Nonlinear Analysis, Theory, Methods & Applications, Vol. 19, No. 12, pp. 1143–1154, 1992.
Printed in Great Britain.

0362-546X/92 $5.00 + .00


# UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF $\Delta u + f(u, |x|) = 0$

EIJI YANAGIDA
Department of Information Science, Faculty of Science, Tokyo Institute of Technology, Meguro-ku, Tokyo 152, Japan

(*Received* 15 *April* 1991; *received for publication* 26 *February* 1992)

*Key words and phrases*: Uniqueness, semilinear elliptic equations, positive radial solutions.

## 1. INTRODUCTION

IN THIS paper we consider positive radial solutions of the following semilinear elliptic equation in $\mathbf{R}^n$ $(n \geq 2)$

$$\Delta u(x) + f(u(x), |x|) = 0. \tag{1.1}$$

Since radial solutions $u = u(|x|)$ are of particular interest, we will investigate the ordinary differential equation

$$u_{rr}(r) + \frac{n-1}{r} u_r(r) + f(u(r), r) = 0 \tag{1.2}$$

where $r = |x|$. The purpose of this paper is to study uniqueness of solutions of equation (1.2) under the conditions

$$u_r(a) = 0 \tag{1.3}$$

$$u(r) > 0 \qquad \text{for } r \in [a, b) \tag{1.4}$$

$$u(b) = 0 \tag{1.5}$$

where $0 \leq a < b < \infty$. It should be noted that, for some class of $f$, any positive solution of equation (1.1) with the boundary conditions (1.3) and (1.5) is necessarily radial [1].

Equations (1.1) and (1.2) arise in various fields of pure and applied mathematics such as Riemannian geometry, astrophysics and population genetics. The existence of solutions of (1.2)–(1.5) was studied by many authors and is well understood nowadays (see [2] and references cited therein). However uniqueness of solutions of (1.2)–(1.5) is very difficult to prove, in particular when $f$ is superlinear, even if $f$ is of a simple form [3–9]. Among them, Kwong [5] proved uniqueness for $f = -u + u^p$ by applying the Sturm oscillation theory. Later his result was generalized in [10–13] with simpler proofs mainly in the case where $f$ does not depend on $r$. In this paper we consider the case where $f$ may depend on $r$ and prove the uniqueness under quite simple and general conditions on $f$.

In Section 2, we introduce an initial value problem for equation (1.2), and then we describe some conditions on $f$ and give our main results. We also give several examples of $f$ to which our method is applicable. In Section 3, we give proofs of the main results by using some lemmas. In Section 4, we describe several properties of solutions of the initial value problem. Sections 5 and 6 are devoted to proofs of the lemmas which are used in the proofs of the main results.

## 2. ASSUMPTIONS AND MAIN RESULTS

Throughout this paper we assume $n \geq 2$. Let $\Omega$ be a domain defined by

$$\Omega := \{(t, r) \mid t \in (0, \infty),\ r \in [a, b]\}$$

and assume $f(t, r) \in C^1(\bar{\Omega})$. If we define

$$f(t, r) := f(t, b) + (r - b)f_r(t, b) \qquad \text{for } r > b,$$

and

$$f(t, r) := f(0, r) + tf_t(0, r) \qquad \text{for } t < 0$$

then we can assume $f(t, r) \in C^1((-\infty, +\infty) \times [a, +\infty))$.

Let $v = v(r; \alpha)$ denote a solution of the initial value problem

$$\begin{gathered} v_{rr} + \frac{n-1}{r} v_r + f(v, r) = 0, \qquad r > a \\ v(a; \alpha) = \alpha > 0, \qquad v_r(a; \alpha) = 0 \end{gathered} \tag{2.1}$$

or equivalently,

$$\begin{gathered} (r^{n-1}v_r)_r + r^{n-1}f(v, r) = 0, \qquad r > a \\ v(a; \alpha) = \alpha > 0, \qquad v_r(a; \alpha) = 0. \end{gathered} \tag{2.2}$$

For these equations, we need not restrict $n$ to integer values. It is clear that any solution $u(r)$ of (1.2)–(1.5) is also a solution of (2.1) and (2.2) with $u(0) = \alpha$ and $u(b) = 0$.

Let $z(\alpha)$ be defined by

$$z(\alpha) := \sup\{r \in (a, \infty) \mid v(s; \alpha) > 0 \text{ for } s \in [a, r)\}.$$

If $v(r; \alpha)$ has a finite zero, $z(\alpha)$ denotes the smallest zero of $v(r; \alpha)$, and if $v(r; \alpha) > 0$ for all $r \geq a$, then $z(\alpha) = \infty$.

Let $\Omega^0$ and $\Omega^-$ be subdomains of $\Omega$ defined by

$$\Omega^0 := \{(t, r) \in \Omega \mid rf_r(t, r) + 2f(t, r) > 0\}$$

$$\Omega^- := \Omega - \Omega^0$$

respectively, and let $y(t, r)$ be a function on $\Omega^0$ defined by

$$y(t, r) := \frac{tf_t(t, r) - f(t, r)}{tf_r(t, r) + 2f(t, r)}.$$

For uniqueness of a solution of (1.2)–(1.5), we assume that the following conditions are satisfied:

(C.1) $tf_t(t, r) - f(t, r) > 0$ for all $(t, r) \in \Omega$; and

(C.2) $y(t, r)$ is nonincreasing in $t$ and nondecreasing in $r$ for all $(t, r) \in \Omega^0$.

