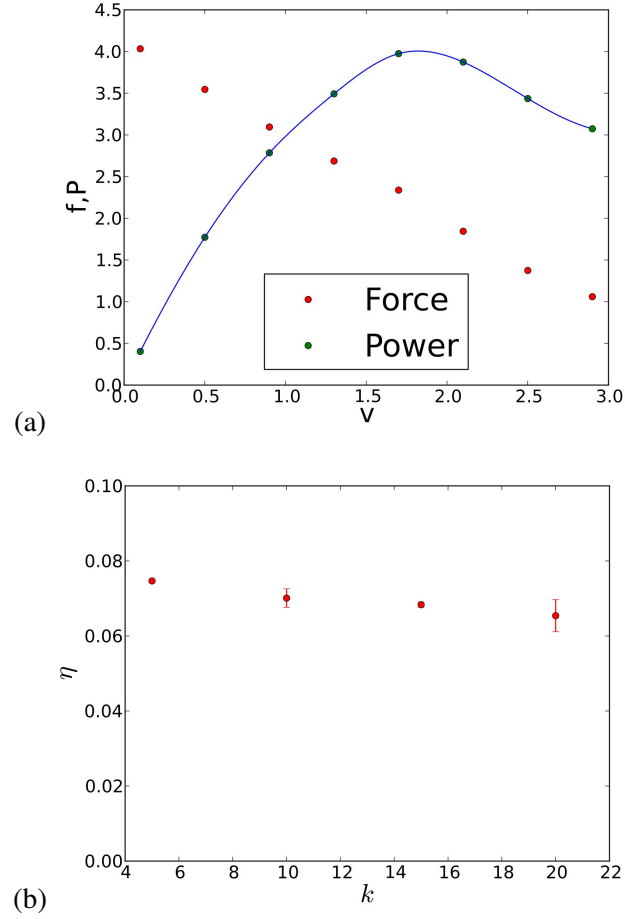


Figure 2.9: (a) The force and power versus velocity for the one dimensional model described in the text. Here the spring coefficient  $k = 15$ . The line going through the power is a cubic spline fit used to more accurately determine the maximum value of the power. The velocity and power are both in units of  $10^{-6}$ . (b) A plot of the efficiency of the motor as a function of the spring constant  $k$ . Four separate runs, each of  $3 \times 10^{10}$  steps were used to determine the error bars, for each data point shown.

be substantially increased by choosing larger  $k$  at the expense of lowering the cross section. The energy delivered in one cycle should scale as  $E_c = kL^2$ . This can be increased to be substantially larger than thermal energies but at the expense of a long relaxation time proportional to  $\exp(E_c/k_B T)$ . As noted earlier, it should be possible to make  $E_c$  about  $7k_B T$ , with reasonable parameter estimates.

## 2.8 Conclusions

Here we have analyzed the viability of converting photons to mechanical energy using a device composed of a canted polymer brush tethered to a lower plate but able to bind its other ends to sites on an upper plate. Photons can dissociate these ends from binding sites. By a combination of analytical and numerical arguments we showed that in steady state, this produces net mechanical power.

The system is inspired by biological motors such as myosin II that bind to actin and dissociate via the binding of ATP. The analysis used here could also be applied to such systems, however in reality they contain many more stages. In general, these kind of systems are classified as “thermal ratchets” [?], where the system can be thought of as moving in a washboard potential in the presence of thermal noise. Though that description can be very useful in understanding the general principles behind the operation of such motors, in the present case we are trying to model the system in more detail than such models can afford. Instead we have described the system using Langevin dynamics and also two coupled Fokker-Planck equations similar in spirit but not identical to previous approaches [?, ?]. The difference here is that in

order for potentials to be a sensible model for a polymer tethered to a single point, they cannot be periodic. Such modelling allows us to see how varying microscopic parameters affect the power and force characteristics.

The analytical results on the Unbound Equilibration Model and extensive one dimensional simulations, show that the force applied by the device can be made arbitrarily large at the expense of having exponentially long relaxation times. At a given photon flux, the production of large forces from single polymers imply the need for a very low cross section of interaction between the photon and the bound end plus binding site, as long relaxation times are required. Therefore there is a trade off between this force and the speed the device can move. This may be circumvented to some extent by stacking transparent devices of this kind, so that even though the cross-section of interaction of an individual photon with a given layer is low, it will eventually be captured by one layer.

