Algebraic Geometry

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These notes were originally taken for the course on Algebraic Geometry at the University of New South Wales. I have since added and rearranged some of the material.

Texts for this course:

- Atiyah-McDonald. Introduction to Commutative Algebra.
- Chapter 1 of Hartshorne. Algebraic Geometry.
- Shafarevich (Good for examples). Basic Algebraic Geometry.

Thanks to all my friends with whom I took this course. They greatly helped me understand this material and would have been completely lost otherwise. Special thanks to Dominic Matan for guiding us through.

Contents

§1 Week 1, Lecture 1

§1.1 Motivation

Algebraic geometry is the study of vanishing points of sets of polynomials which we call algebraic varieties. We start by presenting examples that motivate the study of algebraic geometry.

1. **Algebra and Topology**
   Consider the vanishing set S of the polynomial $y^2 = (x-1)(x-2)$

   $$S = \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)\}$$

   We would like to determine what S is geometrically. To do so, recall the Fundamental Theorem of Algebra.

   Fundamental Theorem of Algebra: The field of complex numbers is algebraically closed. So every polynomial in one variable x with complex coefficients can be written as the product of a complex constant and linear polynomials $x + a$ where $a \in \mathbb{C}$.

   Any complex number can be written in the form $(x-1)(x-2)$. Hence, the set S is the set of all square roots of all complex numbers. Topologically, S is two copies of the complex line $\mathbb{C}$ glued together along a line. In other words, the set of solutions to S is a sphere $\equiv$ a compact Riemann surface of genus 0. (Genus = number of holes in the manifold).

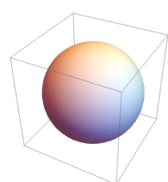
   

   Figure 1: $y^2 = (x-1)(x-2)$

   Let us now instead look at $y^3 = (x-1)(x-2)(x-3)$. Topologically this is a torus (genus 1).

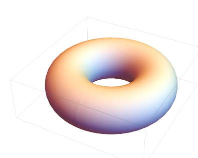
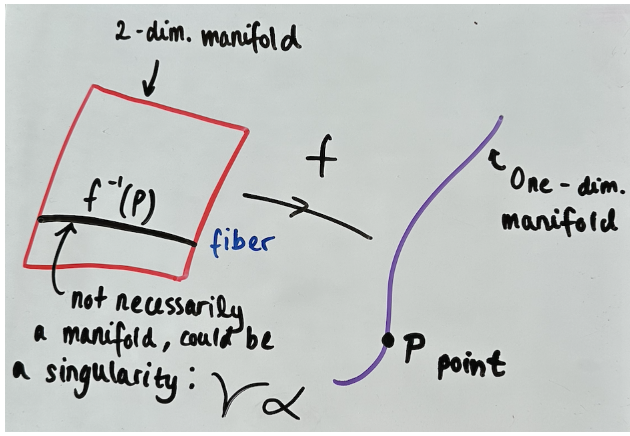
   

   Figure 2: $y^3 = (x-1)(x-2)(x-3)$

   In a more general case, consider $f(x,y) = 0$ and let $d = deg(f)$. If we can say what the genus of f is, then we understand f geometrically. We pose the following question: does the degree of the polynomial fix the genus of the vanishing set? When the solution space is smooth, the answer is yes: g is completely determined by degree of the polynomial.

   $$g = \frac{(d-1)(d-2)}{2}$$

2. **Algebraic Geometry and Differential Geometry**
The central objects of study in differential geometry are smooth manifolds. Let M and N be two manifolds, and let $f : M \to N$ be a morphism (smooth).



The category of manifolds is too small to understand the fibers. Algebraic geometry gives us a larger category to understand what they are. We will see that fibers are what we call varieties.

3. **Algebraic Geometry and Number Theory**

Fermat's Last Theorem: For any integer $n > 2$, the equation $a^n + b^n = c^n$ has no positive integer solutions.

Consider $x^3 + y^3 = 1$. How many rational solutions are there? The language used to prove this is in algebraic geometry. The proof uses schemes (however, we will not cover schemes in this course).

Any modern research in algebraic number theory is done in the language of algebraic geometry.

4. **Complex Manifolds**

Over the complex numbers, (projective) algebraic geometries are in one-to-one correspondence with (compact) complex manifolds. And, on complex manifolds we perform differential geometry using analysis. So, the correspondence gives us a connection between algebra and analysis. This correspondence is a result of the 20th century called **GAGA** (algebraic geometry and analytic geometry). This course will work its way to prove a theorem called Riemann-Roch. The initial problem was a purely analytic problem, and it was solved through this correspondence. Riemann-Roch implies Gauss-Bonnet, an important result in topology.

5. **Physics**
The language of string theory is based on algebraic geometry.

§1.2 Prerequisite Knowledge

- **Topological Space**. It is the most general type of mathematical space that allows for the definition of limits, continuity, and connectedness. Examples: Euclidean spaces, metric spaces and manifolds.

  Definition 1.1 (Topological Space)**.** Let X be a (possibly empty) set. The elements of X are usually called points, though they can be any mathematical object. Let $\mathcal{N}$ be a function assigning to each x (point) in X a non-empty collection $\mathcal{N}(x)$ of subsets of X. The elements of $\mathcal{N}(x)$ will be called neighbourhoods of x with respect to $\mathcal{N}$ (or, simply, neighbourhoods of x). The function $\mathcal{N}$ is called a **neighbourhood topology** if the axioms below are satisfied; and then X with $\mathcal{N}$ is called a **topological space**.

  (i) If N is a neighborhood of x (i.e., $N \in \mathcal{N}(x)$), then $x \in N$. In other words, each point of the set X belongs to everyone of its neighborhoods with respect to $\mathcal{N}$.

  (ii) If N is a subset of X and includes a neighborhood of x, then N is a neighborhood of x. I.e., every superset of a neighborhood of a point $x \in X$ is again a neighborhood of x.

  (iii) The intersection of two neighborhoods of x is a neighborhood of x.

  (iv) Any neighborhood N of x includes a neighborhood M of x such that N is a neighborhood of each point of M.

  Example. In $\mathbb{R}$, a neighborhood of a real number x would be an open interval containing x.

  Definition 1.2 (Open subset)**.** A subset U of X is defined to be **open** if U is a neighborhood of all points in U.

  Definition via open sets. A topology on a set X may be defined as a collection τ of subsets of X, called open sets, satisfying the following axioms:

  (i) The empty set and X itself belong to τ.

  (ii) Any arbitrary (finite or infinite) union of members of τ belong to τ.

  (iii) The intersection of any finite number of members of τ belongs to τ.

  The set τ is commonly called a **topology** on X.

  Example. Given X, the trivial topology on X is $\tau = \{\emptyset, X\}$. The discrete topology on X is the power set of $X, \tau = \mathcal{P}(X)$.

- **Subspace Topology.** A subspace of a topological space X is a subset S of X which is equipped with a topology induced from that of X called the **subspace topology**.

  Definition 1.3. Given a topological space $\{X, \tau\}$ and a subset S of X, the subspace topology on S is defined by

  $$\tau_S = \{S \cap U | U \in \tau\}$$

  That is, a subset of S is open in the subspace topology if and only if it is the intersection of S with an open set in (X, τ). If S is equipped with the subspace topology then it is a topological space in its own right, and is called a **subspace** of (X, τ).

- **Continuous Function between Topological Spaces.** A function $f : X \to Y$ between topological spaces is called continuous if for every $x \in X$ and every neighborhood N of $f(x)$ there is a neighborhood M of x such that $f(M) \subseteq N$. Equivalently, f is continuous if the inverse image of every open set is open.

- **Homeomorphism.** A bijection that is continuous and whose inverse is also continuous. In topology, homeomorphic spaces are essentially identical. A function $f : X \to Y$ between two topological spaces is a homeomorphism if it has the following properties:

  - f is a bijection,
  - f is continuous,
  - f^{-1} is continuous (f is an open mapping).

- **Manifolds.** A topological space that locally resembles Euclidean space near each point. More precisely, an n-dimensional manifold, or n-manifold for short, is a topological space with the property that each point has a neighborhood that is homeomorphic to an open subset of n-dimensional Euclidean space.

  Examples.
  1-dim manifolds: lines, circles, not self-crossing curves.
  2-dim manifolds: surfaces such as the plane, the sphere, the torus, Klein bottle, and real projective plane.

- **Modules.** The use of modules was pioneered by one of the most prominent mathematicians of the first part of this century, Emmy Noether, who led the way in demonstrating the power and elegance of this structure. We study modules over rings.

  Definition 1.4 (Module)**.** Suppose that R is a ring, and 1 is its multiplicative identity. A (left) R-module is an abelian group $(M, +)$ equipped with a scalar multiplication map $R \times M \to M : (r, m) \mapsto r \cdot m$ such that the following axioms hold for all $m, m' \in M$ and $r, r' \in R$:

  (i) $1 \cdot m = m$ (identity law)
  (ii) $r \cdot (r' \cdot m) = (rr') \cdot m$ (associative law)
  (iii) $r \cdot (m + m') = r \cdot m + r \cdot m'$ and $(r + r') \cdot m = r \cdot m + r' \cdot m$ (distributive laws)

- **Short exact sequences.** An **exact sequence** is a sequence of R-module homomorphisms

  $$\cdots \xmapsto{\alpha_{n-2}} X_{n-1} \xmapsto{\alpha_{n-1}} X_n \xmapsto{\alpha_n} X_{n+1} \xmapsto{\alpha_{n+1}} \cdots$$

  such that $\ker \alpha_n = \mathrm{im}\ \alpha_{n-1}$ for all n. A **short exact sequence** is an exact sequence of the form

  $$0 \to A \overset{\psi}{\mapsto} B \overset{\varphi}{\mapsto} C \to 0$$

  We say that B is an extension of C by A.

- **Noetherian Ring, Noetherian Module.** A Noetherian module is a module that satisfies the ascending chain condition on its submodules. That is, any ascending chain of submodules terminates.

A Noetherian ring is a ring that satisfies the ascending chain condition on left and right ideals; if the chain condition is satisfied only for left ideals or for right ideals, then the ring is said left-Noetherian or right-Noetherian respectively. That is, every increasing sequence $I_1 \subseteq I_2 \subseteq \cdots$ of left (or right) ideals has a largest element; that is, there exists an n such that: $I_n = I_{n+1} = \cdots$.

Note. All fields are Noetherian since they only have 0 and the field itself as ideals.

- **Hilbert's Basis Theorem.** If R is a Noetherian ring, then $R[x]$ is a Noetherian ring. Corollary: If R is a Noetherian ring, then $R[x_1, \cdots, x_n]$ is a Noetherian ring.
- **Finitely Generated Algebra.** A finitely generated module that is also a ring such that the definitions of addition and multiplication coincide.

§1.3 Affine Algebraic Sets

Notation. Throughout this course we assume all rings are commutative with a 1. Additionally, we will use the following notation:

- k is a field.
- $\mathbb{A}_k^n = \{(a_1, \cdots, a_n) : a_i \in K\}$ is n-dimensional affine space (an affine space is what is left of a vector space after one has forgotten which point is the origin).
- $k[t_1, \cdots, t_n]$ is the polynomial ring.

Definition 1.5 (Vanishing Set)**.** Defined for all $T \subseteq k[t_1, \cdots, t_n]$,

$$V(T) := \{(a_1, \cdots, a_n) \in \mathbb{A}_k^n \mid f(a_1, \cdots, a_n) = 0, \ \forall f \in T\}$$

Remark. Any field k is Noetherian, and so, by Hilbert's Basis Theorem $k[t_1, \cdots, t_n]$ is Noetherian. Noetherian rings have the property that any ideal of it is finitely generated. So, if I is an ideal of $k[t_1, \cdots, t_n]$, it has to be finitely generated.

Definition 1.6 (Radical of an Ideal)**.** Denoted $\sqrt{I}$. It is an enlargement of an ideal I. We say $g \in \sqrt{I}$ if $g^m \in I$ for some $m \in \mathbb{N}$.

Properties of $V(\cdot) : \mathcal{P}(k[t_1, \cdots, t_n]) \to \mathcal{P}(\mathbb{A}_k^n)$:

(i) By the remark above, any ideal of $k[t_1, \cdots, t_n]$ is finitely generated. The vanishing set of an ideal is the same as the vanishing set of its generators. So for $f_1, \cdots, f_m \in k[t_1, \cdots, t_n]$,

$$V(\langle f_1, \cdots f_n \rangle) = V(f_1, \cdots f_n)$$

So the vanishing set of any ideal is the vanishing set of a finite number of polynomials.

(ii) $V(I) = V(\sqrt{I})$.

(iii) $V(IJ) = V(I) \cup V(J)$.

(iv) $V(I + J) = V(I) \cap V(J)$.

(v) If $I \subseteq J$, then $V(J) \subseteq V(I)$. It is inclusion-reversing.

Remark. (iii) The notion of product of ideals only makes sense for finite products. It's similar to how we cannot take the infinite union of closed sets and get a closed set.

Remark. (v) The inclusion does not respect equality. Here is an example: consider $I = \langle x^2 + y^2 + 1 \rangle \subseteq \mathbb{R}[x, y]$. If we apply V we get $V(I) = \emptyset = V(\mathbb{R}[x, y])$. This happens if you work over non-algebraically closed fields (in our example, $\mathbb{R}$ is not algebraically closed).

Definition 1.7 (Affine Algebraic Set)**.** A set $X \subseteq \mathbb{A}_k^n$ such that $X = V(I)$ for some ideal I of $k[t_1, \cdots, t_n]$.

Examples. $V(k[t_1, \cdots, t_n]) = \emptyset$, $V(0) = \mathbb{A}_k^n$. The affine algebraic subsets of the affine line $\mathbb{A}_k^1$ are $\mathbb{A}_k^1$ and finite collections of points (since $V(f_1, \cdots, f_n) =$ finite number of points).

§1.4 Interlude: Why we assume k is an algebraically closed field

Throughout this course, we will often want k to be algebraically closed. This is an important assumption for many of the definitions we will see later on to be well-defined. Let us look at a specific example that demonstrates how this assumption is vital.

To start, we define an **algebraic plane curve** as the points (x, y) that satisfy a non-constant polynomial $f(x, y) = 0$. Here f is a polynomial with coefficients in a fixed field k and we assume the coordinates (x, y) are in k. We define the degree of f as the **degree of the curve**. We are in fact quite familiar with algebraic plane curves. Conics are curves of degree 2, some examples includes circles ($x^2 + y^2 = a^2$), ellipses ($x^2/a^2 + y^2/b^2 = 1$), and parabolas ($y^2 = 4ax$). Cubics are algebraic curves of degree 3, for example $y^2 = x^2(x + 1)$. The polynomial ring $k[x, y]$ is a unique factorization domain (UFD), that is, any polynomial has a unique factorization $f = f_1^{k_1} \cdots f_r^{k_r}$ (up to constant multiples) as a product of irreducible factors f_i. Then the algebraic curve X given by $f = 0$ is the union of the curves X_i given by $f_i = 0$. A curve is **irreducible** (or **affine variety**) if its equation is an irreducible polynomial. The decomposition $X = X_1 \cup \cdots \cup X_r$ just obtained is called a decomposition of X into irreducible components.

Fermat's Last Theorem states that there are no three positive integers x, y, z satisfying $x^n + y^n = z^n$ for any integer $n > 2$. If we are then working over the field $\mathbb{R}$, the only solutions to $x^2 + y^2 = 0$ is the trivial solution $(0, 0)$. By the above definition, we would call the point $(0, 0)$ a "curve" of degree 2. But $(0, 0)$, is also a solution to $x^{2n} + y^{2n}$ for any n. So the degree of the "curve" is not well-defined. Additionally, $x^2 + y^2 = 0$ is irreducible, but $x^6 + y^6 = 0$ not. Again, it is unclear whether the curve is a reducible one or not. This problem we see here can be dealt with by working over algebraically closed fields. Recall, a field F is **algebraically closed** if every noncontant polynomial in $F[x]$ has a root in F.

Here is an important lemma that shows us how curves are well-defined in algebraically closed fields:

Lemma 1.8. *Let k be an arbitrary field, $f \in k[x, y]$ an irreducible polynomial, and $g \in k[x, y]$ an arbitrary polynomial . If g is not divisible by f then the system of equations $f(x, y) = g(x, y)$ has only a finite number of solutions.*

This lemma tells us that if the field k is algebraically closed and f does not divide g, the two polynomials will have different algebraic curves. An algebraically closed field k is infinite; and if f is nonconstant, the curve with equation $f(x,y)=0$ has infinitely many points. So an irreducible polynomial $f(x,y)$ is uniquely determined, up to a constant multiple, by the curve $f(x,y)=0$. Now the notions of degree of a curve and of irreducible curves are well-defined.

An important theorem in intersection theory is Bézout's theorem which says the number of points of intersection of two distinct irreducible algebraic curves equals the product of their degrees. By the lemma, this number is finite and this theorem only holds in algebraically closed fields.

§1.5 Zariski Topology and Varieties

We now wish to turn $\mathbb{A}^n_k$ into a topological space using the notion of affine algebraic set. To do this, we need a definition of open and closed subsets.

Definition 1.9 (Zariski Topology)**.** A subset $X \subseteq \mathbb{A}^n_k$ is Zariski closed if X is an affine algebraic set. Meaning, $X = V(f_1, \cdots f_n)$ for some $f_i \in k[t_1, \cdots, t_n]$. The complement of X is Zariski open.

Examples. The subsets $\mathbb{A}^n_k$ and $\emptyset$ are both open and closed. In the affine line, $\mathbb{A}^1_k$ and points are closed.

Remark. Any affine algebraic set X has an induced Zariski topology as a subspace topological space.

Definition 1.10 (Quasi-affine Algebraic Set)**.** An open subset of an affine algebraic set is called a quasi-affine algebraic set.

Definition 1.11 (Irreducible Subset)**.** Given a topological space X, we say $\emptyset \neq Y \subseteq X$ irreducible if Y is not the union of two proper closed subsets of Y.

Example. The affine line $\mathbb{A}^1_k$ is irreducible. Its proper closed subsets are sets of finite number of points and we can't write $\mathbb{A}^1_k$ as the union of a finite number of points.

The following lemma is an important one that tells us that open sets are dense in the Zariski topology.

