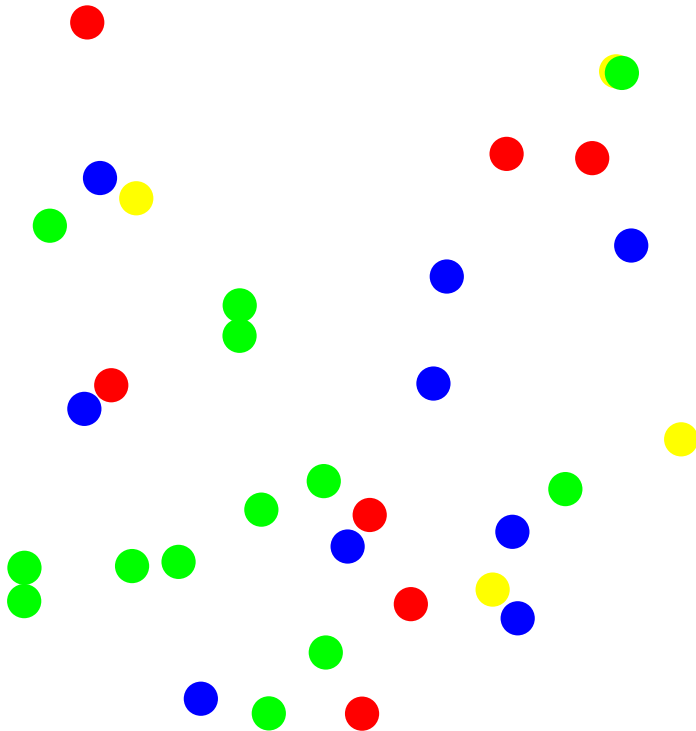

Partitions

Definition

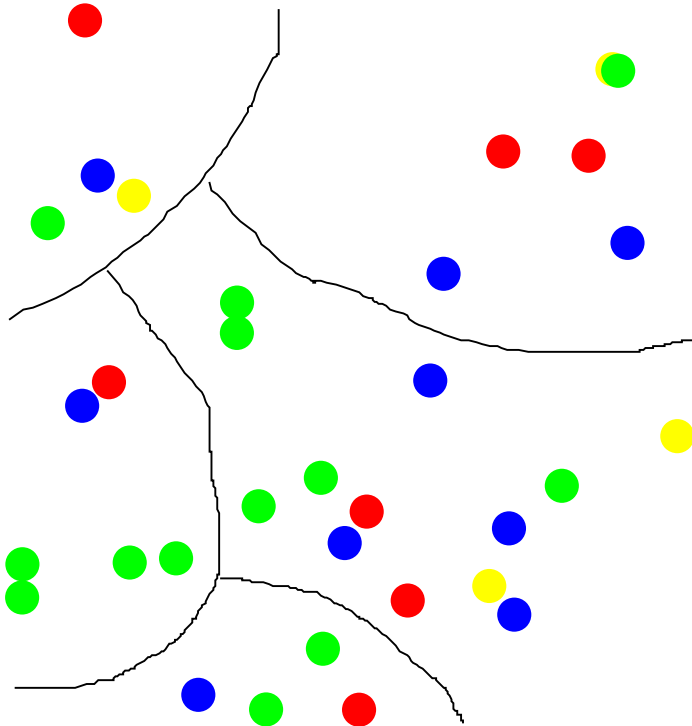
A partition \mathcal{P} of a set A is a set of nonempty, mutually disjoint subsets of A whose union is A .

Examples

- If A is the set of students in this a class, a partition might be the groups who worked on HW4 together (solutions to be posted on Classes, by the way).
- If A is a set of colored points, a partition could be found just by grouping them by color.



or just by designating regions.



Big Idea

Partitions and equivalence relations are the same thing.

Proposition. Let A be a set with equivalence relation R . Then, its equivalence classes $\{[a] : a \in A\}$ form a partition of A .

Proof. We have seen previously that equivalence classes are mutually disjoint. We need only show

$$\bigcup_{a \in A} [a] = A.$$

Since for all a , $[a] \subseteq A$ by definition, we automatically get $\bigcup_{a \in A} [a] \subseteq A$. But for all $x \in A$, $x \in [x] \subseteq \bigcup_{a \in A} [a]$. And so they are equivalent. ■

Proposition. Given a partition \mathcal{P} , the relation $\overset{\mathcal{P}}{\equiv} = \{(x, y) : \exists P \in \mathcal{P}, x, y \in P\}$ is an equivalence relation.

Proof. (reflexive) By definition, every element of A must be in some element of \mathcal{P} , so $x \overset{\mathcal{P}}{\equiv} x$.

(symmetric) If $x \overset{\mathcal{P}}{\equiv} y$, then there exists a set P containing x and y so it contains y and x , and thus $y \overset{\mathcal{P}}{\equiv} x$.

(transitive) Exercise. (Which part of the definition of partition is essential here?) ■

Theorem. Let R be an equivalence relation on a finite set A and suppose all equivalence classes of R have cardinality m . Then the number of equivalence classes is $\frac{|A|}{m}$.

Proof. Really? ■

I know, but this theorem can actually be useful for various counting problems.

Permutations of lists with repeated factors (anagrams)

Here are the anagrams (including nonsense words) of “ketchup”:

