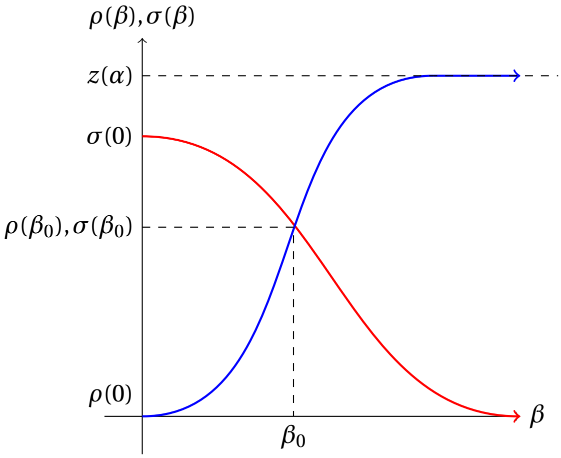
Lemma 3.5. *Let $\alpha \in G \cup N$. There exists a unique $\beta_0 > 0$ such that $\rho\left(\beta_0\right) = \sigma\left(\beta_0\right)$.*

Proof. This follows immediately from property (b) of Lemma 3.4 and correspondence (3.34). This unique intersection can also be seen from a sketch of the two graphs.



□

3.5. MORE ADVANCED PROPERTIES OF SOLUTIONS

We fix $\beta = \beta_0$. This fixes at least $v = v_{\beta_0}$ and $\rho_0 = \rho_{\beta_0}$, which will be used in the following lemma. We will apply the Sturm comparison theorem to equations (3.3) and (2.27) in the following lemma.

Lemma 3.6. *For $\alpha \in G \cup N$, $w(r, \alpha)$ has a unique zero $r_0 \in (0, z(\alpha))$. Furthermore, $w(z(\alpha)) < 0$ for $\alpha \in N$ and if $\alpha \in G$, we have*

$$\lim_{r \to \infty} w(r) = -\infty.$$

Proof. With the simplifications from before, we have

$$\begin{cases} v'' + \dfrac{1}{r} v' + \left[pV(r)u(r)^{p-1} - \lambda \right] v < 0 \quad \text{and } v > 0 \quad \text{on } (0, \rho_0) \quad \text{and} \\ v'' + \dfrac{1}{r} v' + \left[pV(r)u(r)^{p-1} - \lambda \right] v > 0 \quad \text{and } v < 0 \quad \text{on } (\rho_0, z(\alpha)). \end{cases} \tag{3.41}$$

Thus v has a unique zero on $(0, z(\alpha))$. Moreover, $v(0) = \beta_0 \alpha > 0$ and $\lim_{r \to 0} r v'(r) = 0$. Maybe no display to fit on one page. Furthermore, if $\alpha \in N$, then

$$v(z(\alpha)) = z(\alpha) u'(z(\alpha)) < 0$$

by Lemma 3.2 and so v has a unique zero ρ_0 in $[0, z(\alpha)]$.

Let $\tau \in (0, z(\alpha))$ be the first zero of w, which exists by Lemma 3.3. We remember that w satisfies

$$w'' + \frac{1}{r} w' + \left[pV(r)u^{p-1} - \lambda w \right] w = 0 \quad \text{for } r \in (0, z(\alpha)), \tag{3.42}$$

with initial data $w(0) = 1$ and $\lim_{r \to 0} r w'(r) = 0$. Because of this, by # sturm, we know that v oscillates faster than w.

Then $\rho_0 \in (0, \tau)$ and thus

$$v'' + \frac{1}{r} v' + \left[pV(r)u(r)^{p-1} - \lambda \right] v < 0 \quad \text{and } v < 0 \quad \text{on } (\tau, z(\alpha)). \tag{3.43}$$

Since $w(\tau) = 0$ and v has no zero larger than ρ_0, by Sturm we have that w has no further zero in $(\tau, z(\alpha))$ and we can set $r_0 = \tau$. If $z(\alpha) < \infty$, then w has no zero on $(\tau, z(\alpha)]$ and we have $w(z(\alpha)) < 0$.

However, if $z(\alpha) = \infty$, we can apply Lemma 6 of [3, p. 249]. The disconjugacy interal (d, ∞) is such that

$$d < \rho_0 < r_0.$$

DONE: The disconjugacy interval of a differential equation is the largest left-neighbourhood (c, b) of the right most b on which there exists a solution to the differential equation without zeroes. From Sturmian theory, no non-trivial solution can have more than one zero in (c, b). On the other hand, unless $c = a$, with a the left most, any solution of the differential equation with a zero before c must have another zero in (c, b).

Consider the same setting: equations (3.9) and (3.10) satisfying the comparison condition (3.11). In addition, we assume that $U \neq V$ in *any* neighborhood of b. If there exists a solution V of (3.10) with a largest zero at the point ρ, then the disconjugacy interval of (3.9) is a strict superset of (ρ, b).