Lemma 1.12. *Any open subset of an irreducible space is itself irreducible and dense.*

Proof. Let $U \subseteq X$ be a nonempty open subset of an irreducible space X. Suppose $\overline{U} \neq X$. Then $\overline{U}$ and $X \setminus U$ are distinct closed subsets satisfying $\overline{U} \cup (X \setminus U) = X$. This contradicts irreducibility, so $\overline{U} = X$. To show that U is also irreducible, suppose that $U = C \cup C'$ for distinct closed proper subsets of U. Then taking closures in X, we have that $X = \overline{U} = \overline{C} \cup \overline{C'}$. But X is irreducible, so at least one of C or C' has closure all of X. Say, $\overline{C} = X$. Then since C was closed in X we can write $C = \overline{C} \cap U = X \cap U = U$, contradicting minimality. ■

Lemma 1.13. *If Y is an irreducible subset of X, then $\overline{Y}$ is also irreducible.*

Proof. If Y is closed then $\overline{Y} = Y$ and $\overline{Y}$ is therefore irreducible. Suppose now Y is open and irreducible. If $\overline{Y} = C \cup C'$ where C and C' are distinct closed proper subsets, then $Y = \bar{Y} \cap Y = (Y \cap C) \cup (Y \cap C')$. Which is a contradiction to the irreducibility of Y. Hence, $\overline{Y}$ is irreducible. ■

Definition 1.14 (Affine Variety)**.** We call an irreducible affine algebraic set an affine variety. Its open subsets are all quasi-affine algebraic sets.

Definition 1.15 (Subvariety)**.** Let X be a quasi-affine. An irreducible closed subset of $Y \subset X$ is called a subvariety of X.

§1.6 Correspondence between Ideals and Affine Algebraic Sets

Define $I_{(\cdot)} : \mathcal{P}(\mathbb{A}^n_k) \to \mathcal{P}(k[t_1, \cdots t_n])$ as the function that takes in a subset of the affine space and maps it to the ideal generated by all polynomials that vanish on that subset.

$$I_X = \text{the ideal generated by all polynomials that vanish on } X\text{, where } X \subseteq \mathbb{A}^n_k.$$

Properties of $I_{(\cdot)}$:

(i) If $X_1 \subseteq X_2$ then $I_{X_2} \subseteq I_{X_1}$.

(ii) $I_{X_1 \cup X_2} = I_{X_1} \cap I_{X_2}$

(iii) $\overline{X} = V(I_X)$ Zariski closure.

In algebraically closed fields ($k = \bar{k}$), there is a 1-1 correspondence between

$$\underbrace{\text{Prime ideals in } k[t_1, \cdots, k_n]}_{\text{Algebra}} \longleftrightarrow \underbrace{\text{Affine Varieties in } \mathbb{A}^n_k}_{\text{Geometry}}$$

where $V(\cdot)$ takes us from the prime ideal to the affine variety and $I_{(\cdot)}$ looks at the corresponding ideal of an affine variety. The mappings $V(\cdot)$ and $I_{(\cdot)}$ preserve strict inclusions. This is a consequence of the **Nullstellensatz**.

Hilbert Nullstellensatz:

- Weak Version: Let K be a finitely generated k-algebra (every element can be written as a polynomial over k). If K is a field, the it is a finite algebraic extension of the field k. In particular, if k is algebraically closed, then for $\mathcal{M}$ maximal ideal

$$k \simeq \frac{K}{\mathcal{M}}$$

- Full Version: If I is an ideal of $k[t_1, \cdots, t_n]$, the $I_{V(I)} = \sqrt{I}$.

Corollary 1.16. *The maximal ideals in $k[t_1, \cdots, t_n]$ are in 1-1 correspondence with points in affine space:*

$$x = (x_1, \cdots, x_n) \in \mathbb{A}^n \leftrightarrow \langle t_1 - x_1, t_2 - x_2, \cdots, t_n - x_n \rangle := \mathcal{M}_x$$

Proof from assignment:

We say an ideal I in a given (commutative) ring R is irreducible, if

$$(I = \mathfrak{b} \cap \mathfrak{c}, \text{ for any two ideals } \mathfrak{b}, \mathfrak{c}) \Rightarrow (I = \mathfrak{b} \text{ or } I = \mathfrak{c})$$

An ideal is then said to be reducible if it is not irreducible.

(a) Give two examples of a reducible ideal and show that every prime ideal is irreducible.

(b) One generalizes the notion of prime ideal to primary ideals as follows. We say an ideal $I \subset R$ is primary, if

$$(f \cdot g \in I, \text{ for any } f, g \in R) \Rightarrow (f^n \in I \text{ or } g^m \in I \text{ for some } m, n \in \mathbb{N})$$

Show that in a Noetherian ring R, every irreducible ideal is primary.

(c) Use (b) to show that every ideal in a Noetherian ring can be written as the intersection of a finite number of primary ideals.

(d) Use (c) to establish the important fact that every affine algebraic set can be written as a finite union of affine varieties. You may assume that k is algebraically closed.

(a) Consider the ideal generated by 6 in the ring of integers $\mathbb{Z}$.

$$(6) = (3) \cap (2)$$

However, $(6) \neq (3)$ and $(6) \neq (2)$, so it is an example of a reducible ideal. Another example of a reducible ideal is the ideal $(x(x+1))$ in the polynomial ring $\mathbb{Z}[x]$,

$$(x(x+1)) = (x) \cap (x+1)$$

But $(x(x+1)) \neq (x)$ and $(x(x+1)) \neq (x+1)$.

Claim: Every prime ideal is irreducible.
Proof: Let $\mathfrak{p}$ be a prime ideal of the ring R. Assume for the sake of contradiction that $\mathfrak{p}$ is reducible. So $\mathfrak{p} = \mathfrak{b} \cap \mathfrak{c}$, where $\mathfrak{p} \neq \mathfrak{b}$ and $\mathfrak{p} \neq \mathfrak{c}$. Then, there exist elements $x \in \mathfrak{b} \setminus \mathfrak{p}$ and $y \in \mathfrak{c} \setminus \mathfrak{p}$. Fix such x and y. Since $\mathfrak{b}$ and $\mathfrak{c}$ are ideals, $xy \in \mathfrak{b}$ and $xy \in \mathfrak{c}$, which implies $xy \in \mathfrak{b} \cap \mathfrak{c} = \mathfrak{p}$. Since $\mathfrak{p}$ is prime,

$$xy \in \mathfrak{p} \implies x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}$$

However, this contradicts the fact we chose x and y to be elements not in $\mathfrak{p}$. Thus, $\mathfrak{p}$ is irreducible.

(b) Let R be a Noetherian ring and J be an irreducible ideal of R. Suppose $fg \in J$. Further assume that $g^n \notin J$ for any $n \in \mathbb{N}$. We will construct an ascending chain of ideals. Define for $i \in \mathbb{N}$,

$$\mathfrak{p}_i = \{h \in R \mid hf^i \in J\}$$

Clearly, each $\mathfrak{p}_i$ is an ideal since for $x \in \mathfrak{p}_i$ and $y \in R$, we have $yxf^i \in J \Rightarrow xy \in \mathfrak{p}_i$. Moreover, for $m > n$, $\mathfrak{p}_n \subseteq \mathfrak{p}_m$ since if $x \in \mathfrak{p}_n$

$$xf^n \in J \Rightarrow (xf^n)f^{m-n} = xf^m \in J \Rightarrow x \in \mathfrak{p}_m$$

Hence, we have the following chain of ascending ideals

$$\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \mathfrak{p}_3 \subseteq \cdots$$

This chain must terminate since R is Noetherian. So there exists $n \in \mathbb{N}$ such that for $m > n$ $\mathfrak{p}_m = \mathfrak{p}_n$.

Let us now define two new ideals:

$$\mathfrak{b} := J + (g), \ \mathfrak{c} := J + (f^n)$$

It is clear that $J \subseteq \mathfrak{b} \cap \mathfrak{c}$. Let us show that $\mathfrak{b} \cap \mathfrak{c} \subseteq J$. Take $x \in \mathfrak{b} \cap \mathfrak{c}$.

$$x \in \mathfrak{b} \Rightarrow x = j_1 + r_1 g, \quad j_1 \in J, r_1 \in R$$
$$x \in \mathfrak{c} \Rightarrow x = j_2 + r_2 f^n \quad j_2 \in J, r_2 \in R$$

Hence,

$$j_1 + r_1 g = j_2 + r_2 f^n$$

Rearranging,

$$r_1 g - r_2 f^n = j_1 - j_2 \in J$$

Multiplying by f,

$$f(r_1 g - r_2 f^n) \in J$$

$$r_1 f g - r_2 f^{n+1} \in J$$

$fg \in J$, so $r_1 fg \in J$, and so

$$r_2 f^{n+1} \in J$$

$$\Rightarrow r_2 \in \mathfrak{p}_{n+1}$$

By assumption, $\mathfrak{p}_{n+1} = \mathfrak{p}_n$, hence

$$r_2 f^n \in J$$

Recall $x = j_1 + r_2 f^n$, both j_1 and $r_2 f^n$ are in J, so $x \in J$. Hence, $\mathfrak{b} \cap \mathfrak{c} \subseteq J$.

Thus, $J = \mathfrak{b} \cap \mathfrak{c}$. J is an irreducible ideal so either $J = \mathfrak{b}$ or $J = \mathfrak{c}$. We cannot have $J = \mathfrak{b}$ since $g \notin J$ but $g \in \mathfrak{b}$. So, $J = \mathfrak{c} = J + (f^n)$. Hence, $f^n \in J$. This proves J is primary.

(c) Let J be an ideal of a Noetherian ring R. If J is irreducible, then J is primary by part (b), and we are done. Otherwise, J is reducible, and we can write

$$J = \mathfrak{p}_1 \cap \mathfrak{p}_2$$

where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are ideals of R. If both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are irreducible, by part (b) we conclude J is the intersection of two primary ideals. If this is not the case, we again replace $\mathfrak{p}_1$ or $\mathfrak{p}_2$ with the intersections of ideals. For example, if both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are reducible, then

$$J = (\mathfrak{p}_{11} \cap \mathfrak{p}_{12}) \cap (\mathfrak{p}_{21} \cap \mathfrak{p}_{22})$$

We continue repeating these procedure in the case that any of the $\mathfrak{p}_{(\cdot)}$ is reducible. Note that this process must terminate; otherwise, we would have an infinite chain of ideals that never stabilizes

$$p_{i_1} \subseteq p_{i_1 i_2} \subseteq p_{i_1 i_2 i_3} \subseteq \cdots$$

This would contradict the Noetherian ring condition. So, J is the intersection of a finite number of irreducible ideals,

$$J = \bigcap_{i \in I, |I| < \infty} \mathfrak{p}_i$$

By part (b), each $\mathfrak{p}_i$ is primary. So J is the finite intersection of primary ideals.

(d) Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set. Then $X = V(I)$ where I is an ideal of $k[t_1, \cdots, t_n]$. Every field is a Noetherian ring, and so, by Hilbert's Basis Theorem, $k[t_1, \cdots, t_n]$ is a Noetherian ring. By part (c), we can express I as

$$I = \bigcap_{i \in I} J_i$$

where $\{J_i | i \in I\}$ is a finite set of primary ideals ($|I| < \infty$). Hence,

$$X = V(I) = V\left(\bigcap_{i \in I} J_i\right)$$

We use the property of V that for ideals I and J, $V(I \cap J) = V(I) \cup V(J)$ to say

$$X = \bigcup_{i \in I} V(J_i)$$

We now prove the vanishing set of a primary ideal is an affine variety.

Let J be primary and suppose by way of contradiction that $V(J)$ is reducible. That is, there exist distinct nonempty closed proper subsets $V(J_1), V(J_2)$ of $V(J)$ whose union is $V(J)$. In particular this means that $\sqrt{J_1} \neq \sqrt{J} \neq \sqrt{J_2}$. Observe that

$$\begin{aligned}
\sqrt{J} &= \sqrt{J_1 \cap J_2} \\
&= \{f \in k[t_1, \cdots t_n] : f^n \in J_1 \cap J_2 \text{ for some } n\} \\
&= \{f \in k[t_1, \cdots t_n] : f^n \in J_1, f^n \in J_2 \text{ for some } n\} \\
&= \sqrt{J_1} \cap \sqrt{J_2},
\end{aligned}$$

So $\sqrt{J}$ is reducible. Let us show that $\sqrt{J}$ is prime. If $xy \in \sqrt{J}$, then $(xy)^m = x^m y^m \in J$ for some m big enough. Hence, since J is primary, $(x^m)^r = x^{rm} \in J$ for some r or $(y^m)^s = y^{sm} \in J$ for some s. In any case, either $x \in \sqrt{J}$ or $y \in \sqrt{J}$ by definition. Thus, $\sqrt{J}$ is prime, but this contradicts that $\sqrt{J}$ is reducible by (a).

Back to our initial proof. We have

$$X = \bigcup_{i \in I} V(J_i)$$

where each J_i is a primary ideal. By what we just proved, each $V(J_i)$ is an affine variety. Therefore, every affine algebraic set can be written as the finite union of affine varieties.

(e) Note prime ideals are a subset of primary ideals. Hence, in part (d) we also proved that the vanishing set of a prime ideal in $k[t_1, \cdots t_n]$ is an affine variety. Thus, V maps primes to varieties.

Let us first show that the map $I_{(\cdot)} : \mathcal{P}(\mathbb{A}_k^n) \to \mathcal{P}(k[t_1, \cdots t_n])$ maps affine varieties to prime ideals.

Let X be an affine variety of $\mathbb{A}_k^n$. Thus, $X = V(J)$ for some ideal J of $k[t_1, \cdots t_n]$ and $V(J)$ is irreducible. Assume for the sake of contradiction that I_X is not prime. Take $fg \in I_X$ such that $f \notin I_X$ and $g \notin I_X$. Then,

$$(fg) \subseteq I_X$$

The function V is inclusion-reversing, so

$$V(I_X) \subseteq V(fg)$$

Since I and V are inverses of each other

$$X \subseteq V(fg) = V(f) \cup V(g)$$

We can express X as

$$X = (X \cap V(f)) \cup (Y \cap V(g))$$

Since X is an affine variety X is irreducible, so either $X = X \cap V(f)$ or $X = X \cap V(g)$. However, we had assumed $f \notin I_X$ meaning $V(f)$ is a strict subset of X and, likewise, $V(g)$ is a strict subset of X. This is a contradiction. Hence, I maps affine varieties to prime ideals.

Finally, let us show that $I_{(\cdot)}$ and $V(\cdot)$ are inverses of each other. This will prove there is a 1-1 correspondence between prime ideals in $k[t_1, \cdots t_n]$ and affine varieties in $\mathbb{A}_k^n$. Let $X \subseteq \mathbb{A}_k^n$ be an affine variety. Then $V(I_X) = \overline{X} = X$ since X is closed. In the other direction, if J is a prime ideal then $I_{V(J)} = \sqrt{J} = J$ by the Nullstellensatz and since prime ideals are radical.

Thus, $I_{(\cdot)}$ and $V(\cdot)$ are inverses of each other.

§1.7 Exercises

- *Prove for* $f_1, \cdots, f_m \in k[t_1, \cdots, t_n]$,
$$V(\langle f_1, \cdots f_n \rangle) = V(f_1, \cdots f_n)$$
- *Show* $V(I) = V(\sqrt{I})$.
($\subseteq$) Let $x \in V(I)$, then $f(x) = 0$ for all $f \in I$. Clearly, if $g \in \sqrt{I}$, then $g^m \in I$ and $g^m(x) = 0$. So $x \in V(\sqrt{I})$.
($\supseteq$) Let $x \in V(\sqrt{I})$. Then $g(x) = 0$ for all $g \in \sqrt{I}$. $I \subseteq \sqrt{I}$ so $x \in V(I)$.
- *Show* $V(IJ) = V(I) \cup V(J)$.
- *Show* $V(I + J) = V(I) \cap V(J)$.
- *Show if* $I \subseteq J$, *then* $V(J) \subseteq V(I)$.
- *What are the open/closed the subsets of* A_k^n*?*
- *Check the Zariski Topology is a topology on the affine space.*
- *Show if* $X_1 \subseteq X_2$ *then* $I_{X_2} \subseteq I_{X_1}$.
- $I_{X_1 \cup X_2} = I_{X_1} \cap I_{X_2}$
- $\overline{X} = V(I_X)$ *Zariski closure.*
- *Check that if* I *is a prime ideal, then* $\sqrt{I} = I$.

§2 Week 1, Lecture 2

§2.1 Background in Commutative Algebra

2.1.1 Rings of Fractions

Definition 2.1 (Multiplicative Closed Subset of a Ring)**.** Let A be a commutative ring. We call a subset $S \subseteq A$ multiplicative closed if

(i) $1 \in S$

(ii) S is closed under multiplication ($s, s' \in S \Rightarrow ss' \in S$).

Definition 2.2 (Rings of Fractions)**.**

$$S^{-1}A := \left\{ \frac{a}{s} \;\middle|\; a \in A, s \in S \right\} / \equiv$$

where the equivalence relation is given by

$$\frac{a}{s} = \frac{b}{t} \iff u(at - bs) = 0 \text{ for some } \; u \in S$$

The Ring of Fractions has a natural structure of addition and multiplication (hence, it is a ring):

$$\frac{a}{s} + \frac{b}{t} := \frac{at + bs}{st} \quad \text{and} \quad \frac{a}{s} \cdot \frac{b}{t} := \frac{ab}{st}$$

with multiplicative identity $1/1$, and additive identity is $0/1$.

Some facts about rings of fractions:

- $0 \in S$ iff $S^{-1}A = 0$,
- If $A = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$, then $S^{-1}A = \mathbb{Q}$.
- If A is an integral domain (no zero divisors), we can always let $S = A \setminus \{0\}$. $S^{-1}A = \mathrm{Frac}(A)$ is the field of fractions.

Remark. There is a natural ring homomorphism $f : A \to S^{-1}A$ that takes a and maps it to $a/1$. From here on, when we write f we refer to this natural homomorphism.

Proposition 2.3. *If $s \in S$, then $f(s)$ is a unit in $S^{-1}A$.*

Proof. $f(s) = s/1$. Consider $1/s$, then

$$\frac{s}{1}\frac{1}{s} = \frac{s}{s} = \frac{1}{1}$$

since $s - s = 0$. ∎

Proposition 2.4. *If $f(a) = 0$, then $ua = 0$ for some $u \in S$.*

Proof.

$$f(a) = \frac{a}{1} = \frac{0}{1} \Rightarrow \exists u \in S \ s.t. \ u(a \cdot 1 - 1 \cdot 0) = 0$$
$$\Rightarrow ua = 0$$

■

Corollary 2.5. *If A is an integral domain, f is an injection.*

Proof. Note that by the previous proposition we have

$$\ker f = \{a \in A : f(a) = 0\} = \{a \in A : \exists u \in S \ s.t. \ ua = 0\}$$

This set equals zero since there are no zero divisors. Hence, f is injective. ■

Theorem 2.6 (Universal Property of Rings of Fractions)**.** *If $g : A \to B$ is a ring homomorphism such that $g(s)$ is a unit in B for every $s \in S$, then there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that the following diagram commutes:*

$$\begin{tikzcd} S^{-1}A \arrow[dr, dashed, "h"] & \\ A \arrow[u, "f"] \arrow[r, "g"] & B \end{tikzcd}$$

So $h \circ f = g$.

