AN EXPLICATION OF EXISTENCE AND UNIQUENESS RESULTS FOR A NONLINEAR SCHRÖDINGER EQUATION

AN INTRODUCTION TO THE SHOOTING METHOD AND STURM COMPARISON THEOREM

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PHYSICS OF NLS

1.1. DERIVE THE WAVE EQUATION FROM MAXWELL

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \quad (1.1.a) \qquad \nabla \cdot \vec{\mathcal{D}} = 0, \quad (1.1.c)$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t}, \quad (1.1.b) \qquad \nabla \cdot \vec{\mathcal{B}} = 0. \quad (1.1.d)$$

The fields are in three-dimensional Cartesian coordinates, for example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents, we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \quad (1.2.a) \qquad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}. \quad (1.2.b)$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [1, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [2]. This reference work also includes an introduction to the necessary vector calculus.

We use vector calculus and Maxwell's laws to rewrite the curl of the curl:

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) \overset{(1.1.a)}{=} \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathcal{B}}\right) \overset{(1.1.b)}{\underset{(1.2.a)}{=}} -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} \overset{(1.2.b)}{=} -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \text{ and}$$

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \nabla^2 \vec{\mathcal{E}} = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \Delta \vec{\mathcal{E}} \overset{(1.1.c)}{\underset{(1.2.b)}{=}} -\Delta \vec{\mathcal{E}}.$$

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Combining these and using $\mu_0\epsilon_0 = 1/c^2$, we arrive at the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{1.3}$$

1.2. VALIDITY OF PLANE WAVE SOLUTIONS

Stuyding the left and right hand sides of equation (1.3), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{l=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_l^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.4}$$

where k_0 is the wavenumber and ω_0 the frequency. These are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any z-value and the electric field is non-zero in the x and y directions. The wavefronts are spaced by the wavelength λ and the wavenumber k_0 is the reciprocal of the wavelength.

This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane.

We substitute (1.4) in equation (1.3). Note that only Δ_z will be non-zero:

$$\Delta\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation (1.5):

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.5}$$

For a general direction in (x, y, z)-coordinates, define the wavevector

$$\vec{k} = (k_x, k_y, k_z),$$

where $\left|\vec{k}^2\right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$. This satisfies equation (1.3) when $\vec{k}\perp\vec{\mathcal{E}}$ and

$$\mathcal{E}_j = E_c e^{i(\vec{k}\cdot\vec{r}-\omega_0 t)}. \tag{1.6}$$

1.3. DERIVATION OF THE HELMHOLTZ EQUATION

We consider time-harmonic solutions to the scalar wave equation (1.3) of the form

$$\mathcal{E}_j(x,y,z,t) = e^{i\omega_0 t} E(x,y,z) + \text{c.c}, \tag{1.7}$$

which are continuous wave beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, we refer to [3].

Substituting (1.7) in equation (1.3) and taking the derivatives leads to the expression

$$\begin{aligned}\Delta\left(e^{-i\omega_0 t}E\right) &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(e^{-i\omega_0 t}E\right)\\ e^{-i\omega_0 t}\Delta E &= \frac{1}{c^2}(-i\omega_0)^2 E e^{-i\omega_0 t},\end{aligned}$$

where we can divide by $e^{-i\omega_0 t} \neq 0$ and use the dispersion relation (1.5) to arrive at the scalar linear Helmholtz equation for E

$$\Delta E(x,y,z) + k_0^2 E = 0. \tag{1.8}$$

As an example, equation (1.8) is solved by the general-direction plane waves (1.6), where

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

1.4. DERIVATION OF THE LINEAR SCHRÖDINGER EQUATION

REVISE: We write the incoming field $E_0^{\text{inc}}(x,y)$ as a sum of plane waves. Then the electric field $E(x,y,z)$ for non-zero z-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and it is essential to Fourier optics. We have

$$\begin{aligned}E_0^{\text{inc}}(x,y) &= \frac{1}{2\pi}\int_D E_c(k_x,k_y)\, e^{i(k_x x + k_y y)} \mathrm{d}k_x \mathrm{d}k_y, \quad \text{such that}\\ E(x,y,z) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} E_c(k_x,k_y)\, e^{i\left(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z\right)} \mathrm{d}k_x \mathrm{d}k_y,\end{aligned}$$

where D denotes the (circular) laser input beam domain. For laser beams oriented in the z-direction, most of the plane wave modes are nearly parallel to the z-axis, which implies $k_z \approx k_0$. We define $k_\perp^2 = k_x^2 + k_y^2$, such that $k_0^2 = k_\perp^2 + k_z^2$. It is equivalent to $k_0 \approx k_z$ to say that $k_\perp \ll k_z$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{1.9}$$

where $\psi(x,y,z)$ is an envelope (or amplitude) function. The envelope shape may vary over z, in contrast to soliton solutions, see (1.21).

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Substituting (1.9) into the Helmholtz equation (1.8) yields

$$\psi_{zz}(x,y,z)+2ik_0\psi_z+\Delta_\perp\psi=0, \tag{1.10}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that for lasers beams oriented in the z-direction, the wavenumber k_z dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope function $\psi(x,y,z)$ will vary slowly in z and curve even more slowly.

Claim: $|\psi_{zz}| \ll k_0|\psi_z|$ and $|\psi_{zz}| \ll \Delta_\perp\psi$.

REVISE: To see this, we first show that $k_0-k_z \ll 1$. We factor out k_0^2, take the square root on both sides and linearise the square root term of the right hand side:

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1-\frac{k_\perp^2}{k_0^2}\right) \Longrightarrow k_z = k_0\left(1-\frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1-\frac{1}{2}\frac{k_\perp^2}{k_0^2}\right).$$

Finally, we use $k_\perp \ll k_0$ to obtain the intermediate result:

$$k_0-k_z \approx k_0-k_0+\frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} \ll 1.$$

For the first statement of the claim, $|\psi_{zz}| \ll k_0|\psi_z|$, it is equivalent to show that the ratio of $|\psi_{zz}|$ over $k_0|\psi_z|$ is much smaller than 1. We calculate the ratio as follows:

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0-k_z)^2 E_c}{k_0(k_0-k_z)E_c} = \frac{k_0-k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0}\cdot\frac{1}{k_0} \ll 1.$$

For the other statement of the claim, we calculate:

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0-k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0-k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} \ll \frac{1}{4}\frac{k_\perp^2}{k_0^2} \ll 1.$$

Using the approxations in equation (1.10) yields the linear Schrödinger equation:

$$2ik_0\psi_z+\Delta_\perp\psi=0. \tag{1.11}$$

1.5. POLARISATION FIELD

Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}}+\vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E},0,0),\ \vec{\mathcal{P}} = (\mathcal{P},0,0),\ \vec{\mathcal{D}} = (\mathcal{D},0,0),$$

Furthermore, we assume that $\mathcal{E}$ is the continuous wave electric field from (1.7). We write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.12}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials. The constants c_i are actually a function of the frequency ω_0. We rewrite $c_i = \epsilon_0\chi^{(i)}(\omega_0)$, where $\chi^{(i)}$ is the i-th order susceptibility. Then equation (1.12) reads:

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.13}$$

First we consider linear polarisation:

$$\mathcal{P}_{\text{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}.$$

Then the induction electric field $\mathcal{D}$ is given by:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\text{lin}} = \epsilon_0\mathcal{E} + \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E} = \epsilon_0\mathcal{E}\left(1 + \chi^{(1)}(\omega_0)\right) = \epsilon_0 n_0^2(\omega_0)\mathcal{E},$$

where $n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0)$ is the linear index of refraction (or refractive index) of the medium.

