ON EXISTENCE AND UNIQUENESS OF GROUND STATE SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this review, existence and uniqueness of ground state solutions to an NLS-like equation is studied. The nonlinear Schrödinger equation (NLS) is the paraxial approximation of the nonlinear Helmholtz equation (NLH).

1. Introduction

2. Physics of NLS

In this section, first, the nonlinear Schrödinger equation (NLS) will be derived from Maxwell's laws. A brief discussion of the assumptions involved in this derivation and the interpretation of (intermediate) results is given. The discussion is due to Gadi Fibich, for more details, see his book "The Nonlinear Schrödinger Equation". In the end, the NLS is given as

$$2ik_0\psi_z(x,y,z) + \underbrace{\Delta_\perp\psi}_{\text{diffraction}} + \underbrace{k_0^2\frac{4n_2}{n_0}|\psi|^2\psi}_{\text{Kerr nonlinearity}} = 0. \tag{2.1}$$

2.1. **Vector electromagnetic fields: Maxwell's laws.**

The propagation of electromagnetic waves in a medium is governed by Maxwell's laws. (In absence of external charges or currents.) Remember that Maxwell's laws for the electric field $\mathcal{E}$, magnetic field $\mathcal{H}$, induction electric field $\mathcal{D}$ and induction magnetic field $\mathcal{B}$ are given by:

$$\begin{gathered}\nabla\times\vec{\mathcal{E}} = -\frac{\partial\vec{\mathcal{B}}}{\partial t}, \quad \nabla\times\vec{\mathcal{H}} = -\frac{\partial\vec{\mathcal{D}}}{\partial t}, \\ \nabla\cdot\vec{\mathcal{D}} = 0, \quad \nabla\cdot\vec{\mathcal{B}} = 0.\end{gathered} \tag{2.2}$$

These are vector fields: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in x, y, z coordinates. In vacuum, the relations between the electric or magnetic fields and induction fields are given as:

$$\vec{\mathcal{B}} = \mu_0\vec{\mathcal{H}}, \quad \vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} \tag{2.3}$$

2.2. **Wave equation from Maxwell's laws.**

From these relations and the vector identity for the curl of the curl, the wave equation can be derived. In particular,

$$\nabla\times\nabla\times\vec{\mathcal{E}} = \nabla\times(-\frac{\partial\vec{\mathcal{B}}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla\times\vec{\mathcal{B}}), \quad \text{by Maxwell's laws, and} \tag{2.4}$$

$$\nabla\times\nabla\times\vec{\mathcal{E}} = \nabla(\nabla\cdot\vec{\mathcal{E}}) - \nabla^2\vec{\mathcal{E}} = \nabla(\nabla\cdot\vec{\mathcal{E}}) - \Delta\vec{\mathcal{E}}, \quad \text{by vector calculus.} \tag{2.5}$$

Also, calculate the curl of the magnetic field: $\nabla\times\vec{\mathcal{B}} = \mu_0\frac{\partial\vec{\mathcal{D}}}{\partial t}$. Then combining these results:

$$\Delta\vec{\mathcal{E}} - \nabla(\nabla\cdot\vec{\mathcal{E}}) = \mu_0\frac{\partial^2\vec{\mathcal{D}}}{\partial t^2} \tag{2.6}$$

$$\Delta\vec{\mathcal{E}} - \nabla(\frac{1}{\epsilon_0}\nabla\cdot\vec{\mathcal{D}}) = \mu_0\epsilon_0\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{2.7}$$

And using $\nabla\cdot\vec{\mathcal{D}} = \nabla\cdot\epsilon_0\vec{\mathcal{E}} = 0$, this yields the wave equation, where $\mu_0\epsilon_0 = 1/c^2$:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{2.8}$$

2.3. **Scalar wave equation and solutions.**

The components of the vector electric field are decoupled. Thus components of solutions to the vector wave equation satisfy a scalar wave equation:

$$\Delta\mathcal{E}_i = \sum_{i=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_i^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}. \tag{2.9}$$

Decoupled means that the second derivative of any component with respect to time is related to the Laplacian of that component only. There is no term relating $\mathcal{E}_x$ to $\mathcal{E}_y$ or $\mathcal{E}_z$ and vice versa. This can be seen from the following explication of the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \Delta \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathcal{E}_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} + \frac{\partial^2 \mathcal{E}_x}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2} \end{bmatrix} = \frac{1}{c^2} \begin{bmatrix} \frac{\partial^2 \mathcal{E}_x}{\partial t^2} \\ \frac{\partial^2 \mathcal{E}_y}{\partial t^2} \\ \frac{\partial^2 \mathcal{E}_z}{\partial t^2} \end{bmatrix} \tag{2.10}$$

The scalar solutions are of the form:

$$\mathcal{E}_j = E_c \exp^{i(k_0 z - \omega_0 t)} + \text{c.c.}, \tag{2.11}$$

where $k_0^2 = \omega_0^2/c^2$ is the dispersion relation for a plane wave travelling in the positive z-direction. For plane waves propagating in a general direction:

$$\mathcal{E}_j = E_c \exp^{i(\vec{k_0}\cdot\vec{r} - \omega_0 t)} + \text{c.c.}, \tag{2.12}$$

where $\vec{k_0} = k_x^2\hat{\imath} + k_y^2\hat{\jmath} + k_z^2\hat{k}$, $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $|\vec{k_0}|^2 = \omega_0^2/c^2$. In these expressions "c.c." stands for complex conjugate, so that 2.11 actually reads (similar for 2.12):

$$\mathcal{E}_j = E_c \exp^{i(k_0 z - \omega_0 t)} + E_c \exp^{-i(k_0 z - \omega_0 t)} \tag{2.13}$$

as E_c is a constant real number.