It is easy to see from (C.1) that, for each $r \in [a, b]$, there exists $\theta(r) \in [0, \infty]$ such that

$$\begin{cases} f(t, r) \leq 0 & \text{for } t \in [0, \theta(r)] \\ f(t, r) > 0 & \text{for } t \in (\theta(r), \infty). \end{cases} \tag{2.3}$$

Let $F(t, r)$ be defined by

$$F(t, r) := \int_0^t f(t, r)\,\mathrm{d}t.$$

Integrating (C.1) with respect to $t$, we have $tf(t, r) - 2F(t, r) > 0$ for all $(t, r) \in \Omega$. Moreover it follows from (2.3) that there exists $\lambda(r) \in [\theta(r), \infty]$ such that

$$\begin{cases} F(t, r) \leq 0 & \text{for } t \in [0, \lambda(r)], \\ F(t, r) > 0 & \text{for } t \in (\lambda(r), \infty). \end{cases}$$

We shall say that $v(r; \alpha)$ is *admissible* if the following three conditions are satisfied:
(a) $z(\alpha) \leq b$;
(b) $v_r(r; \alpha)$ for $r \in [a, z(\alpha)]$, and

$$\text{(c)} \quad \begin{cases} \alpha > \theta(0) & \text{(i.e. } f(\alpha, 0) > 0\text{) in case } a = 0; \\ \alpha > \theta(a) \text{ and } \alpha \geq \lambda(a) & \text{(i.e. } f(\alpha, a) > 0 \text{ and } F(\alpha, a) \geq 0\text{) in case } a > 0. \end{cases} \tag{2.4}$$

Now we state a main result of this paper.

THEOREM 2.1. Suppose that (C.1) and (C.2) hold. If a solution $v(r; \alpha_0)$ of (2.1) is admissible for some $\alpha_0 > 0$, then the following hold:
(i) $v(r; \alpha)$ is admissible for all $\alpha \in [\alpha_0, \infty)$;
(ii) $z(\alpha)$ is a strictly decreasing function of $\alpha \in [\alpha_0, \infty)$.

The following results are derived from theorem 2.1.

THEOREM 2.2. Suppose that (C.1) and (C.2) hold. Then (1.2)–(1.5) has at most one admissible solution.

THEOREM 2.3. Suppose that (C.1) and (C.2) hold. Suppose further that at least one of the following conditions is satisfied:
(i) $f(t, r) > 0$ for all $(t, r) \in \Omega$;
(ii) $F_r(t, r) \leq 0$ for all $(t, r) \in \Omega$.
Then (1.2)–(1.5) has at most one solution.

Examples of $f(t, r)$ satisfying (C.1) and (C.2) are collected below:

$$\begin{aligned} f &= t^p, & p &> 1 \\ f &= -t + t^p, & p &> 1 \\ f &= t^p/(1 + r^2), & p &> 1 \\ f &= -t + Q(r)t^p, & p &> 1, \end{aligned}$$

where $Q(r)$ and $rQ_r(r)/Q(r)$ are nonincreasing functions of $r \in [a, b]$.

We note that, for these examples, any solution of (1.2)–(1.5) is necessarily admissible. When $f = t^p$, equation (1.2) is called the Lane–Emden equation. The uniqueness was established in [4] and some generalizations were obtained in [7, 8]. We note that $y(t, r)$ is a constant function in this case. When $f = -t + t^p$, equation (1.1) is called a scalar field equation and the uniqueness was discussed in [3, 5, 6, 10–12]. When $f = t^p/(1 + r^2)$, equation (1.1) is called Matukuma's equation and the uniqueness established in [9, 14]. When $f = -u + Q(r)u^p$, equation (1.1) is called a scalar field equation with a potential $Q(r)$. The uniqueness was studied in [9, 11].

## 3. PROOF OF THEOREMS

The following two lemmas are important for the proof of theorem 2.1. These lemmas will be proved in Sections 5 and 6, respectively.

LEMMA 3.1. Suppose that (C.1) and (C.2) hold. If $v(r; \alpha)$ is admissible, then $w(r; \alpha) := (\partial/\partial\alpha)v(r; \alpha)$ satisfies $w(z(\alpha); \alpha) < 0$.

LEMMA 3.2. Suppose that (C.1) and (C.2) hold. If $v(r; \alpha)$ is admissible, then there exists a $\delta > 0$ such that $v(r; \alpha')$ is admissible for every $\alpha' \in [\alpha, \alpha + \delta)$.

By using these lemmas, theorem 2.1 is proved as follows.

*Proof of theorem* 2.1 (i). Contrary to the conclusion, suppose that there exists an $\alpha \in (\alpha_0, \infty)$ such that $v(r; \alpha)$ is not admissible, and put

$$\alpha^* := \inf\{\alpha \in (\alpha_0, \infty) \mid v(r; \alpha) \text{ is not admissible}\} < \infty.$$

By lemma 3.2, $v(r; \alpha^*)$ cannot be admissible. Hence one of the following cases occurs.

