An Explication of Existence and Uniqueness Results for a Nonlinear Schrödinger equation

An Introduction to the Shooting Method and Sturm Comparison Theorem

Bachelor's Thesis

at Delft University of Technology, written by

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Contents

Summary

Summary in English...

1

PHYSICS OF NLS

1.1. DERIVE THE WAVE EQUATION FROM MAXWELL

DONE: The propagation of electromagnetic waves in a medium is governed by Maxwell's laws in absence of external charges or currents. Remember that Maxwell's laws for the electric field $\mathcal{E}$, magnetic field $\mathcal{H}$, induction electric field $\mathcal{D}$ and induction magnetic field $\mathcal{B}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \quad \nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t},$$
$$\nabla \cdot \vec{\mathcal{D}} = 0, \quad \nabla \cdot \vec{\mathcal{B}} = 0.$$

Unless otherwise specified, these are fields in three-dimensional Cartesian coordinates. For example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) are given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \quad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}$$

The notation used here is from [1], for more background on electrodynamics see [2]. This reference work also includes an introduction to the necessary vector calculus.

From these relations and the vector identity for the curl of the curl, a wave equation can be derived. We specifically use $\nabla \cdot \vec{\mathcal{D}} = \nabla \cdot \epsilon_0 \vec{\mathcal{E}} = 0$ and $\nabla \times \vec{\mathcal{B}} = \mu_0 \frac{\partial \vec{\mathcal{D}}}{\partial t}$ to simplify the equation:

$$\nabla \times \nabla \times \vec{\mathcal{E}} = \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \vec{\mathcal{B}}) = -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \quad \text{by Maxwell's laws, and}$$
$$\nabla \times \nabla \times \vec{\mathcal{E}} = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \nabla^2 \vec{\mathcal{E}} = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \Delta \vec{\mathcal{E}} = -\Delta \vec{\mathcal{E}}, \quad \text{by vector calculus.}$$

Combining these and using $\mu_0 \epsilon_0 = 1/c^2$ we arrive at the vector wave equation:

$$\Delta \vec{\mathcal{E}} = \frac{1}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2}. \tag{1.1}$$

1

1.2. VALIDITY OF PLANE WAVE SOLUTIONS

DONE: Stuyding the left and right hand sides of equation (1.1), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{j=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_j^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz (educated guess) for the solutions to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.2}$$

which are so called plane wave solutions. NEW: This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x,y,z')-plane. Taking the necessary derivatives of 1.2 in equation (1.1)

$$\Delta\mathcal{E}_j = \frac{\partial^2}{\partial x^2}\mathcal{E}_j + \frac{\partial^2}{\partial y^2}\mathcal{E}_j + \frac{\partial^2}{\partial z^2}\mathcal{E}_j = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{E}_j$$

$$\frac{\partial^2}{\partial z^2}\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.3}$$

Dispersion (spreading out) is a result of different frequencies propagating at different speeds. Of course, other plane waves exist. In general, let wavevector $\vec{k_0} = (k_x,k_y,k_z)$ satisfy the dispersion relation $\left|\vec{k_0}\right|^2 = \frac{w_0^2}{c^2}$. The wavenumber $\vec{k} = (k_x,k_y,k_z)$ signifies the direction of propagation. For a plane wave with $\vec{k} = (0,0,k_0)$, we say the plane wave travels in the positive z-direction if k_0 is positive.

Not all plane waves are physical (in agreement with Maxwell's laws), for example the wave with electric field $\vec{\mathcal{E}} = (p,p,p)$ with plane wave component $p = E_c e^{i(k_0 z-\omega_0 t)}$.

Claim: this violates Maxwell's law for the divergence of the electric field: $\nabla\cdot\vec{\mathcal{E}} = 0$. Substituting the mentioned plane wave yields

$$\nabla\cdot\vec{\mathcal{E}} = \frac{\partial\mathcal{E}_x}{\partial x} + \frac{\partial\mathcal{E}_y}{\partial y} + \frac{\partial\mathcal{E}_z}{\partial z}$$
$$= 0 + 0 + i k_0 E_c e^{i(k_0 z-\omega_0 t)} \neq 0$$

The nonzero z-component is troublesome in light of the Maxwell divergence law for the electric field.

However, the electric field $\vec{\mathcal{E}} = (p, p, 0)$ does satisfy Maxwell's law. This field is perpendicular to the wavevector $\vec{k} = (0, 0, k_0)$. In fact, this relation holds more generally. Plane waves with wavevector $\vec{k} = (k_x, k_y, k_z)$ are physical when the electric field and wavevector are perpendicular. TODO: show that the general wavevector solves the divergence law when perpendicular to the field... Or maybe more clear through taking the Fourier transform of the divergence law? https://www.theochem.ru.nl/files/dbase/groenenboom-qed-2005.pdf ...

1.3. DERIVE THE HELMHOLTZ EQUATION

NEW: Considering time-harmonic solutions to the scalar wave equation (1.1)

$$\mathcal{E}_j(x, y, z, t) = e^{i\omega_0 t} E(x, y, z) + \text{c.c.} \tag{1.4}$$

These are continuous wave (cw) beam solutions as opposed to pulsed output beams. TODO: include example of pulsed wave expr and sth about pulsed vs continous For more information on the operating principles of lasers, refer to [3].