2.4. **Linear polarisation and interpretation.**

For a plane wave propagating in the positive z direction, in what directions can the electric field point? Suppose $\vec{\mathcal{E}} = (\mathcal{E}_1, 0, 0)$ then $\vec{\mathcal{E}}$ solves the wave equation if

$$\mathcal{E}_1 = E_c \exp^{i(k_0 z - \omega_0 t)} + \text{c.c.} \tag{2.14}$$

which is actually inconsistent with Maxwell's laws. To see this, calculate

$$0 = \nabla\cdot\vec{\mathcal{D}} = \epsilon_0 \nabla\cdot\vec{\mathcal{E}} = \epsilon_0 (\mathcal{E}_1)_x \tag{2.15}$$

which implies the electric field is not localised in x. Then how can one speak of linearly polarised laser beams? The resolution is that in reality, $\mathcal{E}_2$ and $\mathcal{E}_3$ are not actually 0, such that $\vec{\mathcal{E}}$ is linearly polarised in the sense that $\mathcal{E}_1 >> \mathcal{E}_2, \mathcal{E}_3$.

2.5. **Paraxial beams and Helmholtz equation solutions.**

To continue the discussion of solutions to the wave equation, consider a sum of plane waves. In particular, consider a laser beam representated by a sum of plane waves. For this more general situation, consider time-harmonic (monochromatic) solutions given by:

$$\mathcal{E}_j(x, y, z, t) = \exp^{-i\omega_0 t} E(x, y, z) + \text{ c.c.}, \tag{2.16}$$

solving a specific case of the (scalar) wave equation known as the *Helmholtz* equation.

$$\Delta E + k_0^2 E = 0 \tag{Helmholtz}$$

For a laser beam propagating in the z-direction, write the incoming field $E_0^{inc}(x, y)$ as a sum of plane waves:

$$E_0^{inc}(x, y) = \frac{1}{2\pi} \int E_c(k_x, k_y) \exp^{i(k_x + k_y)} \mathrm{d}k_x \mathrm{d}k_y \text{ such that} \tag{2.17}$$

$$E(x, y, z) = \frac{1}{2\pi} \int E_c(k_x, k_y) \exp^{i(k_x + k_y + \sqrt{k_0^2 - k_x^2 - k_y^2} z)} \mathrm{d}k_x \mathrm{d}k_y \tag{2.18}$$

However, since a laser beam is concentrated around the z-axis, $k_z \approx k_0$. Then $k_\perp = k_x^2 + k_y^2 << k_z^2$. In other words, as $k_0^2 := k_x^2 + k_y^2 + k_z^2$, $k_z = \sqrt{k_0^2 - k_x^2 - k_y^2}$. Then $E(x, y, z) = E_c \exp^{i(k_x x + k_y y + k_z z)} = \exp^{ik_0 z} \psi(x, y, z)$ with $\psi = E_c \exp^{i(k_x x + k_y y + (k_z - k_0) z)}$.

This ψ function is an envelope, of which the amplitude varies slowly in z. The envelope satisfies its own Helmholtz equation:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp \psi = 0, \tag{2.19}$$

where $\Delta_\perp = \psi_{xx} + \psi_{yy}$. In this equation, the ψ_{zz} term can be neglected as $\psi_{zz} << k_0\psi_z$, $\psi_{zz} << \Delta_\perp \psi$. The resulting equation is called the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp \psi = 0. \tag{Linear Schrödinger}$$

Split into $E = \exp^{ik_0 z} \psi(x, y, z)$, with ψ an envelope function varying slowly in z. This ψ solves a Helmholtz equation. Neglect ψ_{zz} (paraxial) to obtain the linear Schrödinger equation for ψ.

Now the step needs to be made to the nonlinear Helmholtz equation. The different polarisations need to be described, weakly nonlinear and Kerr nonlinear. Then obtain nonlin HH to apply parax. approx. to obtain NLS. Then make steps to dimensionless NLS, consider solitary waves and specify the ω...

Polarisation, linear polarisation, weakly nonlinear polarisation, Kerr nonlinearity. This all leads to nonlinear Helmholtz, apply paraxial approximation to obtain NLS.

Step over to dimensionless NLS and consider solitary waves.

Fill in details. Then, by considering radially symmetric solitary wave solutions, one obtains:

$$R'' + \frac{1}{r}R' - R + R^3 = 0,$$

with initial condition $R'(0) = 0$ and finite power: $\lim_{r\to\infty} R(r) = 0$. This is the equation for which existence and uniqueness of solutions will be discussed.

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3. Solution sets and notation

The previous chapter leads to the following initial value problem (IVP). What is u? Another important identity is achieved by integrating the IVP over r. The solution sets are disjoint.

$$\begin{cases} -u''(r) - \dfrac{N-1}{r}u'(r) = g(u(r)) & (3.1\text{a}) \\ u(0) = \alpha,\ u'(0) = 0 & (3.1\text{b}) \end{cases}$$

