# An Explication of Existence and Uniqueness Results for a Nonlinear Schrödinger equation

## An Introduction to the Shooting Method and Sturm Comparison Theorem

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# 1

# Physics of NLS

## 1.1. Derive the wave equation from Maxwell

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \quad (1.1.a) \qquad \nabla \cdot \vec{\mathcal{D}} = 0, \quad (1.1.c)$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t}, \quad (1.1.b) \qquad \nabla \cdot \vec{\mathcal{B}} = 0. \quad (1.1.d)$$

The fields are in three-dimensional Cartesian coordinates, for example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in $(x, y, z)$ coordinates. Besides considering no external charges or currents, we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \quad (1.2.a) \qquad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}. \quad (1.2.b)$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [1, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [2]. This reference work also includes an introduction to the necessary vector calculus.

We use vector calculus and Maxwell's laws to rewrite the curl of the curl:

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) \overset{(1.1.a)}{=} \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathcal{B}}\right) \overset{(1.1.b)}{\underset{}{\overset{(1.2.a)}{=}}} -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} \overset{(1.2.b)}{=} -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \text{ and}$$

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \nabla^2 \vec{\mathcal{E}} = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \Delta \vec{\mathcal{E}} \overset{(1.1.c)}{\underset{}{\overset{(1.2.b)}{=}}} -\Delta \vec{\mathcal{E}}.$$

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Combining these and using $\mu_0\epsilon_0 = 1/c^2$, we arrive at the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{1.3}$$

## 1.2. VALIDITY OF PLANE WAVE SOLUTIONS

Stuyding the left and right hand sides of equation (1.3), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{l=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_l^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.4}$$

where $k_0$ is the wavenumber and $\omega_0$ the frequency. These are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any $z$-value and the electric field is non-zero in the $x$ and $y$ directions. The wavefronts are spaced by the wavelength $\lambda$ and the wavenumber $k_0$ is the reciprocal of the wavelength.

This plane wave travels in the positive $z$-direction for positive wavenumber $k_0$ and vice versa. Note that the solution does not depend on $x$ or $y$. As a result, for a fixed $z'$, the electric field $\mathcal{E}$ is constant in the $(x, y, z')$-plane.

We substitute (1.4) in equation (1.3). Note that only $\Delta_z$ will be non-zero:

$$\Delta\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation (1.5):

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.5}$$

For a general direction in $(x, y, z)$-coordinates, define the wavevector

$$\vec{k} = (k_x, k_y, k_z),$$

where $\left|\vec{k}^2\right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$. This satisfies equation (1.3) when $\vec{k}\perp\vec{\mathcal{E}}$ and

$$\mathcal{E}_j = E_c e^{i(\vec{k}\cdot\vec{r}-\omega_0 t)}. \tag{1.6}$$

## 1.3. DERIVATION OF THE HELMHOLTZ EQUATION

We consider time-harmonic solutions to the scalar wave equation (1.3) of the form

$$\mathcal{E}_j(x,y,z,t) = e^{i\omega_0 t}E(x,y,z) + \text{c.c}, \tag{1.7}$$

which are continuous wave beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, we refer to [3].

Substituting (1.7) in equation (1.3) and taking the derivatives leads to the expression

$$\begin{aligned}\Delta\left(e^{-i\omega_0 t}E\right) &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(e^{-i\omega_0 t}E\right)\\ e^{-i\omega_0 t}\Delta E &= \frac{1}{c^2}(-i\omega_0)^2 E e^{-i\omega_0 t},\end{aligned}$$

where we can divide by $e^{-i\omega_0 t} \neq 0$ and use the dispersion relation (1.5) to arrive at the scalar linear Helmholtz equation for $E$

$$\Delta E(x,y,z) + k_0^2 E = 0. \tag{1.8}$$

As an example, equation (1.8) is solved by the general-direction plane waves (1.6), where

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

## 1.4. DERIVATION OF THE LINEAR SCHRÖDINGER EQUATION

REVISE: We write the incoming field $E_0^{\text{inc}}(x,y)$ as a sum of plane waves. Then the electric field $E(x,y,z)$ for non-zero $z$-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and it is essential to Fourier optics. We have

$$\begin{aligned}E_0^{\text{inc}}(x,y) &= \frac{1}{2\pi}\int_D E_c(k_x,k_y)\, e^{i(k_x x + k_y y)}\mathrm{d}k_x\mathrm{d}k_y, \quad \text{such that}\\ E(x,y,z) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} E_c(k_x,k_y)\, e^{i\left(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z\right)}\mathrm{d}k_x\mathrm{d}k_y,\end{aligned}$$

where $D$ denotes the (circular) laser input beam domain. For laser beams oriented in the $z$-direction, most of the plane wave modes are nearly parallel to the $z$-axis, which implies $k_z \approx k_0$. We define $k_\perp^2 = k_x^2 + k_y^2$, such that $k_0^2 = k_\perp^2 + k_z^2$. It is equivalent to $k_0 \approx k_z$ to say that $k_\perp \ll k_z$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{1.9}$$

where $\psi(x,y,z)$ is an envelope (or amplitude) function. The envelope shape may vary over $z$, in contrast to soliton solutions, see (2.5).

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Substituting (1.9) into the Helmholtz equation (1.8) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp\psi = 0, \tag{1.10}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that for lasers beams oriented in the $z$-direction, the wavenumber $k_z$ dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope function $\psi(x,y,z)$ will vary slowly in $z$ and curve even more slowly.

Claim: $|\psi_{zz}| \ll k_0|\psi_z|$ and $|\psi_{zz}| \ll \Delta_\perp\psi$.