Proof. (i) (Existence) Define h via $h(a/s) := g(a)g(s)^{-1}$. We first check it is well defined. Suppose $a/b = c/d$, so $u(ad - bc) = 0$ for some $u \in U$. Applying g to $u(ad - bc) = 0$:

$$g(u)(g(a)g(d) - g(b)g(c)) = 0$$

Since $g(u)$ has an inverse

$$g(a)g(d) - g(b)g(c) = 0$$
$$\Rightarrow g(a)g(d) = g(b)g(c)$$
$$g(a)g(b)^{-1} = g(c)g(d)^{-1}$$
$$\Rightarrow h\left(\frac{a}{b}\right) = h\left(\frac{c}{d}\right)$$

So the function is well-defined. We check it is a ring homomorphism:

- $h(ac/bd) = g(ac)g(bd)^{-1} = g(a)(b)g(b)^{-1}g(d)^{-1} = (g(a)g(b)^{-1})(g(c)g(d)^{-1}) = h(a/b)h(c/d)$
-

$$\begin{aligned} h((ad + bc)/bd) &= g(ad + bc)g(bd)^{-1} \\ &= (g(a)g(d) + g(b)g(c))g(b)^{-1}g(d)^{-1} \\ &= g(a)g(b)^{-1} + g(c)g(d)^{-1} \\ &= h(a/b) + h(b/c) \end{aligned}$$

- $h(1/1) = g(1)g(1)^{-1} = 1$.

(ii) (Uniqueness) Suppose h' satisfies $h' \circ f = g$. Then $h' \circ f = h \circ f$. Hence, for $a \in A$

$$h'(f(a)) = h'(a/1) = h(a/1) = h(f(a))$$

By definition of h: $h(a/1) = g(a)g(1)^{-1} = g(a)$.

■

2.1.2 Localization

Definition 2.7 (Localization)**.** This is an important example of a Ring of Fractions and it consists of the following ideas: [Matan, 2024]

- The ring A_P, where A is a ring, P is a prime ideal of A, $S := A \setminus P$ and

$$A_P := S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in A \setminus P \right\}$$

This is called **localization at P.** This definition assumes S is multiplicatively closed, which is true by the following fact: P is a prime ideal of A if and only if A/P is multiplicatively closed.

Proof. ($\Rightarrow$) Let P be a prime ideal of A. Then $P \neq A$ and so $A \setminus P \neq \emptyset$. Let $x, y \in A \setminus P$. If $xy \notin A \setminus P$, then $xy \in P$. But P being a prime ideal implies x or y is in P. Hence, we must have $xy \in A \setminus P$. Prime ideals do not contain the identity element since they are not equal to the whole ring. So $1 \in A \setminus P$.
($\Leftarrow$) Suppose $A \setminus P$ is a multiplicatively closed subset of A. Then $1 \in A\setminus$ P, so $P \neq A$. Let $xy \in P$. If both x and y are not in P, then $xy \in A \setminus P$. So it must be the case at least one of them is in P. Hence, P is prime. ■

- The map $f : A \to A_p$ which sends $a \in A$ to the element $a/1 \in A_p$.

- The functor $(\cdot)_P = S^{-1}(\cdot) : A\text{-Mod} \to S^{-1}A\text{-Mod}$.
Let S be multiplicatively closed and $f : M \to N$ be a map of A-modules, then $S^{-1} : S^{-1}M \to S^{-1}N$ is given by $S^{-1}f(m/s) = f(m)/s$.

- Taking the ring A and forming fractions with denominators in S.

Proposition 2.8. *The image of a prime ideal P under the localization map is a maximal ideal $\mathcal{M}$ and it is the only maximal ideal in the ring $S^{-1}A$.*

The image of an ideal is not always an ideal. Take for example the ring $\mathbb{Z}$ and the injection of $\mathbb{Z}$ into $\mathbb{Q}$. The ideal $2\mathbb{Z}$ is not an ideal in $\mathbb{Q}$ ($\frac{1}{4} \cdot 2 \notin 2\mathbb{Z}$). So this is a special property of localization.

Proof. We fix a prime ideal P of A. In the localization map $S = A \setminus P$ and P is mapped to

$$\mathcal{M} = S^{-1}P = \left\{ \frac{a}{s} : a \in P, s \notin P \right\}$$

(i) ($\mathcal{M}$ is an ideal of $S^{-1}A$)

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \notin P \right\}$$

Clearly $S^{-1}P \subset S^{-1}A$ and $S^{-1}P$ is additive since it is a ring. We now wish to show that for any $x \in S^{-1}P$, $S^{-1}Ax \subset S^{-1}P$. Let $x = a/s \in S^{-1}P$ and $y = b/t \in S^{-1}A$. Then

$$yx = \frac{ba}{ts}$$

Since $A \setminus P$ is multiplicatively closed $st \notin P$. P is an ideal and $b \in P$, so $ab \in P$. Thus $yx \in S^{-1}P$

(ii) ($\mathcal{M}$ is maximal and the only maximal ideal)
We show that if $\mathcal{M}'$ is an ideal different from $\mathcal{M}$ not contained in $\mathcal{M}$, then $\mathcal{M}'$ must equal the whole ring. Suppose $\mathcal{M}'$ is an ideal such that $\mathcal{M} \neq \mathcal{M}'$ (this covers the case $\mathcal{M} \subset \mathcal{M}'$). Let $x \in \mathcal{M}' \setminus \mathcal{M}$. Then x is of the form a/s where $a \notin P$ and $s \notin P$. But then a/s is a unit in $S^{-1}A$ (s/a is its inverse). Hence, $1 \in \mathcal{M}'$ which means $\mathcal{M}'$ must equal the whole ring $S^{-1}A$.

■

Even though the image of an ideal is not always an ideal, we can always construct an ideal given a ring homomorphism $\phi : R \to S$:

Definition 2.9 (Extension of an Ideal)**.** Let $\phi : R \to S$ be a ring homomorphism, I an ideal of R. The extension of the ideal I is the ideal generated by the image of I under f:

$$I^e := \{\sum_{\text{finite}} s\phi(r) | s \in S, r \in I\} = (f(I))$$

Definition 2.10 (Contraction of an Ideal)**.** Let $\phi : R \to S$ be a ring homomorphism, J an ideal of S. The contraction of the ideal J is

$$J^c := \phi^{-1}(J)$$

Proposition 2.11. *Every ideal in $S^{-1}A$ is an extension of its contraction.*

Proof. Let $f : A \to S^{-1}A$ be the canonical map and let J be an ideal of $S^{-1}A$. We want to prove that $(J^c)^e = J$.
($\subseteq$) Let $x \in (J^c)^e$. We can write x as

$$x = \sum_{\text{finite}} s_i f(r_i)$$

where $s_i \in S^{-1}A$ and $r_i \in J^c$. Let us look at the image of J^c under f:

$$\begin{aligned} f(J^c) &= \{f(r) | r \in J^c\} \\ &= \{f(r) | r \in f^{-1}(J)\} \\ &= \{f(r) | f(r) \in J\} \end{aligned}$$

which is obviously in J. Since J is an ideal, the finite sum is then in J.
($\supseteq$) Let $x/s \in J$. Then $x/1 \in J$ since $1/s$ is a unit in $S^{-1}A$. And so $x \in J^c$. Then

$$\frac{x}{s} = \frac{1}{s}\frac{x}{1} = \frac{1}{s}f(x)$$

■

Proposition 2.12. *There is a 1-1 correspondence between prime ideals in $S^{-1}A$ and prime ideals in A not intersecting S.*

Proof. ■

Definition 2.13 (Local Ring)**.** A ring A with a unique maximal ideal. E.g. $(A_p, \mathcal{M})$.

Localization in algebraic geometry. X affine variety, so I_X is a prime ideal.

$$(k[t_1, \cdots, t_n])_{I_X} = \left\{ \frac{f}{g} \middle| f \in k[t_1, \cdots, t_n], g \notin I_X (\text{not vanish on } X) \right\}$$

2.1.3 Modules of Fractions

Definition 2.14. M is an A-module, $S \subseteq A$ is multiplicatively closed.

$$S^{-1}M := \left\{ \frac{m}{s} \,\middle|\, m \in M, s \in S \right\}$$

Exercise. $S^{-1}M$ is a module over $S^{-1}A$.

Proof. Module is always defined over a ring. We previously showed $S^{-1}A$ has a ring structure with multiplicative identity, so it is an abelian group under addition. Let us start by defining a scalar multiplication map $\cdot : S^{-1}A \times S^{-1}M \to S^{-1}M$. Let $a/s \in S^{-1}A$ and $m/t \in S^{-1}M$. Then

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{a \cdot m}{st}$$

where $a \cdot m$ is multiplication $A \times M \to M$. Just need to check the axioms. ■

Notation. $M_P := S^{-1}M$, where $S = A \setminus P$ for P prime ideal of A.
Some important propositions of $S^{-1}(\cdot)$:

Proposition 2.15. *Exactness of $S^{-1}(\cdot)$: Suppose we have the following short exact sequence of A-modules*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

then

$$0 \to S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \to 0$$

where $S^{-1}f$ maps w'/s to $f(w)/s$, and $S^{-1}g$ is similar.

Corollary 2.16. $S^{-1}(\ker f) = \ker(S^{-1}f)$, $S^{-1}(\operatorname{Im} f) = \operatorname{Im}(S^{-1}f)$, $S^{-1}(M/N) = S^{-1}M/S^{-1}N$

Proposition 2.17. *Let M be an A-module. The natural map $S^{-1}A \otimes_A M \to S^{-1}M$ where*

$$\frac{a}{s} \otimes_A m \mapsto \frac{am}{s}$$

is an isomorphism of $S^{-1}A$-modules. Restriction of scalars? proof 3.5 AM/

Corollary 2.18. *$S^{-1}A$ is flat! preserves A-mod ses. Localization commutes with tensor products*

$$S^{-1}(M \otimes_A N) \simeq S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$

Definition 2.19 (Local Property)**.** A property $\bar{P}$ is said to be local property of a ring R (or module M) if R_P (or M_P) has $\bar{P}$ for all prime ideal P of R (or M).

Examples: Injectivity, Surjectivity, Flatness.

Proposition 2.20. *Let M be an A-module. The following are equivalent:*

(i) $M = 0$.

(ii) $M_P = 0$ *for all prime ideals P of A.*

(iii) $M_{\mathcal{M}} = 0$ *for all maximal ideals $\mathcal{M}$ of A.*

Proposition 2.21. *Let $\phi : M \to N$ be a module homomorphism. The following are equivalent :*

(i) ϕ *is injective.*

(ii) $\phi_P : M_P \to N_P$ *is injective for all prime ideals P of A.*

(iii) $\phi_{\mathcal{M}} : M_{\mathcal{M}} \to N_{\mathcal{M}}$ *is injective for all maximal ideals $\mathcal{M}$ of A.*

Proposition 2.22. *Let M be an A-module. The following are equivalent:*

(i) M *is a flat A-module.*

(ii) M_P *if a flat M_P-module for all prime ideals P of A.*

(iii) $M_{\mathcal{M}}$ *if a flat $A_{\mathcal{M}}$-module for all maximal ideals $\mathcal{M}$ of A.*

§2.2 Integral Dependences

For the following definitions let A and B be rings with $A \subseteq B$.

Definition 2.23 (Integral Element over a Subring)**.** An element x in B is said to be integral over A if x is the root of a monic polynomial with coefficients in A. So,

$$x^n + a_{n-1}x^{n-1} + \cdots a_0 = 0$$

where $a_i \in A$. If A and B are fields we say x is algebraic over A. Note fields do not require the monic condition because we can always divide by the a_n.

Definition 2.24 (Integral Ring over a Subring / Integral Extension)**.** B is integral over A or B is an integral extension of A if every element of B is integral over A.

Definition 2.25 (Integrally Closed)**.** We say A is integrally closed in B if all integral elements of B over A are in A. $x \in B$ integral $\Rightarrow x \in A$.

Example: $\mathbb{Z} \subset \mathbb{Q}$. Field of fractions check!
Let $f : A \to B$ be a ring homomorphism. Let P be a prime ideal of A.

(i) $f(P)$ is not a prime ideal.

(ii) Extension of a prime is not necessarily prime.

(iii) Not every prime ideal of A is a contraction.

Definition 2.26 (Normal/Int. Closed)**.** If A is an integral domain then A is normal or integrally closed if A is integrally closed in its field of fractions $K(A)$.

§2.3 Exercises

- *Show* $0 \in S$ *if and only if* $S^{-1}A = 0$.
  If $0 \in S$, then all elements in $S^{-1}A$ are equal to each other ($0(at - bs) = 0$ for any $a/b, s/t \in S^{-1}A$). So all elements are equal to zero since it is an element in A.
  If $S^{-1}A = 0$, then $1/1 = 0/1$. Which means that there exists $u \in S$ such that $u(1-0) = 0$. Thus $u = 0 \in S$.

- *Check addition and multiplication in* $S^{-1}A$ *are well-defined.*

-

§3 Week 2, Lecture 1

Proposition 3.1. *If $\mathfrak{b}$ prime ideal of B and we define $\mathfrak{a}$ as the contraction of $\mathfrak{b}$, $\mathfrak{a} := \mathfrak{b}^c$ (which is an ideal of A), then*

$$A/\mathfrak{a} \subseteq B/\mathfrak{b}$$

is integral.

Proposition 3.2 (Stability under Localization)**.** *Let $S \subseteq A$ be multiplicatively closed. There is a natural inclusion $S^{-1}A \subseteq S^{-1}B$ which is integral.*

Proof. We begin by proving the inclusion. Let $a/s \in S^{-1}A$. Then $a \in A \subset B$. Hence, $a/s \in S^{-1}B$.
We now prove the extension is integral. Let $b/s \in S^{-1}B$ ■

Proposition 3.3. *Let $A \subseteq B$ be an integral ring extension and let B be an integral domain. Then,*

$$A \text{ is a field} \iff B \text{ is a field}$$

Proof. ($\Rightarrow$) Suppose A if a field. Since B is an integral domain, it suffices to show that every element of B has a multiplicative inverse. Let $x \in B$, then

$$x^n + a_{n-1}x^{n-1} + \cdots a_0 = 0$$

for some $a_i \in A$. We can rewrite this as

$$x(-x^{n-1} - a_{n-1}x^{n-2} - \cdots - a_1) = a_0$$

a_0 is an element of the field A so it has a multiplicative inverse. Hence,

$$x((-x^{n-1} - a_{n-1}x^{n-2} - \cdots - a_1)a_0^{-1}) = 1$$

x has a multiplicative inverse.
($\Leftarrow$) A is a subring of B which is an integral domain, so A is an integral domain too. It suffices to show every element of A has a multiplicative inverse in A. Let $x \in A$. Then $x \in B$ and so

$$x^n + a_{n-1}x^{n-1} + \cdots a_0 = 0$$

for some $a_i \in A$. ■

Corollary 3.4. *Let $A \subseteq B$ be integral, $\mathfrak{q}$ an ideal of B, and define $\mathfrak{p}$ as the contraction of $\mathfrak{q}$ ($\mathfrak{p} := \mathfrak{q}^c = \mathfrak{q} \cap A$). Then,*

$$\mathfrak{q} \text{ if maximal in } B \iff \mathfrak{p} \text{ is maximal in } A$$

Theorem 3.5 (Going-Up Theorem)**.** *Let $A \subseteq B$ be an integral ring extension. Fix a prime ideal $\mathfrak{p} \subseteq A$. Then $\mathfrak{p}$ is the contraction of some ideal $\mathfrak{q}$ in B. $\mathfrak{p} = \mathfrak{q} \cap A$ for some $\mathfrak{q} \subseteq B$.*

Proof. Consider the commutative map of A-modules: ■

Corollary 3.6. *Let $\bar{\mathfrak{q}}$ be a maximal ideal of $B_{\mathfrak{p}}$. Then $\bar{\mathfrak{q}} \cap A_{\mathfrak{p}}$ is a maximal ideal.*

Theorem 3.7 (Going-Up Theorem 2)**.** *Let $A \subseteq B$ be integral. Assume $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_n$ be a chain of prime ideals in A. $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_n$ $m < n$ s.t. $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. Then there exist $q_{m+1} \subseteq \cdots \subseteq q_n$ s.t $\mathfrak{p}_i = \mathfrak{q}_i \cap A$ for $m_1 \leq i \leq n$.*

Two finiteness conditions for Modules:
Let $A \subseteq B$ be an integral extension of rings. Then,

$$B \text{ is a finitely generated } A\text{-algebra} \Longleftrightarrow B \text{ is a finitely generated } A\text{-module}$$

The left implication ($\Leftarrow$) is always true. The integral extension implies the right arrow ($\Rightarrow$).

Definition 3.8 (Finite type & Finite)**.** Let $A \subseteq B$ be an integral extension of rings.

- We say B is finite type over A if B is a finitely generated A-algebra.
- We say B is finite over A if B is a finitely generated A-module.

So if $A \subseteq B$ is an integral extensions, then

$$B \text{ is finite type over } A \Longleftrightarrow B \text{ is finite over } A$$

Proposition 3.9. *Let $A \subseteq B$. The following are equivalent:*

(i) x is integral over A.

(ii) $A[x]$ is a finitely generated module over A.

Example of why we need finitely generated:

Proof. $(i) \Rightarrow (ii)$ Suppose x is integral over A, then

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

for some $a_i \in A$. We prove $V = \{1, x, \cdots, x^{n-1}\}$ generates $A[x]$ over A. Let p be a polynomial in $A[x]$,

$$p = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0, \quad b_i \in A$$

If $\deg(b_k x^k) < n$ then clearly $b_k x^k \in Span(V)$. Otherwise,

$$b_k x^k = b_k x^{k-n} x^n = b_k x^{k-n}(-a_{n-1}x^{n-1} - \cdots - a_0)$$

We can repeat this process to leave us with polynomial of degree less than n.
$(ii) \Rightarrow (i)$ Consider $\phi_x =$ "multiplication by x" $\in End(A[x])$ and $\phi_x(A[x]) \subseteq A[x]$. $\star$ Lemma $\Rightarrow \phi_x^n + a_{n-1}x^{n-1} + \cdots a_0 = 0$ for $a_i \in A$. Applying this to 1:

$$\phi_x^n(1) + a_{n-1}x^{n-1}(1) + \cdots a_0 = 0$$

$$x^n + a_{n-1}x^{n-1} + \cdots a_0 = 0$$

So x is algebraic over A. ■

Lemma 3.10 (★)**.** *This is a generalization of a linear algebra fact, so let us look at that first.*
Let k be a field, let V be a finite dimensional vector space of size n over k. Note k is a field so its only ideals are 1 and V. Let $\phi \in End(V)$, meaning, ϕ is an $n \times n$ matrix M.
Note that the $n \times n$ matrices form a vector space of size n^2. So $\{I, M, M^2, \cdots, M^{n^2}\}$ is a linearly dependent set. So, there exist nonzero $\lambda_i \in k$ such that

$$\lambda_0 I + \lambda_1 M + \cdots \lambda_{n^2} M^{n^2} = 0$$

We can divide by λ_{n^2} because we are working over a field.
Let M be a finitely generated A-module, let I an ideal of A, and let $\phi \in End(M)$ such that $\phi(M) \subseteq I \cdot M$. Then ϕ satisfies a monic polynomial equation with coefficients in I

$$\phi^n + a^{n-1}\phi^{n-1} + \cdots + a_0 = 0$$

for some $a_i \in I$ and $n := rank(M)$.