*Case* 1. $z(\alpha^*) > b$.

*Case* 2. $z(\alpha^*) \leq b$ and $v_r(r; \alpha^*) > 0$ for some $r \in (a, z(\alpha^*))$.

In case 1, $v(r; \alpha^*) > 0$ for $r \in [a, b]$. This implies, by continuity of $v(r; \alpha)$ with respect to $\alpha$, $v(r; \alpha) > 0$ for $r \in [a, b]$ if $|\alpha - \alpha^*| > 0$ is sufficiently small. However this contradicts the admissibility of $v(r; \alpha)$ for $\alpha \in [\alpha_0, \alpha^*)$.

In case 2, by continuity of $v_r(r; \alpha)$ with respect to $\alpha$, we have $v_r(r; \alpha^*) > 0$ for some $r \in (a, z(\alpha))$ if $|\alpha - \alpha^*| > 0$ is sufficiently small. However this contradicts the admissibility of $v(r; \alpha)$ for every $\alpha \in [\alpha_0, \alpha^*)$. Thus it is shown that $v(r; \alpha)$ is admissible for all $\alpha \in [\alpha_0, \infty)$. ■

*Proof of theorem* 2.1 (ii). Let $\varepsilon > 0$ be a sufficiently small number. By definition, we have

$$v(r; \alpha + \varepsilon) = v(r; \alpha) + \varepsilon w(r; \alpha) + o(\varepsilon).$$

Since $w(z(\alpha); \alpha) < 0$, we have $v(z(\alpha); \alpha + \varepsilon) < 0$. By the intermediate value theorem, this implies $z(\alpha + \varepsilon) < z(\alpha)$. Hence $z(\alpha)$ is a strictly decreasing function of $\alpha \in [\alpha_0, \infty)$. ■

Next we give proofs of theorems 2.2 and 2.3 by using theorem 2.1.

*Proof of theorem* 2.2. Suppose that there exist two distinct admissible solutions, say $u_1(r)$ and $u_2(r)$, of (1.2)-(1.5) and assume without loss of generality that $u_1(a) < u_2(a)$. By virtue of (ii) of theorem 2.1 that $z(\alpha)$ is a strictly decreasing function of $\alpha \geq u_1(a)$, hence we obtain $z(u_2(a)) < b$, contradicting $z(u_2(a)) = b$. ■

*Proof of theorem* 2.3. It is sufficient to show that $v(r; \alpha)$ is necessarily admissible if $z(\alpha) \leq b$. First assume $f(t, r) > 0$ for all $(t, r) \in \Omega$. Integrating (2.2) over $[a, r]$, we obtain

$$r^{n-1}v_r(r; \alpha) = -\int_a^r s^{n-1}f(v(s; \alpha), s)\,\mathrm{d}s. \tag{3.1}$$

Since $f(t, r) > 0$ for all $(t, r) \in \Omega$, we obtain $v_r(r; \alpha) < 0$ for $r \in (a, z(\alpha)]$. Since $f(\alpha; a) > 0$ and $F(\alpha; a) > 0$ if $\alpha > 0$, $v(r; \alpha)$ must be admissible.

Next assume $F_r(t, r) \leq 0$ for all $(t, r) \in \Omega$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\{v_r^2/2 + F(v, r)\} &= v_{rr}v_r + f(v, r)v_r + F_r(v, r)\\ &= -(n-1)v_r^2/r + F_r(v, r).\end{aligned}$$

Since $F_r(v, r) \leq 0$ for $r \in (a, z(\alpha))$ and since $v_r(r; \alpha) \not\equiv 0$ if $z(\alpha) - r > 0$ is sufficiently small, we obtain

$$v_r(r; \alpha)^2/2 + F(v(r; \alpha), r) > v_r(z(\alpha); \alpha)^2/2 \qquad \text{for } r \in [a, z(\alpha)). \tag{3.2}$$

Putting $r = a$, we obtain $F(\alpha, a) > 0$ and hence $f(\alpha, a) > 0$.

Since $f(\alpha, a) > 0$, it follows from (3.1) that $v_r(r; \alpha) < 0$ if $r - a > 0$ is sufficiently small. Suppose here that there exists an $r_1 \in (a, z(\alpha)]$ such that $v_r(r_1; \alpha) > 0$. Then there must exist an $r_2 \in (a, r_1)$ such that $v_r(r_2; \alpha) = 0$ and $v_{rr}(r_2; \alpha) \geq 0$. At this point, we have $f(v(r_2; \alpha), r_2) = -v_{rr}(r_2; \alpha) \leq 0$. Hence $F(v(r_2; \alpha), r_2) \leq 0$ so that $v_r(r_2; \alpha)^2/2 + F(v(r_2; \alpha), r_2) \leq 0$. However this contradicts (3.2). Hence $v_r(r; \alpha) \leq 0$ for all $r \in [a, z(\alpha)]$. Thus $v(r; \alpha)$ is necessarily admissible. ■

## 4. PRELIMINARIES

In this section we collect several lemmas which will be needed to prove lemmas 3.1 and 3.2.

