Lecture 10

§22 Induction

- To prove the statement $P(n)$ about $n \in \mathbb{N}$ by using *induction* on n:
  (1) Check the very first statement $P(0)$. This is called the *basic step.*
  (2) Prove that for any $k \geq 1$, $P(k-1)$ is true implies $P(k)$ is true (*inductive step*). The assumption that $P(k-1)$ is true is called the *induction hypothesis.*
- Examples
  (1) Let n be a natural number. Then
  $$\sum_{k=0}^{n} k = \frac{n(n+1)}{2},$$
  $$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$
  $$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$
  (2) Let n be a natural number. Then
  $$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1.$$
  (3) A sequence $\{a_n; n \in \mathbb{N}\}$ is defined as following: $a_0 = -1$, $a_{n+1} = 2a_n - a_n^2$ for all $n \in \mathbb{N}$. Prove that
  $$a_n = -2^{2^n} + 1.$$
  (4) The *Fibonacci numbers* $F_0, F_1, \cdots$ are defined as following: $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that for all $n \geq 1$,
  $$\sum_{k=0}^{n} (-1)^k F_k = (-1)^n F_{n-1} + 1.$$
  Remark: In this example, the basic step is $n = 1$.
  (5) Prove that 3 divides $n^3 - n$ for any natural number n.
  (6) Prove that for all $n \geq 5$, we have $2^n \geq n^2$.
  (7) Prove that $n! \leq n^n$ for all positive integer n.
- There are variations of the regular induction:
  (1) Sometimes two or more basic steps are needed.
  (2) Sometimes we need stronger induction hypothesis. The strongest hypothesis is that we assume all the statements $P(0)$, $P(1)$,$\cdots$, $P(k-1)$ are all true. In this case we call it *strong induction.*
- Examples
  (8) Consider a sequence $b_0, b_1, \cdots$, defined as following: $b_0 = 2$, $b_1 = 5$, and $b_n = 5b_{n-1} - 6b_{n-2}$ for all $n \geq 2$. Prove that $b_n = 2^n + 3^n$ for all $n \in \mathbb{N}$.

(9) Prove that every positive integer can be written as a sum of distinct powers of 2.

HW5(b) (Due 3/2/2016)

- 22.4(c),(d)
- 22.5(c),(d)
- 22.16(e)
- 22.18