ketchup	ketchpu	ketcuhp	ketcuph	ketcphu	ketcuph	ketchup	ketcpu	ketcuph	ketchup	ketcpu	ketcuph
ketuchp	ketucph	ketuhcp	ketuhpc	ketupch	ketuphc	ketpchu	ketpcuh	ketphcu	ketphuc	ketpuch	ketpuhc
kecthup	kecthpu	kectuhp	kectuph	kectphu	kectpuh	kechtup	kechtpu	kechutp	kechupt	kechptu	kechput
kecuthp	kecutph	kecuhtp	kecuhtp	kecupth	kecupht	kecpthu	kecptuh	kecphtu	kecphtu	kecputh	kecpuht
kehtcup	kehtcpu	kehtucp	kehtupc	kehtpcu	kehtpuc	kehctup	kehctpu	kehcutp	kehcupt	kehceptu	kehcuput
kehutcp	kehutpc	kehuctp	kehuept	kehuptc	kehupct	kehptcu	kehptuc	kehpcut	kehpcut	kehputc	kehpuct
keutchp	keutcph	keuthcp	keuthpc	keutpch	keutphc	keuthp	keuctph	keuchtp	keuchpt	keucpth	keucpht
keuhtcp	keuhtpc	keuhctp	keuhept	keuhptc	keuhpct	keuptch	keupthc	keupcth	keupcht	keuphtc	keuphct
keptchu	keptcu	keptcu	keptcuh	keptuch	keptuhc	kepcthu	kepctuh	kepchtu	kepchut	kepcuth	kepcuht
kephtcu	kephtuc	kephtcu	kephcut	kephutc	kephuct	keputch	keputhc	kepucth	kepucht	kepuhtc	kepuhct
ktechup	ktechpu	ktecuhp	ktecuph	ktecphu	ktecuph	ktehcup	ktehcpu	ktehucp	ktehupc	ktehpcu	ktehpuc
kteuchp	kteucph	kteuhcp	kteuhpc	kteupch	kteuphc	ktepchu	ktepcuh	ktephcu	ktephuc	ktepuch	ktepuhc
ktcehup	ktcehpu	ktceuhp	ktceuph	ktcephu	ktcepuh	ktcheup	ktchepu	ktchuep	ktchupe	ktchpeu	ktchpue
ktcuehp	ktcueph	ktcuhep	ktcuhep	ktcupeh	ktcuphe	ktcpehu	ktcpeuh	ktcpheu	ktcpheue	ktcpueh	ktcpuhe
kthecup	kthecpu	ktheucp	ktheupc	kthepcu	kthepuc	kthceup	kthcepu	kthcuel	kthcupe	kthcpeu	kthcpue
kthuecp	kthuepc	kthucep	kthucpe	kthupec	kthupce	kthpecu	kthpeuc	kthpceu	kthpcue	kthpuce	kthpuce
ktuechp	ktuecph	ktuehcp	ktuehpc	ktuepch	ktuephc	ktuechp	ktuceph	ktuchep	ktuchpe	ktucpeh	ktucphe
ktuhecp	ktuhepc	ktuhcep	ktuhcpe	ktuhpec	ktuhpce	ktupech	ktupehc	ktupceh	ktupche	ktuphec	ktuphce
ktpechu	ktpecuh	ktpehcu	ktpehuc	ktpeuch	ktpeuhc	ktpehu	ktpeuh	ktpcheu	ktpchue	ktpcueh	ktpcuhe
ktphecu	ktpheuc	ktphceu	ktphcuc	ktphuac	ktphuac	ktpuech	ktpuehc	ktpuceh	ktpuceh	ktpuhec	ktpuhce
kcethup	kcethpu	kcetuhp	kcetuph	kcetphu	kcetpuh	kcetup	kcetpu	kcetup	kcetup	kcetup	kcetup
kceuthp	kceutph	kceuhtp	kceuhpt	kceupth	kceupht	kcepthu	kceptuh	kcepthu	kcepthu	kcepthu	kcepthu
kctehup	kctehpu	kcteuhp	kcteuph	kctephu	kctepuh	kctheup	kcthepu	kcthuep	kcthupe	kcthpeu	kcthpue
kctuehp	kctueph	kctuhep	kctuhpe	ktupeh	ktupeh	ktpehu	ktpeuh	ktcpheu	ktcpheue	ktcpueh	ktcpuhe
kchetup	kchetpu	kcheutp	kcheupt	kcheptu	kcheput	kchteup	kchtepu	kchtuep	kchtupe	kchtpeu	kchtpue
kchuetp	kchuept	kchutep	kchutpe	kchupet	kchupte	kchpetu	kchpeut	kchpteu	kchptue	kchpuet	kchpute
kcuethp	kcuetph	kcuehtp	kcuehpt	kcuethp	kcuethp	kcutehp	kcuteph	kcuthep	kcuthep	kcutpeh	kcutphe
kcuhetp	kcuhept	kcuhtep	kcuhtpe	kcuhpct	kcuhpct	kcupeth	kcupeht	kcupteh	kcupthe	kcuphet	kcuphte
kcpethu	kcpetuh	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu
kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu	kcpethu
khetcup	khetcpu	khetucp	khetupc	khetpcu	khetpuc	khetcup	khetcpu	khetcup	khetcup	khetcup	khetcup
kheutcp	kheutpc	kheuctp	kheuept	kheuptc	kheupct	kheptcu	kheptuc	khepctu	khepcut	kheputc	khepuct
khtecup	khtecpu	khteucp	khteupc	khtepcu	khtepuc	khtceup	khtcepu	khtcuel	khtcupe	khtcpeu	khtcpue
khtuecp	khtuepc	khtucep	khtucpe	khtupec	khtupce	khtpecu	khtpeuc	khtpceu	khtpcue	khtpuce	khtpuce
khcetup	khcetpu	khceutp	khceupt	khceptu	khceptu	khcteup	khctepu	khctuep	khctupe	khctpeu	khctpue
khcuetp	khcuept	khcutep	khcutpe	khcupet	khcupte	khcpetu	khcpeut	khcpteu	khcptue	khcpuet	khcpute
khuetcp	khuetpc	khuectp	khuecpt	khueptc	khuepct	khutecp	khutepc	khutcep	khutcepe	khutpec	khutpce
khucetp	khucept	khuctep	khuctpe	khucpet	khucpte	khupetc	khupect	khuptec	khuptce	khupcet	khupcte
khpetcu	khpetuc	khpectu	khpecut	khpeutc	khpeuct	khptecu	khpteuc	khptceu	khptcuc	khptuec	khptuce
khpcetu	khpcetuc	khpcetu	khpcetuc	khpcetuc	khpcetuc	khpuetc	khpuect	khputec	khputce	khpuetuc	khpuete
kuetchp	kuetcph	kuethcp									