Proof. Let $M = \langle x_1, \cdots, x_n \rangle$. $\phi(x_i) = \sum_j a_{ij} x_j$. ∎

Lemma 3.11 (Nakayama's Lemma)**.** *This is a consequence of* ★*.*

- *(1st version) M is a finitely generated A-module, I is an ideal of A. Assume $M = I \cdot M$, then there exists $x \in A$ such that $xM = 0$ and $(x - 1) \in I$.*
- *(2nd version) Further assume A is local with $\mathcal{M}$ being the maximal ideal and $I = \mathcal{M}$, then $M = 0$.*

Proposition 3.12 (Nakayama II)**.** *Let $(A, \mathcal{M})$ be local and let M be a finitely generated A-module. Let $\langle \bar{x}_1, \cdots, \bar{x}_n \rangle = \frac{M}{\mathcal{M}M}$ as $A/\mathcal{M}$ Vector space. Then $M = \langle x_1, \cdots, x_n \rangle$ as A-module where $x_i \in M$ such that*

$$x_i \mapsto \bar{x}_i$$

under $M \mapsto \frac{M}{\mathcal{M}M}$

Proof. ∎

Corollary 3.13. *If $(x_1, \cdots, x_n) \subseteq B$ where each x_i integral over A. $A[x_1, \cdots, x_n]$ if finitely generated A-module.*

Corollary 3.14. *$A \subseteq B$ integral. If B is finitely generated A-algebra, then B is finite over A.*

Finite/Finite Type/Integral Ring Homomorphism: If B is a finitely generated A-algebra, then B is finite over A.

Prime ideals in $\mathbb{A}^2$: irreducible polynomials $\to$ affine curve.

Weak Nullstellensatz k finitely generated k-algebra, $k/\mathcal{M}$ field $\simeq k$.

§3.1 Nullstellensatz

Theorem 3.15 (Weak Nullstellensatz)**.** *Let k be a field, A a finitely generated k-algebra. Let $\mathcal{M}$ be a maximal ideal of A. Then the field $A/\mathcal{M}$ is a finite algebraic extension of k. In particular, if k is closed then $A/\mathcal{M} \simeq k$*

Corollary 3.16. *If k is an algebraically closed field, $I \in k[t_1, \cdots, t_n]$ an ideal, then $V(I) = \emptyset$ if and only if $1 \in \sqrt{I}$.*

This is also know as the weak Nullstellensatz **Maximal ideals of the polynomial ring over algebraically closed field K.** If $\mathcal{M}$ is a maximal ideal of $k[t_1, \cdots, t_n]$, where k is an algebraically closed field, then $\mathcal{M}$ is of the form $\mathcal{M} = \langle t_1 - a_1, \cdots, t_n - a_n \rangle$.

Find a map $k[t_1, \cdots, t_n]/\mathcal{M} \to k$.

Theorem 3.17 (Full Nullstellensatz)**.** *$I_{V(I)} = \sqrt{I}$. I prime $I = \sqrt{I}$. $I_{V(I)} = I = \langle f \rangle$.*

Other immediate consequences:

- Preservation of strict inclusion under $V(\cdot)$ for prime ideals $I \subsetneq J \Rightarrow V(J) \subsetneq V(I)$.
- The maximal ideals are in one to one correspondence with point in the affine space:

$$\text{maximal ideal of } k[t_1, \cdots, t_n]\text{: } \langle t_1 - a_1, \cdots, t_n - a_n \rangle \longleftrightarrow (a_1, \cdots, a_n)$$

Exercise. Follows from weak Nullstellensatz.

Theorem 3.18 (Nullstellensatz)**.**

§4 Week 2, Lecture 2

For the next few lectures we set up the definitions of functions on varieties. In this lecture we cover the following important result:

Given an affine variety X, there exists a 1-1 correspondence between subvarieties of X and prime ideals in the "Ring of Regular Functions on X".

We add some remarks on last lecture:

Definition 4.1 (Noetherian Topological Space)**.** A topological space in which closed subsets satisfy the descending chain condition.

A Noetherian Ring satisfies the ascending chain condition. So why do we have a descending chain condition in the definition of Noetherian Topological Space? Well, on the one hand we have the open sets satisfy the ascending chain condition. On the other hand, if we have an ascending chain of ideals $p_1 \subseteq p_2 \subseteq \cdots$, then

$$V(p_1) \supseteq V(p_2) \supseteq \cdots$$

These are the closed sets.

Remark. The Zariski Topology on $\mathbb{A}_k^n$ is Noetherian. This is because the operations $V(\cdot)$ and $I_{(\cdot)}$ preserve strict inclusions between prime ideal in the polynomial ring and affine varieties and the fact that the polynomial ring is Noetherian. (Lecture 1, Correspondence between ideals and affine varieties). This finiteness notion is very important.

§4.1 Regular Functions and Maps

Regularity can be thought of as continuity. It is a local property and we can lift it up.

Definition 4.2 (Regular Function)**.** Let $X \subseteq \mathbb{A}_k^n$ be an algebraic set. We say a function $f : X \to k$ is **regular at** $x \in X$ if there exists a Zariski open neighborhood $U \subseteq X$ containing x such that

$$f\Big|_U = \frac{g}{h}\Big|_U$$

where $g, h \in k[t_1, \cdots, t_n]$ and $h(y) \neq 0$ for all $y \in U$. If f is regular at all $x \in X$, then f is a **regular** function on X. We denote $\mathscr{O}(X)$ as the set of regular functions on X.

Definition 4.3 (Regular Map/Morphism)**.** Let X and Y be algebraic sets of $\mathbb{A}_k^n$ and $\mathbb{A}_k^m$ and suppose $f : X \to Y$. The function f is regular if $f = (f_1, \cdots, f_m)$ where $f_i \in \mathscr{O}(X)$.

Definition 4.4 (Isomorphism)**.** A regular map $\varphi : X \to Y$ is called an isomorphism if φ is regular and with a regular inverse.

Example. $\mathscr{O}(\mathbb{A}^n) = k[t_1, t_2, t_3]$.
For affine varieties the notion coincides with polynomials.
Example. If $f : \mathbb{A}^3 \to \mathbb{A}^2$ is of the form $f = (p_1, p_2)$ where $p_i \in k[t_1, t_2, t_3]$.
Example. The projetion map is regular. $f = (t_1, t_2)$

Proposition 4.5. *$k[t_1, \cdots, t_n]$ naturally endows $\mathscr{O}(X)$ with a ring structure.*

Remark. $X = V(xy)$ $x, y \in \mathscr{O}(X)$. $x \cdot y = 0$. Then $\mathscr{O}(X)$ is not an integral domain. So the fraction fields of $\mathscr{O}(X)$ do not make sense. If X is irreducible then $\mathscr{O}(X)$ is an integral domain. We will not form field of fraction so rational functions are restricted to irreducible objects.

The field of fractions of an integral domain is the smallest field in which it can be embedded. Denoted $\mathrm{Frac}(R)$ or $\mathrm{Quot}(R)$. For a commutative ring that is not an integral domain, the analogous construction is called the localization or ring of quotients.

§4.2 Function Fields

Definition 4.6 (Rational Functions for Quasi-affine Varieties)**.** We define

$$K(X) := \{\langle f, U\rangle \mid U \subseteq X \text{ open, } f \in \mathscr{O}(X)\}/\sim$$

where $\langle f, U\rangle \sim \langle g, V\rangle$ if and only if $f|_{U\cap V} = g|_{U\cap V}$.

Proposition 4.7. *$K(X)$ is a field.*

Definition 4.8 (Rational Map)**.** A rational map is a function $f = (f_1, \cdots, f_m) : X \to Y \subseteq \mathbb{A}_k^m$ where each $f_i \in K(X)$.

Recall that if X is an irreducible quasi-affine variety, then for every subvariety Y of X you can find a prime ideal P_Y in $\mathscr{O}(X)$.

Definition 4.9. notation for forming the ring of fractions at the prime ideal corresponding to Y.

$$\mathscr{O}_{X,Y} := (\mathscr{O}(X))_{P_Y}$$

Specific situation we are interested in : When the subvariety Y is a single point $\{x\}$. then $\mathscr{O}_{X,x} := (\mathscr{O}(X))_{P_x}$ This is localization along P_x.

The natural lozalization map $\mathscr{O}(X) \to \mathscr{O}_{X,Y}$ is injective. ie. $\mathscr{O}_{x,y}$ contains more than just regular functions. What are we doing? We have enlarged the ring of regular functions. WE are including those functions that are only regular near Y. Example: $Y = \{x\}$ the the ring contains rational functions that are regular at x.

Lemma 4.10. *The ratio of two regular functions is rational.*

$$f, g \in \mathscr{O}(X) \Rightarrow \frac{f}{g} \in K(X)$$

Proof. Let $X \subseteq \mathbb{A}^n$ quasi-affine variety and let $f, g \in \mathscr{O}(X)$. Then there exist open subsets $U, V \subseteq X$ such that

$$f\Big|_U = \frac{F}{F'}\Big|_U, \quad F, F' \in k[t_1, \cdots, t_n]$$
$$g\Big|_V = \frac{G}{G'}\Big|_V, \quad G, G' \in k[t_1, \cdots, t_n]$$

Taking the ratio of the two functions and restricting to $U \cap V$:

$$\frac{f}{g}\Big|_{U\cap V} = \frac{FG'}{F'G}\Big|_{U\cap V}$$

We almost have what we want. We need to ensure the denominator is nonzero over an open set of choice. Set $W = U \cap V \cap (X \setminus V(G))$ where $V(G)$ is the vanishing set of the polynomial G. F' has no zeroes in U, and it therefore has no zeroes in $U \cap V$. So $F'G$ has no zeroes in W and

$$\frac{f}{g}\bigg|_W = \frac{FG'}{F'G}\bigg|_W$$
$$\Rightarrow f/g|_W \in \mathscr{O}(W)$$
$$\langle f/g, W\rangle \in K(X)$$

■

Corollary 4.11. *If $f, g \in \mathscr{O}(X)$, then $f/g \in \mathscr{O}(X \setminus V(g))$.*

There is a natural map $\mathscr{O}_{X,Y} \to K(X)$.

(i) (Well-defined) If $f_1/g_1 = f_2/g_2 \in (\mathscr{O}(X))_{P_Y}$ $f_i, g_i \in \mathscr{O}(X)$. g_1, g_2 are not identically 0 over Y. $U_1 = X \setminus V(g_1), U_2 = X \setminus V(g_2)$

$$f_1/g_1 \in \mathscr{O}(X \setminus V(g_1)) \Rightarrow (f_1/g_1, U_1) \in K(X)$$

$$f_2/g_2 \in \mathscr{O}(X \setminus V(g_2)) \Rightarrow (f_2/g_2, U_2) \in K(X)$$

$f_1/g_1 = f_2/g_2$ means $f_1 g_2 = f_2 g_1$

$$\Rightarrow \frac{1}{g_1} \in \mathscr{O}(U_1), \frac{1}{g_2} \in K(X)$$

$$\Rightarrow \frac{f_1}{g_1}\bigg|_{U_1 \cap U_2} = \frac{f_2}{g_2}\bigg|_{U_1 \cap U_2}$$

$$\Rightarrow \langle \frac{f_1}{g_1}, U_1\rangle = \langle \frac{f_2}{g_2}, U_2\rangle$$

(ii) (Injective)

Proposition 4.12. *If X is a quasi-affine variety, Frac$(\mathscr{O}(X)) \simeq K(X)$.*

§4.3 Field of Regular Functions on Affine Varieties.

We prove the result: Regular functions on affine varieties are polynomials.

Alternative description for $\mathscr{O}(X)$ in the case that X is an affine variety.

Definition 4.13 (Coordinate Ring)**.** I_X is an ideal of $k[t_1, \cdots, t_n]$,

$$k[X] := \frac{k[t_1, \cdots, t_n]}{I_X}$$

Note!

$$X \text{ irreducible } \iff I_X \text{ prime } \iff k[X] \text{ is an integral domain}$$

Proposition 4.14. *If X is an affine variety, then $k[X] \simeq \mathscr{O}(X)$*

Proof. There exists a 1-1 correspondence between:

$$\text{Maximal ideals in } k[t_1, \cdots t_n] \text{ containing } I_X \longleftrightarrow \text{Maximal ideals in } k[X]$$

■

Definition 4.15 (Pullback Map)**.** For any regular map $\phi : X \to Y$ of quasi-affine algebraic sets X and Y, there is a natural induce k-algebra homomorphism $\phi^* : \mathscr{O}(Y) \to \mathscr{O}(X)$ defined by $\phi^*(f) = f \circ \phi : X \to k$.

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, dashed] & Y \arrow[d, "f"] \\ & k \end{tikzcd}$$

To check this all is needed is to prove the composition of regular maps is regular.

4.3.1 Local Rings

Aim: New definition of local ring and show it is the same as the original definition.

Definition 4.16 (Local Ring)**.** X is quasi-affine and Y is a subvariety of X.

$$\mathscr{O}_{X,Y} := \varinjlim_{\substack{U \supseteq Y \\ \text{open}}} \mathscr{O}(U) = \bigsqcup_{U \supseteq Y \text{open}} \mathscr{O}(U)/ \sim = \{\langle f, U\rangle | f \in \mathscr{O}(U), Y \subseteq U\}/ \sim$$

$f \sim g$ where $f \in \mathscr{O}(U)$ and $g \in \mathscr{O}(V)$ then $f|_{U \cap V} = g|_{U \cap V}$.

Check: Local Rings are Local Rings (have a unique maximal ideal).

Definition of local ring: it has a unique maximal ideal, The local ring at a point is a local ring. $\mathscr{O}_{X,x}$.

$$\mathcal{M}_{X,x} = \{\langle f, U\rangle : f(x) = 0\}$$

Proposition 4.17. *If X is an affine variety, then $(k[X])_{\mathcal{M}_x} \simeq \mathscr{O}_{X,x}$.*

Proof.

$$(k[X])_{\mathcal{M}_x} = \{\frac{f + I_X}{g + I_x} | g + I_x \notin \mathcal{M}_x\} \to^{\beta} \mathscr{O}(x)$$

$$\beta\left(\frac{f + I_x}{g + I_x}\right) = \langle\frac{\alpha(f)}{\alpha(g)}, X \setminus V(g)\rangle$$

1. Check it is well-defined.

2. Injective.

$$\beta(\frac{f}{g}) = \langle 0, U\rangle = \langle\frac{f}{g}, V\rangle$$

$$0 = \frac{f}{g} \in \mathscr{O}(U \cap V)$$

So f/g is zero and continuous on $U \cap V$ and open set so it is zero $\alpha(f) = 0$. α is injective so $f = 0$.

3. Surjective by definition of regular function. ■

Theorem 4.18. *X affine variety.*

$$k[X] \simeq \mathscr{O}(X)$$

Proof. $k[X]$ sits inside of $\mathscr{O}(X)$ which sits inside $\mathscr{O}_{X,x}$ for all $x \in X$. which in turn live inside $K(X)$. So it is in the intersection $\bigcap_{x\in X} \mathscr{O}_{X,x}$ when considered as subsets of $K(X)$. $\simeq \bigcap(k[X])_{\mathcal{M}_x}$ in the fraction field of $k[X]$. Fact frm algbera $\simeq k[X]$. ■

Summary of Lecture:

- For affine variety X we have $k[X] \simeq \mathscr{O}(X)$.
- $\mathscr{O}_{X,x} \simeq (\mathscr{O}(X))_{\mathcal{M}_x} \simeq k[X]_{\mathcal{M}_x}$ the first isomorphism always hold for X quasi-affine, the second one if Q affine.
- The fraction field: $K(\mathscr{O}_{X,x}) \simeq K((\mathscr{O}(X))_{\mathcal{M}_x}) \simeq K(k[X]_{\mathcal{M}_x})$.

§4.4 Exercises

- *Show $\mathscr{O}(\mathbb{A}^n) = k[t_1, t_2, t_3]$.*
- *Prove $K(X)$ is a field.*
- *$k[t_1, \cdots, t_n]$ naturally endows $\mathscr{O}(X)$ with a ring structure.*
- *Check the relation defined on the field of Rational functions is an equivalence relation*

$$K(X) := \{\langle f, U\rangle \mid U \subseteq X \text{ open, } f \in \mathscr{O}(X)\}/\sim$$

  where $\langle f, U\rangle \sim \langle g, V\rangle$ if and only if $f|_{U\cap V} = g|_{U\cap V}$.
- Recall X be a quasi-affine. An irreducible closed subset of X, $Y \subset X$, is called a subvariety of X. If X is irreducible, for every such Y you can find a prime ideal P_Y in $\mathscr{O}(X)$. Check this is true.
- *Prove the composition of regular functions is regular.*
- *Prove Proposition 4.9*
- *Proposition: f regular $\Rightarrow$ f continuous in Zariski topology.*
  Let $f : X \to Y$ be a regular map. It is sufficient to check for functions of the form $f : X \to k$. We want to check $f^{-1}(Z)$ where Z is closed is closed. $k = \mathbb{A}^1_k$. Let's say $k = \overline{k}$. Closed in $\mathbb{A}^1_k$ = finite collection of points. So it's enough to show for a single point $f^{-1}(a)$ is closed. Fact: $X = \bigcup_\alpha U_\alpha$, Z closed in X iff $Z \cap U_\alpha$ open cover closed in U_α for all α. Pick a nbhd U fo some $x \in f^{-1}(a)$, now,

$$f(x) = a = \frac{g(x)}{h(x)}$$

$$g(x) - ah(x) = 0$$

$$(g - ah)x = 0$$

  So

$$\begin{aligned} f^{-1}(a) \cap U &= \{x : (g - ah)(x) = 0\} \cap U \\ &= V(g - ah) \cap U \end{aligned}$$

  So it is closed.
- Show $k[X]$ always sits inside $\mathscr{O}(X)$. Use first isomorphism theorem with identity map.
- Now for prof in other direction when X is an affine variety. Subvarieities of X are 1-1 correspondence with Prime ideals containing I_X. which are 1-1 with prime ideals in $k[A]/I_X$ which are prime ideals in the coordinate ring. We also know that points in X are in 1-1 with max ideals in $k[X]$.
- We have that for $U \subseteq X$ open $K(U) = K(X)$ function fields. $K(U) \subseteq K(X)$ is immediate. If $\langle f, V\rangle \in K(X)$ then $\langle f, V\rangle \sim \langle f, V \cap U\rangle \in K(U)$.