The possible initial values $u(0) = \alpha > 0$ categorise the solutions in three solution sets: solutions that become negative (N), solutions that are positive (P) for all r and solutions that vanish (G, for ground state). In terms of zeroes, solutions with initial condition in N have at least one zero, solutions with initial condition in P have no zeroes and solutions with initial condition in G tend to zero for r to infinity. As these solution sets describe all possible behaviour, the positive r axis is the disjoint union of P, G and N. Lastly, let $z(\alpha)$ describe the smallest zero of the solution. That is, the supremum of $s > 0$ such that the solution is positive for $r \in [0, s)$. Formally:

$$N \coloneqq \{\alpha > 0\colon \text{ there exists } r > 0 \text{ such that } u(r; \alpha) = 0\}$$

$$G \coloneqq \{\alpha > 0\colon\ u(r; \alpha) > 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r; \alpha)\}$$

$$P \coloneqq \{\alpha > 0\colon\ u(r; \alpha) > 0 \text{ for all } r > 0 \text{ and } \alpha \notin G\}$$

$$(0, \infty) = P \cup G \cup N$$

$$z(\alpha) \coloneqq \sup\{s > 0\colon\ u(r; \alpha) > 0 \text{ for all } r \in [0, s)\}$$

Let $w(r; \alpha) = \frac{\partial}{\partial\alpha} u(r; \alpha)$, which satisfies the following IVP: (for $N = 2$)

$$\begin{cases} w''(r) + \dfrac{1}{r}w'(r) - \dfrac{\lambda}{2}w(r) + pV(r)u(r)^{p-1}w(r) = 0 & (3.2\text{a}) \\ w(0) = 1,\ \lim\limits_{r\downarrow 0} rw'(r) = 0 & (3.2\text{b}) \end{cases}$$

4. Existence of ground state

The Maxwell equations and ++certain approximations yield the initial value problem ++ ... yields the initial value problem (IVP) ++ ... Question: is this model well-posed? Firstly, does there exist at least one solution? Secondly, does the model have one solution at most? Lastly, does the solution change continuously with the initial conditions? Each of these aspects is formalised in a theorem. Rigorous proofs of these theorems are studied and elaborated upon. In ++, existence (of solutions) is proven under certain conditions. In ++, uniqueness (of solutions) is proven for slighty different conditions. For continuous dependence on the initial conditions, refer to ++.

The existence of a ground state solution to (3.1) is guaranteed under certain conditions on g, which is the nonlinear term in the initial value problem. In this chapter the existence theorem and setting for this problem are stated and a proof will be given based on two lemmas. Note that $\mathbb{R}_+ = [0, \infty)$.

Theorem A. Let g satisfy conditions (A1) to (A7) in section 4.2. Then there exists a number $\alpha \in (\alpha_0, \lambda)$ such that the solution $u \in C^2(\mathbb{R}_+)$ of (3.1) has the properties:

$$u(r) > 0 \text{ for } r \in [0, \infty)$$

$$u'(r) < 0 \text{ for } r \in (0, \infty)$$

$$\text{and } \lim_{r \to +\infty} u(r) = 0.$$

If, in addition, we assume g satisfies

$$\limsup_{s \downarrow 0} \frac{g(s)}{s} < 0,$$

then there exist constants $C, \delta > 0$ such that

$$0 < u(r) \leqslant Ce^{-\delta r} \quad \text{for } r \in [0, \infty).$$

4.1. Discussion of the theorem.

Restricted set of initial conditions In the initial value problem $\alpha := u(0)$ and $u'(0) = 0$, and $\alpha \in (0, \infty)$ since the solution depends on $r = |x|$ ($x \in \mathbb{R}^N$). That is, the problem allows any positive initial condition, $\alpha = u(\alpha, 0) > 0$. However, the properties of the nonlinear term allow for a smaller scope of initial conditions. The theorem restricts the initial conditions to $\alpha \in (\alpha_0, \beta)$. This restriction is a by-product of the proof... ELABORATE

The dimensions $N \geqslant 2$ are considered. For the case $N = 1$, see [].

4.2. Conditions on g.

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$$\text{Let } g \text{ be locally Lipschitz continuous from } \mathbb{R}_+ \text{ to } \mathbb{R} \text{ with } g(0) = 0. \tag{A1}$$

$$\text{In addition, let } g(u) = 0 \text{ for } u \leqslant 0.$$

$$\text{Let } \kappa > 0 \text{ be finite, where } \kappa := \inf(\alpha > 0, g(\alpha) \geqslant 0). \tag{A2}$$

$$\text{Define } G(t) := \int_0^t g(s)ds.$$

$$\text{Let } \alpha_0 > \kappa, \text{ where } \alpha_0 := \inf(\alpha > 0, G(\alpha) > 0). \tag{A3}$$

$$\text{Let } \lim_{s \downarrow \kappa} \frac{g(s)}{s - \kappa} > 0. \tag{A4}$$

$$\text{Let } g(s) > 0 \text{ for } s \in (\kappa, \alpha_0]. \tag{A5}$$

$$\text{Let } \lambda \leqslant \infty, \text{ where } \lambda := \inf(\alpha > \alpha_0, g(\alpha) = 0). \tag{A6}$$

$$\text{If } \lambda = \infty, \text{ let } \lim_{s \to \infty} \frac{g(s)}{s^l} = 0, \text{ where} \tag{A7}$$

$$l > 0 \text{ if } N = 2, \text{ and } l < \frac{N+2}{N-2} \text{ for } N > 2.$$

Thirdly, note that g is by assumption negative on $(0, \kappa)$ and hence there exists a number $\alpha > 0$ such that $G(\alpha) > 0$. Here

$$G(t) = \int_0^t g(s)ds.$$

Define $\alpha_0 := \inf(\alpha > 0, G(\alpha) > 0)$. Then α_0 exists by the above and $\alpha_0 > \kappa$.