kupetch	kupethc	kupecth	kupecht	kupehtc	kupehct	kuptech	kuptehc	kuptceh	kuptche	kupthec	kupthce
kupceth	kupceht	kupcteh	kupcthe	kupchet	kupchte	kuphetc	kuphect	kuphtec	kuphtce	kuphcet	kuphcte
kpetchu	kpetcuh	kpethcu	kpethuc	kpetuch	kpetuhc	kpecthu	kpectuh	kpechtu	kpechut	kpecuth	kpecuht
kpehtcu	kpehtuc	kpehctu	kpehcut	kpehutc	kpehuct	kpeutch	kpeuthc	kpeucth	kpeucht	kpeuhct	kpeuhct
kptechu	kptecuh	kptehcu	kptehuc	kpteuch	kpteuhc	kptcehu	kptceuh	kptcheu	kptchue	kptcueh	kptcuhe
kpthecu	kptheuc	kpthceu	kpthcue	kpthuec	kpthuce	kptuech	kptuehc	kptuceh	kptuche	kptuhec	kptuhce
kpcethu	kpcetuh	kpcehtu	kpcehut	kpceuth	kpceuhc	kpctehu	kpcteuh	kpctheu	kpcthue	kpctueh	kpctuhe
kpchetu	kpcheut	kpchete	kpchtue	kpchuet	kpchute	kpcueth	kpcueht	kpcuteh	kpcuthe	kpcuhet	kpcuhte
kphetcu	kphetuc	kphectu	kphecut	kpheutc	kpheuct	kphetcu	kphteuc	kphtceu	kphtcue	kphtuec	kphtuce
kphcetu	kphceut	kphcteu	kphctue	kphcu et	kphcute	kphuetc	kphuect	kphutec	kphutce	kphucet	kphucte
kpuetch	kpuethc	kpuecth	kpuecht	kpuehtc	kpuehct	kputech	kputehc	kputceh	kputche	kputhec	kputhce
kpuceth	kpuceht	kpucteh	kpucthe	kpuchet	kpuchte	kpuhetc	kpuhect	kpuhtec	kpuhtce	kpuhct	kpuhcte
ektchup	ektchpu	ektcuhp	ektcuph	ektcp hu	ektcp uh	ekthcup	ekthcpu	ekthucp	ekthupc	ekthpcu	ekthpuc
ektuchp	ektucph	ektuhcp	ektuhpc	ektupch	ektuphc	ektpchu	ektpcu h	ektphe u	ektp huc	ektpuch	ektpu h c
ekcthu p	ekcthu pu	ekctu hp	ekctup h	ekctphu	ekctpu h	ekchtup	ekchtpu	ekchutp	ekchupt	ekchptu	ekchput
ekcuthp	ekcutph	ekcuhtp	ekcu hpt	ekcupth	ekcupht	ekcpthu	ekcptuh	ekcphtu	ekcphut	ekcputh	ekcput h
ekhtcup	ekhtcpu	ekhtucp	ekhtupc	ekhtpcu	ekhtpuc	ekhctup	ekhctpu	ekhcutp	ekhcu pt	ekhceptu	ekhcept
ekhutc p	ekhutc pc	ekhuctp	ekhucpt	ekhuptc	ekhupet	ekhptcu	ekhptuc	ekhpctu	ekhpcut	ekhpuct	ekhpuct
ekutchp	ekutcp h	ekuthcp	ekuthpc	ekutpch	ekutphc	ekucthp	ekuctph	ekuchtp	ekuchpt	ekucpth	ekucpht
ekuhtcp	ekuhtpc	ekuhctp	ekuhcpt	ekuhpte	ekuhpet	ekuptch	ekupthc	ekupcth	ekupcht	ekuphtc	ekuphct
ekptchu	ekptcu h	ekpthcu	ekpthuc	ekptuch	ekptuhc	ekpethu	ekpctuh	ekpchtu	ekpchut	ekpcuth	ekpcuht
ekphtcu	ekphtuc	ekphctu	ekphcut	ekphutc	ekphuct	ekputch	ekputhc	ekpucth	ekpucht	ekpuhtc	ekpuhct
etkchup	etkchpu	etkcu hp	etkcuph	etkcp hu	etkcp uh	etkhcup	etkhcpu	etkhucp	etkhupc	etkhpcu	etkhpuc
etkuchp	etkucph	etkuhcp	etku hpc	etkupch	etkuphc	etkpc hu	etkpcu h	etkphcu	etkphuc	etkpuch	etkpu h c
etckhu p	etckhu pu	etcku hp	etckup h	etckphu	etckpu h	etchkup	etchkpu	etchukp	etchupk	etchpku	etchpuk
etcukhp	etcukph	etcuhkp	etcuhpk	etcupkh	etcuphk	etcpkhu	etcpku h	etcp hku	etcp huk	etcpukh	etcpuhk
ethkcu p	ethkcpu	ethkucp	ethkupc	ethkpcu	ethkpuc	ethckup	ethckpu	ethcukp	ethcupk	ethcpku	ethcpuk
ethukcp	ethukpc	ethuckp	ethucpk	ethupkc	ethupck	ethpkcu	ethpkuc	ethpcku	ethpcuk	ethpukc	ethpuck
etukchp	etukcp h	etukhcp	etukhpc	etukpch	etukphc	etuckhp	etuckph	etuchkp	etuchpk	etucpkh	etucphk
etuhkcp	etuhkpc	etuhckp	etuhcpk	etuhpke	etuhpck	etupkch	etupkhc	etupekh	etupchk	etuphkc	etuphck
etpkchu	etpkcu h	etpkhu c	etpkhuc	etpkuch	etpkuhc	etpkchu	etpkcu h	etpkchu	etpkchu	etpcukh	etpcuhk
etphkcu	etphkuc	etphcku	etphcuk	etphuke	etphuck	etpukch	etpukhc	etpuckh	etpuchk	etpuhkc	etpuhck
eckthup	eckthpu	ecktu hp	ecktup h	ecktp hu	ecktp uh	eckhtup	eckhtpu	eckhutp	eckhupt	eckhptu	eckhput
eckuthp	eckutph	eckuhtp	eckuhpt	eckupth	eckupht	eckpthu	eckpthu	eckphtu	eckphtu	eckputh	eckpuht
ectkhu p	ectkhu pu	ectku hp	ectkup h	ectkphu	ectkpu h	ecthkup	ecthkpu	ecthukp	ecthupk	ecthpku	ecthpuk
ectukhp	ectukph	ectuhkp	ectuhpk	ectupkh	ectuphk	ectpkhu	ectpkuh	ectphku	ectphuk	ectpuhk	ectpuhk
echktup	echktpu	echkutp	echkupt	echkptu	echkput	echtkup	echtkpu	echtukp	echtupk	echtpku	echtpuk
echuktp	echukpt	echutkp	echutpk	echupkt	echuptk	echpktu	echpkut	echptku	echptuk	echpukt	echputk
ecukthp	ecuktp h	ecukhtp	ecukhpt	ecukpth	ecukpht	ecutkhp	ecutkph	ecuthkp	ecuthpk	ecutpkh	ecutphk
ecuhktp	ecuhkpt	ecuhtkp	ecuhtpk	ecuhpkt	ecuhptk	ecupkth	ecupkht	ecuptkh	ecupthk	ecuphkt	ecuphtk
ecpkthu	ecpktuh	ecpkhtu	ecpkhut	ecpkuth	ecpkuht	ecptkhu	ecptkuh	ecpthku	ecpthuk	ecptukh	ecptuhk
ecphktu	ecphkut	ecphtku	ecphtuk	ecphukt	ecphutk	ecpukth	ecpukht	ecputkh	ecputhk	ecpuhkt	ecpuhtk
ehktcup	ehktcpu	ehktucp	ehktupc	ehktpcu	ehktpuc	ehketup	ehketpu	ehkcutp	ehkcupt	ehkceptu	ehkcept
ehkutcp	ehkutpc	ehkuctp	ehkucpt	ehkuptc	ehkupet	ehkptcu	ehkptuc	ehkpctu	ehkpcut	ehkputc	ehkpuct
ehtkcu p	ehtkcpu	ehtkucp	ehtkupc	ehtkpcu	ehtkpuc	ehtckup	ehtckpu	ehtcukp	ehtcupk	ehtcpku	ehtcpuk
ehtukcp	ehtukpc	ehtuckp	ehtucpk	ehtupkc	ehtupck	ehtpkcu	ehtpkuc	ehtpcku	ehtpcuk	ehtpukc	ehtpuck
ehctkup	ehcktpu	ehckutp	ehckupt	ehckptu	ehckput	ehctkup	ehctkpu	ehctukp	ehctupk	ehctpku	ehctpuk
ehcuktp	ehcukpt	ehcutkp	ehcutpk	ehcupkt	ehcuptk	ehcpktu	ehcpkut	ehcptku	ehcptuk	ehcpukt	ehcputk
ehuktcp	ehuktpc	ehuketp	ehukcpt	ehukpte	ehukpct	ehutkep	ehutkpc	ehutckp	ehutcpk	ehutpkc	ehutpck
ehucktp	ehuckpt	ehuctkp	ehuctpk	ehucpkt	ehucptk	ehupkte	ehupket	ehuptkc	ehuptek	ehupckt	ehupctk
ehpktcu	ehpktuc	ehpktcu	ehpkcut	ehpkutc	ehpkuct	ehptkcu	ehptkuc	ehptcku	ehptcuk	ehptukc	ehptuck
ehpcktu	ehpckut	ehpctku	ehpctuk	ehpcukt	ehpcutk	ehpuktc	ehpukct	ehputkc	ehputek	ehpuckt	ehpuctk
euktchp	euktcph	eukthcp	eukthpc	euktpch	euktphc	eukcthp	eukctph	eukchtp	eukchpt	eukcpth	eukcpht
eukhtcn	eukhtnc	eukhctn	eukhent	eukhnct	eukhnct	eukntch	euknthc	eukncth	eukncht	euknhtc	euknhct