LEMMA 4.1. Suppose that (C.1) and (C.2) hold. Then the following hold:

(i) let $(t_0, r_0)$ be any point in $\Omega^0$. If $(t, r) \in \Omega$ satisfies $t \geq t_0$ and $r \leq r_0$, then $(t, r) \in \Omega^0$;

(ii) let $(t_0, r_0)$ be any point in $\Omega^-$. If $(t, r) \in \Omega$ satisfies $t \leq t_0$ and $r \geq r_0$, then $(t, r) \in \Omega^-$.

*Proof of* (i). Suppose that $(t, r) \in \Omega^-$. Let $\mu \in [0, 1]$ be a parameter and put

$$(t(\mu), r(\mu)) := (t_0 + \mu(t - t_0), r_0 + \mu(r - r_0)).$$

If we define $\mu^* := \sup\{\mu \mid (t(\mu), r(\mu)) \in \Omega^0\} \in (0, 1]$, then

$$r(\mu)f_r(t(\mu), r(\mu)) + 2f(t(\mu), r(\mu)) \to +0 \qquad \text{as } \mu \to \mu^* - 0.$$

By (C.1), this implies that $y(t(\mu), r(\mu)) \to \infty$ as $\mu \to \mu^* - 0$. However this contradicts (C.2). ■

*Proof of* (ii). If $(t, r) \in \Omega^0$, this contradicts (i). ■

Let $\Omega^+$ be a subdomain of $\Omega$ defined by

$$\Omega^+ := \{(t, r) \in \Omega \mid 2rf_r(t, r) + 4f(t, r) - (n-2)\{tf_t(t, r) - f(t, r)\} > 0\}.$$

By (C.1) and $n \geq 2$, we have $\Omega^+ \subset \Omega^0$. Moreover the following lemma holds.

LEMMA 4.2. Suppose that (C.1) and (C.2) hold. Then the following hold:
(i) let $(t_0, r_0)$ be any point in $\Omega^+$. If $(t, r) \in \Omega$ satisfies $t \geq t_0$ and $r \leq r_0$, then $(t, r) \in \Omega^+$;
(ii) let $(t_0, r_0)$ be any point in $\Omega - \Omega^+$. If $(t, r) \in \Omega$ satisfies $t \leq t_0$ and $r \geq r_0$, then $(t, r) \in \Omega - \Omega^+$.

*Proof.* This is clear from (C.2) and lemma 4.1. ■

The following lemma is the most technical part of this paper.

LEMMA 4.3. Suppose that (C.1) and (C.2) hold. If $v(r; \alpha)$ is admissible, then $v_r(r; \alpha) < 0$ and $\{rv_r(r; \alpha)/v(r; \alpha)\}_r < 0$ for $r \in (a, z(\alpha))$.

*Proof.* In this proof, we denote $v(r; \alpha)$ simply by $v(r)$. We note that, in view of (2.1), we have

$$\{rv_r(r)/v(r)\}_r = \frac{\{rv_{rr}(r) + v_r(r)\}v(r) - rv_r(r)^2}{v(r)^2}$$
$$= -\frac{rv_r(r)^2 + (n-2)v_r(r)v(r) + rv(r)f(v(r), r)}{v(r)^2}.$$

Hence the inequality $\{rv_r(r)/v(r)\}_r < 0$ is equivalent to

$$r^n v_r(r)^2 + (n-2)r^{n-1}v_r(r)v(r) + r^n v(r)f(v(r), r) > 0.$$

Also, since $v_r(r) \leq 0$ for $r \in [a, z(\alpha)]$, it follows from lemma 4.2 that there exists an $r^+ \in [a, z(\alpha)]$ such that

$$\begin{cases} 2rf_r(v(r), r) + 4f(v(r), r) - (n-2)\{v(r)f_t(v(r), r) - f(v(r), r)\} > 0 & \text{for } r \in (a, r^+), \\ 2rf_r(v(r), r) + 4f(v(r), r) - (n-2)\{v(r)f_t(v(r), r) - f(v(r), r)\} \leq 0 & \text{for } r \in [r^+, z(\alpha)). \end{cases}$$

First let us prove $v_r(r) < 0$ and $\{rv_r(r)/v(r)\}_r < 0$ for $r \in (a, r^+)$. Since $f(\alpha, a) > 0$, it follows from (3.1) that $v_r(r; \alpha) < 0$ if $r - a > 0$ is sufficiently small. Define

$$G(t, r) := (n-2)\int_a^r s^{n-1}tf(t, s)\,\mathrm{d}s.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}r}G(v(r), r)$$
$$= (n-2)r^{n-1}v(r)f(v(r), r) + (n-2)v_r(r)\int_a^r s^{n-1}\{v(r)f_t(v(r), s) + f(v(r), s)\}\,\mathrm{d}s. \quad (4.1)$$

