Lecture 12

§24 Functions

- A relation f is called a *function* provided $(a, b) \in f$ and $(a, c) \in f$ imply $b = c$.
  If f is a function, and $(a, b) \in f$. Then we write $f(a) = b$. a is called an *input* (of f), and b is called the *output* (of f at a).
- Let f be a function. The set of all possible first elements of the ordered pairs in f is called the *domain* of f and is denoted dom f. The set of all possible second elements of the ordered pairs in f is called the *image* of f and is denoted im f.
  Let A, B be sets and f be a function. We say f is a *function/mapping from A to B* provided dom f=A and im $f \subseteq B$. The notations are $f : A \to B$.
- A function is called *one-to-one* provided that, whenever $(x, b), (y, b) \in f$, we must have $x = y$.
  Let f be a function. The inverse relation f^{-1} is a function if and only if f is one-to-one.
  Let f be a function and suppose f^{-1} is also a function. THen dom f =im f^{-1} and im $f =$ dom f^{-1}.
- Let $f : A \to B$. We say that f is *onto* B provided that for every $b \in B$ there is an a such that $f(a) = b$. In other words, im $f = B$.
- Let $f : A \to B$. We call f a *bijection* provided it is both one-to-one and onto.

§26 Composition

- Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$. Then the function $g \circ f$ is a function from A to C defined by
  $$(g \circ f)(a) = g\,[f(a)]$$
  where $a \in A$. The function $g \circ f$ is called the *composition* of g and f.
- Let A, B, C and D be sets and let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then
  $$h \circ (g \circ f) = (h \circ g) \circ f.$$
- Let A be a set. The *identity function* on A is the function id_A whose domain is A, and for all $a \in A$, $\mathrm{id}_A(a) = a$. In other words, $\mathrm{id}_A = \{(a, a) : a \in A\}$.
  Let $f : A \to B$. Then $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$.
  Let f be a bijection from A to B. Then $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.
- Examples
  (1) Which of the following are functions?
    (a) $f = \{(1, 1), (2, 1), (3, 4), (4, -1), (5, 0)\}$
    (b) $g = \{(1, -3), (5, 8), (-3, 4), (1, 5), (-1, 2)\}$
  (2) Is the relation $\{(a, b) : a, b \in \mathbb{N}, a|b\}$ a function?
  (3) Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{-3, -2, -1, 0, 1, 2, 3\}$. Let $f = \{(0, -1), (1, -2), (2, -3), (3, 3), (4, 0)\}$. Is f a function from B to A?
  (4) Find an example of a pair of functions f and g such that $f \circ g$ and $g \circ f$ are both well defined but $f \circ g \neq g \circ f$.

(5) Let $f : \mathbb{Z} \to \mathbb{Z}$ according to the rule $f(x) = 2x + 3$ and $g : \mathbb{Z} \to \mathbb{Z}$ according to the rule $g(x) = x^2 + 4$.
(a) Find $f(g(2))$.
(b) Find $g(f(0))$.
(c) Find a rule for $f \circ g(x)$ and $g \circ f(x)$.

(6) A *Mobius* transformation is a function $\phi : \mathbb{C} \to \mathbb{C}$ of the form

$$\phi(z) = \frac{az + b}{cz + d}, \qquad \forall z \in \mathbb{C},$$

where a, b, c, d are complex numbers. Show that if ϕ and ψ are Mobius transformations, then so is $\phi \circ \psi$.

(7) Let A, B, C be sets and suppose $f : A \to B$ and $g : B \to C$ are both bijections. Prove $g \circ f$ is also a bijection.

HW6(b) (Due 3/28/2016)

- Exercise 24.6 (a), (f)
- Exercise 24.14 (c), (d)
- Exercise 26.1 (b), (f), (i)