[illegible]

ctphkeu	ctphkue	ctpheku	ctpheuk	ctphuke	ctphuek	ctpuken	ctpukhe	ctpuekn	ctpuenk	ctpunke	ctpunek
chketup	chketpu	chkeutp	chkeupt	chkeptu	chkeput	chkteup	chktepu	chktuep	chktupe	chktpeu	chktpue
chkuetp	chkuept	chkutep	chkutpe	chkupet	chkupte	chkpetu	chkpeut	chkpteu	chkptue	chkpuet	chkpute
chektup	chektpu	chekutp	chekupt	chekptu	chekput	chetkup	chetkpu	chetukp	chetupk	chetpku	chetpuk
cheuktp	cheukpt	cheutkp	cheutpk	cheupkt	cheuptk	chepktu	chepkut	cheptku	cheptuk	chepukt	cheputk
chtkeup	chtkepu	chtkuep	chtkupe	chtkpue	chtekup	chtekpu	chteukp	chteupk	chtepku	chtepuk	chtepuk
chtukep	chtukpe	chtuekp	chtuepk	chtupke	chtupek	chtpkeu	chtpkue	chtpeku	chtpeuk	chtpuke	chtpuek
chuketp	chukept	chuktep	chuktpe	chukpet	chukpte	chuektp	chuekpt	chuetkp	chuetpk	chuepkt	chueptk
chutkep	chutkpe	chutekp	chutepk	chutpke	chutpek	chupket	chupkte	chupekt	chupetk	chuptke	chuptek
chpketu	chpkcut	chpkteu	chpktpue	chpkuet	chpkute	chpektu	chpekut	chpetku	chpetuk	chpeukt	chpeutk
chptkeu	chptkue	chpteku	chpteut	chptuke	chptuek	chpuket	chpukte	chpuekt	chpuetk	chputke	chputek
cukethp	cuketph	cukehtp	cukehpt	cukepth	cukepht	cuktehp	cukteph	cukthep	cukthpe	cuktphe	cuktphe
cukhetp	cukhept	cukhtep	cukhtpe	cukhpet	cukhpte	cukpeth	cukpeht	cukpteh	cukpthe	cukphet	cukphte
cuekthp	cuektph	cuekhtp	cuekhpt	cuekpth	cuekpht	cuethkp	cuetkph	cuethkp	cuethpk	cuetpkh	cuetphk
cuehktp	cuehkpt	cuehtkp	cuehtpk	cuehpkt	cuehptk	cuepkth	cuepkht	cueptkh	cuepthk	cuephkt	cuephtk
cutkehp	cutkeph	cutkhep	cutkhpe	cutkpeh	cutkphe	cutekhp	cutekph	cutehkp	cutehpk	cutepkh	cutephk
cuthkep	cuthkpe	cuthkep	cuthpek	cuthpke	cuthpek	cutpkhe	cutpkhe	cutpekh	cutpehk	cutphke	cutphek
cuhketp	cuhkept	cuhktep	cuhktpe	cuhkpet	cuhkpte	cuhektp	cuhektu	cuhetkp	cuhetpk	cuhepkt	cuheptk
cuhtkep	cuhtkpe	cuhtkep	cuhtpek	cuhtpke	cuhtpek	cuhpket	cuhpkte	cuhpckt	cuhpctk	cuhtpke	cuhtpek
cupketh	cupkeht	cupkteh	cupkthe	cupkhet	cupkhte	cupekth	cupekht	cupetkh	cupethk	cupehkt	cupehtk
cuptkeh	cuptkhe	cuptekh	cuptehk	cupthke	cupthek	cuphket	cuphkte	cuphekt	cuphetk	cuphtke	cuphtek
cpkethu	cpketuh	cpkehtu	cpkehut	cpkeuth	cpkeuht	cpktehu	cpkteuh	cpktheu	cpkthue	cpktueh	cpktuhe
cpkhetu	cpkheut	cpkhneu	cpkhutue	cpkhuet	cpkhute	cpkueth	cpkueht	cpkuteh	cpkuthe	cpkuhet	cpkuhte
cpekthu	cpektuh	cpekhtu	cpekhut	cpekuth	cpekuht	cpetkhu	cpetkuh	cpethku	cpethuk	cpetukh	cpetuhk
cpehktu	cpehktu	cpehtku	cpehtuk	cpehukt	cpehukt	cpeukth	cpeukht	cpeutkh	cpeuthk	cpeuhkt	cpeuhkt
cptkehu	cptkeuh	cptkneu	cptkhue	cptkueh	cptkuhe	cptekhu	cptekuh	cptehku	cptehuk	cpteukh	cpteuhk
cpthkeu	cpthkue	cptheku	cptheuk	cpthuke	cpthuek	cptukeh	cptukhe	cptuekh	cptuehk	cptuhke	cptuhke
cphketu	cphkeut	cphkteu	cphktue	cphkuet	cphkute	cphketu	cphketu	cphetku	cphetuk	cpheukt	cpheutk
cphtkeu	cphtkue	cphteku	cphteuk	cphtuke	cphtuek	cphuket	cphukte	cphuekt	cphuatk	cphutke	cphutek
cpuketh	cpukeht	cpukteh	cpukthe	cpukhet	cpukhte	cpuekth	cpuekht	cpuetkh	cpuethk	cpuehkt	cpuehtk
cputkeh	cputkhe	cputekh	cputehk	cputhke	cputhek	cpuhket	cpuhkte	cpuhekt	cpuhetk	cpuhtke	cpuhtek
hketcup	hketcpu	hketucp	hketupc	hketpcu	hketpuc	hkectup	hkectpu	hkecutp	hkecutp	hkecutp	hkecutp
hkeutcp	hkeutpc	hkeuctp	hkeuctp	hkeuptc	hkeupct	hkeptcu	hkeptuc	hkeptcu	hkeptuc	hkeptcu	hkeptuc
hktecup	hktecpu	hkteucp	hkteupc	hktepcu	hktepuc	hktceup	hktcepu	hktcuep	hktcupe	hktcpeu	hktcpue
hktuecp	hktuepc	hktucep	hktucep	hktupcu	hktupcu	hktpecu	hktpecu	hktpecu	hktpecu	hktpecu	hktpecu
hkceutp	hkceutp	hkceutp	hkceutp	hkceutp	hkceutp	hkceutp	hkceutp	hkceutp	hkceutp	hkceutp	hkceutp
hkcutep	hkcutep	hkcutep	hkcutep	hkcutep	hkcutep	hkcutep	hkcutep	hkcutep	hkcutep	hkcutep	hkcutep
hkuetcp	hkuetpc	hkuetcp	hkuetcp	hkuetcp	hkuetcp	hkuetcp	hkuetcp	hkuetcp	hkuetcp	hkuetcp	hkuetcp
hkucetp	hkucetp	hkucetp	hkucetp	hkucetp	hkucetp	hkucetp	hkucetp	hkucetp	hkucetp	hkucetp	hkucetp
hkpetcu	hkpetuc	hkpetcu	hkpetuc	hkpetcu	hkpetuc	hkpetcu	hkpetuc	hkpetcu	hkpetuc	hkpetcu	hkpetuc
hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu	hkpcetu
hektcup	hektcpu	hektucp	hektupc	hektpcu	hektpuc	hektcup	hektcpu	hektcup	hektcpu	hektcup	hektcpu
hektcup	hektcpu	hektucp	hektupc	hektpcu	hektpuc	hektcup	hektcpu	hektcup	hektcpu	hektcup	hektcpu
hetkcup	hetkcpu	hetkucp	hetkupc	hetkpcu	hetkpuc	hetkcup	hetkcpu	hetkcup	hetkcpu	hetkcup	hetkcpu
hetukcp	hetukpc	hetukcp	hetukpc	hetukcp	hetukpc	hetukcp	hetukpc	hetukcp	hetukpc	hetukcp	hetukpc
hecktup	hecktpu	heckutp	heckupt	heckptu	heckput	hecktup	hecktup	hecktup	hecktup	hecktup	hecktup
hecuktp	hecukpt	hecutkp	hecutpk	hecupkt	hecuptk	hecptku	hecptku	hecptku	hecptku	hecptku	hecptku
heuktep	heuktpc	heukctp	heukcpt	heukptc	heukpct	heutkcp	heutkpc	heutckp	heutcpk	heutpck	heutpck
heucktp	heuckpt	heuctkp	heuctpk	heucpkt	heucptk	heupkte	heupktc	heuptkc	heuptck	heupctk	heupctk
hepkctu	hepkctuc	hepkctu	hepkcut	hepkute	hepkuct	heptkcu	heptkuc	heptcku	heptcuk	heptuke	heptuck
hepcktu	hepckut	hepctku	hepctuk	hepcukt	hepcutk	hepuktc	hepukct	heputkc	heputck	hepuctk	hepuctk
htkecup	htkecpu	htkeucp	htkeupc	htkepcu	htkepuc	htkceup	htkcepu	htkcupe	htkcupe	htkcupe	htkcupe
htkuecp	htkuepc	htkuecp	htkuepc	htkuecp	htkuepc	htkuecp	htkuepc	htkuecp	htkuepc	htkuecp	htkuepc
htekcup	htekcpu	htekucp	htekupc	htekpcu	htekpuc	htekcup	htekcpu	htekcup	htekcpu	htekcup	htekcpu

