Lecture 08

§20 Contradiction

- *Proof by contrapositive*: $A \implies B$ is logically equivalent to $\neg B \implies \neg A$, which is called the *contrapositive* of $A \implies B$. To prove "If A then B" by using the method of contrapositive means to prove "If not B, then not A".
- Examples

(1) Let x be an integer. Prove: If x^2 is even, then x is even.

(2) A real number is called *rational* provided that it is the ratio of two integers. A real number is called *irrational* provided that it is not rational. If a and b are two real numbers such that the product ab is an irrational number, then either a or b must be an irrational number.

- Proof by contradiction: To prove a statement B (or a conditional one "If A then B"), we first assume that B is false. Then we show that this assumption leads to a contradiction. We are then lead to conclude that we were wrong to assume that B is false. So B is true.

  The contradiction could be some conclusion contradicting one of our assumptions, or something obviously false like $1 = 0$.
- Examples

(3) Prove: There are no integer solution to the equation $x^2 - y^2 = 2$.

Proof. Write $x^2 - y^2 = (x-y)(x+y)$. □

(4) Prove: $\sqrt{2}$ is not a rational number.

Proof. Assume $\sqrt{2} = a/b$ which is fully reduced.Then $a^2 = 2b^2$ and one can further prove a, b should be both even. □

(5) Prove: If $x \in [0, \frac{\pi}{2}]$, then $\sin x + \cos x \geq 1$.

Proof. If $\sin x + \cos x \leq 1$, take the square and get $1 + 2\sin x \cos x \leq 1$. □

(6) Prove: There are infinitely many prime numbers.

(7) Prove: There are infinitely many primes p such that $p+2$ is not prime.

Proof. Suppose not true. Then there exists a large M, such that for all $p > M$ prime, $p+2$ is also prime. This leads to p, $p+2$, $p+4$,$p+6, \cdots$ are all prime. □

(8) Twenty five boys and twenty five girls sit around a table. Prove that it is always possible to find a person both of whose neighbors are girls.

Proof. Suppose not true. Girls separate boys into at most 12 parts each of which have at least two boys. Boys separate girls into the same number of parts therefore at least one part of girls have at least three girls. Contradiction! □

HW4(b) (Due 2/24/2016)

- 20.11
- 20.14