Lecture 09

§21 Smallest Counterexample

- To prove a statement P by using contradiction: assume there exists a counterexample such that P is false, and then conclude a contradiction. If P is a statement $P(x)$ about **natural numbers** x (or any subset of $\mathbb{Z}$ with lower bounds), we can prove the statement by *smallest counterexample*. The steps are as following:
  - Let x be a smallest counterexample such that $P(x)$ is not true. It must be clear that there can be such an x.
  - Rule out $x = 0$ (or generally the very smallest possibility). This step is call *basic step*.
  - Consider an instance $P(x')$ where $0 \leq x' < x$. Use the facts that $P(x')$ is true but $P(x)$ is false to obtain a contradiction. Typically x' is chosen to be $x - 1$.
- Examples
  (1) A proof that shows each natural number is divisible by 3: Let x be the smallest natural number which is not divisible by 3. x is not 0. Consider $x' = x - 3 < x$. It is divisible by 3. Therefore $x = x' + 3$ is divisible by 3. Contradiction! So each natural number is divisible by 3.
  Which step of the above proof is wrong?
  (2) A proof that shows every nonnegative real number is rational, i.e., can be written as p/q with $p, q \in \mathbb{Z}$ and $q \neq 0$: Let x be the smallest nonnegative real number which is not rational. $x \neq 0$. Consider $x' = x/2$ which satisfies $0 < x' < x$. x' is rational therefore $x = 2x'$ is also rational. Contradiction! So every nonnegative real number is rational.
  Which step of the above proof is wrong?
  (3) Let n be a positive integer. The sum of the first n odd natural numbers is n^2.
- (**Well-Ordering Principle**) Every nonempty set of **natural numbers** contains a least element.
- Proof by the Well-Ordering Principle. First, let X be the set of counterexamples to the statement. We suppose X is noempty, therefore X contains a least element, x. Second, rule out x being the very smallest possibility. Third, consider $x - 1$. The statement is true for $x - 1$. From here we argue to a contradiction, often that x both is and is not a counterexample to the statement.
- Examples
  (4) Let $n \in \mathbb{N}$. If $a \neq 0$ and $a \neq 1$, then
  $$a^0 + a^1 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}.$$
  (5) Prove: For all integers $n \geq 5$, we have $2^n \geq n^2$.
  (6) Prove: For all positive integers, we have $n! \leq n^n$.
  (7) Prove: The *Fibonacci numbers* $F_0, F_1, \cdots$ are defined as following: $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Then $F_n \leq 1.7^n$ for all $n \in \mathbb{N}$.
  (8) Prove: Every integer $n \geq 7$ can be written as $n = 2a + 3b$, where a and b are positive integers.

HW5(a) (Due 3/2/2016)

- 21.2
- 21.5