Substituting (1.4) in equation (1.1) shows that E should satisfy the scalar linear Helmholtz equation

$$\Delta E(x, y, z) + k_0^2 E = 0, \tag{1.5}$$

where k_0 is given by the dispersion relation (1.3). The plane waves **??** solve equation (1.5) with

$$E = E_c e^{i(k_x x + k_y y + k_z z)}$$

with $k_x^2 + k_y^2 + k_z^2 = k_0^2$.

TODO: insert writings on laser beam as superposition of plane waves, each solving HH

1.4. DERIVE THE LINEAR SCHRÖDINGER

NEW: Most of the plane wave modes in **??** are nearly parallel to the z-axis. These paraxial plane waves satisfy

$$k_\perp^2 << k_z^2, \quad k_\perp^2 = k_x^2 + k_y^2.$$

Since $k_0^2 = k_x^2 + k_y^2 + k_z^2 = k_\perp^2 + k_z^2$, we have $k_0^2 \approx k_z^2$.

1.5. POLARISATION FIELD

NEW: Polarisation describes the effect of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

1

In the following, we assume that the electric field is linearly polarised, that is,

$$\vec{\mathcal{E}} = (\mathcal{E}, 0, 0), \ \vec{\mathcal{P}} = (\mathcal{P}, 0, 0), \ \vec{\mathcal{D}} = (\mathcal{D}, 0, 0),$$

where the electric field $\mathcal{E}$ is the cw electric field

$$\mathcal{E}(x, y, z, t) = e^{-i\omega_0 t} E(x, y, z, t) + \text{c.c.}.$$

First we consider linear polarisation effects. The electric fields affects the medium and induces a polarisation proportional to the electric field

$$\mathcal{P} = \mathcal{P}_{\text{lin}} = c\mathcal{E}$$

for some real number c. In fact, we can write

$$\mathcal{P} = \epsilon_0 \chi^{(1)}(\omega_0)\mathcal{E},$$

where $\chi^{(1)}$ is the first-order optical susceptibility, whose value depends on the frequency ω_0. Then the induction electric field is given by

$$\mathcal{D} = \epsilon_0 \mathcal{E} + \mathcal{P}_{\text{lin}} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}, \quad n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0),$$

where n_0 is the linear index of refraction (or refractive index) of the medium.

TODO: Leads to linear Helmholtz with adjusted k_0...

1.6. NONLINEAR POLARISATION...

NEW: The linear polarisation field is an approximation and we wish to study the nonlinear effects too. Consider the nonlinear polarisation field $\mathcal{P}_{\text{nl}}$ as

$$\mathcal{P} = \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}}.$$

TODO: Write about Taylor expansion of the nonlinear term, show that the even terms are not relevant and study the cubic (Kerr) term

1.7. FOCUSING NLS AND SOLITONS

NEW: (General NLS) Substituting $E = e^{ik_0 z}\psi$ in the NLH **??** and applying the paraxial approximation $\psi_{zz} << k_0 \psi_z$, we obtain the nonlinear Schrödinger equation (NLS)

$$2ik_0\psi_z(z, \bar{x}) + \Delta_\perp \psi + k_0^2 \frac{4n^2}{n_0}|\psi|^2\psi = 0. \tag{1.6}$$

(Focusing NLS) The previous results lead to the focusing NLS given by

$$i\psi_z(z, \bar{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0. \tag{1.7}$$

Considering envelopes of constant shape (solitons) with

$$\psi_{\omega}^{\text{soliton}} = e^{i\omega z} R_{\omega}(\bar{x})$$

leads to an equation in $R_{\omega}(\bar{x})$ by the following steps

1. $i\psi_z(z,\bar{x}) = i\left(i\omega e^{i\omega z} R_{\omega}(\bar{x})\right) = -\omega e^{i\omega z} R_{\omega}(\bar{x})$
2. $\Delta\psi = \left(\Delta e^{i\omega z}\right) R_{\omega}(\bar{x}) + e^{i\omega z}\left(\Delta R_{\omega}(\bar{x})\right)$
3. $|\psi|^{2\sigma}\psi = \left|e^{i\omega z} R_{\omega}(\bar{x})\right|^{2\sigma} e^{i\omega z} R_{\omega}(\bar{x}) = |R_{\omega}(\bar{x})|^{2\sigma} e^{i\omega z} R_{\omega}(\bar{x})$
4. such that
5. $e^{i\omega z}\left[-\omega R_{\omega}(\bar{x}) + \Delta R_{\omega}(\bar{x}) + |R_{\omega}(\bar{x})|^{2\sigma} R_{\omega}(\bar{x})\right] = 0$
6. and
7. $\Delta R_{\omega}(\bar{x}) - \omega R_{\omega}(\bar{x}) + |R_{\omega}(\bar{x})|^{2\sigma} R_{\omega}(\bar{x}) = 0$

REFERENCES

[1] G. Fibich, *The nonlinear Schrödinger equation*, Applied Mathematical Sciences, Vol. 192 (Springer, Cham, 2015) p. 862, singular solutions and optical collapse.

[2] D. Griffiths, *Introduction to Electrodynamics*, Pearson international edition (Prentice Hall, 1999).

[3] A. Siegman, *Lasers* (University Science Books, 1986).

2

Conclusion

This is a concluding chapter explaining the scientific and technical implications for society of the research findings in considerable detail.

Acknowledgements

This is an optional chapter containing acknowledgements.