htuekcp	htuekpc	htueckp	htuecpk	htuepkc	htuepck	htuecku	htuekpu	htueckp	htuepck	htuepck	htuepck
htckcup	htckepu	htckuep	htckupe	htckpeu	htckpue	htcekup	htcekpu	htceukp	htceupk	htcepku	htcepuk
htcukep	htcu kpe	htcuekp	htcuepk	htcupke	htcupek	htcpkeu	htcpkue	htcepeku	htcepek	htcpuke	htcepuk
htukecp	htukepc	htukcep	htukcpe	htukpec	htukpce	htuekcp	htuekpc	htueeckp	htueeckp	htuepkc	htuepck
htuckep	htuckpe	htucekp	htucepk	htucpke	htucpek	htupkec	htupkce	htupekc	htupeck	htupcke	htupcek
htpkecu	htpkouc	htpkceu	htpkcue	htpkuec	htpkuce	htpekcu	htpekuc	htpecku	htpecuk	htpeukc	htpeuck
htpkcu	htpkue	htpckeu	htpceuk	htpcuke	htpcuek	htpukec	htpukce	htpuekc	htpueck	htpucke	htpucek
hckcup	hcketpu	hckeutp	hckeu pt	hckeptu	hckep ut	hckteup	hcktepu	hcktuep	hcktupe	hcktp eu	hcktpue
hckuetp	hckuept	hckutep	hckutpe	hckupet	hckupte	hckpetu	hckpeut	hckpteu	hckptue	hckpuet	hckpute
hcektup	hcektpu	hcekutp	hcekutp	hcekptu	hcekput	hcektup	hcektpu	hceutk p	hceutp k	hceptku	hceptuk
hceuktp	hceukpt	hceutkp	hceutpk	hceupkt	hceuptk	hcepktu	hcepkut	hceptku	hceptuk	hcepukt	hceputk
hctkeup	hctkepu	hctkuep	hctkupe	hctkpeu	hctkpue	hctekup	hctekpu	hcteu k p	hcteu p k	hctepku	hctepuk
hctukep	hctukpe	hctuekp	hctuepk	hctupke	hctupek	hctpkeu	hctpkue	hctpeku	hctpek u	hctpuke	hctpuek
hcuketp	hcuk ept	hcuktep	hcuktp e	hcukpet	hcukpte	hcuektp	hcuekpt	hcu etkp	hcu etpk	hcuepkt	hcueptk
hcutkep	hcutkpe	hcutekp	hcutepk	hc utpke	hcutpek	hc upket	hc upkte	hcu ekt	hc upetk	hcuptke	hcuptek
hcpketu	hcpkeut	hcpkteu	hcpktue	hcpkue t	hcpkute	hcpektu	hcpekut	hcpetku	hcpetuk	hcpeukt	hcpeutk
hcptkeu	hcptkue	hcpteku	hcpteuk	hcptuke	hcptuek	hcpuket	hcpukte	hcpuekt	hcpuetk	hcputke	hcputek
huketcp	huketpc	hukectp	hukecpt	hukeptc	hukepct	huktecp	huktepc	huktc ep	huktcpe	huktp ec	huktpce
hukcetp	hukcept	hukctep	hukctpe	hukcpet	hukcpt e	hukpetc	hukpect	hukptec	hukptce	hukpcet	hukpcte
huektcp	huektpc	huekctp	huekcpt	huekptc	huekpct	huetkcp	huetkpc	huetckp	huetcpk	huetpkc	huetpck
huecktp	hueckpt	huectkp	huectpk	hucp kpt	huecptk	huepktc	huepkct	hueptkc	hueptck	huepckt	huepctk
hutkecp	hutkepc	hutkcep	hutkcpe	hutkpec	hutkpce	hutekcp	hutekpc	huteckp	huteckp	hutepkc	hutepck
hutkep	hutckpe	hutcekp	hutcepk	hute pke	hutcepk	hutpkec	hutpkce	hutpekc	hutpeck	hutpcke	hutpcek
hucketp	huckept	hucktep	hucktpe	huckpet	huckpte	hucektp	hucekpt	hucetkp	hucetpk	hucepkt	huceptk
huctkep	huctkpe	huctekp	huctepk	huctpke	huctpek	hucpket	hucpkte	hucpekt	hucpetk	hucptke	hucptek
hupketc	hupkect	hupktec	hupktce	hupkcet	hupkcte	hupektc	hupekct	hupetkc	hupetck	hupeckt	hupectk
huptkec	huptkce	huptekc	hupteck	huptcke	huptcek	hupcket	hupckte	hupcekt	hupcetk	hupctke	hupctek
hpketcu	hpketuc	hpkectu	hpkecut	hpkeutc	hpkeuct	hpktecu	hpkteuc	hpktceu	hpktcue	hpktuec	hpktuce
hpkcet u	hpkceut	hpkc teu	hpkc tue	hpkcuet	hpkcute	hpkuetc	hpkuect	hpkutec	hpkutce	hpkucet	hpkucte
hpektcu	hpektuc	hpekctu	hpekcut	hpekutc	hpekuct	hpetkcu	hpetkuc	hpetcku	hpetcuk	hpetukc	hpetuck
hpecktu	hpeckut	hpectku	hpectuk	hpecukt	hpecutk	hpeuktc	hpeukct	hpeutkc	hpeutck	hpeuckt	hpeuctk
hptkecu	hptkeuc	hptkceu	hptkcue	hptkuec	hptkuce	hptekcu	hptekuc	hptecku	hptecuk	hpteukc	hpteuck
hptckeu	hptckue	hptceku	hptceuk	hptcuke	hptcuek	hptukec	hptukce	hptuekc	hptueck	hptucke	hptucek
hpcketu	hpckeut	hpckteu	hpcktue	hpckuet	hpckute	hpcektu	hpcekut	hpcetku	hpcetuk	hpceukt	hpceutk
hpctkeu	hpctkue	hpcteku	hpcteuk	hpctuke	hpctuek	hpcuket	hpcukte	hpcuekt	hpcuetk	hpcutke	hpcutek
hpuketc	hpukect	hpuktec	hpuktce	hpukcet	hpukcte	hpuektc	hpuekct	hpuetkc	hpuetck	hpueckt	hpuectk
hputkec	hputkce	hputekc	hputeck	hputcke	hputcek	hpucket	hpuckte	hpucekt	hpucetk	hpuctke	hpuctek
uketcp	uketcp	ukethcp	ukethpc	uketpch	uketphc	ukecthp	ukectph	ukechtp	ukechpt	ukecpth	ukecpht
ukehtcp	ukehtpc										