Fourthly, g satisfies these two conditions regarding κ and g on $(\kappa, \alpha_0]$:

$$\lim_{s \downarrow \kappa} \frac{g(s)}{s - \kappa} > 0; \tag{4.1}$$

$$g(s) > 0 \text{ for } s \in (\kappa, \alpha_0]. \tag{4.2}$$

Fifthly, regarding the behaviour of g after α_0, the function may remain positive indefinitely or become negative again. In the former case, let $\lambda = +\infty$ and in the latter case, $\lambda := \inf(\alpha > \alpha_0, g(\alpha) = 0)$. Then by these assumptions, in any case, $\alpha_0 < \lambda \leqslant \infty$.

Lastly, if $\lambda = \infty$, then $\lim\limits_{s \to \infty} \frac{g(s)}{s^l} = 0$, with $l < \frac{N+2}{N-2}$. Note that this is not defined for $N = 2$, in that case, let $l \in \mathbb{R}$.

REMARKS Note that the infimum in (A2) may be infinite. Then $g(u) < 0$ for all $u > 0$. WHY NOT? Hence the explicit requirement.

Example 1. The conditions on g are satisfied for $g(u) = -u + u^3$. COUPLING TO PHYSICS

4.3. **The solutions are defined on the semi-definite interval.**

Let $\alpha \in (\kappa, \lambda)$ and let $u(\alpha, r)$ be the corresponding solution of (3.1). That g is locally Lipschitz continuous implies that the solution $u(\alpha, r)$ is defined on some interval $[0, r_\alpha)$. Remember, the solution may have asymptotes. Boundedness of the solution is a sufficient and necessary condition for $r_\alpha = \infty$. Note that the initial condition is positive, $\alpha > 0$. Claim: the solution $u(\alpha, r)$ is bounded above by the initial condition, $u(\alpha, r) \leqslant u(\alpha, 0) = \alpha$ for $r \geqslant 0$. To see this, multiply the (3.1) by u' and integrate between 0 and r to obtain:

$$-u'' - \frac{N-1}{r}u' = g(u)$$

$$-\int_0^r \left[u'(s)u''(s)\right] ds - \int_0^r \left[\frac{N-1}{s}[u'(s)]^2\right] ds = \int_0^r \left[u'(s)g(u(s))\right] ds$$

$$\text{Use } \frac{d}{dr}\left[(u'(r))^2\right] = 2u'(r)u''(r) :$$

$$-\frac{1}{2}[u'(r)]^2 - (N-1)\int_0^r [u'(s)]^2\frac{ds}{s} = \int_0^r g(u(s))\frac{du}{ds}ds = \int_0^r g(u)du$$

$$-\frac{1}{2}[u'(r)]^2 - (N-1)\int_0^r [u'(s)]^2\frac{ds}{s} = G(u(\alpha,r)) - G(\alpha) \tag{4.3}$$

Note that $u'(0) = 0$ in evaluating the integrand from the first term of the integral. Suppose that $u' > 0$ for r small. Then $u(\alpha, r) > u(\alpha, 0)$ and since G is nondecreasing on (κ, λ) the right hand side is positive $G(u(\alpha, r)) - G(\alpha) > 0$. Then

$$-\frac{1}{2}[u'(r)]^2 - (N-1)\int_0^r [u'(s)]^2\frac{ds}{s} > 0$$

$$\frac{1}{2}[u'(r)]^2 + (N-1)\int_0^r [u'(s)]^2\frac{ds}{s} < 0 \quad \lightning,$$

which is clearly impossible, both terms are positive. Conclusion: the solution is decreasing at first and bounded above by the initial condition. In fact, suppose $u(\alpha, r) \geqslant \alpha$ for some $r_0 > 0$, then again $G(u(\alpha, r_0)) - G(\alpha) > 0$ and the same contradiction is found.

It remains to show $u(\alpha, r)$ has a lower bound. Let $r_0 := \inf(r > 0, u(\alpha, r_0) = 0)$. Suppose $r_0 < \infty$. (Note that $r_0 = \infty \implies u(\alpha, r) > 0$ for all $r > 0$, hence $u(\alpha, r)$ is bounded.) If $u'(\alpha, r_0) = 0$, then $u(\alpha, r) \equiv 0$. Thus $u'(\alpha, r_0) < 0$. Claim: the derivative will decay hyperbolically for $r \geqslant r_0$ as,

$$u'(\alpha, r) = \left(\frac{r_0}{r}\right)^{N-1} u'(\alpha, r_0) \geqslant u'(\alpha, r_0)$$

.

To see this, use condition (A1) in (3.1),

$$-u'' - \frac{N-1}{r}u' = 0,$$

which is valid for $u(\alpha, r) \leqslant 0$. To be safe, let $r_1 = \inf(r > r_0, u(\alpha, r_1) = 0)$ and suppose $r_1 < \infty$. Then $u(\alpha, r) \leqslant 0$ on $[r_0, r_1]$. Now solve for $u' = u'(\alpha, r)$ on $[r_0, r_1]$:

$$
\begin{aligned}
-\frac{d}{dr}u' - \frac{N-1}{r}u' = 0 \\
\frac{du'}{dr} &= -\frac{N-1}{r}u' \\
\frac{du'}{u'} &= -\frac{N-1}{r}dr \\
\ln u'\big|_{r_0}^{r} &= [(N-1)\ln r]_{r}^{r_0} \\
\ln u'(r) - \ln u'(r_0) &= (N-1)\left[\ln r_0 - \ln r\right] \\
\frac{u'(r)}{u'(r_0)} &= \left(\frac{r_0}{r}\right)^{N-1} \\
u'(\alpha, r) = \left(\frac{r_0}{r}\right)^{N-1} u'(\alpha, r_0) \geqslant u'(\alpha, r_0).
\end{aligned}
$$

It follows that $u'(\alpha, r) \leqslant 0$ on $[r_0, r_1]$. Hence $u(\alpha, r) < 0$ on $(r_0, r_1]$, which contradicts the assumption on r_1. Thus $r_1 = \infty$ and $u(\alpha, r) < 0$ for $r > r_0$. Note how $u'(\alpha, r) \uparrow 0$ for $r \to \infty$. Then $u(\alpha, r)$ has some lower bound. Since the solution is bounded, it is defined on $(0, \infty)$.