With this updated induction electric field $\mathcal{D} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}$, we can update the scalar wave equation and Helmholtz equation. Only the dispersion relation is affected by considering linear polarisation:

$$k_0^2 = \frac{\omega_0^2}{c^2} n_0^2(\omega_0). \tag{1.14}$$

We now consider the nonlinear polarisation field $\mathcal{P}_{\text{nl}}$ as the difference between the true polarisation and the linear approximation:

$$\mathcal{P} = \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$\begin{aligned} -\mathcal{P}_{\text{nl}} &= \epsilon_0\chi^{(2)}(-\mathcal{E})^2 + \epsilon_0\chi^{(3)}(-\mathcal{E})^3 + \epsilon_0\chi^{(4)}(-\mathcal{E})^4 + \epsilon_0\chi^{(5)}(-\mathcal{E})^5 + \mathcal{O}\left(\mathcal{P}^6\right) \\ -\mathcal{P}_{\text{nl}} &= \epsilon_0\chi^{(2)}\mathcal{E}^2 - \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 - \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right), \end{aligned}$$

where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\text{nl}}$ and we have only the odd terms:

$$\mathcal{P}_{\text{nl}} = \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^7\right) \tag{1.15}$$

The leading-order term is called the Kerr nonlinearity:

$$\mathcal{P}_{\text{nl}} \approx \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}^3. \tag{1.16}$$

1.6. IMPLICATIONS OF NONLINEAR POLARISATION

Substituting the continuous wave electric field (1.7) into equation (1.16) yields

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}^3 = 3\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \chi^{(3)}(\omega_0)E^3 e^{3i\omega_0 t} + \text{c.c.},$$

where the second term has a frequency of $3\omega_0$ (third harmonic). This has almost no contribution due to the phase-mismatch with the first harmonic. Hence, we approximate

$$\mathcal{P}_{\mathrm{nl}} \approx 3\epsilon_0\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \text{c.c.} = 3\epsilon_0\chi^{(3)}(\omega_0)\mathcal{E}.$$

Then we simplify $\mathcal{P}_{\mathrm{nl}}$ by defining

$$n_2 := \frac{3\chi^{(3)}}{4\epsilon_0 n_0},$$

so that we obtain the simplified expression

$$\mathcal{P}_{\mathrm{nl}} = 4\epsilon_0 n_0 n_2\,|E|^2\,\mathcal{E}.$$

This allows us to write the induction electric field $\mathcal{D}$ as,

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}} = \epsilon_0 n^2\mathcal{E},$$

where

$$n^2 = n_0^2\left(1 + \frac{4n_2}{n_0}\,|E|^2\right) = n_0^2 + 3\chi^{(3)}(\omega_0)\frac{1}{\epsilon_0}|E|^2.$$

For water, $n_2 \sim 10^{-22}$ which justifies neglecting nonlinear effects. With lasers, the nonlinear effect becomes more relevant, but is still weak. For a typical continuous wave laser with $|E| \sim 10^9$, we still have a weak nonlinearity, as $n_2|E| \sim 10^{-4} \ll n_0 \approx 1.33$.

We update equation (1.8) to the scalar nonlinear Helmholtz equation (NLH):

$$\Delta E(x,y,z) + k^2 E = 0, \quad \text{where } k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right). \tag{1.17}$$

We write $E(x,y,z)$ as the product of the z-propagation and an envelope function $\psi(x,y,z)$:

$$E = e^{ik_0 z}\psi$$

and substitute in (1.17) to obtain:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp\psi + 4k_0^2\frac{n_2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.18}$$

Just as in section 1.4, we apply the paraxial approximation, since for laser beams oriented in the z-direction, we have $\left|\psi_{zz}\right| \ll k_0\left|\psi_z\right|, \left|\psi_{zz}\right| \ll \Delta_\perp\psi$. We finally obtain the nonlinear Schrödinger equation (NLS):

$$2ik_0\psi_z(z,\overline{x}) + \Delta_\perp\psi + k_0^2\frac{4n^2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.19}$$

1.7. SOLITON SOLUTIONS

The NLS equation (1.19) can be written as a dimensionless equation. Starting from equation (1.18), we apply the rescaling of coordinates $(x, y, z) \to (\tilde{x}, \tilde{y}, \tilde{z})$ defined by:

$$\tilde{x} = \frac{x}{r_0} \quad \tilde{y} = \frac{y}{r_0} \quad \tilde{z} = \frac{z}{2L_{\text{diff}}},$$

where r_0 is the input beam width and L_{diff} is the diffraction length. We refer to chapter 2 of [1] for more information on the geometrical optics of lasers. There, we also find that $L_{\text{diff}} = k_0 \cdot r_0^2$. To rescale $\tilde{\psi}$, we define:

$$\tilde{\psi} = \frac{\psi}{E_c}, \quad \text{where } E_c := \max_{x,y} \left|\psi_0(x, y)\right|.$$

Through the rescaling we obtain the dimensionless NLH for $\tilde{\psi}$:

$$\frac{f^2}{4}\tilde{\psi}_{\tilde{z}\tilde{z}}(\tilde{z}, \tilde{x}, \tilde{y}) + i\tilde{\psi}_{\tilde{z}} + \Delta_\perp \tilde{\psi} + \nu \left|\tilde{\psi}\right|^2 \tilde{\psi} = 0,$$

that depends on a nonparaxiality parameter f and a nonlinearity parameter ν:

$$f = \frac{1}{r_0 k_0} = \frac{r_0}{L_{\text{diff}}}, \quad \nu = r_0^2 k_0^2 \frac{4n_2}{n_0} E_c^2.$$

Here the approximation of paraxiality is valid for small $f \ll 1$ and this leads to the dimensionless NLS equation (1.20), where the tildes have been dropped for brevity.

$$i\psi_z(z, x, y) + \Delta_\perp \psi + \nu \left|\psi\right|^2 \psi = 0. \tag{1.20}$$

Radial solitary-wave solutions to (1.20) were considered in [4] with ψ of the form:

$$\psi_\omega^{\text{solitary}}(r, z) = e^{i\omega z} R_\omega(r), \tag{1.21}$$

where ω is a real number and R_ω is the real solution of

$$-\omega R_\omega + \Delta_\perp R_\omega(r) + R_\omega^3 = 0.$$

This can be solved in general by, for example,

$$R_\omega(r) = \sqrt{\omega} R\left(\sqrt{\omega} r\right).$$

However, taking $\omega = 1$ leads to the simplest soliton equation

$$R''(r) + \frac{1}{r}R' - R + R^3 = 0, \quad 0 < r < \infty, \tag{1.22}$$

subject to initial condition $R'(0) = 0$ and integrability condition $\lim_{r\to\infty} R(r) = 0$. The (numerical) solution is known as the Townes profile, which is positive and monotonically decreasing in r.