pekhtcu	pekhtuc	pekhctu	pekhcut	pekhute	pekhuct	pekutch	pekuthc	pekucth	pekucht	pekuhtc	pekuhct
petkchu	petkcu	petkctu	petkhuc	petkuch	petkuhc	petckhu	petckuh	petckhu	petchuk	petcukh	petcuhk
pethkcu	pethkuc	pethctu	pethcuk	pethukc	pethuck	petukch	petukhc	petuckh	petuchk	petuhkc	petuhck
peckthu	pecktuh	peckhtu	peckhut	peckuth	peckuht	pectkhu	pectkuh	pecthku	pecthuk	pectukh	pectuhk
pechktu	pechkut	pechctu	pechtuk	pechukt	pechutk	pecukth	pecukht	pecutkh	pecuthk	pecuhkt	pecuhtk
pehkctu	pehkuc	pehkctu	pehkcut	pehkute	pehkuct	pehtkcu	pehtkuc	pehtcku	pehtcuk	pehtukc	pehtuck
pehcktu	pehckut	pehcktu	pehctuk	pehcutk	pehcutk	pehuktc	pehukct	pehutkc	pehutck	pehuctk	pehuctk
peuktcu	peukthc	peukctu	peukcht	peukhtc	peukhct	peutkch	peutkhc	peutckh	peutchk	peuthkc	peuthck
peuckth	peuckht	peuctkh	peucthk	peuchkt	peuchtk	peuhkte	peuhkct	peuhtkc	peuhtck	peuhckt	peuhctk
ptkechu	ptkecu	ptkectu	ptkehuc	ptkeuch	ptkeuhc	ptkcehu	ptkceuh	ptkcheu	ptkchue	ptkcueh	ptkcuhe
ptkhecu	ptkheuc	ptkhctu	ptkhcuc	ptkhuec	ptkhuce	ptkuech	ptkuehc	ptkuceh	ptkuche	ptkuhec	ptkuhce
ptekchu	ptekcu	ptekctu	ptekhuc	ptekuch	ptekuhc	pteckhu	pteckuh	ptechku	ptechuk	ptecukh	ptecuhk
ptehkcu	ptehkuc	ptehctu	ptehcuk	ptehukc	ptehuck	pteukch	pteukhc	pteuckh	pteuchk	pteuhkc	pteuhck
ptckehu	ptckeu	ptckctu	ptckhue	ptckueh	ptckuhe	ptcekhu	ptcekuh	ptcehku	ptcehuk	ptceuhk	ptceuhk
ptchkeu	ptchkue	ptchectu	ptcheuk	ptchuke	ptchuek	ptcukeh	ptcukhe	ptcuekh	ptcuehk	ptcuhek	ptcuhek
pthkecu	pthkeuc	pthkctu	pthkcuc	pthkuec	pthkuce	pthekcu	pthekuc	pthecku	pthecuk	pthuekc	pthueck
pthckeu	pthckue	pthcectu	pthceuk	pthcuke	pthcukc	pthukec	pthukce	pthuekc	pthueck	pthucke	pthucek
ptukech	ptukehc	ptukctu	ptukche	ptukhec	ptukhce	ptuekch	ptuekhc	ptueckh	ptuechk	ptuehkc	ptuehck
ptuckeh	ptuckhe	ptucectu	ptucehk	ptuchke	ptuchek	ptuhkec	ptuhkce	ptuhekc	ptuheck	ptuhcke	ptuhcek
pckethu	pcketuh	pckhtu	pckehut	pckeuht	pckeuht	pcktehu	pckteuh	pcktheu	pckthue	pcktueh	pcktuhe
pckhetu	pckheut	pckhteu	pckhtue	pckhuet	pckhute	pckueth	pckueht	pckuteh	pckuthe	pckuhet	pckuhte
pcekthu	pcektuh	pcehtu	pcekhut	pcekuth	pcekuht	pcetkhu	pcetkuh	pcethku	pcethuk	pcetukh	pcetuhk
pcehtku	pcehkut	pcehtcu	pcehtuk	pcehukt	pcehutk	pceukth	pceukht	pceutkh	pceuthk	pceuhkt	pceuhkt
pctkehu	pctkeuh	pctkctu	pctkhue	pctkueh	pctkuhe	pctekhu	pctekuh	pctehku	pctehuk	pcteuhk	pcteuhk
pcthkeu	pcthkue	pcthectu	pctheuk	pcthuke	pcthuek	pctukeh	pctukhe	pctuekh	pctuehk	pctuhke	pctuhck
pchketu	pchkeut	pchkctu	pchkue	pchkute	pchkute	pchektu	pchekut	pchetku	pchetuk	pcheukt	pcheutk
pchtkeu	pchtkue	pchtectu	pchteuk	pchtuke	pchtuek	pchuket	pchukte	pchuekt	pchuetk	pchutke	pchutek
pcuketh	pcukeht	pcukctu	pcukthe	pcukhet	pcukhte	pcuekth	pcuekht	pcuetkh	pcuethk	pcuehkt	pcuehtk
pcutkeh	pcutkhe	pcutectu	pcutehk	pcuthke	pcuthek	pcuhket	pcuhkte	pcuhekt	pcuhetk	pcuhtke	pcuhtek
phketcu	phketuc	phkctu	phkecut	phkeutc	phkeuct	phktecu	phkteuc	phktceu	phktcuc	phktuec	phktuce
phkcetu	phkceut	phkctcu	phkctue	phkcuet	phkcute	phkuetc	phkuect	phkutec	phkutce	phkucet	phkucte
phektcu	phektuc	phekctu	phekcut	phekutc	phekuct	phetkcu	phetkuc	phetcku	phetcuk	phetukc	phetuck
phecktu	pheckut	phectcu	phectuk	phecukt	phecutk	pheuktc	pheukct	pheutkc	pheutck	pheuctk	pheuctk
phtkecu	phtkeuc	phtkctu	phtkcuc	phtkuec	phtkuce	phtekcu	phtekuc	phtecku	phtecuk	phteukc	phteuck
phtckeu	phtckue	phtcectu	phtceuk	phtcuke	phtcukc	phtukec	phtukce	phtuekc	phtueck	phtucke	phtucek
phcketu	phckeut	phckctu	phcktue	phckuet	phckute	phcektu	phcekut	phcetku	phcetuk	phceukt	phceutk
phctkeu	phctkue	phctctu	phctuek	phctuke	phctuek	phcuket	phcukte	phcuekt	phcuethk	phcutke	phcutek
phuketc	phukect	phukctu	phuktee	phukcet	phukcte	phuektc	phuekct	phuetc	phuetc	phuectk	phuectk
phutkec	phutkce	phutectu	phuteck	phutcke	phutcek	phuckte	phuckte	phuectk	phuctek	phuctke	phuctek
puketch	pukethc	pukctu	pukecht	pukehct	pukehct	puktech	puktehc	puktceh	puktche	pukthec	pukthce
pukceth	pukceht	pukctcu	pukcthe	pukchet	pukchte	pukhetc	pukhect	pukhtec	pukhtce	pukhctc	pukhcte
puektch	puekthc	puekctu	puekcht	puekhtc	puekhct	puetkch	puetkhc	puetckh	puetchk	puethkc	puethck
pueckth	pueckht	puectkh	puecthk	puechkt	puechtk	puehkte	puehktc	puehtkc	puehtck	puehckt	puehctk
putkech	putkehc	putkctu	putkche	putkhec	putkhce	putekch	putekhc	puteckh	putechk	putehkc	putehck
putckeh	putckhe	putcectu	putcehk	putchke	putchek	puthkec	puthkce	puthekc	putheck	puthcke	puthcek
pucketh	puckeht	puckctu	puckthe	puckhet	puckhte	pucektc	pucekht	pucetkh	pucethk	pucehkt	pucehtk
puctkeh	puctkhe	puctctu	puctehk	pucthke	pucthek	puchket	puchkte	puchekt	puchetk	puchtke	puchteck
puhketc	puhkect	puhkctu	puhktee	puhkctc	puhkcte	puhektc	puhekct	puhetkc	puhetck	puheckt	puhectk
puhtkec	puhtkce	puhtctu	puhteck	puhtcke	puhtcek	puhket	puhkte	puhcekt	puhctek	puhctke	puhctek