Therefore it appears that there is no theoretical obstacle to prevent the photo-mechanical conversion of energy in this manner, however it represents a significant experimental challenge. That being said, we have shown through simulations that the photomechanical effect of this device persists even when binding sites are placed randomly on the upper plate, which suggests that the precision required for production may indeed be afforded some flexibility.

It has been pointed out [?] that there is an important distinction between artificial molecular switches and artificial molecular machines, the former being a fraction of a penny, and the latter being extremely challenging to create. The approach investigated here is closer to biological motors than other proposals, and should be more forgiving about randomness, either in fabrication or due to thermal motion, than approaches that require precise chemical synthesis

of molecules capable of sophisticated conformational changes [?]. However, its experimental realization is still quite clearly a formidable task.

## **Chapter 3**

# **Emergence of Metachronal Waves in Active Microtubule Arrays**

Metachronal waves refer to the synchronization of thin, flexible appendages that result in large-scale wavelike formations. These appear in biological systems at the macroscopic scale (e.g. the motion of millipede legs) and at the microscopic scale (e.g. cilia in air pathways). On the microscopic level, metachronal waves are essential components of several critical biological processes, from motility in microorganisms to mucus clearance in human bronchial tubes [?, ?]. If cilia are unable to effectively move and synchronize, the results are often severe – especially if the disorder is genetic [?]. Research into physical explanations for cilia beating [?], and of spontaneous metachronal behavior in cilia is ongoing and still not well understood [?, ?], although many have suggested that this phenomenon can be explained from hydrodynamic coupling between cilia [?, ?, ?, ?].

Recently, in some remarkable experiments, Sanchez et al. demonstrated metachronal

wave behavior in an *in vitro* system[?, ?]. Microtubules (MTs) aggregated into bundles of length 10 – 100 $\mu$ m due to the addition of polyethylene glycol [?]. Many of these bundles attached at one end to a fixed boundary forming dense arrays. When exposed to a solution containing clusters of kinesin and ATP, sustained metachronal wave behavior between MT bundles (similar to that displayed by cilia and flagella) was observed. MT bundles were constrained to move between two glass slides. It is surprising that a system with such few ingredients could develop complex behavior that so closely resembles biological systems, which are made up from a much more complicated machinery. Proteomic analysis indicate that eukaryotic cilia are composed of many hundreds of proteins [?].

Some important details of this *in vitro* system are still unclear, most notably whether the MTs in this experiment are unipolar or of mixed polarity. Opposite polarity MTs will move past each other, causing separation into unipolar bundles [?, ?]. We present arguments and simulations for unipolarity in S-??. The surprising mechanism for the motion of unipolar bundles mentioned here has not previously been given [?, ?], and we believe that the agreement between our model and experiments provides further evidence to support our proposed explanation.

In this article, we first outline a mechanism by which the metachronal wave formation observed by Sanchez et al. can be understood. The mechanism described here is in most ways identical to the model used to describe and simulate cytoplasmic streaming in *Drosophila* oocytes, and the fact that it can be adapted as such is in many ways a testament to its predictive power. A fair amount of attention has been paid in recent years to the understanding of how metachronal waves form in such arrays [?, ?, ?, ?]. However, such models often rely on assumptions about individual MT (or cilia) beat patterns and/or on phenomenology. The

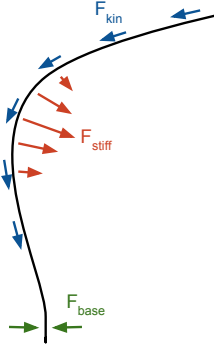


Figure 3.1: Conceptual illustration of forces acting on a single polymer that are not due to hydrodynamic interactions. The blue vectors indicate the buckling forces due to kinesin walkers (tangent to polymer), the red vectors show the direction and relative magnitude of stiffness forces (in the direction of  $d^4\mathbf{r}/ds^4$ ), and the green arrows indicate a restorative force keeping the base of the polymer approximately perpendicular to the binding surface.

model we propose makes no such assumptions (beyond some minor simplifications), relying on first-principles fluid mechanics calculations. This is important, as it is not clear why one would additionally want to impose oscillatory behavior on individual MTs given the lack of a well defined internal structure.

