Contents

7 Stochastic Linear Control

In this section we consider the stochastic reference tracking problem. It is required to move the states and manipulated variables of the system, shown in (85), to the set point (x_{sp}, u_{sp}) by manipulating the input variables u.

$$\begin{aligned} x_{t+1} &= f(x_t, u_t) + w_{t+1} \\ y_{t+1} &= g(x_{t+1}) + v_{t+1} \end{aligned} \tag{85}$$

We assume uncorrelated zero mean additive Gaussian noise in both the state function f and the observation function g with known covariances W and V respectively. Clearly it is not possible to achieve perfect control (zero offset at steady state) because of the noise terms, specifically w_t. For this reason we need to relax the set point goal a little bit. We will be content if our controller is able to achieve Definition 7.1.

Definition 7.1. Stochastic Reference Tracking Goal: Suppose we have designed a controller and set $\delta > 0$ as a controller benchmark. If there exists some positive number $t^* < \infty$ such that $\forall\ t > t^*$ the controller input causes $\mathbb{E}[(x_t - x_{sp})^T Q(x_t - x_{sp}) + (u_t - u_{sp})^T R(u_t - u_{sp})] < \delta$ we will have satisfied the Stochastic Reference Tracking Goal given δ.

In this section we limit ourselves by only considering controllers developed using a single linear model of the underlying, possibly nonlinear, system functions f and g. The linearised model control is based upon is shown in (86) and is subject to the same noise as (85).

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_{t+1} \\ y_{t+1} &= Cx_{t+1} + v_{t+1} \end{aligned} \tag{86}$$

We will endeavour to develop predictive controllers using the Graphical Models of Section 5 and 6.

7.1 Current Literature

7.2 Unconstrained Stochastic Control

Theorem 7.1. Unconstrained Optimisation Equivalence Suppose we have two real valued objective functions $f(x_0, \mathbf{u})$ and $g(x_0, \mathbf{u})$ and we are required to minimise them over the same space where they are defined: $\mathbf{u} \in \mathcal{U}$ and $x_0 \in \mathcal{X}$. Furthermore, suppose there exists a real number $k > 0$ such that $\forall \mathbf{u} \in \mathcal{U}$ we have that $g(x_0, \mathbf{u}) + k = f(x_0, \mathbf{u})$. Finally, assume the existence and uniqueness of the global minimiser for each problem. Then the global minimiser $\mathbf{u}^*$ of $g(x_0, \mathbf{u})$ also minimises $f(x_0, \mathbf{u})$.

Proof. This proof hold over differentiable and non-differentiable objective functions. Suppose not i.e. there exists $\mathbf{u}_g \in \mathcal{U}$ such that $g(x_0, \mathbf{u}_g) < g(x_0, \mathbf{u})\ \forall \mathbf{u} \in \mathcal{U}$ but $f(x_0, \mathbf{u}_g) \not< f(x_0, \mathbf{u})$ $\forall \mathbf{u} \in \mathcal{U}$. This implies that for $\mathbf{u}_f \in \mathcal{U}$ the global minimiser of f we have $f(x_0, \mathbf{u}_f) \leq f(x_0, \mathbf{u}_g)$.

Consider the case where $f(x_0, \mathbf{u}_f) = f(x_0, \mathbf{u}_g)$. This implies that both $\mathbf{u}_f$ and $\mathbf{u}_g$ are global minimisers of f and contradicts our assumption that the global minimiser is unique.

Consider the case where $f(x_0, \mathbf{u}_f) < f(x_0, \mathbf{u}_g)$. Since $g(x_0, \mathbf{u}) + k = f(x_0, \mathbf{u})\ \forall \mathbf{u} \in \mathcal{U}$ this implies that $g(x_0, \mathbf{u}_f) < g(x_0, \mathbf{u}_g)$. But this contradicts our assumption that $\mathbf{u}_g$ is the global minimiser of g.

It must then hold that $f(x_0, \mathbf{u}_g) < f(x_0, \mathbf{u})\ \forall \mathbf{u} \in \mathcal{U}$. Therefore the global minimiser $\mathbf{u}_g$ of $g(x_0, \mathbf{u})$ also minimises $f(x_0, \mathbf{u})$. □

Theorem 7.2. LQR and LQG Objective Function Difference Consider the LQR and LQG Objective Functions in (87) and (88) respectively.

$$
\begin{aligned}
J_{LQR}(x_0, \mathbf{u}) &= \frac{1}{2}\sum_{k=0}^{N-1}\left(x_k^T Q x_k + u_k^T R u_k\right) + \frac{1}{2}x_N^T P_f x_N && (87)\\
&\text{with } x_{t+1} = Ax_t + Bu_t \\
J_{LQG}(x_0, \mathbf{u}) &= \mathbb{E}\left[\frac{1}{2}\sum_{k=0}^{N-1}\left(x_k^T Q x_k + u_k^T R u_k\right) + \frac{1}{2}x_N^T P_f x_N\right] && (88)\\
&\text{with } x_{t+1} = Ax_t + Bu_t + w_{t+1}
\end{aligned}
$$

Suppose x_0 is the state estimate supplied by the Kalman Filter given the latest observation in the stochastic case. In the deterministic case we have that $x_0 = \mathbb{E}[x_0] = \mu_0$ because we exactly observe the state. Given any input sequence $\mathbf{u} \in \mathcal{U}$, where $\mathcal{U}$ is the shared admissible input space, we have that $J_{LQR}(x_0, \mathbf{u}) + \frac{1}{2}\sum_{k=0}^{N}\operatorname{tr}(Q\Sigma_k) = J_{LQG}(x_0, \mathbf{u})$ where $\Sigma_{t+1} = W + A\Sigma_t A^T$ and Σ_0 is the covariance matrix of the current state given by the Kalman Filter.