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2

Existence of ground state

2.1. Initial value problem and nonlinearity

In this chapter, we will study an existence proof for the initial value problem

$$-u''(r) - \frac{N-1}{r}u'(r) = g(u(r)), \quad \text{on } 0 < r < \infty, \tag{2.1}$$

satisfying initial conditions and an integrability condition

$$\begin{cases} u(0) = \alpha, \\ u'(0) = 0 \\ \lim_{r\to\infty} u(r) = 0. \end{cases} \tag{2.2}$$

The existence proof will be based on [1], which generalises earlier results. One of these is the uniqueness result [2], which was later generalised in [3], which forms the basis for the next chapter.

The proof will be by a shooting method, where we categorise the solutions based on their asymptotic behaviour. Furthermore, solutions to the initial value problem equation (2.1) are also positive radial solutions to the more general problem

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \tag{2.3}$$

where $g(u)$ is a given nonlinear function. This partial differential equation is relevant to many areas of mathematical physics.

The solutions $R(r)$ to equation (1.22) are solutions $u(r)$ to (2.1) with $N = 2$ and

$$g(u) = -u + u^3.$$

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2.2. DEFINITIONS OF SOLUTION SETS

A **ground state solution** is strictly decreasing everywhere and has no finite zeroes. Yet, the solution should vanish in the limit as $r \to \infty$.

We define the set G of ground state initial conditions as

$$G := \left\{ \alpha > 0 \,\middle|\, u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}. \tag{2.4}$$

We consider two alternatives: either (i) the derivative vanishes, or (ii) the solution vanishes. We define the set P of initial conditions with a vanishing derivative as

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.5}$$

We define the set N of initial conditions with a vanishing solution as

$$N := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u(r_0,\alpha) = 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.6}$$

These solution sets are disjoint by definition and we write the union of initial conditions as $I = P \mathbin{\dot\cup} G \mathbin{\dot\cup} N$.

2.3. INTERVAL OF DEFINITION

Existence of local unique solutions is guaranteed by the Picard-Lindelöf theorem, see for example [4, Theorem. 2.2].

In these circumstances, boundedness of the solution $u(r,\alpha)$ is a sufficient condition for the solution to be defined on the maximal interval $[0,\infty)$. This is also called the *blow-up alternative*. Either (i) for some $r_0 \geqslant 0$ we have

$$\left|u(r_0,\alpha)\right| > M, \qquad \text{for all } M \geqslant 0,$$

or (ii) the solution is defined for all $r \geqslant 0$.

In this section, we will derive an upper and a lower bound for $u(r,\alpha)$. Since the solution is initially decreasing, possibly the initial condition α is an upper bound.

Lemma 1. $u(r,\alpha) \leqslant u(0,\alpha) = \alpha$ *for* $r \geqslant 0$.

Proof. In this proof, we write $u(r) = u(r,\alpha)$ for brevity. We start with (2.1) and multiply by $u'(r)$. Then we integrate from 0 to r to obtain

$$-\int_0^r \left[u'(s)u''(s)\right] ds - \int_0^r \left[\frac{N-1}{s}[u'(s)]^2\right] \mathrm{d}s = \int_0^r \left[u'(s)g(u(s))\right] \mathrm{d}s. \tag{2.7}$$

We use the chain rule simplify the first term in (2.7) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}[u'(r)^2] = 2u'(r)u''(r) \overset{(2.2)}{\iff} \frac{1}{2}[u'(r)]^2 = \int_0^r \left[u'(s)u''(s)\right] \mathrm{d}s.$$

Then, we rewrite the right hand side of (2.7) using the fundamental theorem of calculus

$$\int_0^r \left[u'(s)g(u(s))\right] \mathrm{d}s = \int_0^r \left[\tfrac{\mathrm{d}u}{\mathrm{d}s} g(u(s))\right] \mathrm{d}s = \int_{u(0)}^{u(r)} g(u)\,\mathrm{d}u = G(u(r)) - G(u(0)).$$

Finally, using $u(0) = \alpha$, we have rewritten (2.7) as

$$-\frac{1}{2}\left[u'(r)\right]^2 - (N-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = G(u(r)) - G(\alpha). \tag{2.8}$$

This expression will be useful in the following proof by contradiction. We suppose by contradiction that

$$u(r_0) > \alpha, \quad \text{for some } r_0 > 0. \tag{2.9}$$

By the assumptions on $g(u)$, we have $g(u) > 0$ on I. As a result, $G(u)$ is increasing on I. Using assumption (2.9) and $\alpha > \kappa$, we deduce that

$$G(u(r_0)) > G(\alpha) \iff G(u(r_0)) - G(\alpha) > 0.$$

This contradicts (2.8), as the left hand side is clearly non-positive. □

We now proceed to show that $u(r,\alpha)$ has a lower bound. Let r_0 be the first zero of $u(r,\alpha)$

$$r_0 := \inf\{\, r > 0 \mid u(r,\alpha) = 0 \,\}. \tag{2.10}$$

Suppose that $r_0 < \infty$. Alternatively, $r_0 = \infty$ would imply that $u(r,\alpha) > 0$ for all $r > 0$ and we have a lower bound.

Lemma 2. *Suppose $r_0 < \infty$. Then for $r \geqslant r_0$, we have*

$$u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{N-1} u'(r_0,\alpha) \geqslant u'(r_0,\alpha). \tag{2.11}$$

Proof. We consider the sign of $u'(r_0,\alpha)$. Firstly, if $u'(r_0,\alpha) = 0$ then u and u' vanish simultaneously in r_0. Then from (2.1) we have

$$u'' = 0, \quad \text{with } u(r_0) = u'(r_0) = 0,$$

which is solved by

$$u(r) = c_1 r + c_2,$$

where we must have $c_1 = c_2 = 0$ to satisfy the conditions at r_0, so that $u \equiv 0$. But this contradicts $u(0,\alpha) = \alpha > 0$. Hence, u and u' cannot vanish simultaneously for $\alpha > 0$.