§5 Week 3, Lecture 1

Recap of last lecture:

1. Let X be an affine variety and $X^0 \subseteq X$ a quasi-affine variety. The definition of field of rational functions does not require closeness so we can have both $K(X)$ and $K(X^0)$. What's more is $K(X^0) = K(X)$. This is due to the fact that open sets are dense in the Zariski topology.

   Additionally, when X is an affine variety $K(X) \simeq Frac(k[X]) \simeq Frac(\mathscr{O}(X))$. For quasi-affine varieties, rational functions are simply quotients of polynomials.

2. Morphisms of quasi-affine algebraic sets $f : X \to Y$ induces a k-algebra homomorphism $f^* : \mathscr{O}(Y) \to \mathscr{O}(X)$. This is the pullback map defined as $f^*(g) = g \circ f \in \mathscr{O}(X)$.

Today's focus: we want to analyze properties of this k-algebra homomorphisms.
Properties of the Pullback map:

(i) **Condition on which f^* is injective:**

$$f^* \text{ is injective} \iff \overline{f(X)} = Y \text{ (the closure of } f \text{ is dominant)}$$

Proof:
($\Leftarrow$) Assume $\overline{f(X)} = Y$. Note $f : X \to Y$, in this example $g \in \mathscr{O}(Y)$ ($g : Y \to k$). Suppose

$$\begin{aligned}
f^*(g) = 0 &\Leftrightarrow g \circ f = 0 \in \mathscr{O}(X) && \text{by definition of } f^* \\
&\Leftrightarrow g \in I_{f(X)} && \text{since } I_{f(X)} \text{ is the ideal that vanishes on } f(X) \\
&\Leftrightarrow (g) \subseteq I_{f(X)} && \text{the ideal generated by } g \text{ is contained in } I_{f(X)} \\
&\Leftrightarrow V(I_{f(X)}) \subseteq V((g)) && \text{property of vanishing sets} \\
&\Leftrightarrow V(I_{f(X)}) \subseteq V(g) && \text{Vanishing set of an ideal = vanishing set of its generators} \\
&\Leftrightarrow \overline{f(X)} \subseteq V(g) && \text{by the following property } T \subseteq \mathbb{A}_k^n\ \overline{T} = V(I_T) \\
&\Leftrightarrow Y \subseteq V(g) && \text{by the assumption } \overline{f(X)} = Y \\
&\Leftrightarrow g = 0
\end{aligned}$$

($\Rightarrow$) Assignment 1.

(ii) **Condition on which f^* is surjective:**

$$f^* \text{ onto} \Rightarrow f \text{ is closed (the image of } f \text{ is a closed subset of } Y)$$

Proof in Assignment 1.

Proposition 5.1. *(i) Let K_X and K_Y be finitely generated k-algebras. For every k-algebra homomorphism $\phi : K_Y \to K_X$, there are affine algebraic sets X and Y and a unique regular map $f : X \to Y$ such that $\phi = f^*$.*

(ii) ϕ is an isomorphism if and only if f is an isomorphism.

(iii) If K_Y and K_X are integral domains, then X and Y in (i) are affine varieties.

Proof. (i)

(ii)

(iii)

■

§5.1 Equivalence of Algebra and Geometry

There is an equivalence of categories:

$$\text{Affine Varieties} \longleftrightarrow \text{Integral domains, finitely generated } k\text{-algebras}$$
$$\text{Morphisms of affine varieties } (X \to Y) \longleftrightarrow k\text{-algebra homomorphisms } (k[X] \to k[Y])$$

Pullback map ($\to$) is injective. The correspondence is up to isomorphism.

§5.2 Category Theory Review

§6 Week 3, Lecture 2

- There exists an equivalence of categories up to isomorphism between

  Affine Varieties & Morphisms $\leftrightarrow$ Integral Domain, finitely generated k-algebras and k-alg homs.

  One way of the directions is given by the pullback map. The other direction is through $K_Y = \frac{k[t_1, \cdots, t_n]}{I_Y}$ where I_Y is prime. Then $Y := V(I_Y) \subseteq \mathbb{A}^n$ and $K_Y = k[V(I_Y)] = k[Y]$. $f : X \to Y$ and $f^* = \phi$.

- Rational functions: $f \in K(X)$, where X is a quasi-affine variety. Then $f = \langle \phi_U, U \rangle$, where $\phi_U \in \mathscr{O}(U)$ germs of regular functions.

- The quasi-affine variety X always sits in an affine variety: $X \subseteq \overline{X}$ and the function field of X, $k(X)$, is equal to the function field of $\overline{X}$, $k(\overline{X})$, which equals $Frac(k[\overline{X}])$.

- We have a notion of **domain of** f, Dom(f):

$$f|_{\text{Dom}(f)} \in \mathscr{O}(\text{Dom}(f))$$

§6.1 Equivalence of Categories Contravariant Functor.

Definition 6.1 (Domain of a Rational Map)**.** The domain of a rational map is the largest open set for which the function is defined on.

Proposition 6.2. *The domain of a rational map on a quasi-affine variety (irreducible) is open and irreducible.*

Proof. Consider the rational map $\phi = (\phi_1, \cdots, \phi_n) : X \dashrightarrow \mathbb{A}^n$, where $\phi_i \in k(X)$. Since each $\phi_i : X \dashrightarrow k$ is a rational map we have the domain of each ϕ_i is open and irreducible (any open subset of an irreducible space is irreducible and dense). The domain of the function ϕ is defined on the intersection of the domains of the ϕ_is. So

$$\text{Dom}(\phi) = \bigcap_i \text{Dom}(\phi_i) \subseteq X$$

is open and irreducible (the intersection of irreducible sets is irreducible). ■

Definition 6.3 (Dominant Rational Map)**.** A rational map $\phi : X \dashrightarrow Y$ is dominant if its image is dense in the codomain. So,

$$\overline{\phi(\text{Dom}(f))} = Y.$$

Proposition 6.4. *If $\phi : X \dashrightarrow Y$ is a dominant rational map, then $\phi^* : k(Y) \to k(X)$ is well-defined.*

Take $g \in K(Y)$, $g : Y \to k$, then g is regular over open $V \subseteq Y$. Let U be an open subspace of the domain of ϕ. Then $\phi(U)$ is dense in Y. Therefore, we have

$$\phi(U) \cap V \neq \emptyset.$$

We have $\phi : U \to Y$ is regular, and therefore continuous, so $\phi^{-1}|_U(V)$ is open an nonempty. By construction we have

$$\phi^* g \in \mathscr{O}(\phi^{-1}|_U(V))$$

and so

$$\langle \phi^* g, \phi^{-1}|_U(V)\rangle \in K(X)$$

Now we have the pullback map well-defined for rational functions. Previously we had defined it for regular functions. Check it is a k-alg homomorphism.

Theorem 6.5. *There exists a contravariant functor between*

$$\textit{Quasi-affine Varieties} \leftrightarrow \textit{Finitely generated extension over } k$$

$$\textit{Dominant Rational Maps } \phi : X \to Y \leftrightarrow k\textit{-algebra homomorphisms } \phi^* : k(Y) \to k(X)$$

which induces an equivalence of categories.

What is a finitely generated field extension? It means it has a finite transcendence basis.

Proof.

1. Show every f.g. field extension K corresponds to a function field $k(X)$ for some affine variety X. Let $\{x_1, \cdots, x_n\}$ be a transcendence basis.
2. Show every k-algebra homomorphism $\phi : k(Y) \to k(X)$ there exists a dominant rational map $\phi : X \to Y$ such that $\phi^* = \varphi$

■

Definition 6.6 (Birational)**.** Quasi-affine varieties X and Y are birational if there exists a rational map $\phi : X \dashrightarrow Y$ with rational inverse.

Corollary 6.7. *For X, Y quasi-affine varieties the following are equivalent:*

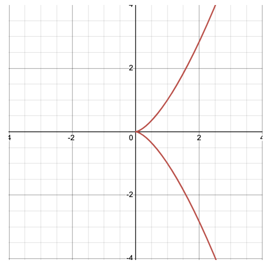
(i) X and Y are birational.

(ii) There exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $U \simeq V$ via ϕ.

(iii) $k(X) \simeq k(Y)$.

Proof. $(i) \Rightarrow (ii)$: X and Y are quasi-affine and therefore open. The result is immediate.
$(ii) \Rightarrow (iii)$: immediate from result of previous lectures $U \simeq V$ implies $k(U) \simeq k(V)$. And thus, $k(X) \simeq k(Y)$.
$(iii) \Rightarrow (i)$: Follows from construction in the previous theorem. ■

Example. $X = V(y^2 - x^3) \subseteq \mathbb{A}^2$, $k[t_1, t_2]$.



Parametrize by $X = \{(t^2, t^3) : t \in \mathbb{R}\}$ and define $\phi : X \dashrightarrow k$ by

$$\phi(x, y) := \frac{y}{x}$$

So $\phi(t^2, t^3) = t$. It is birational $t \mapsto (t^2, t^3)$. So X is birational to $\mathbb{A}^1$. $\operatorname{Frac}(k[x, y]/(y^2 - x^3)) \simeq k(t)$. Smoothness and singularities are invariant under isomorphism.

§6.2 Projective Space

In mathematics, the concept of a projective space originated from the visual effect of perspective, where parallel lines seem to meet at infinity. A projective space may thus be viewed as the extension of a Euclidean space, or, more generally, an affine space with points at infinity, in such a way that there is one point at infinity of each direction of parallel lines.

Using linear algebra, a projective space of dimension n is defined as the set of the vector lines (that is, vector subspaces of dimension one) in a vector space V of dimension n + 1. Equivalently, it is the quotient set of V $\setminus\{0\}$ by the equivalence relation "being on the same vector line". As a vector line intersects the unit sphere of V in two antipodal points, projective spaces can be equivalently defined as spheres in which antipodal points are identified.

Projective spaces are widely used in geometry, as allowing simpler statements and simpler proofs. For example, in affine geometry, two distinct lines in a plane intersect in at most one point, while, in projective geometry, they intersect in exactly one point. Also, there is only one class of conic sections, which can be distinguished only by their intersections with the line at infinity: two intersection points for hyperbolas; one for the parabola, which is tangent to the line at infinity; and no real intersection point of ellipses.

So, a projective space of dimension n can be defined as the set of vector lines (vector subspaces of dimension one) in a vector space of dimension n + 1. A projective space can also be defined as the elements of any set that is in natural correspondence with this set of vector lines.

This set can be the set of equivalence classes under the equivalence relation between vectors defined by "one vector is the product of the other by a nonzero scalar". In other words, this amounts to defining a projective space as the set of vector lines in which the zero vector has been removed.

Definition 6.8 (Projective Space)**.** "The space of lines". It is the set of equivalence classes

$$P^n := (\mathbb{A}^{n+1} \setminus \{0\})/\sim$$

where $(x_0, \cdots, x_n) \sim (y_0, \cdots, y_n)$ if and only if $(y_0, \cdots, y_n) = a(x_0, \cdots, x_n)$ for some $a \in k \setminus \{0\}$. The equivalence class of $(x_0, \cdots, x_n)$ is represented by $(x_0 : \cdots : x_n)$ and is called the set of homogeneous coordinates for $P = (x_0, \cdots, x_n)$.

Observation: For every $i \in [n]$, there is a equivalence class $U_i = \{(x_0 : \cdots : x_n) | x_i \neq 0\} \subseteq \mathbb{P}^n$. Define $\phi_i : U_i \to \mathbb{A}^n$ via

$$(x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right)$$

There is also an inverse ϕ_i^{-1}:

$$(y_1, \cdots, y_n) \mapsto (y_1 : \cdots : y_{i-1} : 1 : y_{i+1} : \cdots : y_n)$$

So ϕ_i is a bijection.

Aim: Endow $\mathbb{P}^n_k$ with a Zariski Topology.

§6.3 Quasi-Projective Varieties

Definition 6.9 (Homogeneous Polynomial)**.** $F \in k[t_0, \cdots, t_n]$ is called homogeneous if $F = \sum_{i=1}^r F_i$ for some $r \in \mathbb{N}$, where the F_is are monomials of equal degree.

Example. $x^5 + 2x^3y^2 + 9xy^4$ is a homogeneous polynomial of degree 5. The polynomial $x^3 + 3x^2y + z^7$ is not homogeneous.

Observation: Homogeneous polynomials do not define functions on $\mathbb{P}^n_k$. But $V(F)$ is a well-defined subset of $\mathbb{P}^n_k$ since

$$F(a_0, \cdots a_n) = F(\lambda(a_0, \cdots, a_n) = \lambda F(a_0, \cdots a_n) = 0$$

for F homogeneous.

Definition 6.10 (Homogeneous Ideal)**.** An ideal I of $k[t_1, \cdots, t_n]$ is called homogeneous if it is generated by homogeneous polynomials.

Facts from commutative algebra:

- If I, J are homogeneous ideals, then IJ, $I + J$, $\sqrt{I}$ are homogeneous ideals.
- If I is a homogeneous ideal, the I is prime if for f, g homogeneous polynomials with $fg \in I \Rightarrow f \in I$ or $g \in I$.

Definition 6.11 (Algebraic subset of $\mathbb{P}^n_k$/ Projective algebraic sets)**.** $V(I)$ for some homogeneous I.

Each open or closed subset of $\mathbb{P}^n_k$ inherits a topology as a subspace topology.

Proposition 6.12. *Each $\phi_i : U_i \to \mathbb{A}^n$ is a homeomorphism with respect to the Zariski Topology.*

Proof. Show ϕ_0 and ϕ_0^{-1} preserve closed subsets.
Take closed subset $X \subseteq U_0$. Then $X = V(I)$, where $I = \langle F_1, \cdots, F_r \rangle$ and the F_is are homogeneous in $k[x_0, \cdots x_n]$. Define ■

§6.4 Exercises

- Prove:
  - If I, J are homogeneous ideals, then IJ, $I+J$, $\sqrt{I}$ are homogeneous ideals.
  - If I is a homogeneous ideal, the I is prime if for f, g homogeneous polynomials with $fg \in I \Rightarrow f \in I$ or $g \in I$.

§7 Week 4, Lecture 1

Theorem 7.1 ($\star$)**.** *If X is projective, Y quasi-projective, $f : X \to Y$ regular, then f is closed (the image of f is closed).*

Theorem 7.2 ($\star\star$)**.** *X projective variety in $\mathbb{P}^n_k$, $\mathscr{O}(X) \simeq k$. So functions are of the form $f : X \to k$ or you can think of it as $f : X \to \mathbb{A}^1_k$.*

§8 Week 4, Lecture 2

Last lecture: we use the Segre embedding to endow product spaces with a topology. We also covered introduced the following important theorems:

- ⋆ If X is projective, Y quasi-projective, $f : X \to Y$ regular, then f is closed (the image of f is closed).
- ⋆⋆ X projective variety in $\mathbb{P}^n_k$, $\mathscr{O}(X) \simeq k$. So functions are of the form $f : X \to k$ or you can think of it as $f : X \to \mathbb{A}^1_k$.

Today: we want a more practical definition of closed space in the product space than the one defined by the Segre embedding. We would like a definition to be of the form of a vanishing of polynomials in the product space. We will see that it can be defined as the vanishing of bihomogeneous polynomials. We will also prove the two theorems introduced last lecture and introduce blow ups.

Definition 8.1 (Bihomogeneous polynomials)**.** Consider the polynomial ring $k[u_0, \cdots, u_n, v_0, \cdots, v_m]$, where the u_is are the homogeneous coordinates of $\mathbb{P}^n$ and the v_is of $\mathbb{P}^m$. We say f in this ring is bihomogeneous in $\{u_i\}, \{v_i\}$ if each monomial of f has the same degree in u_i and v_j. We write $f(u_0, \cdots, u_n, v_0, \cdots, v_m)$.

Example. $u_0^2 v_1 - v_2 u_0 u_1$ is bihomogeneous.

Remark. We can turn any bihomogeneous polynomial f in $k(u_0, \cdots, u_n, v_0, \cdots, v_m)$ into a bihomogeneous polynomial of equal degree in both u_i and v_j: suppose $\deg_{u_i} f = r$ and $\deg_{v_j} f = s$ with $r > s$. If $v_j \neq 0$, then $v_j^{r-s} f$ is bihomogeneous of equal degree in both u_i and v_j.

Example. Take $f = u_0 v_2^2 + u_1 v_1^2$, so $\deg_{u_i} f = 1$ and $\deg_{v_j} f = 2$. Then,

$$u_i^{2-1} f = (u_i u_0) v_2^2 + (u_i u_1) v_1^2$$

is bihomogeneous and of equal degree.

Proposition 8.2. *The vanishing of any bihomogeneous polynomial is closed.*

Proof. Whenever $v_i \neq 0$, $V(v_i^{r-s} f) = V(f)$. If x vanishes on f, then it clearly vanishes on $v_i^{r-s} f$, so $V(f) \subseteq V(v_i^{r-s} f)$. If $x = (x_0, \cdots, x_n, \bar{x}_0, \cdots, \bar{x}_m)$ vanishes on $v_i^{r-s} f$, then $\bar{x}_i f(x) = 0$, so $f(x) = 0$. Define the open subset V_i of the affine space to be the set of points where $v_i \neq 0$. By what we have just shown,

$$V(f)|_{V_i} = V(v_i^{r-s} f)_{V_i}$$

For each $i \in [m]$, define $T_i := V(v_i^{r-s} f)_{V_i}$. So that

$$V(f) = \bigcup_i T_i$$

We claim the right hand-side is closed in W (the image under the Segre embedding). This is something to check! ■

§8.1 Closed Subsets of W

Let X be closed in $\mathbb{P}^n$ and Y be closed in $\mathbb{P}^m$. Is the image of $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ closed in W?

