- Define Lyapunov function $E(r)$.
- Rewrite IVP to seperate $-\frac{1}{r}u'(r)$.
- Calculate $E'(r)$.
- Use IVP to simply expression.
- Conclude from sign of $E'(r)$ that $E(r)$ is nonincreasing.
  - Note that $V'(r) \leqslant 0$ and $u'(r)^2 \geqslant 0$, $r \geqslant 0$ and $u(r)^{p+1} \geqslant 0$.
- Evaluate $E(0)$. Analyse the sign of $E(0)$ as a function of α.
  - Solve $E(0) = 0$ for $\bar{\alpha}$ and note $\alpha < \bar{\alpha} \implies E(0) < 0$ and vice versa.
- Now analyse the behaviour of solutions with $\alpha < \bar{\alpha}$ using proof by contradiction.
  - $\alpha < \bar{\alpha} \in N$ would imply $u(z(\alpha)) = 0$ and $E(z(\alpha)) \geqslant 0$ contradicting $E < 0$.
  - $\ldots \in G$ would imply $\lim_{r\to\infty} E(r) = 0$ contradicting $E < 0$.
- Conclusion: $\alpha \in P$.

- As $\alpha \in G \cup N$, $E(r) \geqslant 0$ for $r \in (0, z(\alpha))$.
- Even in $z(\alpha)$, $E(z(\alpha)) \geqslant 0$ (SHOW).
- By IVP and Lemma 5.1: $u''(0) < 0$.
- Using an argument from Chapter 4: $u''(0)$ and $u'(0) = 0$ would imply $u \equiv \alpha$.
- On the other hand $u''(0) > 0$ and $u'(0) = 0$ would imply $u(r) > u(0) = \alpha$ for $r > 0$ and small.
- This contradicts $E'(r) \leqslant 0$? (SHOW)
- Conclusion: $u''(0) < 0$ and $u'(r) < 0$ for $r > 0$ and small.
- To show $u'(r) < 0$ for $r \in (0, z(\alpha))$:
  - Suppose by contradiction that there exists r_0 such that $u'(r_0) = 0$.
  - By the IVP: $u''(r_0) = \lambda u(r_0) - V(r_0)u(r_0)^p$.
  - From the previous concavity argument, $u''(r_0) \geqslant 0$ and $u''(r_0) = 0$ would imply $u \equiv u(r_0)$.
  - Then $u''(r_0) > 0$ which can be used in analysis of $E(r_0)$.
  - Evaluate $E(r_0) = \frac{1}{2}u'(r_0)^2 - \frac{\lambda}{2}u(r_0)^2 + \frac{1}{p+1}V(r_0)u(r_0)^{p+1} \geqslant 0$.
  - An interesting property of $u(r_0)$ by IVP, $u''(r_0) > 0$ and $u'(r_0) = 0$ follows
    - $u(r_0) \leqslant \left[\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}} < \left[\frac{\lambda}{V(0)}\frac{p+1}{2}\right]^{\frac{1}{p-1}} \iff -\frac{\lambda}{2}u(r_0)^2 + \frac{1}{p+1}V(r_0)u(r_0)^{p+1} > 0 \iff E(r_0) > 0$
    - Which contradicts $E(r) \geqslant 0$ for $r \in (0, z(\alpha))$.
  - Hence $u'(r) < 0$ for $r \in (0, z(\alpha))$.
- If $\alpha \in N$ then $u'(z(\alpha)) = 0$ then $u \equiv 0$.
  - Concavity prevents $u'(z(\alpha)) > 0$. (SHOW)
  - What does this yield if $\alpha \in G$? (SHOW)
- So for $\alpha \in N$: $u'(r) < 0$ for $r \in (0, z(\alpha)]$.