REVISE: To see this, we first show that $k_0 - k_z \ll 1$. We factor out $k_0^2$, take the square root on both sides and linearise the square root term of the right hand side:

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right) \implies k_z = k_0\left(1 - \frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right).$$

Finally, we use $k_\perp \ll k_0$ to obtain the intermediate result:

$$k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} \ll 1.$$

For the first statement of the claim, $|\psi_{zz}| \ll k_0|\psi_z|$, it is equivalent to show that the ratio of $|\psi_{zz}|$ over $k_0|\psi_z|$ is much smaller than 1. We calculate the ratio as follows:

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0-k_z)^2 E_c}{k_0(k_0-k_z)E_c} = \frac{k_0-k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0}\cdot\frac{1}{k_0} \ll 1.$$

For the other statement of the claim, we calculate:

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0-k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0-k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} \ll \frac{1}{4}\frac{k_\perp^2}{k_0^2} \ll 1.$$

Using the approxations in equation (1.10) yields the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp\psi = 0. \tag{1.11}$$

## 1.5. POLARISATION FIELD

Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E},0,0),\ \vec{\mathcal{P}} = (\mathcal{P},0,0),\ \vec{\mathcal{D}} = (\mathcal{D},0,0),$$

Furthermore, we assume that $\mathcal{E}$ is the continuous wave electric field from (1.7). We write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.12}$$

where the $c_i$ are real for all $i$. Note that $c_0 = 0$ except in ferro-electric materials. The constants $c_i$ are actually a function of the frequency $\omega_0$. We rewrite $c_i = \epsilon_0\chi^{(\mathrm{i})}(\omega_0)$, where $\chi^{(\mathrm{i})}$ is the $i$-th order susceptibility. Then equation (1.12) reads:

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.13}$$

First we consider linear polarisation:

$$\mathcal{P}_{\mathrm{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}.$$

Then the induction electric field $\mathcal{D}$ is given by:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} = \epsilon_0\mathcal{E} + \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E} = \epsilon_0\mathcal{E}\left(1 + \chi^{(1)}(\omega_0)\right) = \epsilon_0 n_0^2(\omega_0)\mathcal{E},$$

where $n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0)$ is the linear index of refraction (or refractive index) of the medium.

With this updated induction electric field $\mathcal{D} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}$, we can update the scalar wave equation and Helmholtz equation. Only the dispersion relation is affected by considering linear polarisation:

$$k_0^2 = \frac{\omega_0^2}{c^2} n_0^2(\omega_0). \tag{1.14}$$

We now consider the nonlinear polarisation field $\mathcal{P}_{\mathrm{nl}}$ as the difference between the true polarisation and the linear approximation:

$$\mathcal{P} = \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$\begin{aligned} -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}(-\mathcal{E})^2 + \epsilon_0\chi^{(3)}(-\mathcal{E})^3 + \epsilon_0\chi^{(4)}(-\mathcal{E})^4 + \epsilon_0\chi^{(5)}(-\mathcal{E})^5 + \mathcal{O}\left(\mathcal{P}^6\right) \\ -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\mathcal{E}^2 - \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 - \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right), \end{aligned}$$

where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\mathrm{nl}}$ and we have only the odd terms:

$$\mathcal{P}_{\mathrm{nl}} = \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^7\right) \tag{1.15}$$

The leading-order term is called the Kerr nonlinearity:

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}^3. \tag{1.16}$$

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## 1.6. IMPLICATIONS OF NONLINEAR POLARISATION

Substituting the continuous wave electric field (1.7) into equation (1.16) yields

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}^3 = 3\chi^{(3)}(\omega_0)\,|E|^2\, E e^{i\omega_0 t} + \chi^{(3)}(\omega_0) E^3 e^{3i\omega_0 t} + \text{c.c.},$$

where the second term has a frequency of $3\omega_0$ (third harmonic). This has almost no contribution due to the phase-mismatch with the first harmonic. Hence, we approximate

$$\mathcal{P}_{\mathrm{nl}} \approx 3\epsilon_0 \chi^{(3)}(\omega_0)\,|E|^2\, E e^{i\omega_0 t} + \text{c.c.} = 3\epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}.$$

Then we simplify $\mathcal{P}_{\mathrm{nl}}$ by defining

$$n_2 := \frac{3\chi^{(3)}}{4\epsilon_0 n_0},$$

so that we obtain the simplified expression

$$\mathcal{P}_{\mathrm{nl}} = 4\epsilon_0 n_0 n_2 \,|E|^2\, \mathcal{E}.$$

This allows us to write the induction electric field $\mathcal{D}$ as,

$$\mathcal{D} = \epsilon_0 \mathcal{E} + \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}} = \epsilon_0 n^2 \mathcal{E},$$

where

$$n^2 = n_0^2\left(1 + \frac{4n_2}{n_0}\,|E|^2\right) = n_0^2 + 3\chi^{(3)}(\omega_0)\frac{1}{\epsilon_0}|E|^2.$$

For water, $n_2 \sim 10^{-22}$ which justifies neglecting nonlinear effects. With lasers, the nonlinear effect becomes more relevant, but is still weak. For a typical continuous wave laser with $|E| \sim 10^9$, we still have a weak nonlinearity, as $n_2|E| \sim 10^{-4} \ll n_0 \approx 1.33$.

We update equation (1.8) to the scalar nonlinear Helmholtz equation (NLH):

$$\Delta E(x,y,z) + k^2 E = 0, \quad \text{where } k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right). \tag{1.17}$$

We write $E(x,y,z)$ as the product of the $z$-propagation and an envelope function $\psi(x,y,z)$:

$$E = e^{ik_0 z}\psi$$

and substitute in (1.17) to obtain:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp \psi + 4k_0^2 \frac{n_2}{n_0}\left|\psi\right|^2 \psi = 0. \tag{1.18}$$

Just as in section 1.4, we apply the paraxial approximation, since for laser beams oriented in the $z$-direction, we have $\left|\psi_{zz}\right| \ll k_0\left|\psi_z\right|, \left|\psi_{zz}\right| \ll \Delta_\perp \psi$. We finally obtain the nonlinear Schrödinger equation (NLS):

$$2ik_0\psi_z(z,\overline{x}) + \Delta_\perp \psi + k_0^2 \frac{4n^2}{n_0}\left|\psi\right|^2 \psi = 0. \tag{1.19}$$