Proof. Expanding the LQG objective function and noting that $\mathbf{u}$ is deterministic we have (89). Note that the conditional expectations in the expansion originate from the graphical model in Figure 23 (due to the first order Markov assumption).

$$
\begin{aligned}
J_{LQG}(x_0, \mathbf{u}) &= \frac{1}{2}\mathbb{E}\left[x_0^T Q x_0 + u_0^T R u_0\right] + \frac{1}{2}\mathbb{E}\left[x_1^T Q x_1 + u_1^T R u_1 | x_0\right] + ... \\
&+ \frac{1}{2}\mathbb{E}\left[x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} | x_{N-2}\right] + \frac{1}{2}\mathbb{E}\left[x_N^T P_f x_N | x_{N-1}\right] \\
&= \frac{1}{2}\mathbb{E}\left[x_0^T Q x_0\right] + \frac{1}{2}u_0^T R u_0 + \frac{1}{2}\mathbb{E}\left[x_1^T Q x_1 | x_0\right] + \frac{1}{2}u_1^T R u_1 + ... \\
&+ \frac{1}{2}\mathbb{E}\left[x_{N-1}^T Q x_{N-1} | x_{N-2}\right] + \frac{1}{2}u_{N-1}^T R u_{N-1} + \frac{1}{2}\mathbb{E}\left[x_N^T P_f x_N | x_{N-1}\right]
\end{aligned} \tag{89}
$$

We know that $x_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$ because the current state estimate comes from the Kalman Filter. This means that we can evaluate the first expected value in (89) using Theorem 2.5 as shown in (90).

$$
\mathbb{E}\left[x_0^T Q x_0\right] = \operatorname{tr}(Q\Sigma_0) + \mu_0^T Q \mu_0 \tag{90}
$$

Now we turn out attention to the second expected value in (89). First note that because we have x_0 and $\mathbf{u}$ we can use the result from Section 5.2 to predict (optimally) the distribution of

x_1. Therefore we know that $x_1 \sim \mathcal{N}(A\mu_0 + Bu_0, W + A\Sigma_0 A^T)$. Now we let $\mu_1 = A\mu_0 + Bu_0$ and $\Sigma_0 = W + A\Sigma_0 A^T$. Then by using Theorem 2.5 as before we have (91).

$$\mathbb{E}\left[x_1^T Q x_1 | x_0\right] = \mathrm{tr}(Q\Sigma_1) + \mu_1^T Q \mu_1 \tag{91}$$

Note that $\mathrm{tr}(Q\Sigma_1)$ does not depend on u_0 but only on the initial state estimate x_0 which is independent of the future inputs $\mathbf{u}$. Notice that we can continue in this manner to simplify the LQG objective function to (92).

$$\begin{aligned}
&J_{LQG}(x_0, \mathbf{u}) = \frac{1}{2}\sum_{k=0}^{N-1}\left(\mu_k^T Q \mu_k + u_k^T R u_k\right) + \frac{1}{2}\mu_N^T P_f \mu_N + \frac{1}{2}\sum_{k=0}^{N}\mathrm{tr}(Q\Sigma_k) \\
&\text{with } \mu_{t+1} = A\mu_t + Bu_t \\
&\text{and } \Sigma_{t+1} = W + A\Sigma_t A^T
\end{aligned} \tag{92}$$

Now note that except for the last term $J_{LQG}(x_0, \mathbf{u})$ is exactly the same as $J_{LQR}(x_0, \mathbf{u})$. The conclusion follows because $\frac{1}{2}\sum_{k=0}^{N}\mathrm{tr}(Q\Sigma_k)$ is independent of $\mathbf{u}$. □

Theorem 7.3. Solution of the Finite Horizon LQG control problem We wish to solve the LQG control problem within the framework of this dissertation. The full problem is shown in (93).

$$\begin{aligned}
&\min_{\mathbf{u}} V(x_0, \mathbf{u}) = \mathbb{E}\left[\frac{1}{2}\sum_{k=0}^{N-1}\left(x_k^T Q x_k + u_k^T R u_k\right) + \frac{1}{2}x_N^T P_f x_N\right] \\
&\text{subject to } x_{t+1} = Ax_t + Bu_t + w_t \\
&\text{and } y_t = Cx_t + v_t
\end{aligned} \tag{93}$$

We assume that we have the Kalman Filter state estimate for x_0. We use Theorem 7.2 to prove that given x_0 and $\forall \mathbf{u} \in \mathcal{U}$ we have that $J_{LQR}(x_0, \mathbf{u}) + \frac{1}{2}\sum_{k=0}^{N}\mathrm{tr}(Q\Sigma_k) = J_{LQG}(x_0, \mathbf{u})$ with $\frac{1}{2}\sum_{k=0}^{N}\mathrm{tr}(Q\Sigma_k) \in \mathbb{R}$ a constant depending only on x_0. Thus we can use Theorem 7.1 to prove that we only need to solve for the optimal controller input $\mathbf{u}^0$ using the LQR objective function. Thus we can use Theorem 2.10 to find $\mathbf{u}$.

As we have mentioned before, the Separation Theorem states that the solution of the LQG control problem is achieved by using the Kalman Filter to optimally estimate the current state and then feeding that state estimate into the optimal LQR controller. It is reassuring that Theorem 7.3 confirms this result.

Theorem 7.4. Solution of the Infinite Horizon LQG control problem Using either the Separation Theorem [15] or using, with some minor adjustments, Theorems 7.1 and 7.2 it is possible to show that the infinite horizon LQG problem is solved in a similar manner. The Kalman Filter state estimate is used in conjunction with the infinite horizon LQR solution.

7.3 Chance Constrained Stochastic Control

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