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\{r^n v_r(r)^2 + (n-2)r^{n-1}v_r(r)v(r)\}$$

$$= 2r^n v_r(r)\{v_{rr}(r) + (n-1)v_r(r)/r\} + (n-2)r^{n-1}v(r)\{v_{rr}(r) + (n-1)v_r(r)/r\}$$

$$= -2r^n v_r(r) f(v(r), r) - (n-2)r^{n-1}v(r)f(v(r), r). \tag{4.2}$$

Here

$$r^n f(v(r), r) = a^n f(v(r), a) + \int_a^r s^{n-1}\{nf(v(r), s) + sf_r(v(r), s)\}\,\mathrm{d}s, \tag{4.3}$$

because

$$\int_a^r s^{n-1}\{nf(v(r), s) + sf_r(v(r), s)\}\,\mathrm{d}s = \int_a^r \frac{\partial}{\partial s}\{s^n f(v(r), s)\}\,\mathrm{d}s$$

$$= r^n f(v(r), r) - a^n f(v(r), a).$$

Thus, combining (4.1)–(4.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\{r^n v_r(r)^2 + (n-2)r^{n-1}v_r(r)v(r) + G(v(r), r) + 2a^n F(v(r), a)\}$$

$$= -v_r(r)\int_a^r s^{n-1}\{2sf_r(v(r), s) + 4f(v(r), s) - (n-2)(v(r)f_t(v(r), s) - f(v(r), s))\}\,\mathrm{d}s. \tag{4.4}$$

Suppose here that there exists a $c \in (a, r^+)$ such that $v_r(r) < 0$ for $r \in (a, c)$ and $v_r(c) = 0$. If $r \in (a, c)$, the integrand of the right-hand side of (4.4) is positive, i.e. the right-hand side of (4.4) is positive. Hence, integrating (4.4) over $[a, r]$, we obtain

$$r^n v_r(r)^2 + (n-2)r^{n-1}v_r(r)v(r) + G(v(r), r) + 2a^n F(v(r), a)$$

$$> 2a^n F(\alpha, a) \geq 0 \qquad \text{if } r \in (a, c]. \tag{4.5}$$

Here, if $(t, r) \in \Omega^+$, then

$$G(t, r) = \int_a^r (n-2)s^{n-3}s^2 tf(t, s)\,\mathrm{d}s$$

$$= [s^{n-2}s^2 tf(t, s)]_a^r - \int_a^r s^{n-2}t\{s^2 f_r(t, s) + 2sf_r(t, s)\}\,\mathrm{d}s$$

$$< r^n tf(t, r) - a^n tf(t, a).$$

Since $tf(t, a) > 2F(t, a)$, we obtain

$$G(t, r) + 2a^n F(t, a) < r^n tf(t, r) \qquad \text{for } (t, r) \in \Omega^+. \tag{4.6}$$

Hence it follows from (4.5) and (4.6) that

$$r^n v_r(r)^2 + (n-2)r^{n-1}v_r(r)v(r) + r^n v(r)f(v(r), r) > 0 \qquad \text{for } r \in (a, c).$$

This means that $\{rv_r(r)/v(r)\}_r < 0$ for $r \in (a, c)$ so that $cv_r(c)/v(c) < av_r(a)/v(a) = 0$. However this contradicts $v_r(c) = 0$. Thus we have shown that $v_r(r) < 0$ for $r \in (a, r^+)$. Then, by (4.4) and (4.6), we have $\{rv_r(r)/v(r)\}_r < 0$ for $r \in (a, r^+)$. Moreover, if $r^+ \in (a, z(\alpha))$, we have $v_r(r) < 0$ and $\{rv_r(r)/v(r)\}_r < 0$ at $r = r^+$.

Next, let us prove $v_r(r) < 0$ and $\{rv_r(r)/v(r)\}_r < 0$ for $r \in (r^+, z(\alpha))$. By lemma 4.2, if $t \in [0, v(r^+)]$ and $r \in [r^+, z(\alpha))$, then

$$2rf_r(t, r) + 4f(t, r) - (n - 2)\{tf_t(t, r) - f(t, r)\} \leq 0.$$

Integrating this with respect to $t$ and multiplying it by $r^{n-1}$, we obtain

$$2r^nF_r(t, r) + 4r^{n-1}F(t, r) - (n - 2)r^{n-1}\{tf(t, r) - 2F(t, r)\} \leq 0$$
$$\text{if } t \in [0, v(r^+)] \text{ and } r \in [r^+, z(\alpha)).$$

Hence, by (4.2), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}&\{r^nv_r(r)^2 + (n - 2)r^{n-1}v_r(r)v(r) + 2r^nF(v(r), r)\}\\ &= -2r^nv_r(r)f(v(r), r) - (n - 2)r^{n-1}v(r)f(v(r), r)\\ &\quad + 2nr^{n-1}F(v(r), r) + 2r^nv_r(r)f(v(r), r) + 2r^nF_r(v(r), r)\\ &= 2r^nF_r(v(r), r) + 4r^{n-1}F(v(r), r) - (n - 2)r^{n-1}\\ &\quad \times \{v(r)f(v(r), r) - 2F(v(r), r)\} \leq 0 \qquad \text{for } r \in (r^+, z(\alpha)).\end{aligned} \tag{4.7}$$

On the other hand,

$$r^nv_r(r)^2 + (n - 2)r^{n-1}v_r(r)v(r) + 2r^nF(v(r), r)|_{r = z(\alpha)} = z(\alpha)^nv_r(z(\alpha))^2 \geq 0.$$

Hence, integrating (4.7) over $[r, z(\alpha)]$ and using $f(t, r) > 2F(t, r)$, we obtain

$$r^nv_r(r)^2 + (n - 2)r^{n-1}v_r(r)v(r) + r^nv(r)f(v(r), r) > 0 \qquad \text{for } r \in [r^+, z(\alpha)). \tag{4.8}$$