## 1.7. SOLITON SOLUTIONS

The NLS equation (1.19) can be written as a dimensionless equation. Starting from equation (1.18), we apply the rescaling of coordinates $(x, y, z) \to (\tilde{x}, \tilde{y}, \tilde{z})$ defined by:

$$\tilde{x} = \frac{x}{r_0} \quad \tilde{y} = \frac{y}{r_0} \quad \tilde{z} = \frac{z}{2L_{\text{diff}}},$$

where $r_0$ is the input beam width and $L_{\text{diff}}$ is the diffraction length. We refer to chapter 2 of [1] for more information on the geometrical optics of lasers. There, we also find that $L_{\text{diff}} = k_0 \cdot r_0^2$. To rescale $\tilde{\psi}$, we define:

$$\tilde{\psi} = \frac{\psi}{E_c}, \quad \text{where } E_c := \max_{x,y} \left|\psi_0(x, y)\right|.$$

Through the rescaling we obtain the dimensionless NLH for $\tilde{\psi}$:

$$\frac{f^2}{4}\tilde{\psi}_{\tilde{z}\tilde{z}}(\tilde{z}, \tilde{x}, \tilde{y}) + i\tilde{\psi}_{\tilde{z}} + \Delta_\perp \tilde{\psi} + \nu \left|\tilde{\psi}\right|^2 \tilde{\psi} = 0,$$

that depends on a nonparaxiality parameter $f$ and a nonlinearity parameter $\nu$:

$$f = \frac{1}{r_0 k_0} = \frac{r_0}{L_{\text{diff}}}, \quad \nu = r_0^2 k_0^2 \frac{4n_2}{n_0} E_c^2.$$

Here the approximation of paraxiality is valid for small $f \ll 1$ and this leads to the dimensionless NLS equation (1.20), where the tildes have been dropped for brevity.

$$i\psi_z(z, x, y) + \Delta_\perp \psi + \nu \left|\psi\right|^2 \psi = 0. \tag{1.20}$$

Radial solitary-wave solutions to (1.20) were considered in [4] with $\psi$ of the form:

$$\psi_\omega^{\text{solitary}}(r, z) = e^{i\omega z} R_\omega(r), \tag{1.21}$$

where $\omega$ is a real number and $R_\omega$ is the real solution of

$$-\omega R_\omega + \Delta_\perp R_\omega(r) + R_\omega^3 = 0.$$

This can be solved in general by, for example,

$$R_\omega(r) = \sqrt{\omega} R\left(\sqrt{\omega} r\right).$$

However, taking $\omega = 1$ leads to the simplest soliton equation

$$R''(r) + \frac{1}{r}R' - R + R^3 = 0, \quad 0 < r < \infty, \tag{1.22}$$

subject to initial condition $R'(0) = 0$ and integrability condition $\lim_{r\to\infty} R(r) = 0$. The (numerical) solution is known as the Townes profile, which is positive and monotonically decreasing in $r$.

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# 2

# EXISTENCE OF GROUND STATE

## 2.1. INITIAL VALUE PROBLEM AND NONLINEARITY

In this chapter, we will study an existence proof for the initial value problem

$$-u''(r) - \frac{n-1}{r}u'(r) = f(u(r)), \quad \text{on } 0 < r < \infty, \tag{2.1}$$

satisfying initial conditions and an integrability condition

$$\begin{cases} u(0) = \alpha, \\ u'(0) = 0 \\ \lim\limits_{r\to\infty} u(r) = 0. \end{cases} \tag{2.2}$$

The existence proof will be based on [1], which generalises earlier results. One of these is the uniqueness result [2], which was later generalised in [3], which forms the basis for the next chapter.

The proof will be by a shooting method, where we categorise the solutions based on their asymptotic behaviour. Furthermore, solutions to the initial value problem equation (2.1) are also positive radial solutions to the more general problem

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

where $f(u)$ is a given nonlinear function. This partial differential equation is relevant to many areas of mathematical physics.

The solutions $R(r)$ to equation (1.22) are solutions $u(r)$ to (2.1) with $n = 2$ and

$$f(u) = -u + u^3.$$

## 2.2. DEFINITIONS OF SOLUTION SETS

A **ground state solution** is strictly decreasing everywhere and has no finite zeroes. Yet, the solution should vanish in the limit as $r \to \infty$.

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We define the set $G$ of ground state initial conditions as

$$G := \left\{ \alpha > 0 \;\middle|\; u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}. \tag{2.4}$$

We consider two alternatives: either (i) the derivative vanishes, or (ii) the solution vanishes. We define the set $P$ of initial conditions with a vanishing derivative as

$$P := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.5}$$

We define the set $N$ of initial conditions with a vanishing solution as

$$N := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u(r_0,\alpha) = 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.6}$$

We note that the sets $P$ and $N$ are disjoint by definition. Either the derivative vanishes first, or the solution vanishes first.

We will show that the sets $P$ and $N$ are non-empty, and open. Then, there exist initial conditions that belong to neither $P$ nor $N$. Solutions that belong to neither $P$ nor $N$ are *everywhere positive and decreasing*

$$\begin{cases} u(r,\alpha) > 0 & \text{for } r \geqslant 0, \text{ and} \\ u'(r,\alpha) < 0 & \text{for } r > 0. \end{cases} \tag{2.7}$$

Lastly, we will show that under certain assumptions, such an element belongs to $G$.

## 2.3. ASSUMPTIONS ON $f$

We assume that $f$ is locally Lipschitz continuous from $\mathbb{R}_+ \to \mathbb{R}$ and satisfies $f(0) = 0$. Additionally, we assume that hypotheses (H1)–(H5) are satisfied. Firstly,

$$f(\kappa) = 0, \text{ for some } \kappa > 0. \tag{H1}$$

Secondly, defining $F(t)$ as the integral of $f(t)$

$$F(t) := \int_0^t f(s)\,\mathrm{d}s, \tag{2.8}$$

there exists an initial condition $\alpha > 0$ such that $F(\alpha) > 0$. We define

$$\alpha_0 := \inf\{\alpha > 0 \mid F(\alpha) > 0\}. \tag{H2}$$

Thirdly, the right-derivative of $f(s)$ at $\kappa$ is positive

$$f'(\kappa^+) = \lim_{s\downarrow\kappa} \frac{f(s) - f(\kappa)}{s - \kappa} > 0, \tag{H3}$$

and fourthly, we have

$$f(s) > 0 \quad \text{for } s \in \left(\kappa, \alpha_0\right]. \tag{H4}$$

We define

$$\lambda := \inf\left\{\, \alpha > \alpha_0 \;\middle|\; f(\alpha) = 0 \,\right\}, \tag{2.9}$$

and note that $\alpha_0 < \lambda \leqslant \infty$. In the situation where $\lambda = \infty$, we assume

$$\lim_{s\to\infty} \frac{f(s)}{s^l} = 0, \quad \text{with } l < \frac{n+2}{n-2}. \tag{H5}$$