We now present a model for the simulation of the Sanchez et al. system. A similar method has been used successfully to simulate cytoplasmic streaming in *Drosophila* oocytes[?], and is based on theoretical work completed several decades ago regarding the calculation of Stokes flows created by a point force (stokeslet) near no-slip boundaries[?, ?]. A conceptual explanation of this mechanism is given below, and further details regarding theory and implementation are given in supplementary materials ?? and ??.

An illustration of how MT bundles are simulated is given in Figure 3.1. Each MT bundle is modeled as a chain of monomers (i.e. polymer) which are held an approximately



fixed distance from one another by a spring force. The base of each polymer is anchored to a single point, and the polymer at the base is kept roughly perpendicular to the anchoring surface. Let the polymer be described by the curve  $\mathbf{r}(s)$ , where  $\mathbf{r}(0)$  is the location of the polymer base, and  $s$  is the arc length. We give the polymer a stiffness by implementing an energetic cost of bending proportional to curvature squared, which implies a local force at  $s$  proportional to  $d^4\mathbf{r}/ds^4$ . Additionally, monomers feel a “buckling” force due to the drag from the walking kinesin  $F_{kin} = -f_k d\mathbf{r}/ds$ , which is parallel to the polymer and toward the polymer base.  $f_k$  will depend linearly on the speed of the kinesin and the solvent viscosity. This force continually adds energy to the system (making it active), and has been shown to be a good representation of the average drag force due to kinesin walking along the microtubule away from the polymer base [?].

This kind of model for a single chain was first employed to understand glide assay dynamics in two dimensions [?]. In three dimensions, periodic waves develop whose dynamics have been analyzed in detail [?], and related theoretical work has recently also been performed [?]. However, scaling can be used to get the relevant length and timescales [?]. The average radius of curvature depends on the strength of the buckling force  $f_k$ , and the elastic constant of a filament characterizing its stiffness  $k_{stiff}$ . The radius of curvature over quite a wide range of parameters can be shown to be  $R = (k_{stiff}/(\beta f_k))^{1/3}$ , where  $\beta \approx 0.05$ . Likewise, the angular frequency is  $\omega = f_k/(vR)$ , where  $v$  is the hydrodynamic drag coefficient per unit length. Although there is a fairly large experimental uncertainty in parameters used to model a *Drosophila* oocyte, this model finds quite good agreement with the experimental time and length scales.  $R$  was predicted to be  $25 - 54\mu\text{m}$ , close to the  $16.3 \pm 2.2\mu\text{m}$  observed. Likewise,

the time scale was predicted to be  $203 - 1094$ s, which is in the observed range of  $370 \pm 42$ s. It is interesting that the length and time scales observed by Sanchez et al. are also quite close to these numbers, and that the frequency of biological cilia beating is often three orders of magnitude higher than this.

Polymers also feel hydrodynamic forces. As the force from the kinesin causes the polymers to buckle, we begin to see complex motion. Each monomer acts as a point force (stokeslet) in the surrounding fluid. This force a monomer exerts on the fluid is simply the sum of all of the other forces on the monomer: because the Reynolds number is nearly zero, there are no inertial terms, meaning the force is transferred perfectly from the monomer to the fluid. As this is a Stokes flow, the flow contributions from all stokeslets add linearly, and we can (in principle) calculate the flow everywhere. However, we do not need to calculate the fluid velocity everywhere – only at points with monomers. Therefore, the evolution can be calculated via a pairwise sum over all monomers (see the supplemental materials S-??).

We also assume all polymer motion is two dimensional with a constant value of  $z$ , which is physically sensible when considering the geometry of the Sanchez et al. experiments. In this experiment, MT bundles were observed between glass slides, with a height  $H$ , of approximately  $10\mu\text{m}$ , creating a narrow channel for which fluid can flow. For this reason, we adopt a two dimensional geometry. In addition, the no-slip boundaries of the plates have a large impact on the hydrodynamic forces between monomers[?, ?], which we state explicitly in the supplemental materials S-??. Other close-range contact forces were also used (repulsion from anchoring surface, monomer-monomer repulsion), and these are explained in the supplementary materials S-??.

We can now address at the qualitative level the mechanism by which we propose the metachronal waves observed by Sanchez et al. form. As kinesin walk away from the polymer bases, the polymers will tend to buckle. If a polymer is isolated, this buckling will lead to unstable motion (corkscrew motion or irregular beating)[?]. When placed in an array, however, nearby polymers will exert hydrodynamic forces on one another that tend to synchronize their motion. If these hydrodynamic forces are sufficiently strong, this can cause a transition from disordered motion to aligned MTs and correlated motion.