Secondly, if $u'(r_0,\alpha) > 0$ we also reach a contradiction. By (2.1) with $u(r_0,\alpha) = 0$, we see that u'' and u' have opposite signs in r_0. Then either:

$$\begin{cases} \text{(i) } u'' > 0 \quad \text{and} \quad u' < 0 \quad \text{in } r_0, \quad \text{or} \\ \text{(ii) } u'' < 0 \quad \text{and} \quad u' > 0 \quad \text{in } r_0. \end{cases} \tag{2.12}$$

The latter case implies that $u(r,\alpha) < 0$ in a left neighborhood of r_0, which contradicts $u(r,\alpha) > 0$ on $[0, r_0)$. Thus, we have $u'(r_0,\alpha) < 0$.

In the following, we extend $g(u) = 0$ for $u \leqslant 0$. Then for $u(r,\alpha) \leqslant 0$ the IVP (2.1) reads

$$-u''(r,\alpha) - \frac{N-1}{r} u'(r,\alpha) = 0, \tag{2.13}$$

We solve (2.13) for $u' = u'(r,\alpha)$ and seperate the variables, resulting in

$$\frac{\mathrm{d}u'}{u'} = -\frac{N-1}{r}\,\mathrm{d}r.$$

We integrate the expression from r_0 to r and evaluate the limits to get

$$\ln u'\big|_{r_0}^{r} = \left[(N-1)\ln r\right]_r^{r_0} \iff \ln u'(r) - \ln u'(r_0) = (N-1)\left[\ln r_0 - \ln r\right].$$

Then, we rewrite the expression to arrive at the desired result

$$\frac{u'(r)}{u'(r_0)} = \left(\frac{r_0}{r}\right)^{N-1} \iff u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{N-1} u'(r_0,\alpha) \geqslant u'(r_0,\alpha). \tag{2.14}$$

□

Since $u'(r_0,\alpha)$ is non-positive, $u'(r,\alpha)$ will remain non-positive and converge to zero. This implies that $u(r,\alpha)$ is bounded from below by $L = \lim_{r\to\infty} u(r,\alpha)$.

We have found an upper and a lower bound for $u(r,\alpha)$ and as such, it is defined on the maximal interval $(0,\infty)$. For all $r > 0$, we have $L \leqslant u(r,\alpha) \leqslant \alpha$.

2.4. Lemma 1

Lemma 3. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function such that $g(0) = 0$. Let $u(r,\alpha_1)$ be a solution to initial value problem (2.1) with $\alpha_1 \in (0,\infty)$ such that*

$$\begin{cases} u(r,\alpha_1) > 0 & \text{for all } r \geqslant 0 \quad \text{and} \\ u'(r,\alpha_1) < 0 & \text{for all } r > 0. \end{cases}$$

Then the number $l = \lim_{r\to\infty} u(r,\alpha_1)$ satisfies $g(l) = 0$.

Proof. Let α_1 be as assumed in the lemma and consider the limit as $r \to \infty$ of equation (2.1)

$$\lim_{r\to\infty}\left[u''(r,\alpha_1) + \frac{N-1}{r} u'(r,\alpha_1)\right] = -g(l). \tag{2.15}$$

By the assumptions of the lemma on $u(r,\alpha_1)$ we have that $u(r,\alpha_1)$ is monotone decreasing and hence $0 \leqslant u(r,\alpha_1) \leqslant \alpha_1$ for all $r > 0$. By the monotone convergence theorem, the limit $l = \lim_{r\to\infty} u(r,\alpha_1)$ exists, is finite and non-negative.

We consider the limit as $r \to \infty$ in equation (2.8)

$$\lim_{r\to\infty}\left[-\frac{1}{2}\left[u'(r,\alpha_1)\right]^2-(N-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}\right]=\lim_{r\to\infty}G(u(r,\alpha_1))-G(\alpha_1).$$

We use $\lim_{r\to\infty}G(u(r,\alpha_1))=G(l)$ and rearrange the terms

$$\lim_{r\to\infty}\frac{1}{2}[u'(r,\alpha_1)]^2+(N-1)\int_0^\infty[u'(\alpha_1,s)]^2\frac{ds}{s}=G(\alpha_1)-G(l). \tag{2.16}$$

We note that the right hand side of (2.16) is finite. Then both terms of the left hand side should be finite too. This can only be true when $u'(r,\alpha_1)$ converges. Remember that $u(r,\alpha_1)$ is bounded, so if the derivative converges, it must converge to zero

$$\lim_{r\to\infty}u'(r,\alpha_1)=0.$$

Now, we return to equation (2.15) and use $\lim_{r\to\infty}u'(r,\alpha_1)=0$ to obtain

$$-\lim_{r\to\infty}\left[u''(r,\alpha_1)\right]=g(l).$$

However, we found that $u'(r,\alpha_1)$ needs to be bounded and even converge to 0. Hence, $u''(r,\alpha_1)$ needs to bounded and converge to 0.

$$\lim_{r\to\infty}u''(r,\alpha_1)=0.$$

The desired result follows: $g(l)=0$. □

The following lemma is non-obvious, as the nonlinearity $g(u)$ has more than one zero. Both $g(0)=0$ and $g(\kappa)=0$. We will show that an additional assumption on $g(u)$ ensures that $l\neq\kappa$, whenever $g(l)=0$.

Lemma 4. *In addition to lemma 3, if $g(u)$ satisfies*

$$\lim_{s\downarrow\kappa}\frac{g(s)}{s-\kappa}>0$$

then $l\neq\kappa$. Hence, $g(l)=0$ is only satisfied by

$$l=\lim_{r\to\infty}u(r,\alpha_1)=0.$$

Proof. Suppose to the contrary that $l=\kappa$. We will use the substitution

$$v(r)=r^{(1/2)(N-1)}\left[u(r,\alpha_1)-\kappa\right]$$

in equation (2.1) to obtain a differential equation in $v(r)$. In the remainder of the proof, we will abbreviate $u(r,\alpha_1)=u(r)$. We note that $v(r)>0$ by definition, as the assumption is that $u(r)>\kappa$ for $r>0$ and $u(r)\downarrow\kappa$.