We simplify the situation. Suppose X is the vanishing of a homogeneous polynomial in u_is, so $X = V(G(u_0, \cdots, u_n))$. Can also think of it as bihomogeneous in $(u_0, \cdots, u_n, v_0, \cdots, v_n)$. When X is restricted to V_i:

$$X|_{V_i} = V(G'(u_i v_j)_{i,j})$$

where $G'(u_i v_j)_{i,j}$ is homogeneous. Similarly,

$$Y|_{W_k} = V(H'(u_i v_j)_{i,j})$$

where $H'(u_i v_j)_{i,j}$ is homogeneous too. Thus,

$$X \times Y|_{V_i \cap W_k} = \{(u_0 : \cdots : u_n : v_0 : \cdots : v_n) | G'(u_0 : \cdots : u_n : v_0 : \cdots : v_n) = 0, H'(u_0 : \cdots : u_n : v_0 : \cdots : v_n) = 0\}$$

which is closed. By the same argument as before, under the Segre embedding $X \times Y$ is closed in W.

From now on, we want a definition of closeness independent of the Segre embedding.

Definition 8.3 (Closed subspace of $\mathbb{P}^n \times \mathbb{P}^m$)**.** We say $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed if Z is the zero of a system of bihomogeneous polynomials in $(u_0 : \cdots : u_n : v_0 : \cdots : v_n)$.

Remark: In $\mathbb{A}^n \times \mathbb{P}^m$ we endow a topology induced by the one on $\mathbb{P}^n \times \mathbb{P}^m$. So $Z \subset \mathbb{A}^n \times \mathbb{P}^m$ is closed if it the zero of a system of polynomials $f(x_1, \cdots, x_n, y_0, \cdots, y_m)$ that is homogeneous in y_i.

Claim: $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a topological space with subspace topology.

Recall this is the theorem we wanted to prove last lecture:

Definition 8.4 (Graph of a Regular Map)**.** Given a regular map $f : X \to Y$, we call $\Gamma_f = \{(x, y) \in X \times Y, y = f(x)\}$ the graph of f. Graph can be defined for any X and Y, we do not require X to be projective.

Definition 8.5 (Diagonal Graph)**.** The graph of the identity map. $\Gamma_i = \{(x, x) \in X \times X\}$.

Proposition 8.6. *The graph of a regular map f, $\Gamma_f \subset X \times Y$ is closed.*

Proof. Since $f : X \to Y \subseteq \mathbb{P}^n$, it suffices to show $\Gamma_f \subseteq X \times \mathbb{P}^n$ is closed. Let $i : \mathbb{P}^n \to \mathbb{P}^n$ be the identity map. Consider $(f, i) : X \times \mathbb{P}^n \to Y \times \mathbb{P}^n \subseteq \mathbb{P}^m \times \mathbb{P}^n$ that maps $(x, y) \mapsto (f(x), y)$. Note, that Γ_f (what we are looking for) is precisely the preimage of the diagonal under the map we have defined:

$$(f, i)^{-1}(\Gamma_i) = \Gamma_f$$

This is because $(f, i)^{-1}(\Gamma_i) = f, i)^{-1}(\Gamma_i \cap (f(x) \times \mathbb{P}^n) = (f, i)^{-1}((f(x), f(x))) = (x, f(x)) = \Gamma_f$. Because (f, i) is regular (the cross product of regular maps is regular, check), and therefore continuous, it suffices to show:

Claim: $\Gamma_i \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is closed. Check! ∎

Proposition 8.7. *X projective, Y quasi-projective variety, then $pr_2 : X \times Y \to Y$ is closed. Shafarevich p 58-59.*

§8.2 Proofs of Theorems $\star$ and $\star\star$.

Theorem $\star$: If X is projective, Y quasi-projective, $f : X \to Y$ regular, then f is closed (the image of f is closed).

Proof. By 8.6, Γ_f is closed in $X \times Y$, and by 8.7 pr_2 is a closed map. Consider $pr_2(\Gamma_f)$, which we know by these two propositions is closed:

$$\begin{aligned} pr_2(\Gamma_f) &= pr_2((x, f(x)) \\ &= f(x) \end{aligned}$$

So the image of the graph of f under the second projection is the image of X under f. Hence, f is closed. ■

Corollary 8.8. *Every regular map $f : X \to Y$, X projective variety, Y quasi-affine, f is constant.*

Proof. By the theorem $\star$, the image of any regular map is closed, so it can either be a finite set of points or the whole affine line. But is cannot be more than one point because X is irreducible. So it is either a single point or the affine line. Now can the image be the whole affine line? No , because $f : X \to \mathbb{A}^1_k$, there is an embedding of $\mathbb{A}^1_k$ in $\mathbb{P}^1_k$ and $\mathbb{A}^1_k$ is open os $\mathbb{P}^1_k$ so the composition of f with the injection is regular and its image is open. Contradiction. So the image of f can only be single point. ■

Corollary 8.9. *Let $X \subseteq \mathbb{P}^n_k$ (not a point) be a projective variety, F a non-constant homogeneous polynomial in $k[t_1, \cdots, t_n]$, then*

$$V(F) \cap X \neq 0$$

This is cool! This tells us that in projective space any two projective curves intersect. Any two projective planes intersect, any two projective lines intersect. This is not the case in the affine space, given two polynomials we have no information on whether they intersect. There is no connection between the algebra and geometry. This is beginning of intersection theory in projective spaces. Bezout's theorem, tells us precisely the number of intersections later on. To do so, we first need to formally define other definitions.

Proof. Suppose $V(F) \cap X = 0$. Pick two distinct points $x_1, x_2 \in X$ and a homogeneous polynomial G with the same degree as F such that $G(x_1) = 0$ and $G(x_2) \neq 0$ (check we can find such G).

Now define the regular map $\psi := (F : G) : X \to \mathbb{P}^1$ (meaning ψ maps $x \in X \mapsto (F(x) : G(x))$). By theorem $\star$ we know $\psi(X)$ is closed since X is irreducible. So $\psi(X)$ is $\mathbb{P}^1$ or a point. We had assumed the vanshing of F did not intersect with X so both x_1 and x_2 do not vanish on F. Hence,

$$\psi(x_1) = (F(x_1) : G(x_1)) = (F(x_1) : 0) = (1 : 0)$$

$$\psi(x_2) = (F(x_2) : G(x_2)) = (F(x_2) : G(x_2)) \neq (0 : 1)$$

But $(0 : 1) \notin \psi(X)$ so $\psi(X)$ cannot be $\mathbb{P}^1$. But $(1 : 0)$ and $(a : G(x_1)) \in \psi(X)$ where $a \neq 0$ and $G(x_1) \neq 0$, so $\psi(X)$ is not a point either. Contradiction. ■

§8.3 Rational Functions on Projective Space

Definition 8.10 (Rational Functions on Quasi-Projective Varieties)**.** For X quasi-projective variety,

$$K(X) := \{\langle\psi, U\rangle | U \subseteq X \text{ open}, \psi \in \mathscr{O}(U)\}/\sim$$

Fact: Theorem 3.4 Hartshorne. If X is a quasi-projective variety, then

$$K(X) \simeq \left\{ \frac{f}{g} \middle| f, g \text{ homogeneous polynomials of equal degree} \right\}$$

Remark. We can prove theorem ⋆⋆ with this fact.

Definition 8.11 (Rational Map)**.** A mapping $\phi : X \to Y \subseteq \mathbb{P}^n$, $\phi = (\phi_1, \cdots, \phi_n)$ where $\phi_i \in K(X)$ is a rational map. So $\phi = (f_1, \cdots, f_n)$ where the f_is are homogeneous polynomials of equal degree (check!).

§8.4 Blow Ups at Points

Definition 8.12 (Blow up of the Affine Space at the Origin)**.** Consider the first projection $pr_1 : \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$. Fix coordinates $(x_1, \cdots, x_n) \in \mathbb{A}^n$ and $(y_1 : \cdots : y_n) \in \mathbb{P}^{n-1}$. Restrict pr_1 to closed subsets of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the following equations

$$\{x_i y_j = x_j y_i | 1 \leq i < j \leq n\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

Denote the closed subset $\{x_i y_j = x_j y_i | 1 \leq i < j \leq n\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ by $BL_{\underline{0}}(\mathbb{A}^n)$. This is the blow up at $\underline{0} = (0, \cdots, 0)$. Let $\sigma : BL_{\underline{0}}(\mathbb{A}^n) \to \mathbb{A}^n$ be the map from the closed subset to $\mathbb{A}^n$ via pr_1, this is the **blow up map**.

Definition 8.13 (Exceptional Set)**.** We call the exceptional set $\sigma^{-1}(0, \cdots, 0)$.

Proposition 8.14. *(i) $\sigma : BL_{\underline{0}}(\mathbb{A}^n) \setminus \sigma^{-1}(0, \cdots, 0) \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism. In particular, the blow up map σ is birational (we have an isomorphism over open subset).*

(ii) The exceptional set is isomorphic to $\mathbb{P}^{n-1}$, $E \simeq \mathbb{P}^{n-1}$.

(iii) The points of $\sigma^{-1}(\underline{0})$ are in 1-1 correspondence with lines through 0. More precisely, for all lines $l = \{t(a_1, \cdots, a_n) : t \in k\}$,

$$\sigma^{-1}(l) = \tilde{l} \cup \sigma^{-1}(\bar{0})$$

where $\tilde{l}$ is a line in $BL_{\underline{0}}(\mathbb{A}^n)$ passing through $P_l = (0, \cdots, 0, a_1 : \cdots a_n)$.

(iv) $BL_{\underline{0}}(\mathbb{A}^n)$ is irreducible.

Proof. (i) It suffices to show that $\sigma|_{BL_{\underline{0}}(\mathbb{A}^n) \setminus E}$ is a bijection with a regular inverse. Suppose $x_1 \neq 0$, then

$$\sigma^{-1}(x_1, \cdots, x_n) = \{(x_1, \cdots, x_n, y_1 : \cdots : y_n) | x_i y_j = x_j y_i\}$$

and it must satisfy $x_1y_j = x_jy_1$ for all j. So each x_j fixes y_j because $\frac{y_j}{y_1} = \frac{x_j}{x_1}$ for $y_1 \neq 0$. So for every $(x_1, \cdots, x_n)$ with $x_1 \neq 0$, there exists

$$\left(1, \frac{y_2}{y_1}, \cdots, \frac{y_n}{y_1}\right)$$

which defines a unique point in homogeneous coordinates

$$\left(1 : \frac{y_2}{y_1} : \cdots : \frac{y_n}{y_1}\right) \in \mathbb{P}^{n-1}$$

And so

$$\begin{aligned}\sigma^{-1}(x_1, \cdots, x_n) &= \left(x_1, \cdots, x_n, 1 : \frac{y_2}{y_1} : \cdots : \frac{y_n}{y_1}\right)\\ &= \left(x_1, \cdots, x_n, 1 : \frac{x_2}{x_1} : \cdots : \frac{x_n}{x_1}\right)\\ &= \left(x_1, \cdots, x_n, x_1 : \cdots : x_n\right)\end{aligned}$$

So when $x_1 \neq 0$, σ^{-1} is regular. Similar when $x_i \neq 0$.

(ii)

$$\begin{aligned}\sigma^{-1}(0, \cdots, 0) &= \{(0, \cdots, 0, y_1 : \cdots : y_n) | 0y_j = 0y_i\}\\ &= \{(0, \cdots, 0, y_1 : \cdots : y_n) | [y_1 : \cdots : y_n] \in \mathbb{P}^{n-1}\}\\ &\simeq \mathbb{P}^{n-1}\end{aligned}$$

(iii) As we saw in (i), if $x_1 \neq 0$ then σ^{-1} takes $(x_1, \cdots, x_n)$ and maps it to $(x_1, \cdots, x_n, x_1 : \cdots : x_n)$. And for another point in the line passing through x in the exceptional set: $t(x_1, \cdots, x_n)$ gets mapped to $(tx_1, \cdots, tx_n, tx_1 : \cdots : tx_n) = (tx_1, \cdots, tx_n, x_1 : \cdots : x_n)$.

(iv)

■

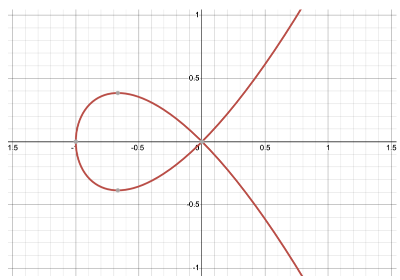
§9 Week 5, Lecture 1

Definition 9.1 (Blow up of Embedded Variety). Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Suppose $\underline{0} \in X$. Then $BL_{\underline{0}}(X) = \sigma^{-1}(X \setminus \{\underline{0}\}) \subseteq BL_{\underline{0}}(\mathbb{A}^n)$.

Remark. Similar in the quasi-affine case.

Definition 9.2 (Blow up at a Point of an Affine Algebraic Set). Let X be an affine algebraic set and $x \in X$.

Example. Blowing up a node
Consider $X = V(y^2 - x^2(x+1))$



§9.1 Dimension

Definition 9.3 (Dimension of Topological Space). Let X be a Topological Space. Given a descending chain of distinct irreducible closed subsets $\emptyset \neq X_n \subset \cdots X_1 \subset X_0$, we define the length to be n and

$$dim(X) := \sup\{\text{lengths of chains of distinct irreducible closed subsets}\}$$

Note. If X is irreducible, then $X_0 = X$.
Example. $dim(\mathbb{A}^1) = 1$.

Proposition 9.4. *If $X = \bigcup_{i=1}^n X_i$, $X_i \subset X$ closed topological spaces, then $dim(X) = \max_i(dim(X_i))$.*

Proof. ■

Definition 9.5 (Height of an Ideal). R is a ring, P a prime ideal, then the height of P is the supremum of the length of ascending chains of distinct prime ideals $P_0 \subset P_1 \subset \cdots \subset P_r = P$.

Definition 9.6 (Dimension of Rings/Krull Dimension).

$$dim(R) := \sup_{P \text{ prime ideal of } R} \text{height}(P)$$

Proposition 9.7. *If X is an affine variety, then*

$$dim(X) = dim(\mathscr{O}(X)) = dim(k[X])$$

Proof. ■

Definition 9.8 (Transcendence Degree)**.** The transcendence degree of a field extension K over k is the maximal number of algebraic independent element in K over k or size of the transcendence basis.

Notation. $tr_k(K)$ or $trdeg_k(K)$. For example, $trdeg(k[t_1, \cdots, t_n]) = n$.

Question: What is the relation between transcendence degree and dimension?
Facts from algebra: for X affine variety,

(i) $trdeg_k(K(X)) = dim(k[X]) = dim(\mathscr{O}(X))$.

(ii) $dim(\mathscr{O}(X)) = dim(\mathscr{O}_{X,x})$ for all $x \in X$.

Corollary 9.9. *If U is a quasi-affine variety, then*

$$dim(U) = trdeg(K(U))$$

Proof. ■

§10 Week 5, Lecture 2

Recap:

- We had to come up with a notion of dimension in order to define singularities.
- We ended up with a good definition of local rings (direct limit). Exercise. check definition coincides.
- From the definition of local ring: If X is a quasi-projective variety, $x \in X$, for all $U, V \subseteq X$,
$$\mathscr{O}_{U,x} \simeq \mathscr{O}_{V,x}.$$
Check this using direct limit definition.
- For U affine variety and open:
$$dim(X) := trdeg_k(K(X)) = trdeg_k(K(U)) = dim_{\text{krull}}(k[U]) = dim(\mathscr{O}_{U,x}) = dim(\mathscr{O}_{X,x})$$
We will often use $\dim X = dim(\mathscr{O}_{X,x})$ and $\dim(k[U])$.

Some important facts about dimension:

1. Let $\langle x_1 \rangle$ be a prime ideal in $k[t_1, \cdots, t_n]$. Then $\text{height}(\langle x_1 \rangle) = 1$. Check!
2. If we fix $X := V(x_1)$, then $X \simeq \mathbb{A}^{n-1}$. We thus have
$$dim(X) + \text{height}(\langle x_1 \rangle) = dim(\mathbb{A}^n).$$
More generally, for any finitely generated integral domain k-algebra and prime ideal P, we have:
$$dim(A/P) + \text{height}(P) = dim(A)$$
This is useful for affine varieties because their coordinate ring satisfies i.d f.g k-alg.
3. When X is an affine variety, $A = k[X]$, and $Y = V(P)$, P prime ideal of A,
$$dim(Y) = dim(k[Y]) = dim(X) - \text{height}(P)$$
because $k[Y] \simeq k[X]/P$.
4. In general, finding heights could be difficult. On the other hand, if we start with $Y \subset X, dim(Y) = dim(X) - 1$, then $\text{height}(I_Y) = 1$, I_Y prime ideal in $k[X]$.

Theorem 10.1. *A Noetherian Integral Domain R is a Unique Factorization Domain (UFD) if and only if every prime ideal P of R of height 1 is principal (it is generated by a single element of R through multiplication by every element of R).*

Check!

Example. Let $C \subset \mathbb{A}^2_k$ be an irreducible curve. The coordinate ring of the affine plane is $k[t_1, t_2]$, so it is a UFD. The irreducible curve should be of dimension 1, so its corresponding prime ideal will have height 1: I_C in $k[t_1, t_2]$, $\text{height}(I_C) = 1$, I_C is principal.
We will see there is a relation between UFDs and singularities.

Theorem 10.2. *In a Noetherian UFD ring, every principal prime ideal is of height 1.*

Check!

Definition 10.3 (Codimension)**.** Let $Y \subseteq X$ closed. The codimension of Y in X is

$$\mathrm{Codim}_X Y := \dim X - \dim Y$$

§10.1 Local Properties

10.1.1 Singularities:

Let k be algebraically closed and let $X = V(f_1, \cdots, f_r) \subseteq \mathbb{A}_k^n$ be an affine variety. Consider the $r \times n$ Jacobian matrix where the i, jth entry is

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i \le r, 1 \le j \le n}$$

This is referred to as $Jac(f)$ where $f := (f_1, \cdots, f_n)$.

Example. $r = 1$

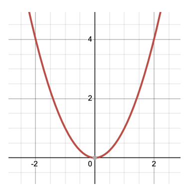
$$Jac(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \end{pmatrix}$$

Definition 10.4 (Smooth/Non-singular point)**.** A point $x \in X$ is smooth or non-singular if $Jac(f)_x$ (evaluated at x) has maximal rank (equal to r). Otherwise, x is said to be singular.

It is not clear that this definition is invariant under isomorphism. We will give a better defintion in the future.

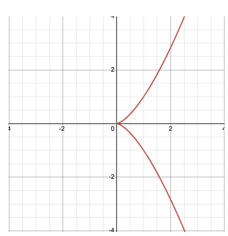
Examples.

(i) $X = V(y - x^2)$ in $k[x, y]$.