Despite the fact that this model was developed to explain and simulate cytoplasmic streaming, its mechanism can be easily adapted for related biological phenomena. Indeed, when the conditions of the Sanchez et al. experiment are simulated in the same way, we observe metachronal waves. It is not clear if this is formally a transition or a more continuous crossover effect, but the results found make strong predictions that should be testable experimentally. In the following, we present the results of these simulations and discuss the required conditions for metachronal wave formation.

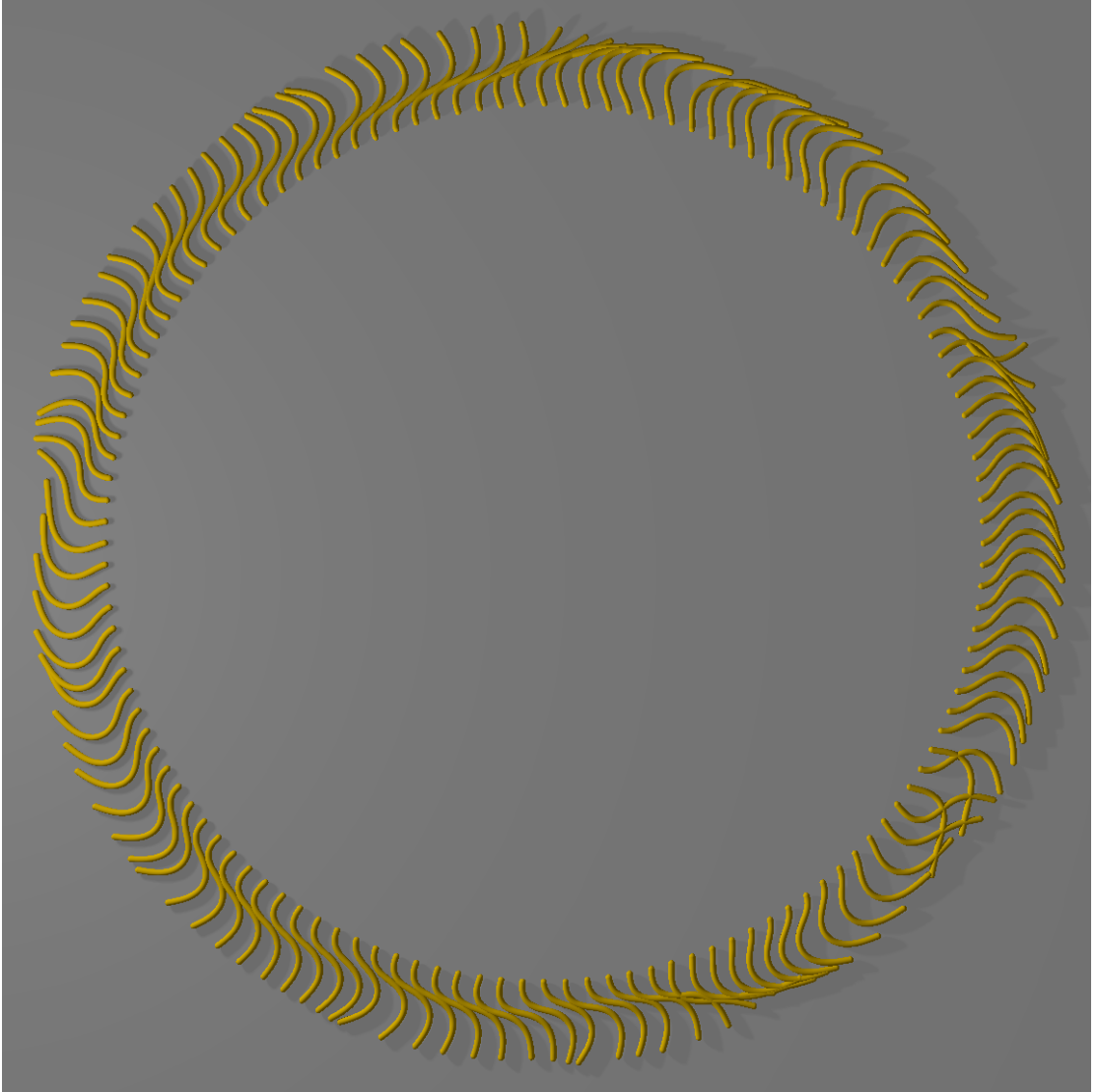
Videos of select simulations are included in the Supplementary Materials. Figure 3.2 shows some still frames of simulated arrays demonstrating metachronal wave behavior in both the planar and circular geometries.