2

We proceed to calculate the first derivative

$$v'(r) = \frac{1}{2}(N-1)r^{(N-3)/2}\left[u(r)-\kappa\right] + r^{(N-1)/2}u'(r),$$

and the second derivative, where we gather the terms by $u(r)$, $u'(r)$ and $u''(r)$ as

$$v''(r) = \frac{1}{4}(N-1)(N-3)r^{(N-5)/2}\left[u(r)-\kappa\right] + (N-1)r^{(N-3)/2}u'(r) + r^{(N-1)/2}u''(r). \quad (2.17)$$

We multiply the IVP (2.1) by $r^{(N-1)/2}$ to obtain

$$-r^{(N-1)/2}u''(r) - (N-1)r^{(N-1)/2}r^{-1}u'(r) = g(u(r))r^{(N-1)/2}. \quad (2.18)$$

We can use this to simplify (2.17)

$$v''(r) = \frac{1}{4}(N-1)(N-3)r^{(N-1)/2}r^{-2}\left[u(r)-\kappa\right] - g(u(r))r^{(N-1)/2}.$$

Now we factor out $v(r) = r^{(N-1)/2}\left[u(r)-\kappa\right]$ to obtain

$$v''(r) = r^{(N-1)/2}\left[u(r)-\kappa\right]\left\{\frac{1}{4}(N-1)(N-3)r^{-2} - \frac{g(u)}{u(r)-\kappa}\right\}.$$

Lastly, we multiply by -1 to obtain the exact expression from [1] as

$$-v''(r) = \left\{\frac{g(u)}{u(r)-\kappa} - \frac{(N-1)(N-3)}{4r^2}\right\}v \quad (2.19)$$

The first term in braces is positive by definition. The second term decreases with r.

Lemma 5. *There exist $\omega > 0$ and $R_1 > 0$ such that*

$$\frac{g(u)}{u(r)-\kappa} - \frac{(N-1)(N-3)}{4r^2} \geqslant \omega \quad \textit{for all } r \geqslant R_1. \quad (2.20)$$

Proof. Note that the sign of $g(u)$ changes in κ, more specifically: $g(u) < 0$ whenever $u(r) < \kappa$ and $g(u) > 0$ whenever $u(r) > \kappa$. Thus, the first term is positive everywhere.

We define

$$M = \max_{r \geqslant 0} \frac{g(u(r))}{u(r)-\kappa},$$

and rewrite (2.20) to obtain

$$\frac{(N-1)(N-3)}{4r^2} \leqslant M - \omega \quad (2.21)$$

We note that choosing any $\omega \leqslant M$ yields a non-negative value for the right hand side of (2.21). In the case that $N = 2$, the left hand side evaluates to a negative number. Then, the inequality holds for all $r \geqslant 0$. In the case that $N = 1$ or $N = 3$, the left hand side evalutes to 0. Then too, the inequality holds for $r \geqslant 0$. In the case that $N \geqslant 4$, we write $(N-1)(N-3)$ as $(N-2)^2 - 1$ and obtain

$$\frac{(N-2)^2 - 1}{4r^2} \leqslant M - \omega$$

Now, depending on the choice of $\omega \leqslant M$, there will be a least $R_1 \geqslant 0$ such that the inequality holds. Since $\omega \leqslant M$ and $r > 0$, we solve and find that for

$$r \geqslant \frac{(N-2)^2 - 1}{2\sqrt{M - \omega}} = R_1$$

the inequality (2.21) holds. This concludes the proof of lemma 5. □

By lemma 5 we have $v''(r) < 0$ for $r \geqslant R_1$, which implies by concavity that

$$v'(r) \downarrow L \geqslant -\infty, \quad \text{as } r \to \infty.$$

We cannot have $L < 0$. Then $v' \downarrow -\infty$ and $v \downarrow -\infty$. But $v > 0$ by assumption. We also cannot have $L \geqslant 0$. This would imply $v \geqslant v(R_1) > 0$ for all $r \geqslant R_1$. But this still agrees with $v'' < 0$ for $r \geqslant R_1$ and hence, $v' \downarrow -\infty$, contradicting $L \geqslant 0$.

Hence $l = \kappa$ is an impossible assumption under the assumptions on g and hence $l \neq \kappa$. This concludes the proof of lemma 4. □

2.5. P IS OPEN

1. P is open, by continuity
2. $u(r,\alpha)$ and $u'(r\alpha)$ depend continuously on the initial condition α
3. N is open by similar argument

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4. for $\alpha \in I_+$ we have

$$r_0 = \inf\{r > 0 \mid u'(r,\alpha) = 0 \text{ and } u(r,\alpha) > 0\}$$

5. such that

$$\begin{cases} u(r,\alpha) > 0 & \text{for all } r \in [0, r_0] \\ u'(r,\alpha) < 0 & \text{for all } r \in (0, r_0). \end{cases} \tag{2.22}$$

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6. Evaluating the IVP (2.1) in r_0 yields

$$u''(r_0,\alpha) = -g(u(r_0,\alpha))$$

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7. we consider two cases:
8. (i) $u''(r_0,\alpha) = 0$, such that

$$u(r_0,\alpha) = \alpha \quad \text{and } u'(r_0,\alpha) = 0.$$

9. by a uniqueness argument: $u(r,\alpha) \equiv \alpha$ which is impossible.

10. (ii) $u''(r_0,\alpha) \neq 0$ and hence $u''(r_0,\alpha) > 0$.

11. this implies existence of $r_1 > r_0$ such that

$$u(r,\alpha) > u(r_0,\alpha) \quad \text{for all } r \in (r_0, r_1].$$

12. now by continuity, we argue that for α^* near α we have

$$\begin{cases} u(r_1,\alpha^*) > u(r_0,\alpha^) \\ \alpha^* > u(r,\alpha^*) > 0 \quad \text{for all } r \in (0, r_1]. \end{cases} \tag{2.23}$$

13. this implies that $\alpha^* \in I_+$ and hence that I_+ is open.

P and N invoke continuous dependence on the initial data as explained in detail in for example [**?**].

Lemma 6. *Under the assumptions of* **??**, *the sets P and N are non-empty, disjoint and open.*

Proof. Consider the following cases: (i) $\alpha_p \in N$, (ii) $\alpha_p \notin P$, (iii) $\alpha_p \in P$ and note they are mutually exclusive. If the initial condition is in N, then the solution vanishes in some point r_0. Then, if the initial condition is not in N, the solution does not vanish anywhere: $u(r,\alpha_p) > 0$ for $r \geqslant 0$. If the initial condition is not in P, then the derivative is negative everywhere. Disproving these two cases yields the properties: the solution is positive everywhere, but the derivative vanishes in some point r_0, which is exactly the definition of P. Hence disproving case (i) and (ii) implies case (iii) applies.