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## 2.4. MAIN THEOREM

**Theorem 2.1.** *Let $f$ be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0,\infty)$ such that $f(0) = 0$ and $f$ satisfies hypotheses $(H1) - (H5)$. Then there exists a number $\alpha \in (\alpha_0, \lambda)$ such that the solution $u(r,\alpha) \in C^2(\mathbb{R}_+)$ of the initial value problem*

$$\begin{cases} -u''(r) - \dfrac{n-1}{r} u'(r) = f(u(r)), \; \textit{for } r > 0, \\ u(0) = \alpha, \quad u'(0) = 0 \end{cases} \tag{2.10}$$

*is an element of solution set $G$ defined in* (2.4)

$$G := \left\{ \alpha > 0 \;\middle|\; u(r,\alpha) > 0 \textit{ and } u'(r,\alpha) < 0 \textit{ for all } r > 0 \textit{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}.$$

*Proof.* We will show in Lemma 2.1-2.3 that solutions to the differential problem (2.10) are defined for $0 < r < \infty$. Furthermore, by Lemma 2.4 solutions with $\alpha \notin \left(P \dot\cup N\right)$ satisfy

$$\lim_{r\to\infty} u(r,\alpha) = 0.$$

Lastly, we will show that solution sets $P$ and $N$ are non-empty and open. In Lemma 2.5 we show that solution set $P$ is non-empty and open. By similar argument, solution set $N$ is open. For the argument that $N$ is non-empty, we refer to "$I_-$ is non-empty" in [1, p. 147].

In conclusion, $G$ is non-empty. □

## 2.5. INTERVAL OF DEFINITION

Existence of local unique solutions is guaranteed by the Picard-Lindelöf theorem, see for example [4, Theorem. 2.2].

In these circumstances, boundedness of the solution $u(r,\alpha)$ is a sufficient condition for the solution to be defined on the maximal interval $[0,\infty)$. This is also called the *blow-up alternative*. Either (i) for some $r_0 > 0$ we have

$$\left|u(r_0,\alpha)\right| > M, \qquad \text{for all } M > 0,$$

and the solution is defined on $[0, r_0)$. Or (ii) for some $M > 0$ we have

$$\left|u(r,\alpha)\right| \leqslant M, \quad \text{for all } r \geqslant 0,$$

and the solution is defined for all $r \geqslant 0$.

2

**Lemma 2.1.** *For any initial condition $\alpha > 0$ and $r > 0$, we have the identity*

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)). \tag{2.11}$$

*Proof.* We multiply the IVP (2.1) by $-u'(r)$. Then we integrate from 0 to $r$ to obtain

$$\int_0^r \left[u'(s)u''(s)\right] ds + \int_0^r \left[\frac{n-1}{s}[u'(s)]^2\right] \mathrm{d}s = -\int_0^r \left[u'(s)f(u(s))\right] \mathrm{d}s. \tag{2.12}$$

We use the chain rule simplify the first term in (2.12) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}[u'(r)^2] = 2u'(r)u''(r) \overset{(2.2)}{\Longleftrightarrow} \frac{1}{2}[u'(r)]^2 = \int_0^r \left[u'(s)u''(s)\right] \mathrm{d}s.$$

Then, we rewrite the right-hand side of (2.12)

$$-\int_0^r \left[u'(s)f(u(s))\right] \mathrm{d}s = \int_r^0 \left[\tfrac{\mathrm{d}u}{\mathrm{d}s} f(u(s))\right] \mathrm{d}s$$

and use the fundamental theorem of calculus

$$\int_{u(r)}^{u(0)} f(u)\,\mathrm{d}u = F(u(0)) - F(u(r)).$$

Finally, using $u(0) = \alpha$, we have rewritten (2.12) as

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)).$$ □

In this section, we will derive an upper and a lower bound for $u(r,\alpha)$. Since the solution is initially decreasing, possibly the initial condition $\alpha$ is an upper bound.

**Lemma 2.2.** *Let $\alpha > \kappa$. Then $u(r,\alpha) \leqslant u(0,\alpha) = \alpha \quad$ for $r \geqslant 0$.*

*Proof.* We suppose by contradiction that

$$\alpha < u(r_0,\alpha) < \lambda, \quad \text{for some } r_0 > 0. \tag{2.13}$$

By (H4) and (2.9), we have $F$ non-decreasing on $(\kappa, \lambda)$. Then,

$$F(\kappa) < F(\alpha) < F(u(r_0,\alpha)) < F(\lambda).$$

In particular, we have

$$F(\alpha) - F(u(r_0,\alpha)) < 0.$$

This contradicts Lemma 2.1, as the left-hand side is clearly non-negative. □

We will show that $u(r,\alpha)$ has a lower bound for $r<\infty$. Let $r_0$ be the first zero of $u(r,\alpha)$

$$r_0 := \inf\{\, r>0 \,|\, u(r,\alpha)=0\,\}. \tag{2.14}$$

If $r_0=\infty$, then we have $u(r,\alpha)>0$ for all $r>0$. When $r_0<\infty$, we have the following bound on the derivative $u'(r,\alpha)$.