4.4. **The shooting method.**

Now any $\alpha \in I$ is defined on $(0, \infty)$. Also, $g(\alpha) > 0$ and therefore $u''(\alpha, 0) < 0$ by the (3.1): $u'(0) = 0$ and $-u''(\alpha, 0) = g(u(\alpha, 0)) = g(\alpha) > 0$. Then for $r > 0$ and small: $u'(\alpha, r) < 0$ and $u(\alpha, r) > 0$. To analyse the behaviour of the solutions for larger r, distinguish two sets of initial conditions: solutions that have a vanishing derivative in some point and are positive up to and including that point, and solutions and vanish in some point, but have negative derivative up to and including that point. These sets are defined below. If these sets are open, nonempty and disjoint, then there exist elements $\alpha^* \in I$ such that $u(\alpha^*, r) > 0$ for all $r \geqslant 0$ and $u'(\alpha^*, r) < 0$ for all $r > 0$. By lemma 4.2 and its proof, the sets have these properties and hence such elements exist. Intuitively, a solution that is positive everywhere and has negative derivative everywhere has a limit for r tending to infinity, and is possibly ground state. However, that also requires that this limit is zero.

Lemma 4.1. *Let g be locally Lipschitz continuous on $\mathbb{R}_+$ such that $g(0) = 0$. Let $\alpha_1 \in (0, \infty)$ be such that $u(\alpha_1, r) > 0$ for all $r \geqslant 0$ and $u'(\alpha_1, r) < 0$ for all $r > 0$. Then the number $l = \lim_{r\to\infty} u(\alpha_1, r)$ satisfies $g(l) = 0$. Furthermore, if g satisfies* (4.1) *and $g(\kappa) = 0$, then $l \neq \kappa$.*

Proof. Let α_1 be as assumed in the lemma and let r tend to infinity in the (3.1):

$$
\lim_{r\to\infty}\left[u''(\alpha_1, r) + \frac{N-1}{r}u'(\alpha_1, r) + g(u(\alpha_1, r))\right] = 0, \tag{4.4}
$$

and note that $l = \lim_{r\to\infty} u(r)$. To prove the first statement of the lemma: the number $l = \lim_{r\to\infty} u(r)$ satisfies $g(l) = 0$, information about the limits of u' and u'' is required. In fact, they both need to converge to zero. Then $g(l) = 0$ and it remains to show that $g \neq \kappa$. <u>Claim:</u> both $u'' \to 0$ and $u' \to 0$ as $r \to \infty$.

Proof of the claim. Remember expression (4.3), where the (3.1) was multiplied by u' and integrated from 0 to r. Now evaluate the limit for r tending to infinity:

$$\lim_{r\to\infty}\left[-\frac{1}{2}[u'(\alpha_1,r)]^2-(N-1)\int_0^r[u'(\alpha_1,s)]^2\frac{ds}{s}\right]=\lim_{r\to\infty}\left[G(u(\alpha,r))-G(\alpha)\right]$$

$$\lim_{r\to\infty}\frac{1}{2}[u'(\alpha_1,r)]^2+(N-1)\lim_{r\to\infty}\int_0^r[u'(\alpha_1,s)]^2\frac{ds}{s}=G(\alpha_1)-\lim_{r\to\infty}G(u(\alpha,r))$$

$$\lim_{r\to\infty}\frac{1}{2}[u'(\alpha_1,r)]^2+(N-1)\int_0^\infty[u'(\alpha_1,s)]^2\frac{ds}{s}=G(\alpha_1)-G(l)$$

$$\lim_{r\to\infty}\frac{1}{2}[u'(\alpha_1,r)]^2+(N-1)\int_0^\infty[u'(\alpha_1,s)]^2\frac{ds}{s}<\infty$$

and thus both terms of the left hand side should be finite, so $u'(\alpha_1,r)$ converges as $r\to\infty$. Remember now that $u(\alpha_1,r)$ is bounded, so the derivative must converge to 0:

$$\lim_{r\to\infty}u'(\alpha_1,r)=0.$$

Now return to (4.4) and use the acquired information:

$$\lim_{r\to\infty}\left[u''(\alpha_1,r)+\frac{N-1}{r}u'(\alpha_1,r)+g(u(\alpha_1,r))\right]=0$$

$$-\lim_{r\to\infty}\left[u''(\alpha_1,r)\right]-\lim_{r\to\infty}\left[\frac{N-1}{r}u'(\alpha_1,r)\right]=\lim_{r\to\infty}g(u(\alpha_1,r))$$

$$-\lim_{r\to\infty}\left[u''(\alpha_1,r)\right]=g(l)$$

But $g(l)$ is finite since u is bounded and thus u'' converges as r tends to infinity. By similar argument the limit is zero. Note that u' is bounded, so u'' has to converge to zero.

$$\lim_{r\to\infty}u''(\alpha_1,r)=0.$$

So the claim is valid and $g(l)=0$. □

It remains to be shown that if g satisfies (4.1) then $l\neq\kappa$. Suppose to the contrary $l=\kappa$. Then introduce the following substitution:

$$v(r)=r^{(1/2)(N-1)}\left[u(r)-\kappa\right],$$

where $u(r)=u(\alpha_1,r)$. Combining this function, its derivatives and the (3.1), one can obtain a differential equation in v. This will then be used to argue that $l=\kappa$ can only lead to contradictions when g satisfies (4.1) and $g(\kappa)=0$.