- To conclude that w has one zero in $(0, z(\alpha))$, use Lagrange identity.
- The proofs for $\alpha \in G$ and $\alpha \in N$ will be done seperately.
- First, suppose $\alpha \in N$.
- Rewrite IVP and *w-d.e.* to the following:
  - $(ru')' + r\left[-\lambda u + Vu^p\right] = 0$
  - $(rw')' + r\left[-\lambda w + pVu^{p-1}w\right] = 0$
- Multiply by w and u respectively, then integrate from 0 to $z(\alpha)$:
  - $\int_0^{z(\alpha)} w(ru')' - u(rw')'dr = \int_0^{z(\alpha)} r\left[pVu^pw - Vu^pw\right]dr$
- By partial integration for left hand side, one obtains:
  - $rwu' \Big|_0^{z(\alpha)} - ruw' \Big|_0^{z(\alpha)} - \int_0^{z(\alpha)} \left[ru'w' - ru'w'\right]dr = (p-1)\int_0^{z(\alpha)} rVu^pwdr$
- Use $u(z(\alpha)) = 0$ to obtain:

- $z(\alpha)w(z(\alpha))u'(z(\alpha)) - z(\alpha)u(z(\alpha))w'(z(\alpha)) = (p-1)\int_0^{z(\alpha)} rVu^p w dr$
- Suppose $w > 0$ on $(0, z(\alpha))$ then left hand side $\leqslant 0$ as $z(\alpha) > 0$, $w(z(\alpha)) \geqslant 0$ (?) and $u'(z(\alpha)) < 0$.
- To resolve this, suppose $w > 0$ on $(0, z(\alpha))$ then $z(\alpha) > 0$, $w(z(\alpha)) > 0$ and $u'(z(\alpha)) < 0 \implies$ l.h.s. < 0. That *is* sufficient to show contradiction with r.h.s. > 0 as ...
- To *actually* resolve this, the initial supposition was correct: $w > 0$ on $(0, z(\alpha))$ implies l.h.s. $\leqslant 0$ as $z(\alpha) > 0$, *importantly* $w(z(\alpha)) > 0$ and $u'(z(\alpha)) < 0$. Why can't $w(z(\alpha)) = 0$? By Sturm comparison! Since $pVwu^{p-1} \neq Vu^p$, the zeroes of w and u will not coincide!
  - Note work remains to be done to clarify this part of the argument. From Śuppose $w > 0$...t́o the contradiction.
- By this contradiction then, $w(r)$ has at least one zero on $(0, z(\alpha))$.
- To conclude the same for $\alpha \in G$, again, assume $w > 0$ on $(0, z(\alpha))$ then still r.h.s. of **??**dentity > 0.
- As for the l.h.s. regard the expression $\frac{u}{w}$...
  - Write $\left(\frac{u}{w}\right)' = \frac{wu' - uw'}{w^2}$.
  - Note that $u(0) > 0$ and $w(0) > 0$ implies $\frac{u}{w}(0) > 0$.
  - Rewrite the **??**dentity to read:
    * $rwu' - ruw' = (p-1)\int_0^{z(\alpha)} rVu^p w dr$
    * $\frac{wu'}{w^2} - \frac{uw'}{w^2} = \frac{p-1}{rw^2}\int_0^{z(\alpha)} rVu^p w dr > 0$
    * $\frac{wu' - uw'}{w^2} > 0 \implies \left(\frac{u}{w}\right)' > 0$
  - So $\left(\frac{u}{w}\right)'$ is increasing.
- Now, there is a hole.. Apparantly, this also implies $w > 0$ yields contradiction.
  - Intuitively, $\left(\frac{u}{w}\right)'$ increasing means that w decays faster than u everywhere.
  - Then, as for $\alpha \in G$ the solution decays to 0, so must w.
  - But, since w decays faster than u, the zero of w must be to the left of $z(\alpha) = \infty$.
  - Hence, w has a zero in $(0, z(\alpha))$.
    * Hooray!
- Might need to formalise this a bit further.

Department of Applied Physics, TU Delft, Lorentzweg 1, 2628CJ, Delft, Netherlands, EU
Email address: j.s.eenhoorn@student.tudelft.nl