Length[Permutations[{"k", "e", "t", "c", "h", "u", "p"}]]

Here are those for “ukelele”:

[illegible]

```
Length[Permutations[{"u", "k", "e", "l", "e", "l", "e"}]]
```

[illegible]

[illegible]

[illegible]

3λukele	3λukeel	3λukleε	3λuklee	3λukeel	3λukele	3λuekle	3λuekel	3λuelke	3λuelek	3λueekl	3λueelk
3λulkee	3λulkee	3λuleke	3λuleek	3λuleke	3λuleek	3λuekel	3λuekle	3λueekl	3λueelk	3λuelke	3λuelek
3λkuele	3λkueel	3λkuleε	3λkulee	3λkueel	3λkuele	3λkeule	3λkeuel	3λkelue	3λkeleu	3λkeeu	3λkeelu
3λklueε	3λkluee	3λkleue	3λkleeu	3λkleue	3λkleeu	3λkeuel	3λkeule	3λkeeu	3λkeelu	3λkelue	3λkeleu
3λeukle	3λeukel	3λeulke	3λeulek	3λeuekl	3λeuelk	3λekule	3λekuel	3λeklue	3λekleu	3λekeul	3λekelu
3λeluke	3λeluek	3λelkue	3λelkeu	3λeleuk	3λeleku	3λeeukl	3λeeulk	3λeekul	3λeeklu	3λeeluk	3λeelku
3λlukee	3λlukee	3λlueke	3λlueek	3λlueke	3λlueek	3λlkueε	3λlkuee	3λlkeue	3λlkeeu	3λlkeue	3λlkeeu
3λleuke	3λleuek	3λlekue	3λlekue	3λleuek	3λleeku	3λleuek	3λleuek	3λlekue	3λlekue	3λleeku	3λleeku
3λeukel	3λeukle	3λeuekl	3λeuelk	3λeulke	3λeulek	3λekuel	3λekule	3λekeul	3λekelu	3λeklue	3λekleu
3λeukl	3λeulk	3λeekul	3λeeklu	3λeeluk	3λeelku	3λeluke	3λeluek	3λelkue	3λelkeu	3λeleuk	3λeleku

Length[Permutations[{"u", "k", "e", "l", "ε", "λ", "3"}]]

5040

What we are really doing is taking the set of all lists (w/o repeats) of length 7 from 7 elements (above).