2.5.1. Assumptions on g

We assume that g is locally Lipschitz continuous from $\mathbb{R}_+ \to \mathbb{R}$ and satisfies $g(0) = 0$. Local Lipschitz continuity is an important condition for the Picard-Lindelöf local existence and uniqueness theorem. Additionally, we assume that hypotheses (H1)–(H5) are satisfied. Firstly,

$$g(\kappa) = 0, \text{ for some } \kappa > 0. \tag{H1}$$

Secondly, defining $G(t)$ as the integral of $g(t)$

$$G(t) := \int_0^t g(s)\,\mathrm{d}s, \tag{2.24}$$

there exists an initial condition $\alpha > 0$ such that $G(\alpha) > 0$. We define

$$\alpha_0 := \inf\{\alpha > 0 \mid G(\alpha) > 0\}. \tag{H2}$$

Thirdly, the derivative $g'(s)$ is positive in κ

$$g'(\kappa) = \lim_{s \downarrow \alpha} \frac{g(s) - g(\kappa)}{s - \kappa} > 0, \tag{H3}$$

and fourthly we have

$$g(s) > 0 \quad \text{for } s \in \left(\kappa, \alpha_0\right]. \tag{H4}$$

We define

$$\lambda := \inf\left\{ \alpha > \alpha_0 \,\middle|\, g(\alpha) = 0 \right\}, \tag{2.25}$$

and note that $\alpha_0 < \lambda \leqslant \infty$. In the situation where $\lambda = \infty$ we make the final assumption

$$\lim_{s\to\infty} \frac{g(s)}{s^l} = 0, \quad \text{with } l < \frac{N+2}{N-2} \tag{H5}$$

Step 1. Solution set P is non-empty.

Let $\alpha_p \in (\kappa, \alpha_0]$. We know that $G(\alpha)$ is increasing for $\alpha > \kappa$ and $G(\alpha_0) = 0$ by assumptions on (**??**) on g . Since $\alpha_p < \alpha_0$, we have $G(\alpha_p) < 0$.

For $\alpha \in N$, there exists a r_0 such that

$$\begin{cases} u(r_0, \alpha) = 0, \quad u'(r_0, \alpha) < 0 \quad \text{for } r \leqslant r_0 \end{cases} \tag{2.26}$$

1. First, we suppose by contradiction that $\alpha_p \in N$.

2. Then by definition, there exists a r_0 such that $u(r_0, \alpha_p) = 0$ and $u'(r_0, \alpha_p) < 0$.

3. Now evaluate **??** in r_0

$$\begin{aligned}
-\frac{1}{2}[u'(r_0)]^2 - (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s} &= G(u(r_0, \alpha_p)) - G(\alpha_p) \\
G(\alpha_p) - G(u(r_0, \alpha_p)) &= \frac{1}{2}[u'(r_0)]^2 + (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s} \\
G(\alpha_p) - G(0) &= \frac{1}{2}[u'(r_0)]^2 + (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s} \\
G(\alpha_p) &= \frac{1}{2}[u'(r_0)]^2 + (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s} \\
&\implies G(\alpha_p) > 0,
\end{aligned}$$

but the assumption $\alpha_p \in (\kappa, \alpha_0]$ implies $G(\alpha_p) \leqslant 0$, a contradiction, hence $\alpha_p \notin N$.

$\alpha_p \notin P$ Next, suppose $\alpha_p \notin P$, then $u(r, \alpha_p) > 0$ for $r \geqslant 0$ and $u'(r, \alpha_p) < 0$ for $r > 0$. That implies $u(r, \alpha_p) \downarrow l \geqslant 0$ as $r \uparrow \infty$ and by lemma 3 the limit is zero, $l = 0$. Similar to the previous argument and the proof of **??**, observe **??** for r tending to infinity, and note the following:

$$l = \lim_{r\to\infty} u(r, \alpha_p), \; G(l) = G(0) = 0,$$

and

$$\lim_{r\to\infty} \left[u'(r, \alpha_p)\right]^2 = 0,$$

as well as the lower bound on the integral term

$$\int_0^\infty \left[u'(\alpha_p, s)\right]^2 \frac{ds}{s} > 0,$$

to conclude that:

$$\begin{aligned}
\lim_{r\to\infty}\left[G(\alpha_p) - G(u(r,\alpha_p))\right] &= \lim_{r\to\infty}\left[\frac{1}{2}[u'(r)]^2 + (N-1)\int_0^r [u'(s)]^2 \frac{ds}{s}\right] \\
G(\alpha_p) - G(l) &= 0 + (N-1)\int_0^\infty [u'(s)]^2 \frac{ds}{s} \\
G(\alpha_p) - G(0) &> 0 \\
\implies G(\alpha_p) &> 0,
\end{aligned}$$

to reach the same contradiction, $\alpha_p \in (\kappa, \alpha_0] \implies G(\alpha_p) \leqslant 0$.

Conclusion Hence $\alpha_p \in P$, and any $\alpha \in (\kappa, \alpha_0]$ is in P, that is, $(\kappa, \alpha_0] \subset P$ and P is nonempty.

Claim: Solution set P is open. Let $u(r,\alpha)$ be a solution to equation (2.1) with $\alpha \in P$. From the above, we know that the solution set P is nonempty. By # (ref), both $u(r,\alpha)$ and $u'(r,\alpha)$ depend continously on α. From the definition of solution set P, solutions with $\alpha \in P$ has the following property: there exists an $r_0 > 0$ such that the derivative vanishes.

$$u'(r_0, \alpha) = 0 \text{ for some } r_0 > 0.$$

Let r_0 be the smallest $r_0 > 0$ for which the derivative vanishes, $u'(r_0,\alpha) = 0$.

$$r_0 := \inf\left\{r > 0 : u(r,\alpha) > 0, u'(r,\alpha) = 0\right\} > 0.$$

Not only does the derivative vanish in r_0. We know the solution to be positive and decreasing up to r_0. Remember, when $\alpha \in P$, we have:

$$\begin{cases} u(r,\alpha) > 0 & \text{for all } r \in [0, r_0] \\ u'(r,\alpha) < 0 & \text{for all } r \in (0, r_0). \end{cases}$$

Evaluating the equation (2.1) in r_0 with the assumption that $u'(r_0) = 0$:

$$\begin{aligned}
u'' + \frac{N-1}{r}u' - u + u^3 &= 0 \\
u''(r_0,\alpha) &= -g(u(r_0,\alpha)).
\end{aligned}$$