This means that $\{rv_r(r)/v(r)\}_r < 0$ for $r \in (r^+, z(\alpha))$. Moreover, since $v_r(r^+)$, we obtain

$$rv_r(r)/v(r) < r^+v_r(r^+)/v(r^+) \leq 0$$

which proves $v_r(r) < 0$ for $r \in (r^+, z(\alpha))$. ∎

*Remark* 4.1. When $b = \infty$ and $z(\alpha) = \infty$, lemma 4.3 still holds if the following conditions are satisfied:

(i) $\Omega = \Omega^0$;

(ii) $\Omega \neq \Omega^0$ and $r^nv_r(r)^2 + (n - 2)r^{n-1}v_r(r)v(r) + 2r^nF(v(r), r) \geq 0$ at $r = \infty$.

## 5. PROOF OF LEMMA 3.1

In this section, we give a proof of lemma 3.1. The idea of the proof is essentially due to Kwong [5].

Differentiating (2.2) with respect to $\alpha$, we obtain

$$\begin{aligned}&(r^{n-1}w_r)_r + r^{n-1}f_t(v, r)w = 0, \qquad r > a\\ &w(a; \alpha) = 1, \qquad w_r(a; \alpha) = 0.\end{aligned} \tag{5.1}$$

We shall study the behavior of $w(r;\alpha)$ by analyzing a linear differential operator $L_v$ defined by

$$L_v(V(r)) := \{r^{n-1}V_r(r)\}_r + r^{n-1}f_t(v,r)V(r).$$

LEMMA 5.1. (i) $L_v(v) = r^{n-1}\{vf_t(v,r) - f(v,r)\}$.
(ii) $L_v(rv_r) = -r^{n-1}\{rf_r(v,r) + 2f(v,r)\}$.
(iii) $L_v(w) = 0$.

*Proof of* (i).

$$\begin{aligned} L_v(v) &= (r^{n-1}v_r)_r + r^{n-1}f_t(v,r)v \\ &= r^{n-1}\{vf_t(v,r) - f(v,r)\}. \quad \blacksquare \end{aligned}$$

*Proof of* (ii).

$$\begin{aligned} L_v(rv_r) &= \{r^{n-1}(rv_r)_r\}_r + r^{n-1}f_t(v,r)(rv_r) \\ &= r^n\left\{v_{rrr} + \frac{n-1}{r}v_{rr} + f_t(v,r)v_r\right\} + 2r^{n-1}v_{rr} + (n-1)r^{n-2}v_r \\ &= r^n\left\{\frac{n-1}{r^2}v_r - f_r(v,r)\right\} - 2r^{n-1}\left\{\frac{n-1}{r}v_r + f(v,r)\right\} + (n-1)r^{n-2}v_r \\ &= -r^{n-1}\{rf_r(v,r) + 2f(v,r)\}. \quad \blacksquare \end{aligned}$$

*Proof of* (iii). This is clear from (5.1). $\blacksquare$

LEMMA 5.2. Suppose that (C.1) and (C.2) hold and define $U(r) := v + \beta rv_r$, where $\beta \geq 0$ is a parameter. If $v(r;\alpha)$ is admissible and if $af_r(\alpha,a) + 2f(\alpha,a) > 0$, then we can choose $\beta > 0$ suitably so that, for some $R_0 \in (a, z(\alpha))$, $U(r)$ satisfies

$$\begin{cases} U(r) > 0 \text{ and } L_v(U(r)) < 0 & \text{for } r \in (a, R_0), \\ U(r) < 0 \text{ and } L_v(U(r)) \geq 0 & \text{for } r \in (R_0, z(\alpha)). \end{cases}$$

*Proof.* Since $(\alpha, a) \in \Omega^0$ and $v_r(r;\alpha) < 0$ for $(a, z(\alpha))$, it follows from lemma 4.1 that there exists $r^*(\alpha) \in (a, z(\alpha)]$, such that

$$\begin{cases} (v(r;\alpha), r) \in \Omega^0 & \text{for } r \in [a, r^*(\alpha)), \\ (v(r;\alpha), r) \in \Omega^- & \text{for } r \in [r^*(\alpha), z(\alpha)). \end{cases}$$

Since $(rv_r/v)_r < 0$ for $r \in (a, z(\alpha))$, for any $\beta > 0$, there exists a $\rho(\beta) \in (a, z(\alpha))$ such that

$$\begin{cases} v + \beta rv_r > 0 & \text{for } r \in (a, \rho(\beta)), \\ v + \beta rv_r < 0 & \text{for } r \in (\rho(\beta), z(\alpha)), \end{cases} \tag{5.2}$$

and $\rho(\beta)$ is a strictly decreasing continuous function of $\beta$ satisfying

$$\begin{cases} \rho(\beta) \to z(\alpha) - 0 & \text{as } \beta \to +0, \\ \rho(\beta) \to a + 0 & \text{as } \beta \to \infty. \end{cases} \tag{5.3}$$

Hence there exists a unique $\beta^* \geq 0$ such that $\rho(\beta^*) = r^*(\alpha)$.