Then we impose an equivalence relation on them, namely two such permutations are equivalent if the l's occupy the same set of positions, and the e's occupy the same set of positions.

$$\begin{aligned} ukel\epsilon\lambda 3 &\equiv ukel3\lambda\epsilon \equiv uk\epsilon\epsilon\lambda 3 \equiv uk\epsilon l3\lambda\epsilon \equiv uk3\epsilon\lambda\epsilon \equiv uk3\epsilon\lambda\epsilon \\ &\equiv uke\lambda\epsilon l3 \equiv uke\lambda3\epsilon \equiv uk\epsilon\lambda\epsilon l3 \equiv uk\epsilon\lambda3\epsilon \equiv uk3\lambda\epsilon\epsilon \equiv uk3\lambda\epsilon\epsilon \end{aligned}$$

or if x and y are such permutation

$$x \equiv y \Leftrightarrow \{i: x_i \text{ is "ell"}\} = \{i: y_i \text{ is "ell"}\} \wedge \{i: x_i \text{ is "ee"}\} = \{i: y_i \text{ is "ee"}\}$$

The size of each equivalence class is thus the number of permutations of the “ee” positions (3!) and the “ell” positions (2!) multiplied.

$$\frac{7!}{3! \cdot 2!}$$

420

Example

6 tennis players in a club want to play singles (one-on-one) matches. How many ways can matchups be made?

Example

6 tennis players in a club want to play singles (one-on-one) matches. How many ways can this be done?

Start with arrangements of the 6 players. There are $6!$ possibilities. Assume that for any given list $(x_1, x_2, x_3, x_4, x_5, x_6)$, x_1 will play x_2 , x_3 will play x_4 and x_5 will play x_6 .

We form an equivalence relation by noting the matchups are the same if we make any number of switches in the individual matchups (2 choices for each of 3 pairs = 2^3) or any permutations of the pairings themselves ($(3)_3 = 3!$ arrangements of pairs). Thus each equivalence class has $2^3 \cdot 3! = 48$ elements. Thus the total number of possibilities is

$$\frac{6!}{48}$$

$$48$$

$$15$$

Combinations

We define

n^k = the number of lists of length k from n elements.

$(n)_k$ = the number of lists of length k from n elements without repeats.

$\binom{n}{k}$ = the number of subsets of size k from a set of n elements.

Formula

Notice that sets could be considered equivalence classes of lists with distinct elements where the relation is “is a rearrangement of”.

Since k elements can be arranged $k!$ ways, we have

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{(n-k)! k!}$$

Corollary

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Examples

1. What is the coefficient of x^3 in $(2x + 1)^6$?

2.

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} =$$

Example

Show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0.$$

This means by rearrangement,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

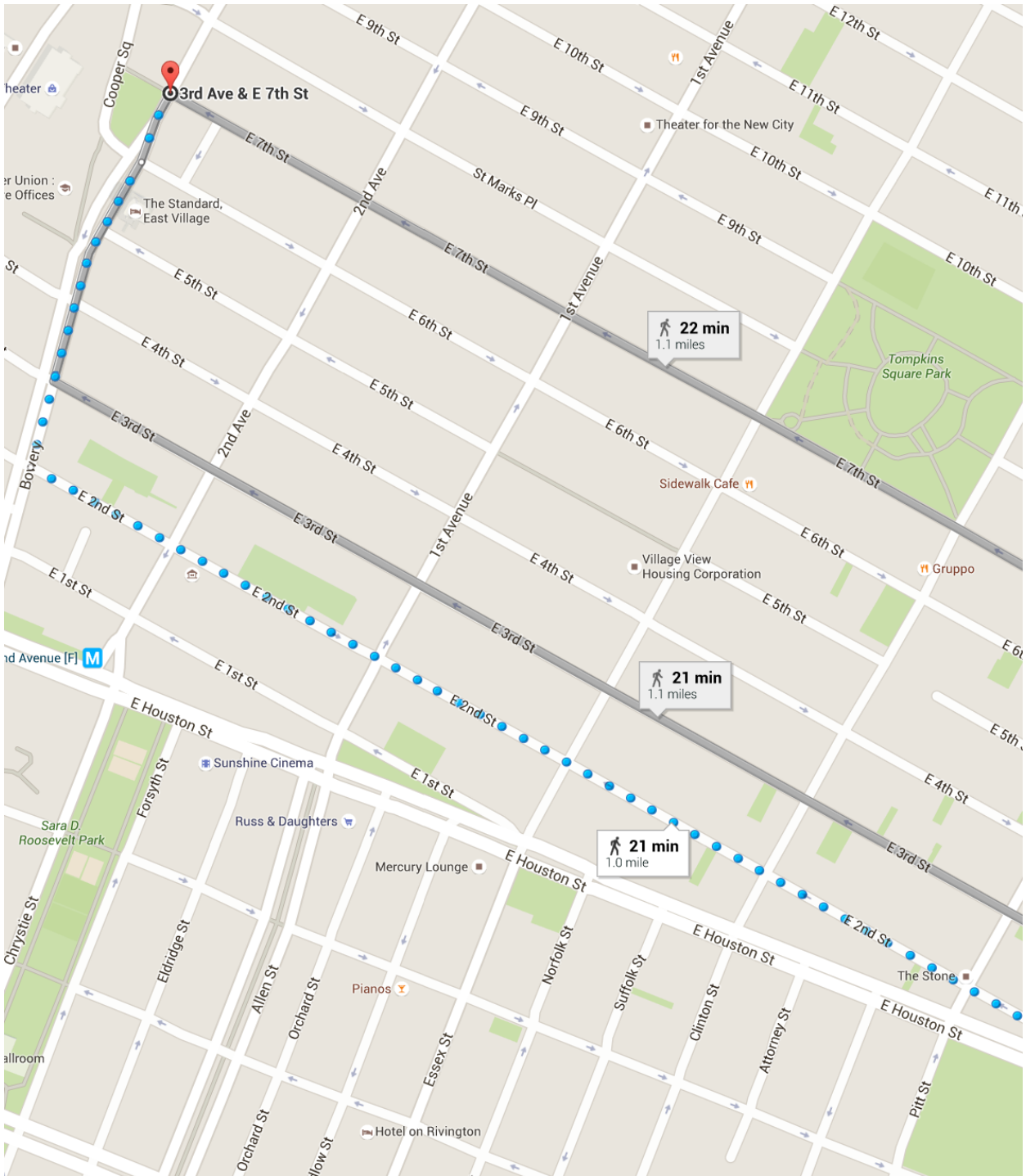
Is there a combinatorial proof of this?

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Exercise

How many (sane) ways can one walk from Houston and Ave D to 3rd Ave and 7th St?



Total ways of traversing a 6×5 grid:

$$\binom{11}{5}$$

462

If we exclude 5th between Ave C and Ave B (There is a school blocking the through-street), we get

$$\binom{11}{5} - \binom{4}{1} \binom{6}{2}$$

402