Cases for $u''(r_0,\alpha)$ Consider the following cases: $u''(r_0,\alpha) = 0$ or $u''(r_0,\alpha) \neq 0$. Suppose by contradiction that $u''(r_0,\alpha) = 0$. Then, since the solution attains values strictly between α and 0... the only candidate value is $u(r_0,\alpha) = 0$, such that $g(u(r_0,\alpha)) = 0$. That is, since $u(r,\alpha) \in (0,\alpha)$ we have .. Remember the zeroes of $g(u(r,\alpha))$ are 0 and κ. **Solution must have** $u(r_0,\alpha) \equiv \kappa$ But the solution is positive everywhere r_0: $\alpha \in P \implies 0 < u(r_0,\alpha) < \alpha$. In conclusion the only value such that $u''(r_0,\alpha) = -g(u(r_0,\alpha)) = 0$ is $u(r_0,\alpha) = \kappa$. $u(r_0,\alpha) \equiv \kappa$ **as per example** However, let $u \equiv \kappa$ be a solution to the equation (2.1). # (Example of $g(u) = -u + u^3$.) Explicitly, $\kappa = 1$, and evaluating $u \equiv 1$ in the equation (2.1) yields $0 + 0 - 1 + 1^3 = 0$. Hence the solution $u \equiv \kappa$ solves the equation (2.1) and $u(r_0, \alpha) = \kappa$ and $u'(r_0, \alpha) = 0$. **Contradiction is found** But by Picard-Lindelöf, the solutions to the equation (2.1) are unique! Now the solution $u \equiv \kappa$ does not satisfy the earlier assumptions: $u'(r, \alpha) < 0$ on $(0, r_0)$, a contradiction! So the assumption $u''(r_0, \alpha) = 0$ must be false $\lightning$. **Other case for** $u''(r_0, \alpha)$ On the other hand, if $u''(r_0, \alpha) \neq 0$ then $u''(r_0, \alpha) > 0$. Concavity forces this. Also, $u''(r_0, \alpha) = -g(u(r_0, \alpha)) > 0 \implies u(r_0, \alpha) < \kappa$. In $r = 0$, evaluating # IVP, where $u'(0, \alpha) = 0$ and $u(0, \alpha) = \alpha$, $u''(0, a) = -g(\alpha) < 0$, (from the graph of g) so u is concave down. **Second derivative changes sign once** Now, we know that for larger r, the derivative vanishes $u''(r, \alpha) = 0$ for some r_0, that is, $u'(r_0, \alpha) = 0$. Then the solution has precisely one inflection point between 0 and r_0, that is, the second derivative changes sign exactly once. **Conclusion** $u''(r_0, \alpha) > 0$ In conclusion, $u''(r_0, \alpha) > 0$. # PICTURE. The combination of $u''(r_0, \alpha) > 0$ and $u'(r_0, \alpha) = 0$ implies that the solution is increasing for r larger than r_0 but small enough. That is, there exists $r_1 > r_0$ such that $u(r, \alpha) > u(r_0, \alpha)$ for all $r \in (r_0, r_1]$. **First mention of ε-tube** Now consider an ε-tube around $u(r, \alpha)$ which guarantees that $u(r_1, \beta) > u(r_0, \beta)$ and $0 < u(r, \beta) < \beta$ for all $r \in (0, r_1]$. By continuous dependence on the initial data, this ε-tube implies $u(r, \beta)$ solves the # IVP with β also in P. # PICTURE of ε-tube. Define the average of $u(r_1, \alpha)$ and $u(r_0, \alpha)$ as $\overline{u_\alpha} := \frac{1}{2}\left[u(r_1, \alpha) - u(r_0, \alpha)\right] > 0$. **Continuous dependence implies...** By continuous dependence on the initial data $\exists \delta > 0 : |u(r, \alpha) - u(r, \beta)| < \varepsilon$ whenever $|\alpha - \beta| < \delta$. Let $\varepsilon = \overline{u_\alpha}$. Then $\exists\, \delta_\alpha > 0$ such that $|u(r, \alpha) - u(r, \beta)| < \overline{u_\alpha} = \ldots$ whenever $|\alpha - \beta| < \delta_\alpha$. $u(r_1, \beta) > u(r_0, \beta)$ fails when $u(r_1, \beta) \leqslant u(r_0, \beta)$, which is when $u(r_1, \beta)$ is minimal and $u(r_0, \beta)$ is maximal, that is, when $u(r_1, \beta) = u(r_1, \alpha) - \varepsilon$ and $u(r_0, \beta) = u(r_0, \alpha) + \varepsilon$. Another extreme case is when $u(r, \beta) \leqslant 0$ for some $\tilde{r}$ in $(0, r_1]$. ε-tube around $u(r, \alpha)$. Note $u(r_1, \alpha) - \varepsilon > u(r_0, \alpha) + \varepsilon$ **The tube guarantees** $\implies u(r_1, \alpha) - u(r_0, \alpha) > 2\varepsilon$. Dividing both sides by 2 yields: $\varepsilon = \overline{u_\alpha} := \frac{1}{2}\left[u(r_1, \alpha) - u(r_0, \alpha)\right] > 0$ (1) $u(r_0, \beta) - u(r_0, \alpha) < \varepsilon$ (2) $u(r_1, \alpha) - u(r_1, \beta) < \varepsilon \ldots u(r_0, \beta) < \frac{1}{2}\overline{u_\alpha}$ $u(r_1, \beta) > \frac{1}{2}\overline{u_\alpha} \implies u(r_0, \beta) < \frac{1}{2}\overline{u_\alpha} < u(r_1, \beta) \implies u(r_1, \beta) - u(r_0, \beta) > \frac{1}{2}\overline{u_\alpha} > 0$. In other words, the ε-tube definitely guarantees $u(r_1, \beta) > u(r_0, \beta)$. If the tube also guarantees $0 < u(r, \beta) < \beta$ then $\beta \in P$. **About open sets, examples** Remember that a set is open if it contains a ball around each of its points. Conversely, a set is closed if it is not open. An example would be the interval $[0, 2)$ that contains a ball around any value near 2. However, 0 is in the interval, yet a ball around zero includes negative values, which are not in the interval. Conclusion: $[0, 2)$ is not open. **Formal definition of open** That is, for any $\alpha \in P$ there exists $\varepsilon > 0$ such that $B_\varepsilon(\alpha) \subset P$. Here $B_\varepsilon(\alpha)$ denotes an ε-ball around α, which is defined as

$$B_\varepsilon(\alpha) := \left\{\beta \in I : |\beta - \alpha| < \varepsilon\right\}$$

Open intervals Equivalently, P is open if for every $\alpha \in P$ there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha + \varepsilon) \subset P$. **Let $\alpha \in P$ leads to $g(u(r, \alpha)) = 0$ or κ and ...** Now, to show that P is open, let α be an initial condition in P. A valid assumption, as P is nonempty. The definition of P yields the following properties of $u(r,\alpha)$:

$$\begin{cases} u(r,\alpha) > 0 \text{ on } [0,r_0] & (2.27a)\\ u'(r,\alpha) < 0 \text{ on } (0,r_0) & (2.27b)\\ u(r_0,\alpha) > 0 & (2.27c)\\ u'(r_0,\alpha) = 0 & (2.27d)\end{cases}$$