By virtue of lemma 5.1, we have

$$L_v(U) = r^{n-1}\{\{vf_t(v, r) - f(v, r)\} - \beta\{rf_r(v, r) + 2f(v, r)\}\}.$$

Here if $r \in [r^*(\alpha), z(\alpha))$, then $rf_r(v, r) + 2f(v, r) \leq 0$. Hence

$$L_v(U) > 0 \qquad \text{for } r \in [r^*(\alpha), z(\alpha)). \tag{5.4}$$

If $r \in [a, r^*(\alpha))$, then $L_v(U)$ can be written as

$$L_v(U) = r^{n-1}\{rf_r(v, r) + 2f(v, r)\}\{y(v, r) - \beta\}. \tag{5.5}$$

Since $v_r(r; \alpha) < 0$ for $r \in (a, z(\alpha))$, it follows from (C.2) that $y(v(r; \alpha), r)$ is a nondecreasing function of $r \in [a, r^*(\alpha))$.

Let $C(\alpha)$ be a graph of $y = y(v(r; \alpha), r)$, where $r$ ranges over $[a, r^*(\alpha))$. If $\beta > 0$ is sufficiently large, it follows from (5.3) that the point $(\rho(\beta), \beta)$ is located above $C(\alpha)$. On the other hand, $(\rho(\beta^*), \beta^*)$ is located below $C(\alpha)$. Since $C(\alpha)$ is smooth, there exists a $\beta \in (\beta^*, \infty)$ such that the point $(\rho(\beta), \beta)$ is on $C(\alpha)$. Namely $U(r) = 0$ and $L_v(U) = 0$ at $r = R_0 := \rho(\beta) > 0$. Moreover, by (5.4) and (5.5), $U(r)$ satisfies

$$\begin{cases} L_v(U) < 0 & \text{for } r \in (a, R_0), \\ L_v(U) \geq 0 & \text{for } r \in (R_0, r^*(\alpha)). \end{cases}$$

This and (5.2) proves the lemma. ■

*Remark* 5.1. When $b = \infty$ and $z(\alpha) = \infty$, lemma 5.2 still holds if the condition in remark 4.1 is satisfied.

LEMMA 5.3. Let $V_1(r)$ and $V_2(r)$ be any twice differentiable functions and let $r_1$ and $r_2$ be arbitrary numbers in $[a, z(\alpha)]$. Then

$$[r^{n-1}(V_1 V_{2r} - V_{1r} V_2)]_{r_1}^{r_2} = \int_{r_1}^{r_2} \{V_1 L_v(V_2) - V_2 L_v(V_1)\}\, dr.$$

*Proof*. Integration by parts yields

$$\begin{aligned} &\int_{r_1}^{r_2} \{V_1 L_v(V_2) - V_2 L_v(V_1)\}\, dr \\ &\quad = \int_{r_1}^{r_2} V_1\{(r^{n-1}V_{2r})_r + r^{n-1}f_t(v, r)V_2\}\, dr - \int_{r_1}^{r_2} V_2\{(r^{n-1}V_{1r})_r + r^{n-1}f_t(v, r)V_1\}\, dr \\ &\quad = [r^{n-1}V_1 V_{2r} - r^{n-1}V_{1r}V_2]_{r_1}^{r_2}. \quad ■ \end{aligned}$$

LEMMA 5.4. Suppose that (C.1) holds. If $z(\alpha) \le b$, then $w(r;\alpha)$ has at least one zero in $(a, z(\alpha))$.

*Proof.* By virtue of lemmas 5.1 and 5.3, we have

$$[r^{n-1}(vw_r - v_r w)]_a^{z(\alpha)} = \int_a^{z(\alpha)} \{vL_v(w) - wL_v(v)\}\,dr$$
$$= -\int_a^{z(\alpha)} r^{n-1} w\{vf_t(v,r) - f(v,r)\}\,dr.$$

Suppose here that $w(r;\alpha) > 0$ for all $r \in (a, z(\alpha))$. Then, by (C.1) the right-hand side is negative. On the other hand, since $v_r \le 0$ and $v = 0$ at $r = z(\alpha)$, the left-hand side is nonnegative. This is a contradiction. ∎

LEMMA 5.5. Suppose that (C.1) and (C.2) hold and that $v(r;\alpha)$ is admissible. Suppose further that $af_r(\alpha, a) + 2f(\alpha, a) > 0$ and let $R_0$ be as in lemma 5.2. Then $R_0 < R_1$, where $R_1$ is the first zero of $w(z;\alpha)$.

*Proof.* Let $U$ be as in lemma 5.2. If we suppose $R_0 \ge R_1$, then $w > 0$ and $L_v(U) < 0$ for $r \in [a, R_1)$. Hence, by lemmas 5.1 and 5.3, we have

$$[r^{n-1}(wU_r - w_r U)]_a^{R_1} = \int_a^{R_1} wL_v(U)\,dr < 0.$$

Since $w > 0$, $w_r = 0$, and $U_r = v_r + \beta v_{rr} \le 0$ at $r = a$, and since $w = 0$, $w_r < 0$, and $U \ge 0$ at $r = R_1$, the left-hand side must be nonnegative. This is a contradiction. ∎

Now let us complete the proof of lemma 3.1. It is sufficient to show that $R_1$ is the unique zero of $w(r;\alpha)$ in $[a, z(\alpha)]$. Suppose, on the contrary, that there exists a zero in $(R_1, z(\alpha)]$ and denote by $R_2$ the smallest zero in $(R_1, z(\alpha)]$.