We characterize the behavior of each system using the correlation function

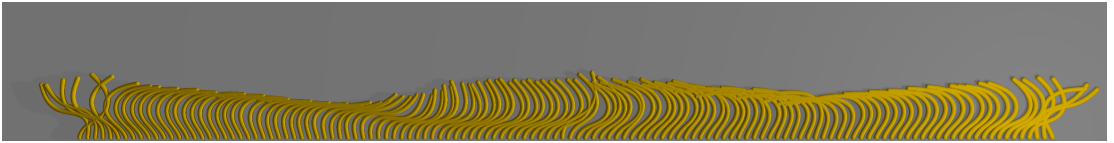
$$C(\Delta i, \Delta t) = \langle \Delta x(i + \Delta i, t + \Delta t) \Delta x(i, t) \rangle, \quad (3.1)$$

where

$$\Delta x(i, t) = x(i, t) - \langle x(i, t) \rangle.$$



(a)



(b)

Figure 3.2: Simulated metachronal wave formation for 128-polymer arrays in (a) circular and (b) planar geometries. In both cases,  $k_{o\text{seen}} = 0.1$ ,  $k_{stiff} = 10.0$ ,  $H = 1$ .

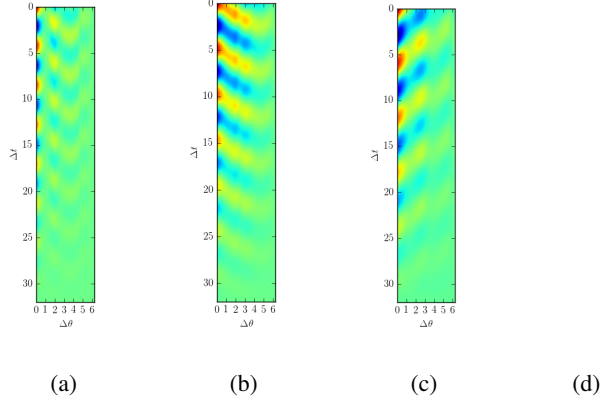


Figure 3.3: Full correlation functions for circular geometry with  $H = 1$  and  $k_{stiff} = 10$ , with  $k_{Oseen} = 0.1, 0.2$ , and  $0.3$  (a-c, respectively). The correlation function at  $\Delta i = 0$  for all of these values of  $k_{Oseen}$  are shown in (d).

Figs. 3.3, 3.4, and 3.5 show correlation functions for planar and a circular geometries (for the circular geometry,  $\theta$  is the position variable rather than  $x$ ). In the following, we will discuss these and examine how the system responds to changes in  $k_{Oseen}$ ,  $k_{stiff}$ , and height  $H$ . It should be noted that changes in the viscosity or kinesin velocity and density (that affect  $f_k$ ), can be absorbed into a rescaling of time, and of  $k_{stiff}$ .

The strength of the interaction tensor,  $k_{Oseen}$ , has a dramatic effect on the type of wave behavior seen, or whether it is observed at all. This strength is a function of the hydrodynamic effects of kinesin walking along microtubules, and will depend on their density and speed, as explained in detail in Ref. [?]. Figure 3.3 shows the correlation results of three 128-polymer simulations in the same circular geometry shown in Figure 3.2(a) for three different values of  $k_{Oseen}$ . There is an overall strengthening of the metachronal behavior as  $k_{Oseen}$  is increased from 0.1 to 0.2. The sign of the slope reflects the initial conditions of the system. Long lived waves travel predominantly in a single direction over long times scales resulting in a slope of the crests

(a)(b) (d)

Figure 3.4: Full correlation functions for planar geometry with  $H = 1$  and  $k_{Oseen} = 0.1$ , with  $k_{stiff} = 5.0, 10.0$ , and  $20.0$  (a-c, respectively). The correlation function at  $\Delta i = 0$  for all of these values of  $k_{stiff}$  are shown in (d).