2

**Lemma 2.3.** *Suppose that there exists $r_0>0$ such that*

$$\begin{cases} u(r_0,\alpha)=0 \\ u'(r_0,\alpha)<0. \end{cases} \tag{2.15}$$

*If we have $f(u)=0$ for $u\leqslant 0$, then for $r\geqslant r_0$ we have*

$$u'(r,\alpha)=\left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha)\geqslant u'(r_0,\alpha). \tag{2.16}$$

*Proof.* For $u(r,\alpha)\leqslant 0$ the IVP (2.1) reads

$$-u''(r,\alpha)-\frac{n-1}{r}u'(r,\alpha)=0, \tag{2.17}$$

We solve (2.17) for $u'=u'(r,\alpha)$ and seperate the variables, resulting in

$$\frac{\mathrm{d}u'}{u'}=-\frac{n-1}{r}\mathrm{d}r.$$

We integrate the expression from $r_0$ to $r$ and evaluate the limits

$$\ln u'\big|_{r_0}^{r}=\left[(n-1)\ln r\right]_r^{r_0} \iff \ln u'(r)-\ln u'(r_0)=(n-1)\left[\ln r_0-\ln r\right].$$

Then, we rewrite the expression to arrive at the desired result

$$\frac{u'(r)}{u'(r_0)}=\left(\frac{r_0}{r}\right)^{n-1} \iff u'(r,\alpha)=\left(\frac{r_0}{r}\right)^{n-1}u'(r_0,\alpha)\geqslant u'(r_0,\alpha). \qquad \square$$

In conclusion, the solution $u(r,\alpha)$ is bounded for bounded $r$. More specifically, in the case of everywhere positive solutions, we have

$$0<u(r,\alpha)\leqslant\alpha \quad \text{for all } r>0.$$

Alternatively, for solutions with $u(r_0,\alpha)=0$ and $u'(r_0,\alpha)<0$ by Lemma 2.3 we have

$$u(r,\alpha)\geqslant\int_{r_0}^{r}\left(\frac{r_0}{s}\right)^{n-1}u'(r_0,\alpha)\,\mathrm{d}s>-\infty \quad \text{for } r>r_0, \tag{2.18}$$

such that for $n=2$, we have

$$u(r,\alpha)\geqslant r_0 u'(r_0,\alpha)\left(\ln r-\ln r_0\right) \tag{2.19}$$

and for $n>2$, we have

$$u(r,\alpha)\geqslant\frac{r_0^{n-1}u'(r_0,\alpha)}{2-n}\left(r^{2-n}-r_0^{2-n}\right). \tag{2.20}$$

2

## 2.6. ASYMPTOTICS OF POSITIVE DECREASING SOLUTIONS

In this section, we will show that everywhere positive decreasing solutions $u(r,\alpha)$ vanish in the limit as $r\to\infty$.

**Lemma 2.4.** *Let $f:\mathbb{R}^+\to\mathbb{R}$ be a locally Lipschitz continuous function such that $f(0)=0$. Let $u(r,\alpha_1)$ be a solution to initial value problem (2.1) with $\alpha_1\in(0,\infty)$ such that*

$$\begin{cases} u(r,\alpha_1)>0 & \text{for all } r\geqslant 0, \quad \text{and} \\ u'(r,\alpha_1)<0 & \text{for all } r>0. \end{cases} \tag{2.21}$$

*Then the number $l:=\lim_{r\to\infty} u(r,\alpha_1)$ satisfies $f(l)=0$.*

*If additionally, $f(u)$ satisfies (H3), then $l=0$.*

*Proof step 1.* By assumption (2.21) on $u(r,\alpha_1)$ and the monotone convergence theorem, we have $0\leqslant l<\alpha_1$. Then $f(l)<f(\alpha_1)$. We consider the limit as $r\to\infty$ of the IVP (2.1)

$$\lim_{r\to\infty}\left[-u''(r,\alpha_1)-\frac{n-1}{r}u'(r,\alpha_1)\right]=f(l)<\infty. \tag{2.22}$$

We restate equation (2.11)

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2+(n-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}=F(\alpha_1)-F(u(r,\alpha_1))$$

and note that the right hand side is finite. We write

$$(n-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}=F(\alpha_1)-F(u(r,\alpha_1))-\frac{1}{2}\left[u'(r,\alpha_1)\right]^2$$

and note that the left hand side is increasing and bounded above. Hence,

$$\int_0^\infty u'(s,\alpha_1)^2\frac{\mathrm{d}s}{s}<\infty.$$

We write

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2=F(\alpha_1)-F(u(r,\alpha_1))-(n-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}.$$

Then $\lim_{r\to\infty}u'(r,\alpha_1)^2$ exists. Since $u'(r,\alpha_1)<0$ and $u(r,\alpha_1)$ is bounded, we have

$$\lim_{r\to\infty}u'(r,\alpha_1)=0. \tag{2.23}$$

Now, we return to equation (2.22) and use $\lim_{r\to\infty}u'(r,\alpha_1)=0$ to obtain

$$-\lim_{r\to\infty}\left[u''(r,\alpha_1)\right]=f(l).$$

We have (2.23) and hence, we have

$$\lim_{r\to\infty} u''(r,\alpha_1) = 0.$$

The desired result follows: $f(l) = 0$. □

*Proof step 2.* The nonlinearity $f(u)$ has more than one zero. Both $f(0) = 0$ and $f(\kappa) = 0$. We will show that under assumption (H3), only $l = 0$ satisfies the IVP (2.1).

Suppose to the contrary that $l = \kappa$. We will use the substitution

$$v(r) = r^{(1/2)(n-1)}\left[u(r,\alpha_1) - \kappa\right] \tag{2.24}$$

in equation (2.1) to obtain a differential equation in $v(r)$. In the remainder of the proof of this lemma, we will abbreviate $u(r,\alpha_1) = u(r)$. We note that $v(r) > 0$ by definition, since we have $u(r) \downarrow \kappa$.