Calculate the derivatives of v:

$$v(r) = r^{(N-1)/2}\left[u(r) - \kappa\right]$$

$$v'(r) = \frac{1}{2}(N-1)r^{(N-3)/2}\left[u(r) - \kappa\right] + r^{(N-1)/2}u'(r)$$

$$\begin{aligned} v''(r) &= \frac{1}{4}(N-1)(N-3)r^{(N-5)/2}\left[u(r) - \kappa\right] \\ &\quad + \frac{1}{2}(N-1)r^{(N-3)/2}u'(r) + \frac{1}{2}(N-1)r^{(N-3)/2}u'(r) \\ &\quad + r^{(N-1)/2}u''(r) \end{aligned}$$

$$v''(r) = \frac{1}{4}(N-1)(N-3)r^{(N-5)/2}\left[u(r) - \kappa\right] + (N-1)r^{(N-3)/2}u'(r) + r^{(N-1)/2}u''(r)$$

and take out any integer powers of r: (then all terms carry $r^{(N-1)/2}$)

$$v(r) = r^{(N-1)/2}\left[u(r) - \kappa\right]$$

$$v'(r) = \frac{1}{2}(N-1)r^{(N-1)/2}r^{-1}\left[u(r) - \kappa\right] + r^{(N-1)/2}u'(r)$$

$$v''(r) = \frac{1}{4}(N-1)(N-3)r^{(N-1)/2}r^{-2}\left[u(r) - \kappa\right] + \underline{(N-1)r^{(N-1)/2}r^{-1}u'(r) + r^{(N-1)/2}u''(r)}$$

and multiply the (3.1) by $r^{(N-1)/2}$:

$$-u''(r) - (N-1)r^{-1}u'(r) = g(u(r))$$

$$-r^{(N-1)/2}u''(r) - (N-1)r^{(N-1)/2}r^{-1}u'(r) = g(u(r))r^{(N-1)/2} \tag{*}$$

to see that the last two (underlined) terms of $v''(r)$ are equal (up to a minus sign) to the left hand side of (*). That means we can write $-g(u(r))r^{(N-1)/2}$ in the expression for $v''(r)$:

$$v''(r) = \frac{1}{4}(N-1)(N-3)r^{(N-1)/2}r^{-2}\left[u(r) - \kappa\right] - g(u(r))r^{(N-1)/2}.$$

Now take out a factor $v(r) = r^{(N-1)/2}\left[u(r) - \kappa\right]$ and multiply by -1 to obtain:

$$v''(r) = r^{(N-1)/2}\left[u(r) - \kappa\right]\left\{\frac{1}{4}(N-1)(N-3)r^{-2} - \frac{g(u)}{u(r) - \kappa}\right\}$$

$$v''(r) = v\left\{\frac{(N-1)(N-3)}{4r^2} - \frac{g(u)}{u(r) - \kappa}\right\}$$

$$-v''(r) = \left\{\frac{g(u)}{u(r) - \kappa} - \frac{(N-1)(N-3)}{4r^2}\right\}v$$

Also $v(r) > 0$ for $r \geqslant 0$ by definition of $v(r)$. The r-term is positive and increasing and the term in brackets is positive and decreasing. Thus $v(r)$ is positive. As mentioned, this differential equation in v is what will be used to show that $l = \kappa$ is impossible under the assumptions on g. Before diving into the cases, a lower bound for the term in brackets will be calculated. Remember that by assumption $u(r) \downarrow \kappa$ as $r \uparrow \infty$ and g satisfies (4.1). Claim: there exist positive numbers ω and R_1 such that:

$$\frac{g(u)}{u(r) - \kappa} - \frac{(N-1)(N-3)}{4r^2} \geqslant \omega \quad \text{for all } r \geqslant R_1$$

Proof of the claim. By assumption, $g(\kappa) = 0$ and using the definition of the derivative:

$$\lim_{u(r)\downarrow\kappa} \frac{g(u(r))}{u(r)-\kappa} = \lim_{u(r)\downarrow\kappa} \frac{g(u(r))-g(\kappa)}{u(r)-\kappa} \overset{(u(r)-\kappa=h)}{=} \lim_{h\downarrow 0} \frac{g(\kappa+h)-g(\kappa)}{h} = g'(\kappa^+)$$

and $g'(\kappa^+) > 0$ by conditions on g. Let $\epsilon > 0$. Then there exists $R_\epsilon > 0$ such that

$$r \geqslant R_\epsilon \implies \left| \frac{g(u(r))}{u(r)-\kappa} - g'(\kappa) \right| \leqslant \frac{\epsilon}{2}$$

$$\implies \frac{g(u(r))}{u(r)-\kappa} \geqslant g'(\kappa) - \frac{\epsilon}{2} \tag{A}$$

Note also that there exists $R_\theta > 0$ such that

$$r \geqslant R_\theta \implies \left| \frac{(N-1)(N-3)}{4r^2} \right| \leqslant \frac{\epsilon}{2}$$

$$\implies \frac{(N-1)(N-3)}{4r^2} \geqslant -\frac{\epsilon}{2}. \tag{B}$$

$$\text{Addition yields: } \frac{g(u(r))}{u(r)-\kappa} - \frac{(N-1)(N-3)}{4r^2} \geqslant g'(\kappa) - \epsilon \tag{A+B}$$