(a)(b) (d)

Figure 3.5: Full correlation functions for circular geometry with  $k_{stiff} = 10$ ,  $k_{Oseen} = 0.2$ , and  $H = 1.0, 0.5$ , and  $0.1$  (a-c, respectively). The correlation function at  $\Delta i = 0$  for all of these values of  $H$  are shown in (d).

of the correlation function that can either be positive or negative. Similar crests are seen in the analysis of the real experimental data [?]. With this circular geometry, the correlation function must be periodic, which is why it rises again when  $i$  becomes large.

The polymer stiffness  $k_{stiff}$  also has an interesting effect on metachronal wave formation. Figure 3.4 shows the correlation functions for  $k_{stiff} = 5.0, 10.0$ , and  $20.0$  in a planar geometry. While Figs. 3.4(a-b) are qualitatively similar, we do see an apparent decrease in the metachronal wavelength. Figs. 3.4(c-d) show that if the polymer is made too stiff, no metachronal behavior is observed at all. In general, planar geometry appears to cause more coherence in the motion of the different bundles, and the correlation function is dominated by motion at the longest lengths and time scales.

The distance between plates,  $H$ , has a considerable effect on the dynamics as well. Longer range, more coherent motion is observed when  $H$  is larger, and short range, less coherent motion when  $H$  is small. See Figure 3.5. This is to be expected due to the strong screening effect that these boundary conditions impose. Smaller  $H$  reduces the hydrodynamic coupling, causing

a decrease in coherence.

When comparing these results to those of Sanchez et al., we find that the basic features agree. The videos included in the supplemental materials qualitatively mimic the experimental videos, and the experimental correlation analysis agrees quite well with the simulations. More importantly, this agreement between theory and experiment was reached from first principles. We only use a handful of forces in our simulations, and each force has a physical justification for being used.

There are potential shortcomings of this model that may result in some differences between experiment and theory. The first is that the experiments observe bundles of microtubules that taper away from their base. The hydrodynamics are not expected to be uniform along the length of a chain. In addition these bundles will, for short enough times, behave like rigid material, but for longer times, because they are connected through walking kinesin molecules, will behave more as individual microtubules with a greatly reduced elastic constant. On the time scales of the motion, we expect to be in the latter regime. However the details of the hydrodynamics and elasticity in these bundles is still not understood experimentally. In fact, as we mentioned earlier, the polarity of individual MT's is not known experimentally, and arguments for their unipolarity are given in the supplementary information, S-???. But still, the basic mechanism of dynamic buckling due to kinesin drag, and metachronal waves being generated by hydrodynamic coupling is robust over a wide parameter range, so we believe that these complications, aside from unipolarity, will not alter the basics of our explanation.

At a more technical level, there are other things that may make a slight difference to the results here. The bundles are constrained to move only in the  $xy$ -plane, and while it is

true that MT motion is nearly 2-dimensional, there is some room in the  $z$ -direction that MT bundles can occupy. Additionally, this model does not account for the fluid boundary condition at the anchoring surface. This may introduce some errors if a monomer becomes close ( $\sim H$ ) to the anchoring plane. However, because of the screening effects of the plates, this should not alter the behavior at distances large compared to the plate separation. We have tested for this by adding image charges to the planar case, and found that their effects on correlations are small, as expected.

In conclusion, we have developed a model for the spontaneous formation of wave-like behavior in active polymer arrays that only requires two ingredients: semi-flexible chains tethered to a surface, and motors walking from their bases to their tips. The hydrodynamics in their confined geometry gives rise to metachronal waves that appear remarkably similar to what is observed experimentally [?, ?]. There is no need to posit additional mechanisms that force individual bundles to oscillate. This all happens as a consequence of Newton's laws and fluid mechanics, allowing us to gain a better understanding of how metachronal waves form with considerable predictive power. As such, we have examined new parameter spaces and have demonstrated boundaries between different types of metachronal behavior and regimes in which no metachronal behavior exists. It would be of great interest to test these predictions experimentally. Given the simplicity and robust nature of this mechanism, and the ubiquity of microtubules and kinesin in cells, it gives one further impetus to look for other places in biology where this kind of behavior can be found.



## **Chapter 4**

## **Conclusion**

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## **Appendix A**

### **Some Ancillary Stuff**

Ancillary material should be put in appendices, which appear after the bibliography.