We proceed to calculate the first derivative $v'(r)$

$$v'(r) = \frac{1}{2}(n-1)r^{(n-3)/2}\left[u(r)-\kappa\right] + r^{(n-1)/2}u'(r),$$

and the second derivative $v''(r)$, where we gather the terms by $u(r)$, $u'(r)$ and $u''(r)$

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-5)/2}\left[u(r)-\kappa\right] + (n-1)r^{(n-3)/2}u'(r) + r^{(n-1)/2}u''(r). \tag{2.25}$$

We multiply the IVP (2.1) by $r^{(n-1)/2}$ to obtain

$$-r^{(n-1)/2}u''(r) - (n-1)r^{(n-1)/2}r^{-1}u'(r) = f(u(r))r^{(n-1)/2}. \tag{2.26}$$

We can use this to simplify (2.25) to

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-1)/2}r^{-2}\left[u(r)-\kappa\right] - f(u(r))r^{(n-1)/2}.$$

Now we factor out $v(r) = r^{(n-1)/2}\left[u(r)-\kappa\right]$ to obtain

$$v''(r) = r^{(n-1)/2}\left[u(r)-\kappa\right]\left\{\frac{1}{4}(n-1)(n-3)r^{-2} - \frac{f(u)}{u(r)-\kappa}\right\}.$$

Lastly, we multiply by $-1$ to obtain the exact expression from [1] as

$$-v''(r) = \left\{\frac{f(u)}{u(r)-\kappa} - \frac{(n-1)(n-3)}{4r^2}\right\}v. \tag{2.27}$$

In proof step 3, we will show that there exist $\omega > 0$ and $R_1 > 0$, such that

$$\frac{f(u)}{u(r)-\kappa} - \frac{(n-1)(n-3)}{4r^2} \geqslant \omega \quad \text{for all } r \geqslant R_1. \tag{2.28}$$

We have $v''(r) < 0$ for $r \geqslant R_1$, which implies by

$$v'(r) = v'(R_1) + \int_{R_1}^{r} v''(s)\,\mathrm{d}s$$

that

$$v'(r) \downarrow L \geqslant -\infty, \quad \text{as } r \to \infty.$$

Suppose that $L < 0$, then $v(r) \to -\infty$ as $r \to \infty$. However, by (2.24) we have $v > 0$.

Then $L \geqslant 0$. This implies $v'(r) \geqslant 0$ for $r \geqslant R_1$. But then $v(r) \geqslant v(R_1) > 0$ for $r \geqslant R_1$. By (2.28) and (2.27), we have

$$-v''(r) \geqslant \omega v(R_1) > 0,$$

such that $v'(r) \to -\infty$ as $r \to \infty$. This contradicts $L \geqslant 0$. Hence, we have $l = 0$. □

*Proof step 3.* The first term (2.28) is non-negative and decreasing by (H3). We will write

$$M(r) := \frac{f(u)}{u(r)-\kappa} > 0, \tag{2.29}$$

and rewrite (2.28) to obtain

$$M(r) \geqslant \frac{(n-1)(n-3)}{4r^2} + \omega. \tag{2.30}$$

We choose $2\omega = \max_{r>0} M(r)$ and choose $R_1 > 0$ such that

$$\frac{(n-1)(n-3)}{4r^2} \leqslant \frac{1}{2} M(r) \quad \text{for } r \geqslant R_1.$$

□

## 2.7. $P$ IS NON-EMPTY AND OPEN

In this section we will show that $P$ is non-empty and open. The proof that $N$ is open is similar to the proof given for $P$. For the proof that $N$ is non-empty, we refer to "$I_-$ is non-empty" in [1, p. 147].

**Lemma 2.5.** *Solution set $P$ as defined in* (2.5)

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0,\alpha) = 0 \textit{ and } u(r,\alpha) > 0 \textit{ for all } r \leqslant r_0 \right\}$$

*is non-empty and open.*

*Proof step 1.* We will show that solution set $P$ is non-empty. Let $\alpha \in (\kappa, \alpha_0]$. We refer to (H1) and (H2) for the definitions of $\kappa$ and $\alpha_0$.

First, we suppose by contradiction that $\alpha \in N$. By the definition of $N$ in (2.6), there exists a number $r_0 > 0$ such that

$$\begin{cases} u(r_0, \alpha) = 0, \\ u'(r, \alpha) < 0 \quad \text{for } \ r \leqslant r_0. \end{cases} \tag{2.31}$$

We restate equation (2.11) from Lemma 2.1 for $r = r_0$ and use $F(u(r_0, \alpha)) = F(0) = 0$

$$\frac{1}{2}\left[u'(r_0, \alpha)\right]^2 + (n-1)\int_0^{r_0} u'(s, \alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha). \tag{2.32}$$

The left hand side of (2.32) is positive. For $\alpha \in (\kappa, \alpha_0]$, we have $F(\alpha) < 0$. Hence $\alpha \notin N$.

Next, we suppose that $\alpha \notin P$. Thus $\alpha \notin (P \mathbin{\dot\cup} N)$. We have the situation of (2.7)

$$\begin{cases} u(r, \alpha) > 0 \quad \text{for } r \geqslant 0, \text{ and} \\ u'(r, \alpha) < 0 \quad \text{for } r > 0, \end{cases}$$

which is the setting of Lemma 2.4. Thus, we have $l = 0$ and by equation (2.23), we have

$$\lim_{r \to \infty} u'(r, \alpha) = 0.$$

Then equation (2.32) evaluates to

$$(n-1)\int_0^{\infty} u'(s, \alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) < 0,$$

but the left hand side is positive. We have $(\kappa, \alpha_0] \subset P$, since $\alpha$ was chosen arbitrarily. □

*Proof step 2.* We will show that $P$ is open. Let $\alpha \in P$. There exists

$$r_0 := \inf\left\{ r > 0 \,\middle|\, u'(r, \alpha) = 0 \text{ and } u(r, \alpha) > 0 \right\}$$

such that by the definition of $P$ in (2.5)

$$\begin{cases} u(r, \alpha) > 0 \quad \text{for all } r \in [0, r_0], & (2.33a) \\ u'(r, \alpha) < 0 \quad \text{for all } r \in (0, r_0). & (2.33b) \end{cases}$$

Evaluating the IVP (2.1) in $r_0$ yields

$$u''(r_0, \alpha) = -f(u(r_0, \alpha)).$$

Suppose that $u''(r_0, \alpha) = 0$. Then $-f(u(r_0, \alpha)) = 0$. The zeroes of $f(u)$ are $f(\kappa) = 0$ and $f(0) = 0$. Thus, $u(r_0, \alpha) = \kappa$ by (2.33a).