First we consider the case where $af_r(\alpha, a) + 2f(\alpha, a) > 0$. By lemma 5.2, $w < 0$ and $L_v(U) \ge 0$ for $r \in (R_1, R_2)$. Hence, by lemmas 5.1 and 5.3, we obtain

$$[r^{n-1}(wU_r - w_r U)]_{R_1}^{R_2} = \int_{R_1}^{R_2} wL_v(U)\,dr \le 0.$$

Since $w = 0$, $w_r < 0$, and $U < 0$ at $r = R_1$, and since $w = 0$, $w_r > 0$, and $U < 0$ at $r = R_2$, the left-hand side must be positive. This is a contradiction.

Next we consider the case where $af_r(\alpha, a) + 2f(\alpha, a) \le 0$. In this case, by lemma 5.1, $L_v(rv_r) \ge 0$ for $r \in [a, z(\alpha)]$. Hence, by lemmas 5.1 and 5.3, we obtain

$$[r^{n-1}\{w(rv_r)_r - w_r(rv_r)\}]_{R_1}^{R_2} = \int_{R_1}^{R_2} wL_v(rv_r)\,dr \le 0.$$

Since $w = 0$, $w_r < 0$, and $rv_r < 0$ at $r = R_1$, and since $w = 0$, $w_r > 0$, and $rv_r \le 0$ at $r = R_2$, the left-hand side must be positive. This is a contradiction. ∎

## 6. PROOF OF LEMMA 3.2

By lemma 3.1, if $\alpha' - \alpha > 0$ is sufficiently small, then $z(\alpha') < z(\alpha)$. Since $\alpha' > \alpha > \theta(a)$, we have $f(\alpha', a) > 0$. Hence, by (3.1), $v_r(r; \alpha') < 0$ if $r - a > 0$ is sufficiently small. By continuity of $v_r(r; \alpha')$ with respect to $\alpha'$, there exists a $\delta > 0$ for any $\varepsilon > 0$ such that $v_r(r; \alpha') < 0$ for $r \in [a + \varepsilon, z(\alpha') - \varepsilon]$ if $0 < \alpha' - \alpha < \delta$.

In case $(0, z(\alpha)) \in \overline{\Omega^+}$, it follows from lemma 4.1 that $(0, z(\alpha')) \in \overline{\Omega^+}$. Hence, by (4.4) and (4.6), $v_r(r; \alpha') < 0$ for $r \in (a, z(\alpha'))$. In case $(0, z(\alpha)) \in \bar{\Omega} - \overline{\Omega^+}$, we take $\varepsilon > 0$ so small that $(0, z(\alpha') - \varepsilon) \in \bar{\Omega} - \overline{\Omega^+}$. Then, by (4.8), $v_r(r; \alpha') < 0$ for $r \in (z(\alpha') - \varepsilon, z(\alpha'))$. Thus, in either case, it is shown that $v_r(r; \alpha') < 0$ for $r \in (a, z(\alpha'))$. This proves the admissibility of $v(r; \alpha')$.

*Acknowledgement*—The author would like to express his appreciation to Professors N. Kawano and S. Yotsutani for their valuable comments.

## REFERENCES

1. Gidas B., Ni W.-M. & Nirenberg L., Symmetries and related properties via the maximum principle, *Communs math. Phys.* **68**, 209–243 (1979).
2. Ni W.-M., Some aspects of semilinear elliptic equations on $\mathbf{R}^n$, in *Nonlinear Diffusion Equations and their Equilibrium States* II (Edited by W.-M. Ni, L. A. Peletier and J. Serrin), pp. 171–205. Springer, New York (1988).
3. Coffman C. V., Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Archs ration. Mech. Analysis* **46**, 81–95 (1972).
4. Fowler R. H., Further studies of Emden's and similar differential equations, *Quart. J. Math.* **2**, 259–288 (1931).
5. Kwong M. K., Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$, *Archs ration. Mech. Analysis* **105**, 243–266 (1989).
6. McLeod K. & Serrin J., Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, *Archs ration. Mech. Analysis* **99**, 115–145 (1987).
7. Ni W.-M., Uniqueness of solutions of nonlinear Dirichlet problems, *J. diff. Eqns* **50**, 289–304 (1983).
8. Ni W.-M. & Nussbaum R. D., Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$, *Communs pure appl. Math.* **38**, 67–108 (1985).
9. Yanagida E., Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in $\mathbf{R}^n$, *Archs ration. Mech. Analysis* **115**, 257–274 (1991).
10. Chen C.-C. & Lin C.-S., Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, *Communs partial diff. Eqns* **16**, 1549–1572 (1991).
11. Kwong M. K. & Li Y., Uniqueness of radial solutions of semilinear elliptic equations (to appear).
12. Kwong M. K. & Zhang L., Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus, *Diff. Integral Eqns* **4**, 583–599 (1991).
13. McLeod K., Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$ II (preprint).
14. Yanagida E., Structure of positive radial solutions of Matukuma's equation, *Japan J. indust. appl. Math.* **8**, 165–173 (1991).