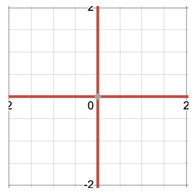
$Jac(f) = \begin{pmatrix} -2x & 1 \end{pmatrix}$. The point $(0, 0)$ is smooth since $Jac(f)_{(0,0)} = \begin{pmatrix} 0 & 1 \end{pmatrix}$ has rank 1.

(ii) $X = V(y^2 - x^3)$ in $k[x, y]$.



$Jac(f) = \begin{pmatrix} -3x^2 & -2y \end{pmatrix}$. The point $(0, 0)$ is singular since $Jac(f)_{(0,0)} = \begin{pmatrix} 0 & 0 \end{pmatrix}$ has rank 0. There are too many tangent directions.

(iii) $X = V(xy)$ in $k[x, y]$ (reducible).



$$Jac(f) = \begin{pmatrix} y & x \end{pmatrix}. \text{ The point } (0,0) \text{ is singular since } Jac(f)_{(0,0)} = \begin{pmatrix} 0 & 0 \end{pmatrix} \text{ has rank } 0.$$

Upshot: If X is a hypersurface (defined by one polynomial), $X = V(F) \in \mathbb{A}^n$. Then,

$$x \in X \text{ is singular} \Leftrightarrow \partial_{x_i} f = 0 \text{ for all } i$$

Singularity is related to excess tangent directions. Let us define what this means. We are going to define tangent space at a given point, and this will lead us to an intrinsic definition of singularity.

10.1.2 Differentials and Tangent Space

Definition 10.5 (Differential of a Polynomial at a Point)**.** Given a polynomial $F \in k[t_1, \cdots, t_n]$, we define the differential of F at $x = (x_1, \cdots, x_n) \in \mathbb{A}^n$:

$$d_x F := \sum_{i=1}^{n} \frac{\partial F}{\partial t_i}(x)(t_i - x_i)$$

Remark. Every polynomial F has an expansion at $x \in \mathbb{A}^n$.

$$F(t) = F^{(0)}(t-x) + F^{(1)}(t-x) + \cdots + F^{(k)}(t-x)$$

where each $F^{(i)}(t-x)$ is homogeneous of degree i. Then the differential of F at x is exactly $F^{(1)}(t-x)$.

Linearity Properties of Differential: Check!

- $d_x(F+G) = d_xF + d_xG$.
- $d_x(FG) = F(x)d_xG + G(x)d_xF$.

Definition 10.6 (Tangent Space at a Point)**.** Let X be an affine algebraic set equal to $V(f_1, \cdots, f_r) \subset \mathbb{A}^n_k$. The tangent space of $x \in X$ is

$$T_{X,x} := V_{\mathbb{A}^n}(d_xf_1, \cdots d_xf_r)$$

The tangent space is defined as solutions to a linear system. It has a vector space structure. Why? Consider the tangent space at $0 \in X$: $p \in T_{X,0}$ if and only if p satisfies the system of homogeneous linear equations:

$$d_0f_1 = 0, \ d_0f_2 = 0, \ \cdots \ d_0f_r = 0.$$

This is equivalent to
$$p \in \ker(Jac(f)_0).$$
So it is a hyperplane through the origin, and thus, a vector space. If $x \neq 0$, then is a hyperplane translated by x, and so it is endowed with the structure of a vector space with x being the additive identity. Also note that, by rank-nullity we have

$$\dim T_{X,0} = \dim\ker(\mathrm{Jac}(f)_0) = n - rk(\mathrm{Jac}(f)_0) \geq n - r$$

Hence, the dimension of the tangent space is equal to $n - r$ exactly when the Jacobian has maximal rank, meaning, x is smooth. We remark this in the following proposition.

Proposition 10.7. *$x \in X$ is smooth if and only if $T_{X,x} = n - r$*

Examples.

(i) $X = V(y - x^2) \subseteq \mathbb{A}^2$. We found before that $(0, 0)$ is smooth. So we expect the dimension of the tangent space at 0 to have dimension 1:

$$d_0(y - x^2) = \partial_x(y - x^2)_0(x - 0) + \partial_y(y - x^2)_0(y - 0) = y$$
$$\Rightarrow T_{X,0} = V(y) \simeq \mathbb{A}^1.$$

(ii) $X = V(xy)$,
$$d_0(xy) = 0$$
$$T_{X,0} = V(0) = \mathbb{A}^2$$

10.1.3 Differentials over Affine Algebraic Sets

Proposition-Definition 10.8 (Differential of an Element in the Coordinate Ring)**.** *Let X be an affine algebraic set, $x \in X \subseteq \mathbb{A}^n$, for all $g \in k[X]$, $d_x g$ defines a linear function on $T_{X,x}$. By definition of coordinate ring, we know that there exists a polynomial $G \in k[t_1, \cdots, t_n]$ such that $G\big|_X = g$. Define*
$$d_x g := (d_x G)\big|_{T_{X,x}}$$
the differential at an element of the coordinate ring.

There is no unique choice of G, so is this well-defined? If $G|_X = (G + F)|_X$ for all $F \in I_X = \langle f_1, \cdots, f_r\rangle$. We have $F = s_1 f_1 + \cdots + s_r f_r$, $s_i \in I_X$. Then the differential at x of F is
$$d_x F = d_x s_1 f_1(x) + d_x s_2 f_2(x) + \cdots + d_x s_n f_n(x)$$
Check $d_x g$ is a linear function on $T_{X,x}$.

$$\Rightarrow d_x F = s_1 d_x f_1(x) + s_2 d_x f_2(x) + \cdots + s_n d_x f_n(x)$$

So,
$$d_x F|_{T_{X,x}} = 0$$

Hence, it is well-defined.

Proposition 10.9. *Let X be affine algebraic, $x \in X$, $\mathcal{M}_x$ ideal of $k[X]$, then*

$$d_x : \frac{\mathcal{M}_x}{\mathcal{M}_x^2} \to T^*_{X,x}$$

is an isomorphism of k-vector spaces. Here $T^*_{X,x}$ is the dual of the tangent space at x.

We start by proving a small claim:
We already know by 10.8 that there is a map $d_x : k[X] \to T^*_{X,x}$. We claim this map defines a group homomorphism under addition.

1. $d_x(f+g) = d_x f + d_x g$. Check!
2. $d_x(fg) = f(x) d_x g + g(x) d_x f$. Check!

What we want to do now is say that if we restrict this map to $\mathcal{M}_x/\mathcal{M}_x^2$, then we get a surjection.

Note that $T_{X,x} \subseteq \mathbb{A}^n$ is an affine algebraic set. So every regular function on $T_{X,x}$ is defined by $\alpha|_{T_{X,x}}$ for some $\alpha \in k[t_1, \cdots, t_n]$. In particular, this holds for linear functions on $T_{X,x}$. Now assume α is linear in $(t_i - x_i)$ and $\alpha(x) = 0$. i.e α is a linear combination of $(t_i - x_i)$. Define $G := \alpha$. By construction we have

$$d_x G = \alpha (= G)$$

$$G(x) = 0$$

Set $g = G|_X$, so $d_x g = d_x g|_{T_{X,x}}$. We know $g(x) = 0 \Rightarrow g \in \mathcal{M}_x$, and $d_x : \mathcal{M}_x \to T^*_{X,x}$ is onto.

Since $g \in \mathcal{M}_x$, the expansion of g at $x$is of the form:

$$g^{(0)} + g^{(1)} + g^{(2)} + \cdots$$

So $d_x g = 0$ if and only if $g^{(i)} = 0$ if and only if $g \in \mathcal{M}_x^2$. By the first isomorphism theorem

$$\mathcal{M}_x/\mathcal{M}_x^2 \simeq T^*_{X,x}$$

Fact: $\frac{\mathcal{M}_x}{\mathcal{M}_x^2}$is a module over $\frac{k[X]}{\mathcal{M}_x} \simeq k$. Check! So it is an isomorphism over k.

§11 Week 7, Lecture 1

§11.1 Regularity

Definition 11.1 (Regular Ring)**.** We call a Noetherian local ring $(A, \mathcal{M})$ regular if

$$dim_{\text{krull}}(A) = dim\left(\frac{\mathcal{M}}{\mathcal{M}^2}\right)$$

as a vector space over $A/\mathcal{M}$

Proposition 11.2. *Let $(A, \mathcal{M})$ be a noetherian local ring. The minimal number of generators for $\mathcal{M}$ is $dim_{A/\mathcal{M}}\left(\frac{\mathcal{M}}{\mathcal{M}^2}\right)$*

Definition 11.3 (Regular Quasi-Projective Variety)**.** We say the quasi-projective variety X is regular if $\mathscr{O}_{X,x}$ is a regular ring. So

$$dim(\mathscr{O}_{X,x}) = dim_{A/\mathcal{M}}\left(\frac{\mathcal{M}_{X,x}}{\mathcal{M}_{X,x}^2}\right)$$

§12 Week 7, Lecture 2

Recall that for k algebraically closed:

(1) X affine algebraic variety of $\mathbb{A}^n$. Then

$$x \in X \text{ smooth} \iff \text{if } \mathrm{Jac}(f_1, \cdots, f_r) \text{ has maximal rank} \iff \dim T_{X,x} = n - r$$

$$T^*_{X,x} \simeq \frac{\mathcal{M}_{X,x}}{\mathcal{M}^2_{X,x}}$$

(2) X quasi-projective variety. Then

$$x \in X \text{ is regular if } \mathscr{O}_{X,x} \text{ is regular}$$
$$\mathscr{O}_{X,x} \text{ is regular} \iff \dim \mathscr{O}_{X,x} = \dim \tfrac{\mathcal{M}_{X,x}}{\mathcal{M}^2_{X,x}}$$

In particular, $x \in X$ is regular if the number of local parameters (the minimum number of generators of the maximal ideal) at x is equal to $\dim X$.

(3) Fact: A regular ring is a UFD.

Proposition 12.1. *For X affine variety,*

$$x \in X \textit{ is smooth} \iff x \in X \textit{ is regular}$$

Proof. We use an inductive style argument for this proof:

(i) Assume X is a hypersurface: $X = V(f) \subseteq \mathbb{A}^n_k$, f irreducible, X affine.
We know $x \in X$ is smooth if and only if $\mathrm{Jac}(f)$ has rank 1, which is if and only if $\dim T_{X,x} = n - 1$. What is $\dim X$?
Since $f \in k[t_1, \cdots, t_n]$ is irreducible and $k[t_1, \cdots, t_n]$ is a UFD, hence $\langle f \rangle$ is prime. One of the fundamental properties of a UFD ring is that any principal prime ideal has height equal to 1. The other fundamental property of UFD rings is that any prime ideal of height 1 is principal. This will come in useful when we look at divisors. Therefore, $\mathrm{height}(\langle p \rangle) = 1$. So,

$$\begin{aligned} \dim X = \dim k[X] &= \dim \left(\frac{k[t_1, \cdots, t_n]}{\langle f \rangle} \right) \\ &= n - \mathrm{height}(\langle f \rangle) && \text{fact from algebra} \\ &= n - 1 \end{aligned}$$

Thus far we have:

$$x \in X \text{ smooth} \iff \dim T_{X,x} = \dim \frac{\mathcal{M}_{X,x}}{\mathcal{M}^2_{X,x}} = n-1 = \dim X = \dim \mathscr{O}_{X,x} \iff x \in X \text{ regular}$$

(ii) Assume X is the vanishing of two polynomials: $X = V(f_1, f_2) \subseteq \mathbb{A}^n$. Define $X_i = V(f_i)$, $i = 1, 2$, and assume $X_i \nsubseteq X_j$. Then,

$$\begin{aligned} x \in X \text{ is smooth} &\longleftrightarrow \operatorname{Jac}(f) = \begin{pmatrix} d_x f_1 \\ d_x f_2 \end{pmatrix} \text{ has rank } n-2 \\ &\longleftrightarrow d_x f_1 \neq 0 \text{ and } d_x f_2 \neq 0 \text{ and } \dim T_{X,x} = n-2 \\ &\longleftrightarrow x \in X_i \text{ smooth and } \dim T_{X,x} = n-2 \\ &\longleftrightarrow \mathscr{O}_{X_i,x} \text{ is regular and } \dim T_{X,x} = n-2 \qquad \text{by Step 1} \end{aligned}$$

Claim: $\mathscr{O}_{X_i,x}$ is regular if and only if $\mathscr{O}_{X,x}$ is regular.
Proof: ($\Leftarrow$) If $\mathscr{O}_{X_i,x}$ is not regular for some i, then $d_x f_i = 0$. Then, by step 1, $\dim T_{X,x} \geq n-1$. But $\dim X \leq n-2$. There is an excess in the tangent direction. So $\mathscr{O}_{X,x}$ is not regular (because $\dim T_{X,x} = \dim X$ is regular).
($\Rightarrow$) Suffices to show $\dim X = n-2$. To do so, consider $f_2|_{X_1} \in k[X_1]$, and note that since $X = V_{X_1}(f_2)$ is irreducible, $\langle f_2|_{X_1} \rangle \subseteq k[X_1]$ is prime. Next, consider the localization $k[X_1] \to \mathscr{O}_{X_1,x}$. Denote the image of $(f_2|_{X_1})$ by $\bar{f}_2$.
Claim: $\langle \bar{f}_2 \rangle \subseteq \mathscr{O}_{X,x}$ is prime.
Proof of Claim done in Week 1. There exists a 1-1 correspondence in $S^{-1}A$ and prime ideals in A that do not meet S. In our case $f_2|_{X_1}$ is not in our multiplicatively closed subset, so its extension is prime. Here

$$S := k[X_1] \setminus \mathcal{M}_x \subseteq k[X_1]$$

$$S = \{f \in k[X_1] | f(x) \neq 0\}$$

$$f_2(x) = 0 \Rightarrow (f_2|_{X_1}) \cap S \neq 0$$

$$\Rightarrow \langle \bar{f}_2 \rangle \text{ is prime}$$

Now, as $\mathscr{O}_{X_1,x}$ is regular, it is a UFD. With $\langle \bar{f}_2 \rangle$ being principal and prime, it follows that height$(\langle \bar{f}_2 \rangle) = 1$.

Again, from the 1-1 correspondence, we find that

$$\operatorname{height}(\langle f_2|_{X_1} \rangle) = 1$$

Since

$$K[X] \simeq \frac{k[X_1]}{\langle f_2|_{X_1} \rangle} = \dim k[X] = \dim X_1 - \operatorname{height}(\langle f_2|_{X_1} \rangle) = (n-1) - 1) = n-2$$

Inductive argument finishes the proof. ■

§12.1 Indeterminacy of Rational Maps

Definition 12.2 (Locus of Indeterminacy of a Rational Map)**.** Let $X \subseteq \mathbb{P}^m$ be a quasi-projective variety of dimension n. Let $\phi : X \dashrightarrow \mathbb{P}^r$ be a rational map (say $\phi = [\phi_1 : \cdots : \phi_n]$, where $\phi_i \dashrightarrow k$). We refer to the complement at the domain of ϕ as the locus of indeterminacy of ϕ. I.e the common zeroes of finite number of homogeneous restricted to X.

$$X \setminus \operatorname{dom}(\phi) = V(\phi_1, \cdots, \phi_n)$$

Theorem 12.3. $\phi : X \dashrightarrow \mathbb{P}^r$ *as above. Assume* X *is regular. Then*

$$\dim(\textit{locus of indeterminacy}) \leq n-2 \iff \textit{Codim}_X(\textit{Indeterminacy}) \geq 2$$

Proof. By definition, $\phi = (F_0 : \cdots : F_r)$ Rational map, where F_i are homogeneous polynomials. Note that we can cancel common factors without changing ϕ.
Fix $x \in V(F_0, \cdots, F_r)$. We may also assume F_is have no common factors in $\mathscr{O}_{X,x}$. Now let U be an affine neighborhood of $x \in X_i$. ■

What does this theorem mean?
Consider 2 quasi-projective varieties U and V. Assume U is regular and there exists a rational map $\phi : X \to V$. Suppose ϕ is not regular along a closed subset $D \subseteq U$ of codimension 1. Meaning,

$$\phi|_{U \setminus D} : U \setminus D \to V \text{ is regular}$$

Question: Can we extend $\phi|_{U \setminus D}$ to a regular map $\phi|_{U \setminus D'} : U \setminus D' \to V$ where $D' \subset D$ and $\operatorname{codim}_U(D') \geq 2$?
Answer: No.
Example. $U = \mathbb{P}^1$ and $V = \mathbb{A}^1$. The extension in theorem is only valid if V is projective.

Corollary: Any rational map $\phi \dashrightarrow \mathbb{P}^n$ from a smooth quasi-projective curve X, extends to a morphism. In particular, for two smooth projective curves X and Y:

$$X \simeq^{bir} Y \Rightarrow X \simeq Y$$

Nonexample. $X = V(y^2 - x^3)$.

§13 Week 8, Lecture 1

§13.1 Finite Morphisms

Definition 13.1 (Finite Ring Homomorphism)**.** Let A and B be commutative rings with unity. Let $\phi : A \to B$ be a ring homomorphism. We say ϕ is finite if and only if B is finite as an algebra over A via ϕ. Equivalently, ϕ is finite if and only if there exists a finite number of $b_1, \cdots, b_n$ such that every $b \in B$ can be written as

$$b = \sum_{i=1}^{n} \phi(a_i) b_i$$

where $a_i \in A$.

Definition 13.2 (Finite Morphisms of Affine Varieties)**.** If $\phi : U \to V$ is a morphism of affine varieties, we say ϕ is finite if the pullback map $\phi^* : \mathscr{O}(V) \to \mathscr{O}(U)$ is finite as a ring homomorphism. So $\phi(U)$ is finite over $\phi^*(\mathscr{O}(V))$.

Example. $k = \mathbb{C}$. Define $X := V(x^n - t) \subseteq \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. Consider $pr_2 : \mathbb{A}^2 \to \mathbb{A}^1$ and $\phi : X \to \mathbb{A}^1$.

$$k[X] = \frac{k[x,t]}{\langle x^n - t \rangle} \simeq k[t] \oplus k[t]x \oplus \cdots \oplus k[t]x^{n-1}$$

and $k[t] \simeq \phi^*(k[t]) \subset K[X]$. This is a finite map.

Remark. Let $\phi : X \to Y$. We proved in the assignment that ϕ is closed. The image is irreducible, so it is either a point or all of Y. If the image is a point, then it is not finite. So ϕ has to be onto.

Remark. Dimension is preserved under finite maps. We have surjective map: $\dim Y \leq \dim X$. We use the fact: $R \subset S$ rings, S finitely generated R-module, then $\dim R = \dim S$.

$$\phi \text{ dominant} \Rightarrow \phi^* \text{ injective}$$

Remark. A finite map has finite fibres.