2

Then, the differential equation (2.1) with

$$\begin{cases} u(r_0,\alpha) = \kappa, \\ u'(r_0,\alpha) = 0, \\ u''(r_0,\alpha) = 0 \end{cases}$$

is solved by $u \equiv \kappa$, and by uniqueness of solutions this contradicts $u(0,\alpha) = \alpha > \kappa$. Hence $u''(r_0,\alpha) \neq 0$. Since $u'(r,\alpha) < 0$ for $r < r_0$ and $u'(r_0,\alpha) = 0$, we have

$$u''(r_0,\alpha) > 0.$$

Then there exists $r_1 > r_0$, such that

$$u(r,\alpha) > u(r_0,\alpha) \quad \text{for all } r \in (r_0, r_1].$$

Since $u(r,\alpha)$ is pointwise continuous in $\alpha$, we have

$$\forall \epsilon > 0 \, \exists \, \delta > 0 : \left|\alpha - \beta\right| < \delta \implies \left|u(r,\alpha) - u(r,\beta)\right| < \epsilon.$$

We define

$$\epsilon := \frac{1}{2}\left(u(r_1,\alpha) - u(r_0,\alpha)\right).$$

For $\delta_{r_0} > 0$ sufficiently small, we have

$$\left|u(r_0,\alpha) - u(r_0,\beta)\right| < \epsilon,$$

and for $\delta_{r_1} > 0$ sufficiently small, we have

$$\left|u(r_1,\alpha) - u(r_1,\beta)\right| < \epsilon.$$

Let $\delta = \min\left\{\delta_{r_0}, \delta_{r_1}\right\} > 0$. Then, for $\left|\alpha - \beta\right| < \delta$, we have

$$\begin{cases} u(r_1,\beta) > u(r_0,\beta) \\ \beta > u(r,\beta) > 0 \quad \text{for all } r \in (0, r_1]. \end{cases} \tag{2.34}$$

Thus $\beta \in P$ and $P$ is open. $\square$

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# 3

# UNIQUENESS OF GROUND STATE

## 3.1. INTRODUCTION

In this chapter, we study a paper from 1972 by Charles V. Coffman [1]. The paper proves uniqueness of the positive radially symmetric (ground state) solution $u = \phi_1 \in C^2 \cap L^4$ for the equation

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^3. \tag{3.1}$$

Note that all function spaces consist of real valued functions on $\mathbb{R}^3$. Furthermore, radial symmetry is with respect to the origin only.

The existence of such a function $\phi_1$ was shown in [2], where $\phi_1 = v_1(|x|)$ solves (3.1). In fact, there exist functions $v_n(|x|) \in C^2([0,\infty))$, $n = 1,2,\ldots$, such that for each $n$, $v_n$ has exacly $n-1$ isolated zeroes in $[0,\infty)$, decays exponentially as $r \to \infty$. This was shown in [3, 4]

Moreover, Theorem 3.1 of [1] improves the result of [5], which also studied (3.1) (in the context of variational calculus). In [5], they show that the Lagrangian associated with (3.1) is zero in its first variation and the second variation is positive if $\lambda_1 > 1$. The latter is shown only through approximations. Theorem 3.1 of [1] shows that the Rayleigh quotient $J$ associated with (3.1)

$$J(u) = \frac{\left(\int |\nabla u|^2 + u^2\, dx\right)^2}{\int u^4\, dx} \tag{3.2}$$

is indeed minimal for $u = \phi_1$ and for

$$u(x) = k\phi_1(x + x_0) \tag{3.3}$$

for any $k \neq 0$ and $x_0 \in \mathbb{R}^3$.

## 3.2. PRELIMINARY RESULTS FOR THE INTEGRAL EQUATION

The problem (3.1) subject to $u \in L^4$ is equivalent to the integral equation in $L^4$

$$\begin{cases} u(x) = \int g(x-y)u^3(y)\,\mathrm{d}y, \quad \text{where} \\ g(x) = (4\pi)^{-1}|x|^{-1}e^{-|x|}. \end{cases} \tag{3.4}$$



Here $g(x)$ is the Yukawa (screened Coulomb) potential. This potential is associated with the equation

$$\Delta u - u = 0. \tag{3.5}$$

We consider radially symmetric solutions to (3.5), $r = |x|$, which solve

$$\frac{\mathrm{d}}{\mathrm{d}r^2}(ru) = ru. \tag{3.6}$$

Hence, the Yukawa potential $u(r)$ is of the form $ru = e^{-r} \iff u(r) = r^{-1}e^{-r}$.

The following two subsections discuss (mostly) standard results regarding the Sobolev space $H^1$ and the convolution operator $\tau : u \to g * u$.

### 3.2.1. SOME RESULTS REGARDING $H^1$

First, concerning the space $H^1$, we have the following results:

a) $C_0^\infty$ is dense in $H^1$.

b) If $u \in H^1$, then $v = |u| \in H^1$ and

$$|u|_{1,2} = |v|_{1,2}.$$

c) If $u \in H^1$, then $u \in L^4$ and

$$|u|_{0,4} \leq 2^{-1/4}|u|_{1,2}. \tag{3.7}$$

d) Let $V$ denote the subspace of $H^1$ consisting of radially symmetric functions. The embedding $V \to L^4$ is compact.

### 3.2.2. SOME RESULTS REGARDING THE CONVOLUTION OPERATOR

e) If $u \in L^{4/3}$, then $v = g * u \in H^1 \subseteq L^4$, $\int u\,v\,\mathrm{d}x > 0$ unless $u = 0$, and $v$ is a weak solution of

$$-\Delta v + v = u. \tag{3.8}$$

f) If $u \in L^1 \cap L^\infty$, then $v = g * u$ has bounded continuous first derivatives and

$$\lim_{|x|\to\infty} v(x) = 0$$

.

g) If $u \in L^1 \cap L^\infty \cap C^1$, then $v = g * u \in C^2$ and $v$ satisfies (3.8).

h) Let $X$ and $Y$ denote the subspaces of $L^{4/3}$ and $L^4$ respectively, consisting of radially symmetric functions. Then $Y = X^*$ and $\tau : X \to Y$ is compact.