(chapter numbering of equations, not section.) Using equation (2.27d) in the initial value problem yields:

$$u''(r_0,\alpha) = -g(u(r_0,\alpha)).$$

Uniqueness argument Consider two cases, $u''(r_0,\alpha) = 0$ and $u''(r_0,\alpha) \neq 0$. The former case requires $g(u(r_0,\alpha)) = 0$. From earlier analysis, the zeroes of $g(u(r,\alpha))$ are 0 and κ. But property (1) contradicts $u(r,\alpha) = 0$. By a uniqueness argument $u(r,\alpha) = \kappa$ also fails. **Why $u \equiv \kappa$ fails** Hence, $u''(r_0,\alpha) \neq 0$. To see this, consider $u \equiv \kappa$. This satisfies property (3) and (4) and solves the IVP. By uniqueness of solutions # REF, Boundary values, $u \equiv \kappa$ is the only solution that satisfies (3) and (4). But this contradicts property (2), $u'(r) < 0$ on $(0,r_0)$. As a consequence, $\kappa \notin P$ and $u''(r,\alpha)$ and $u'(r,\alpha)$ never vanish for the same r if $\alpha \in P$. **Let β be in ε-ball around** $\alpha \in P$ Let $\beta \in B_\varepsilon(\alpha)$, that is, let β within ε-distance of α, that is, let $|\beta - \alpha| < \varepsilon$. Then for β to be a solution in P, the same properties need to be derived. All in all, show that $u(r,\beta)$ is positive up to some r_0, and there exists r_1 such that $u(r_1,\beta) > u(r_0,\beta)$.

Continuous dependence on initial data Define $\overline{u_\alpha} := \frac{1}{2}u(r_1,\alpha) - \frac{1}{2}u(r_0,\alpha)$. Let $\varepsilon = \overline{u_\alpha}$. Then by continuous dependence on initial data, $\exists\delta > 0$ s.t. $|\alpha - \beta| < \delta \implies |u(r,\alpha) - u(r,\beta)| < \overline{u_\alpha}$. This also implies:

$$\text{Statement 1: } |u(r_0,\beta) - u(r_0,\alpha)| < \overline{u_\alpha},$$
$$\text{Statement 2: } |u(r_1,\beta) - u(r_1,\alpha)| < \overline{u_\alpha}$$

Start over again, properties of $\alpha \in P$ Let $\alpha \in P$. Remember that P is non-empty by the previous results. By definition of P we have:

$$r_0 = \inf\{r > 0,\ u'(\alpha,r) = 0,\ u(\alpha,r) > 0\} > 0$$

and

$$\begin{cases} u(\alpha,r) > 0 & \text{for all } r \in [0,r_0]\\ u'(\alpha,r) < 0 & \text{for all } r \in (0,r_0).\end{cases}$$

Split into two cases about $u''(r_0,\alpha)$, $u \equiv \kappa$ **is impossible** By equation (2.1) we have $u''(\alpha,r_0) = -g(u(\alpha,r_0))$. Consider as the first of two cases, $u''(\alpha,r_0) = 0$. Then $g(u(\alpha,r_0)) = 0$. The only zero of g where $0 < u(\alpha,r_0) < \alpha$ is κ, i.e. $u(\alpha,r_0) = \kappa$. But since $u'(\alpha,r_0) = 0$ and $u''(\alpha,r_0) = 0$, $u(\alpha,r) \equiv \kappa$, which is impossible. $u''(r_0,\alpha) \neq 0$ **leads to** $u(r,\alpha) > u(r_0,\alpha)$ Consider the second case, $u''(\alpha,r_0) \neq 0$. Then because the derivative was negative up to r_0 and has now vanished, $u''(\alpha,r_0) > 0$. This implies the derivative is positive to the right of r_0 and there exists a $r_1 > r_0$ such that $u(\alpha,r) > u(\alpha,r_0)$ for all $r \in (r_0,r_1]$. # FIGURE. **Continuous dependence implies** $\alpha^* \in P$ By continuous dependence on the initial condition, let α^* be sufficiently close to α then for all $r \in (0, r_1]$ the following hold:

$$\begin{cases} u(\alpha^*, r_1) > u(\alpha^*, r_0) \\ \alpha^* > u(\alpha^*, r) > 0 \end{cases}$$

which can be interpreted as: for α^* sufficiently small, $u(\alpha^*, r_1)$ is still above $u(\alpha^*, r_0)$, the derivative vanishes in some point $r_0^* \in (0, r_1]$ and the solution does not vanish on $(0, r_1]$. Then these properties together imply $\alpha^* \in P$ and hence P is open. **About open sets** P is open if for any initial condition α in P, there exists a real number $\epsilon_\alpha > 0$ such that points whose distance from α is less than ϵ_α are also in P. **Critical case for** $u(r_0, \beta) < u(r_1, \beta)$Of these absolute inequalities, the most critical case (that allows $u(r_0, \beta) > u(r_1, \beta)$) is when $u(r_0, \beta)$ is maximal within the ε-tube and $u(r_1, \beta)$ is minimal within the ε-tube. That is, if $u(r_0, \beta) < u(r_1, \beta)$ for this case, it is true for any combination of $u(r_0, \beta)$ and $u(r_1, \beta)$. Hence we have the following to prove: **ε-tube again, many equations** Drawing of ε tube around $u(r, \alpha)$. Required: $u(r_1, \alpha) - \varepsilon > u(r_0, \alpha) + \varepsilon \implies u(r_1, \alpha) - u(r_0, \alpha) > 2\varepsilon$.

$$\begin{aligned} \text{Statement 1: } u(r_0, \beta) - u(r_0, \alpha) < \varepsilon = \frac{1}{2}u(r_1, \alpha) - \frac{1}{2}u(r_0, \alpha), \\ \text{Statement 2: } u(r_1, \alpha) - u(r_1, \beta) < \varepsilon = \frac{1}{2}u(r_1, \alpha) - \frac{1}{2}u(r_0, \alpha). \end{aligned}$$

$$\begin{aligned} 2u(r_0, \beta) - 2u(r_0, \alpha) < u(r_1, \alpha) - u(r_0, \alpha), \\ 2u(r_1, \alpha) - 2u(r_1, \beta) < u(r_1, \alpha) - u(r_0, \alpha). \end{aligned}$$

$$\begin{aligned} 2u(r_0, \beta) - u(r_0, \alpha) < u(r_1, \alpha), \\ u(r_1, \alpha) - 2u(r_1, \beta) < -u(r_0, \alpha). \end{aligned}$$

$$\begin{aligned} u(r_0, \beta) < \frac{1}{2}\big(u(r_1, \alpha) + u(r_0, \alpha)\big), \\ u(r_1, \beta) > \frac{1}{2}\big(u(r_1, \alpha) + u(r_0, \alpha)\big). \end{aligned}$$

$$\begin{aligned} \implies u(r_0, \beta) < \frac{1}{2}\Delta u_\alpha < u(r_1, \beta). \\ \implies u(r_1, \beta) - u(r_0, \beta) > \frac{1}{2}\Delta u_\alpha > 0. \end{aligned}$$

Conclusion The combination of $u(r_1, \beta) > u(r_0, \beta)$ and $u(r, \beta) \in (0, \beta) \implies \beta \in P$. □

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