One way to interpret the above is to remember that V oscillates faster than U and note that any solution U with a zero in (ρ, b)

Indeed, if we had $\rho_0 \leqslant d$, we could find a solution $\tilde{w}$ linearly independent of w such that $\tilde{w}(\tilde{r}) = 0$ for some $\tilde{r} > d \geqslant \rho_0$. But then $w + \tilde{w}$ is a solution of # w ivp with two zeroes in (ρ_0, ∞) and thus, v should have another zero in that interval. By this contradiction, we have $w(r_0) = 0$ with $r_0 \in (d, \infty)$ and Lemma 6 of [3, p. 249] implies that

$$\lim_{r \to \infty} w(r) = -\infty.$$

□

Lemma 3.7. *Let $\alpha \in G$. There exists $\epsilon > 0$ such that $(\alpha, \alpha + \epsilon) \subset N$.*

Proof. DONE: We remember that $w(r)$ has a unique zero r_0 by Lemma 3.6 and since $\alpha \in G$, we have $w(r) \to -\infty$ as $r \to \infty$. This implies by Lemma 6 of [3, p. 249] that $r_0 \in (d, \infty)$, where (d,∞) is the disconjugacy interval of initial value problem (3.3). We refer to # for the definition of disconjugacy interval. We can choose r_1 and r_2

$$d < r_1 < r_0 < r_2,$$

such that by (**??**), there exists $\epsilon > 0$ such that for all $\tilde{\alpha} \in (\alpha, \alpha+\epsilon)$, we have

$$\tilde{u}(r_1) > u(r_1) \quad \text{and} \quad \tilde{u}(r_2) < u(r_2),$$

3

where $\tilde{u}(r) = u(r,\tilde{\alpha})$. Hence, the graphs of u and $\tilde{u}$ intersect in some $r_3 \in (r_1, r_2)$, where r_3 may depend on the choice of $\tilde{\alpha}$. Next, we will show that there exists $\tilde{r} \in (r_3,\infty)$ such that $\tilde{u}(\tilde{r}) = 0$. Then, we would have $\tilde{\alpha} \in N$.

Proof step 2. To see that $\tilde{u}(\tilde{r}) = 0$ for some $\tilde{r} \in (r_3,\infty)$, we suppose by contradiction that $\tilde{u}(r) > 0$ for all $r > r_3$. We will show that $\tilde{u}(r) < u(r)$ for all $r > r_3$. □

Proof step 3. To see that $\tilde{u}(r) < u(r)$ for $r > r_3$, we suppose by contradiction that $\tilde{u}(r_4) = u(r_4)$ for some $r_4 > r_3$ and $u - \tilde{u} > 0$ on (r_3, r_4). □

The function $z := u - \tilde{u}$ satisfies

$$z'' + \frac{1}{r}z' + \left[V(r)\frac{u^p - \tilde{u}^p}{u - \tilde{u}} - \lambda \right] z = 0 \quad \text{on } (r_3, r_4). \tag{3.44}$$

Since $z(r_3) = z(r_4) = 0$ by assumption and

$$\frac{u^p - \tilde{u}^p}{u - \tilde{u}} < p u^{p-1} \tag{3.45}$$

on (r_3, r_4), we can apply Sturm's theorem # ref. w oscillates faster than z Let y be any solution of w ivp linearly independent of w. ... There is no positive solution of w ivp on (d,∞), contradicting ...

Hence $z(r) > 0$ for all $r > r_3$.

Sturm comparing fractions

Integrating sturm comparands

Hence $z(r) \equiv u(r) - \tilde{u}(r) \to \infty$ as $r \to \infty$. This is impossible, since $0 < \tilde{u}(r) < u(r)$ on (r_3,∞) and $u(r) \to 0$ as $r \to \infty$. Therefore, $\tilde{u}(r)$ must vanish at some point $\tilde{r} \in (r_3,\infty)$ and the proof is complete.

□

Lemma 3.8. *Let $\alpha^* \in N$. Then $[\alpha^*,\infty) \subset N$ and $z : [\alpha^*,\infty) \to (0,\infty)$ is monotone decreasing.*

Proof. N is open subset of $(0,\infty)$ ...

z is continuous ...

By Lemma 3.6, we have $w(z(\alpha^*)) < 0$. Then, there exists $\epsilon > 0$ such that

$$(\alpha^*, \alpha^* + \epsilon) \subset N \quad \text{and} \quad u(z(\alpha^*), \alpha) < 0 \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon).$$

The intermediate value theorem implies existence of an $r \in (0, z(\alpha^*))$ so that $u(r,\alpha) = 0$. We note that

$$z(\alpha) \leqslant r < z(\alpha^*) \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon)$$

and z is decreasing, since ...

We define

$$\overline{\alpha} := \sup\left\{\alpha > \alpha^* : \left[\alpha^*, \alpha\right) \subset N \text{ and } z : \left[\alpha^*, \alpha\right) \to (0,\infty) \text{ is decreasing}\right\}.$$

By contradiction, we suppose that $\overline{\alpha} < \infty$. Then there exists ... □

3.6. PROOF OF MAIN THEOREM

Proof of Theorem 3.1. # general proof, referring to lemma's □

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