2. Physics of NLS

In this section, first, the nonlinear Schrödinger equation (NLS) will be derived from Maxwell's laws. A brief discussion of the assumptions involved in this derivation and the interpretation of (intermediate) results is given. In the end, the NLS is given as

$$2ik_0\psi_z(x,y,z) + \underbrace{\Delta_\perp\psi}_{\text{diffraction}} + \underbrace{k_0^2\frac{4n_2}{n_0}|\psi|^2\psi}_{\text{Kerr nonlinearity}} = 0. \tag{2.1}$$

The propagation of electromagnetic waves in a medium is governed by Maxwell's laws. (In absence of external charges or currents.) Remember that Maxwell's laws for the electric field $\mathcal{E}$, magnetic field $\mathcal{H}$, induction electric field $\mathcal{D}$ and induction magnetic field $\mathcal{B}$ are given by:

$$\begin{gathered}\nabla\times\vec{\mathcal{E}} = -\frac{\partial\vec{\mathcal{B}}}{\partial t}, \quad \nabla\times\vec{\mathcal{H}} = -\frac{\partial\vec{\mathcal{D}}}{\partial t}, \\ \nabla\cdot\vec{\mathcal{D}} = 0, \quad \nabla\cdot\vec{\mathcal{B}} = 0.\end{gathered} \tag{2.2}$$

These are vector fields: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in x, y, z coordinates. In vacuum, the relations between the electric or magnetic fields and induction fields are given as:

$$\vec{\mathcal{B}} = \mu_0\vec{\mathcal{H}}, \quad \vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} \tag{2.3}$$

From these relations and the vector identity for the curl of the curl, the wave equation can be derived. In particular,

$$\begin{aligned}\nabla\times\nabla\times\vec{\mathcal{E}} &= \nabla\times(-\frac{\partial\vec{\mathcal{B}}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla\times\vec{\mathcal{B}}), &&\text{by Maxwell's laws, and}\\ \nabla\times\nabla\times\vec{\mathcal{E}} &= \nabla(\nabla\cdot\vec{\mathcal{E}}) - \nabla^2\vec{\mathcal{E}} = \nabla(\nabla\cdot\vec{\mathcal{E}}) - \Delta\vec{\mathcal{E}}, &&\text{by vector calculus.}\end{aligned}$$

Also, calculate the curl of the magnetic field: $\nabla\times\vec{\mathcal{B}} = \mu_0\frac{\partial\vec{\mathcal{D}}}{\partial t}$. Then combining these results:

$$\begin{gathered}\Delta\vec{\mathcal{E}} - \nabla(\nabla\cdot\vec{\mathcal{E}}) = \mu_0\frac{\partial^2\vec{\mathcal{D}}}{\partial t^2}\\ \Delta\vec{\mathcal{E}} - \nabla(\frac{1}{\epsilon_0}\nabla\cdot\vec{\mathcal{D}}) = \mu_0\epsilon_0\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}.\end{gathered}$$

And using $\nabla\cdot\vec{\mathcal{D}} = \nabla\cdot\epsilon_0\vec{\mathcal{E}} = 0$, this yields the wave equation, where $\mu_0\epsilon_0 = 1/c^2$:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{2.4}$$

Scalar solutions, plane waves propagating in positive z-direction and general direction, complex conjugates, wavenumber.

Polarisations, linear polarisation inconsistent with Maxwell, but in leading order still possible: $E_1 << E_2, E_3$.

Helmholtz equation, laser beam as sum of plane waves, mostly parallel to z-axis (paraxial).

Split into $E = \exp^{ik_0 z} \psi(x, y, z)$, with ψ an envelope function varying slowly in z. This ψ solves a Helmholtz equation. Neglect ψ_{zz} (paraxial) to obtain the linear Schrödinger equation for ψ.

Polarisation, linear polarisation, weakly nonlinear polarisation, Kerr nonlinearity. This all leads to nonlinear Helmholtz, apply paraxial approximation to obtain NLS.

Step over to dimensionless NLS and consider solitary waves.

Fill in details. Then, by considering radially symmetric solitary wave solutions, one obtains:

$$R'' + \frac{1}{r}R' - R + R^3 = 0,$$

with initial condition $R'(0) = 0$ and finite power: $\lim_{r\to\infty} R(r) = 0$. This is the equation for which existence and uniqueness of solutions will be discussed.