## 3.3. MINIMISATION OF $J$

This section first states that a solution $u \in L^4$ must belong to $H^1$. For $u \in L^4$, $u \neq 0$, we define $\sigma(u)$ by

$$\big(\sigma(u)\big)(x) = c \int g(x-t) u^3(t)\, dt \tag{3.9}$$

[...]

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**Lemma 3.1.** *If $u$ is an admissible solution, then $\sigma(u)$ is admissible and*

$$J(\sigma(u)) \leqslant J(u) \tag{3.10}$$

*with equality only if $\sigma(u) = u$. Moreover, $\sigma(u) \in L^\infty$ and $v = \sigma^2(u)$ has bounded continuous derivatives and satisfies*

$$\lim_{|x|\to\infty} v(x) = 0; \tag{3.11}$$

*finally $\sigma^3(u) \in C^2$.*

*Proof.* □

The following two lemmata are corollaries of Lemma 3.1.

**Lemma 3.2.** *If $v \in L^4$ is a solution of (3.4) then $v \in C^2$, $v$ has bounded first derivatives, and $v$ satisfies (3.11).*

**Lemma 3.3.** *If $u$ is any (radially symmetric) admissible function, then there is a (raaially symmetric) admissible function $v \in C^2$ which is positive, has bounded first derivatives and satisfies (3.11) and*

$$J(v) \leqslant J(u). \tag{3.12}$$

*Moreover, unless $u$ itself has the same properties and is a solution of (3.4) (to within a positive factor), then $v$ can be chosen so that inequality (3.12) is strict.*

*Proof.* □

**Theorem 3.1.** *Let*

$$\lambda_1 = \inf\{J(u) : u \text{ admissible}\}.$$

*There exists a $\phi_1 \in V$ with*

$$J(\phi_1) = \lambda_1.$$

*For $u \in H^1$, $J(u) > \lambda_1$ unless $u$ is of the form (3.3).*

The proof of Theorem 3.1 shows the desired results of the paper except for the last statement. This requires 3.2 of the next section.

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## 3.4. UNIQUENESS THEOREM

The radially symmetric solutions of (3.1) are of the form

$$u(x) = |x|^{-1} w(|x|),$$

where $w(r)$ $(r = |x|)$ solves

$$w'' - w + r^{-2} w^3 = 0. \tag{3.13}$$

We refer to 3.4.1 for the details.

To prove the uniqueness of ground state solution $\phi_1$ for (3.1), it suffices to prove that (3.13) has at most one positive solution satisfying the following boundary conditions

$$0 < \lim_{r\to 0} r^{-1} w(r) < \infty, \quad \lim_{r\to\infty} w(r) = 0. \tag{3.14}$$

The problem (3.13) is transformed to an initial value problem where

$$\lim_{r\to 0} r^{-1} w(r) = a > 0. \tag{3.15}$$

The basic facts regarding the problem (3.13), (3.15) are summarised in Lemma 4.1. The proofs are omitted in [1].

**Lemma 3.4.** *For each $a > 0$ the equation (3.13) has a unique solution $w = w(r, a)$ which is of class $C^2$ on $(0,\infty)$ and satisfies (3.15). The partial derivatives $\partial w(r, a)/\partial a$ and $\partial w'(r, a)/\partial a$ exist for all positive $r$ and $a$. Furthermore, $\partial w(r, a)/\partial a$ coincides on $(0,\infty)$ with the solution $\delta = \delta(r, a)$ of the regular initial value problem*

$$\begin{cases} \delta'' - \delta + 3r^{-2} w^2 \delta = 0, \\ \delta(0) = 0, \quad \delta'(0) = 1, \end{cases} \tag{3.16}$$

*with $w = w(r, a)$; $\partial w'(r, a)/\partial a = \delta'(r, a)$.*

It is clear that a solution of (3.13) which satisfies (3.14) belongs to the one-parameter family $w = w(r, a)$, $a > 0$; we therefore formulate our uniqueness result as follows.

**Theorem 3.2.** *There is at most one positive value of $a$ for which*

$$w(r, a) > 0, \quad 0 < r < \infty \tag{3.17}$$

*and*

$$\lim_{r\to\infty} w(r, a) = 0. \tag{3.18}$$

Theorem 4.1 is implied by the following lemma.

**Lemma 3.5.** *(i) If $a > 0$ and $w(r, a) > 0$ on $(0, z_1)$ with $w(z_1, a) = 0$, then $\delta(z_1, a) < 0$.*

*(ii) If $a > 0$ and $w(r, a)$ satisfies (4.5) and (4.6) then*

$$\lim_{r\to\infty} e^{-r} \delta(r, a) < 0. \tag{3.19}$$

*Proof.* □

By studying the zeroes of $w(r,a)$, we can show that $A$ (the set of $a>0$ such that $w(r,a)$ has at least one zero in $(0,\infty)$) has a left endpoint.

### 3.4.1. DERIVATION OF EQUATION FOR RADIALLY SYMMETRIC SOLUTIONS

Consider radially symmetric solutions to (3.1). Then $u(x) = u(|x|) = u(r)$ . This transforms (3.1) to the ODE (2.1), restated here for $N=3$



$$u'' + \frac{2}{r}u' - u + u^3 = 0 \tag{3.20}$$

Furthermore, substituting $u(r) = r^{-1}w(r)$, we calculate the derivatives of $u$ as

1. $u'(r) = -r^{-2}w(r) + r^{-1}w'(r)$
2. $u''(r) = 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r)$.

We substitute in (3.20) to obtain

$$\begin{aligned} u''(r) + \frac{2}{r} - u(r) + u^3(r) \\ = 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r) + \frac{2}{r}\left(-r^{-2}w(r) + r^{-1}w'(r)\right) \\ - r^{-1}w(r) + r^{-3}w^3(r) = 0, \end{aligned} \tag{3.21}$$

which is simplified to

$$r^{-1}\left(w'' - w + r^{-2}w^3\right) = 0.$$

In conclusion, since $r \neq 0$, we obtain

$$w'' - w + r^{-2}w^3 = 0. \tag{3.22}$$

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