Definition 13.3 (Finite Morphisms)**.** Let X and Y be quasi-projective varieties. A morphism $\phi : X \to Y$ is said to be finite if there exists an affine open covering $\{V_i\}$ of Y such that

(i) $U_i := \phi^{-1}(V_i) \simeq$ affine variety.

(ii) $\phi|_{U_i} : U_i \to V_i$ is finite.

Facts from Algebra:

- Integral is a transitive property. If $R \subseteq S$ is integral and $S \subseteq S'$ is integral, then S' is integral over R.
- Finite Type + Integral = Finite.
- We say R is integrally closed in S if $s \in S$ integral over R, implies $s \in R$.

- R integral domain, integrally closed = R is integrally closed in $Frac(R)$.
- Given $R \subset S$, we define the integral closure of R in S by $R[s_1, s_2, \cdots]$.
- Integrally closed is a local property.
- Any UFD is integrally closed. (local regular ring is integrally closed)
- Any integrally closed local ring R with dimension 1 is always regular.

Theorem 13.4. *Every non-constant morphism $\phi : X \to Y$ of regular projective (irreducible) curves X and Y, is finite.*

§13.2 Global Geometry

§13.3 Prime Divisors

Definition 13.5 (Prime Divisor)**.** Let X be a quasi-projective variety. A prime divisor of X is any subvariety of $\mathrm{codim}_X = 1$.

Definition 13.6 (Prime Divisor)**.** Let X be a quasi-projective variety. A divisor on X is a finite formal sum of prime divisors with coefficients in $\mathbb{Z}$.

$$D = \sum_{i \in I} a_i D_i$$

where D_i are prime, $a_i \in \mathbb{Z}$, and $|I| < \infty$.

Notation. Div(X) is the set of divisors on X. It forms an abelian group.

§14 Week 8, Lecture 2

Definition 14.1 (Effective Divisor)**.** We say $D = \sum a_i D_i$ is effective and write $D \geq 0$ if $a_i \geq 0$ for all i. We write $D_1 \geq D_2$ if and only if $D_1 - D_2 \geq 0$.

We wish to come up with a notion of valuation, $\mu_Y(f)$, for any nonzero $f \in k(X)$:

Proposition 14.2. *Every prime divisor on a smooth variety is locally principle.*

Proof. Let X be a regular quasi-projective variety. Assume we are working over an algebraically closed field k. Let $Y \subset X$ be a subvariety of codimension 1 (Y is a prime divisor). ■

Definition 14.3 (Divisor of f)**.** Fix non-zero $f \in K(X)$, where X is a smooth quasi-projective variety. The principal divisor associated to f is

$$(f) := \sum \mu_Y(f) D$$

Also denoted $div(f)$. Note this is finite.

Remark. If $f_1, f_2 \in k(X)$ are non-zero: $(f_1 f_2) = (f_1) + (f_2)$ by (i).

Remark. If $f = g/h$, then $(f) = (g) - (h)$ by (i).

Remark. (f) is precisely: sums of zeroes - sums of poles, where coefficients being the order of vanishings and poles.

Proposition 14.4. *Principal divisors form a subgroup of $Div(X)$.*

Definition 14.5 (Class Group)**.** Class group on X:

$$Cl(X) := Div(X) \backslash \sim$$

For any $D_1, D_2 \subset Div(X)$, we say D_1 is linearly equivalent to D_2 and write $D_1 \sim D_2$ if $D_1 - D_2 = (f)$ for some nonzero $f \in K(X)$.

2 Fundamental Examples:

(i) $X = \mathbb{A}^n$, every prime divisor Y defines a prime ideal I_Y of height equal to 1 in $k[t_1, \cdots, t_n]$. By UFD properties,

§14.1 Pullback of Divisors

Definition 14.6 (Pullback of Regular Projective Curves)**.** Let $\phi : X \to Y$ be a surjective morphism of regular projective curves. Since X and Y are curves, their prime divisors (the codimension 1 subvarieties) are exactly points. Let $y \in Y$ as a prime divisor in Y. We define the pullback $\phi^*(y)$ as follows:
Let U be an open neighborhood of y equipped with the local u parameter adapted to y:

$$\langle u \rangle = I_y \text{ ideal of } \mathscr{O}(U)$$

Consider $\phi^{-1}(y) = \{x_i\}$ = finite number of points.

Each x_i is a prime divisor in X, so $\mu_{x_i}(\phi^* u)$ makes sense.
Now define pullback of y as

$$\phi^*(y) = \sum \mu_{x_i}(\phi^*(u))x_i \in Div(X)$$

More generally, for any $D = \sum a_i D_i \in Div(Y)$, we define

$$\phi^* D = \sum a_i \phi^* D_i$$

§14.2 Local Parameter adapted to Y.

$Y \subseteq X$ codim 1, X is regular.

(i) Goal: To describe Y locally as the vanishing of a single polynomial. So for every $y \in Y$, there is an open neighborhood U of y such that $Y|_U = V(u)$ for some $u \in \mathscr{O}(u)$.

(ii) Let U be an affine neighborhood of x.

(iii) Then $Y \cap U$ is closed in U and has codimension 1.

(iv) Fact: $I_{Y\cap U}$ is a prime ideal of $\mathscr{O}(U) = k[U]$ and it is of height 1.

(v) Consider the extension $I^e_{Y\cap U}$ prime ideal of height 1 in $\mathscr{O}_{U,x}$.

(vi) $\mathscr{O}_{U,x}$ is a UFD because X is regular. Hence, $I^e_{Y\cap U} = \langle g_x \rangle$.

(vii) We want to express g_x as

$$g_x = \frac{f}{h} = u\frac{f_0}{h}$$

where $u \in \mathscr{O}(U)$ and $f_0/h \in \mathscr{O}^\times_{U,x}$ (the multiplicative group). If we have this, then $\langle g \rangle = \langle u f_0/h \rangle = \langle u \rangle$. And so $Y = V_U(u)$.

(viii) Write $g_x = r_u/s_u$, shrink U so that s_u has no zeroes on U.

$$Y \cap U = V_U(f_1, \cdots, f_r) = V_U(f_1/1, \cdots, f_r/1)$$

where $f_i/1 \in I_{Y,x}$ ideal of $\mathscr{O}_{U,x}$. where $I_{Y,x} = \langle g_x \rangle$.

(ix) So $f_i = g_x h_{i,x}$, $h_{i,x} \in \mathscr{O}_{U,x}$.

(x) Shrink U to U'so that $h_{i,x}$ has no zeroes or poles. Then

$$V_{U'}(f_i) = V_{U'}(g_x|_{U'}) = V_{U'}(r_u|_{u'})$$

(xi) Set $u = r_u|_{U'}$.

§15 Week 9, Lecture 1

§15.1 Degree of Finite Morphisms

§15.2 Sheaves over Varieties

Definition 15.1 (Presheaf)**.** A presheaf $\mathscr{F}$ on a topological space X consists of the following data:

(i) $\mathscr{F}(\emptyset) = 0$.

(ii) For each open $U \subseteq X$, $\mathscr{F}(U)$ is an abelian group.

(iii) For any inclusion $V \subseteq V$ of open sets, a group homomorphism

$$\rho_{UV}\mathscr{F}(U) \to \mathscr{F}(V)$$

called a restriction map.

(iv) $\rho_{UU} = id_{\mathscr{F}(U)}$.

Definition 15.2 (Sheaf)**.** Is a presheaf with

(i) If $U \subseteq X$ open and $\{U_I\}$ covers U and $s \in \mathscr{F}(U)$ such that $s|_{U_i} = 0$ for all i, then $s = 0$.

(ii) If $U \subseteq X$ open and $\{U_i\}$ cover U amd $s_i \in \mathscr{F}(U_i)$ satisfy $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a unique $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

Examples:

(i) $\mathscr{O}_X$.

(ii) $B(\mathbb{R})$ is a presheaf but not a sheaf.

(iii) Let A be an abelian group, X topological space $\mathbb{A}(U) = \{f : U \to A\}$ continuous. Discrete topology.

(iv) $C^\infty(\mathbb{R}^n)$.

§16 Week 9, Lecture 2

§16.1 Morphisms of Sheaves

Definition 16.1 (Morphism of Sheaves)**.** A map $\phi : \mathscr{F} \to \mathscr{G}$ is called a morphism of sheaves if for each open $U \subseteq X$, there is a map ϕ_U such that $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is a morphism of abelian groups and commutes. We say ϕ is an isomorphism if it has an inverse.

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\rho_{UV}"'] & \mathscr{G}(U) \arrow[d, "\rho'_{UV}"] \\ \mathscr{F}(V) \arrow[r, "\phi_V"'] & \mathscr{G}(U) \end{tikzcd}$$

Remark. A sheaf morphism $\phi : \mathscr{F} \to \mathscr{G}$ is an isomorphism if it restricts to isomorphisms over an open covering. Note this not saying that sheaf morphisms are determined locally - you have to start with a global sheaf morphism. Check!

§16.2 Varieties as Ringed Spaces (non-examinable)

Definition 16.2 (Ringed Space)**.** A ringed space is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings.

Remark. Using morphisms of sheaves you can define morphisms of ringed spaces:

$$(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$$

with $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a sheaf morphisms.

Definition 16.3 (Abstract Algebraic Variety)**.** An abstract algebraic variety is a ringed space $(X, \mathscr{O}_X)$, where $\mathscr{O}_X$ is a sheaf of integral domains, and equipped with an open covering $\{U_i\}$ such that

$$(U_i, \mathscr{O}_{U_i}) \simeq (Y_i, \mathscr{O}_{Y_i}),$$

with Y_i an affine variety with its sheaf of regular functions $\mathscr{O}_Y$. The isomorphism here is as ringed spaces. In particular, $\mathscr{O}_X(U_i) \simeq k[Y_i]$.

Remark. We have already seen that any quasi-projective variety is an abstract algebraic variety.

Remark. Consider $X := V(x) \subseteq \mathbb{A}^2$ and $X' := V(x^2) \subseteq \mathbb{A}^2$, with regular functions $k[x, y]$. Notice that X and X' are affine algebraic sets and they are topologically the same. So some sort of data is being lost. $\mathscr{O}(X)$ and $\mathscr{O}(X')$ are the same if we just follow the definitions of regular functions. However, as ringed spaces they are defined differently:

$$\left(X, \mathscr{O}_X := \frac{\mathscr{O}_{\mathbb{A}^2}}{\langle x \rangle}\right) \text{ and } \left(X', \mathscr{O}_{X'} := \frac{\mathscr{O}_{\mathbb{A}^2}}{\langle x^2 \rangle}\right)$$

So now we can distinguish between these as ringed spaces.

§16.3 From Divisors to Invertible Sheaves

Let X be a quasi-projective variety and $\mathscr{O}_X$ the sheaf of regular functions.

Definition 16.4 (Sheaf of $\mathscr{O}_X$-modules)**.** We say a sheaf $\mathscr{F}$is a sheaf of $\mathscr{O}_X$-modules on X if for any $U \subseteq X$ open, $\mathscr{F}(U)$ (sections of U) is an $\mathscr{O}_X(U)$-module.

Definition 16.5 (Locally Free)**.** A sheaf of $\mathscr{O}_X$-modules on X is locally free if there exists a covering $\{U_i\}$ on X such that $\mathscr{F}|_{U_i}$ (or $\mathscr{F}(U_i)$) is a freely generated as an $\mathscr{O}_X(U_i)$-module.

Definition 16.6 (Invertible Sheaf)**.** We say a locally free sheaf $\mathscr{F}$ is invertible if there is an open cover $\{U_i\}$ of X such that $\mathscr{F}(U_i)$ is freely generated by one element (i.e. $\mathscr{F}$ is locally free of rank 1). So

$$\mathscr{O}_X(U_i) \simeq \mathscr{F}(U_i)$$

16.3.1 Operations on Sheaves

Remark. Operations on sheaves of $\mathscr{O}_X$-modules are tensor product/ tensor powers (exterior powers) / $\mathrm{Hom}(\cdot,\cdot)$ / ...

Remark. Tensor product of two locally free sheaves is locally free.

Remark. If $\mathscr{F}$ and $\mathscr{G}$ are invertible, then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \simeq \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{F}$.

Proposition 16.7. *Given any invertible sheaf $\mathscr{F}$, there exists and invertible sheaf $\mathscr{G}$ such that*

$$\mathscr{G} \otimes \mathscr{F} \simeq \mathscr{O}_X.$$

We denote $\mathscr{G}$ by $\mathscr{F}^*$ of $\mathscr{F}^{-1}$.

Proof. Not examinable. Sketch. The choice of $\mathscr{G}$ will be the dual of $\mathscr{F}$. Recall the dual of R-module M is

$$M^* = Hom(M, R).$$

Check: If M is free then its dual is free. Now, define the dual of a sheaf: $\mathscr{F}^* = Hom_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. Check this is a sheaf of $\mathscr{O}_X$-modules. So $\mathscr{F}^* \otimes_{\mathscr{O}_X} \mathscr{F} \simeq \mathscr{O}_X$ (check). ∎

Corollary 16.8. *The set of isomorphism classes of invertible sheaves under the tensor product $\otimes_{\mathscr{O}_X}$ form an Abelian group.*

Definition 16.9 (Picard Group)**.** The Picard group is the abelian group of sets of isomorphism classes of invertible sheaves under the tensor product $\otimes_{\mathscr{O}_X}$. It is denoted $\mathrm{Pic}(X)$.

- Identity: $\mathscr{O}_X$.
- Multiplication: $\oplus_{\mathscr{O}_X}$.
- Inverses: $\mathscr{F}^{-1} = \mathscr{F}^\vee = Hom_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.

Theorem 16.10. *Let X be a regular projective variety. There is a group homomorphism*

$$\phi : Div(X) \to Pic(X)$$

$$D \mapsto \mathscr{O}_X(D)(U)$$

that factors through class group. So from any divisor I can construct an invertible sheaf.

$$\begin{array}{ccc} Div(X) & \xrightarrow{\phi} & Pic(X) \\ & \searrow & \uparrow \\ & & Cl(X) \end{array}$$

Set $\mathscr{O}_X(D)$ to be the sheaf on X defined by

$$(\mathscr{O}_X(D))(U) = \{f \in K(U) | (f) + D|_U \geq 0\}.$$

Proof. Set $\phi(D) := \mathscr{O}_X(D)$. Steps to proving ϕ is a group homomorphism:

Step 1. Show ϕ is well-defined: $\phi(D) = \mathscr{O}_X(D)$ is an invertible sheaf. We start with the simpler case: D is a prime divisor.
Let $\{U_i\}$ be an open covering equipped with local parameter u_i for each $D|_{U_i}$. Reminder: each $U_i \simeq$ affine variety, and u_i is a regular function, $u_i \in \mathscr{O}(U_i)$, such that $D|_{U_i} = V(u_i)$. Regularity is used to find these local parameters. The divisor locally is principal. We will use this to show that $\mathscr{O}_X(D)$ is an invertible sheaf.
Claim: $\mathscr{O}_X(D)(U_i) \simeq \mathscr{O}_X(U_i)\langle 1/u_i\rangle$. ($\mathscr{O}_X(D)$ is locally free and of rank 1.)
Proof of claim: Define $D_{U_i} = D|_{U_i}$ for notational ease. The local sections look like $(f)+D_{U_i} \geq 0$, where f is rational so $f = g/h$ for some $g, h \in k[U_i]$. Hence, the principal divisor associated to f is $(f) = (g) - (h)$. Thus,

$$(g) - (h) + D_{U_i} \geq 0.$$

This tells us something about h: h cannot be too big. Recall: when we were defining the notion of valuation of principal divisors we assumed g and h have no common factors. This is an assumption we are going to make again, we may assume that (g) and (h) have no common components. This means that $V(h) \subseteq D_{U_i} = V(u_i)$. We can also assume that h is nonconstant. With that assumption, we can make the following **claim**: we can assume $\langle h\rangle$ in $\mathscr{O}_X(U_i)$ is prime. This is the same as saying h is irreducible in the local ring. **Proof**: for all $x \in D_{U_i}$, consider the image $h/1 \in \mathscr{O}_X$. Check.
So $\langle h\rangle$ is principal and prime $\Rightarrow \langle h\rangle$ is of height 1 from the fact that X is regular. So,

$$\dim V(h) = \dim U_i - 1$$

$$\Rightarrow V(h) = V(u_i)$$

By Nullstellensatz, $h = t \cdot u_i$ for some $t \in \mathscr{O}_X(U_i)$. Moreover, t must have no zeroes over U_i, i.e $t \in \mathscr{O}_x(U_i)^*$. Thus, $1/t$ is also regular over U_i. Hence,

$$f = \frac{g}{h} = \frac{g}{t}\frac{1}{u_i} \Rightarrow f \in \mathscr{O}_X(U_i)\langle 1/u_i\rangle.$$

Step 2. Check $\mathscr{O}_X(D)$ is an invertible sheaf in the more general case: $D = \sum_{k=1}^m a_k D_k$. For each D_k we have local adapted parameters. Repeated application of the arguments

that we have for finding local parameters: we find $u_{i,k}$ corresponds to $D_k|_{U_i}$ and define $v_i := \prod_{k=1}^m u_{i,k}^{a_k}$. Just like before, these will generate $\mathscr{O}_X(D)$.
Claim: $\mathscr{O}_X(D)(U_i) \simeq \mathscr{O}_X(U_i)\langle 1/v_i \rangle$.
Proof: We have $(f) + (D = \sum_{k=1}^m a_k D_k)|_{U_i} \geq 0$. We are going to split this into two parts, the sum of terms with positive coefficients and the terms with negative coefficients: set

$$D = \sum_{k=1}^{l} a_k D_k^+ + \sum_{k=l+1}^{m} a_k D_k^-.$$

Now f is a rational function so it equals g/h for some $g, h \in \mathscr{O}_X(U_i)$. This is the same as saying

$$\left((g) + \sum_{k=1}^{l} a_k D_k^+\right) + \left(\left(\frac{1}{h}\right) + \sum_{k=l+1}^{m} a_k D_k^-\right) \geq 0.$$

Step 3.

$\blacksquare$

§17 Week 10, Lecture 2

§18 Review with Dominic

Quasi-affine variety:

- Open subset of an affine variety (subspace Zariski Topology).
- U open $\subseteq X$ closed $\subseteq \mathbb{A}^n$. U is open **in** X. That means, there exists $V \subseteq \mathbb{A}^n$ open such that $U = X \cap V$. It is the restriction of a bigger open set to X.
- Example:

Given a quasi-affine variety X, we define the rational functions on X as follows:

$$K(X) = \left\{ \langle f, U \rangle \middle| f \in \mathscr{O}(U), U \text{ open set }, U \subseteq X \right\}$$

References

[Matan, 2024] Matan, D. (2024). Red center discussions.