And the claim is valid, let $\omega = g'(\kappa) - \epsilon$ with $\epsilon > 0$ small enough and $R_1 = \max(R_\epsilon, R_\theta)$. ☐

From this, $-v''(r) > 0$ for $r \geqslant R_1$ and thus $v''(r) < 0$ for $r \geqslant R_1$, which implies $v'(r) \downarrow L \geqslant -\infty$ as $r \uparrow \infty$. To see this, remember that $v''(r) < 0$ implies $v(r)$ will be concave down, that is, the tangent line will lie above the function. Even if the derivative $v'(r)$ would be positive for r slightly larger than R_1, since $v''(r) < 0$ indefinitely, the function will remain concave down and the derivative will become negative and stay negative. Consider the following possible limits L: $L < 0$ and $L \geqslant 0$. In the first case, $L < 0$, then $v(r) \to -\infty$ as $r \to \infty$ which is impossible, since $v(r) > 0$ for $r \geqslant 0$ $\lightning$. Then consider $L \geqslant 0$. Note that $v'(R_1) \geqslant 0$. Indeed, suppose $v'(R_1) < 0$. Then since $v''(r) < 0$, the function is concave down and the derivative will only decrease. Then the limit of the derivative can not be $L \geqslant 0$. Thus $v'(R_1) \geqslant 0$. This, by the same argument, implies the derivative will be positive for $r \geqslant R_1$. Suppose the derivative is negative for some $R_2 > R_1$ then the derivative will remain negative for $r \geqslant R_2$ as $v''(r) < 0$. Clearly, $v(r) \geqslant v(R_1) > 0$. This implies that $-v''(r) \geqslant \omega v(R_1) > 0$ and therefore $v'(r) \downarrow -\infty$ as $r \to \infty$. Why? Because if $v''(r) < 0$ indefinitely, the limit of $v'(r)$ can not be $L \geqslant 0$, as the derivative will decrease while $v''(r) < 0$. Since $v''(r) < 0$ indefinitely, the limit of $v'(r)$ will be minus infinity. Again, this contradicts $v(r) > 0$ for $r \geqslant 0$. $\lightning$Conclusion: in any case $v''(r) < 0$ for $r \geqslant R_1$ which implies $v'(r) \downarrow -\infty$, which contradicts $v(r) > 0$ for $r \geqslant 0$ and hence $l = \kappa$ is an impossible assumption under the assumptions on g and hence $l \neq \kappa$. This concludes the proof of the lemma. ☐

4.5. **P and N are nonempty and open.**

Lemma 4.2. *Under the assumptions of* **??** *A, the sets P and N are nonempty, disjoint and open.*

Proof. The sets P and N are disjoint by definition. The order in which the statements of the lemma will be proven is as follows: P is nonempty, P is open, N is open. That N is nonempty is outside the scope of this report. See [**ber**] for the proof.

P is non-empty. To prove that P is non-empty, let $\alpha_p \in (\kappa, \alpha_0]$. Consider the following cases: (i) $\alpha_p \in N$, (ii) $\alpha_p \notin P$, (iii) $\alpha_p \in P$ and note they are mutually exclusive. If the initial condition is in N, then the solution vanishes in some point r_0. Then, if the initial condition is not in N, the solution does not vanish anywhere: $u(\alpha_p, r) > 0$ for $r \geqslant 0$. If the initial condition is <u>not</u> in P, then the derivative is negative everywhere. Disproving these two cases yields the properties:

the solution is positive everywhere, but the derivative vanishes in some point r_0, which is exactly the definition of P. Hence disproving case (i) and (ii) implies case (iii) applies.

First, suppose by contradiction that $\alpha_p \in N$. Then by definition, there exists a r_0 such that $u(\alpha_p, r_0) = 0$ and $u'(\alpha_p, r_0) < 0$. Now evaluate (4.3) in r_0, using $u(\alpha_p, r_0) = 0$, $G(0) = 0$ and that on $(0, r_0]$ the derivative is strictly negative:

$$-\frac{1}{2}[u'(r_0)]^2 - (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s} = G(u(\alpha_p, r_0)) - G(\alpha_p)$$

$$G(\alpha_p) - G(u(\alpha_p, r_0)) = \frac{1}{2}[u'(r_0)]^2 + (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s}$$

$$G(\alpha_p) - G(0) = \frac{1}{2}[u'(r_0)]^2 + (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s}$$

$$G(\alpha_p) = \frac{1}{2}[u'(r_0)]^2 + (N-1)\int_0^{r_0} [u'(s)]^2 \frac{ds}{s}$$

$$\implies G(\alpha_p) > 0,$$

but the assumption $\alpha_p \in (\kappa, \alpha_0]$ implies $G(\alpha_p) < 0$, a contradiction, hence $\alpha_p \notin N$.

Next, suppose $\alpha_p \notin P$, then $u(\alpha_p, r) > 0$ for $r \geqslant 0$ and $u'(\alpha_p, r) < 0$ for $r > 0$. That implies $u(\alpha_p, r) \downarrow l \geqslant 0$ as $r \uparrow \infty$ and by lemma 4.1 the limit is zero, $l = 0$. Similar to the previous argument and the proof of lemma 4.1, observe (4.3) for r tending to infinity, and note the following: $l = \lim_{r\to\infty} u(\alpha_p, r)$, $G(l) = G(0) = 0$, $\lim_{r\to\infty} [u'(\alpha_p, r)]^2 = 0$ as well as the lower bound on the integral term $\int_0^\infty [u'(\alpha_p, s)]^2 \frac{ds}{s} \geqslant 0$, to conclude that:

$$\lim_{r\to\infty} [G(\alpha_p) - G(u(\alpha_p, r))] = \lim_{r\to\infty} \left[\frac{1}{2}[u'(r)]^2 + (N-1)\int_0^{r} [u'(s)]^2 \frac{ds}{s}\right]$$

$$G(\alpha_p) - G(l) = 0 + (N-1)\int_0^{\infty} [u'(s)]^2 \frac{ds}{s}$$

$$G(\alpha_p) - G(0) \geqslant 0$$

$$\implies G(\alpha_p) \geqslant 0,$$

to reach the same contradiction, $\alpha_p \in (\kappa, \alpha_0] \implies G(\alpha_p) < 0$. Hence $\alpha_p \in P$, and any $\alpha \in (\kappa, \alpha_0]$ is in P, that is, $(\kappa, \alpha_0] \subset P$ and P is nonempty. ☐

The proofs for openness of both P and N invoke continuous dependence on the initial data as explained in detail in for example [**codlev**].

P is open. To prove that P is open, note that for $\alpha \in P$,

$$r_0 = \inf\{r > 0,\ u'(\alpha, r) = 0,\ u(\alpha, r) > 0\} > 0$$

and

$$\begin{cases} u(\alpha, r) > 0 & \text{for all } r \in [0, r_0] \\ u'(\alpha, r) < 0 & \text{for all } r \in (0, r_0). \end{cases}$$

From (3.1) follows $u''(\alpha, r_0) = -g(u(\alpha, r_0))$. Consider $u''(\alpha, r_0) = 0$. Then $g(u(\alpha, r_0)) = 0$. The only zero of g where $0 < u(\alpha, r_0) < \alpha$ is κ, i.e. $u(\alpha, r_0) = \kappa$. But since $u'(\alpha, r_0) = 0$ and $u''(\alpha, r_0) = 0$, $u(\alpha, r) \equiv \kappa$, which is impossible.

Consider $u''(\alpha, r_0) \neq 0$. Then because the derivative was negative up to r_0 and has now vanished, $u''(\alpha, r_0) > 0$. This implies the derivative is positive to the right of r_0 and there exists a $r_1 > r_0$ such that $u(\alpha, r) > u(\alpha, r_0)$ for all $r \in (r_0, r_1]$. FIGURE. By continuous dependence on the initial condition, let α^* be sufficiently close to α then for all $r \in (0, r_1]$ the following hold:

$$\begin{cases} u(\alpha^*, r_1) > u(\alpha^*, r_0) \\ \\ \alpha^* > u(\alpha^*, r) > 0 \end{cases}$$

which can be interpreted as: for α^* sufficiently small, $u(\alpha^*, r_1)$ is still above $u(\alpha^*, r_0)$, the derivative vanishes in some point $r_0^* \in (0, r_1]$ and the solution does not vanish on $(0, r_1]$. Then these properties together imply $\alpha^* \in P$ and hence P is open. (P is open if for any initial condition α in P, there exists a real number $\epsilon_\alpha > 0$ such that points whose distance from α is less than ϵ_α are also in P.) □

N is open. To prove that N is open, note that for $\alpha \in N$,

$$r_0 = \inf\{r > 0,\ u(\alpha, r) = 0,\ u'(\alpha, r) < 0\} > 0$$

and

$$\begin{cases} u(\alpha, r) > 0 \quad \text{for all } r \in [0, r_0) \\ \\ u'(\alpha, r) < 0 \quad \text{for all } r \in (0, r_0]. \end{cases}$$

In any case, since $u(\alpha, r_0) = 0$, $g(u(\alpha, r_0)) = g(0) = 0$. In fact, remember $g(u) = 0$ for $u \leqslant 0$. Note that $u'(\alpha, r_0) < 0$. Then there exists $r_1 > r_0$ such that $u(\alpha, r) < u(\alpha, r_0) = 0$ for $r \in (r_0, r_1]$. The (3.1) yields for $r \in (r_0, r_1]$:

$$u''(\alpha, r) = -\frac{N-1}{r} u'(\alpha, r)$$

$$u'(\alpha, r) = \left(\frac{r_0}{r}\right)^{N-1} u'(\alpha, r_0) \geqslant u'(\alpha, r_0)$$

Thus $u'(\alpha, r) \uparrow 0$ for $r \to \infty$. Note that the function does not become positive, r_1 could be extended to infinity. By continuous dependence on the initial condition, let α^* be sufficiently close to α that for all $r \in (0, r_1]$ the following hold:

$$\begin{cases} u(\alpha^*, r_1) < u(\alpha^*, r_0) \\ \\ u'(\alpha^*, r) = \left(\dfrac{r_0^*}{r}\right)^{N-1} u'(\alpha, r_0^*) \end{cases}$$

where r_0^* uniquely satisfies $u(\alpha^*, r_0^*) = 0$ on $(0, r_1]$. This can be interpreted as: for α^* sufficiently small, $u(\alpha^*, r_1)$ is still below $u(\alpha^*, r_0)$ (now not necessarily zero) and the derivative decays hyperbolically after the solution vanishes in some point $r_0^* \in (0, r_1]$. Then these properties together imply $\alpha^* \in N$ and hence N is open. (N is open if for any initial condition α in N, there exists a real number $\epsilon_\alpha > 0$ such that points whose distance from α is less than ϵ_α are also in N.) □

4.6. **Conclusions on existence.**

Under the conditions on g as specified after the theorem, there exists a ground state solution to (3.1) as proven by ODE methods. Remember that the interval of definition of solutions with initial condition $\alpha \in (\kappa, \lambda)$ is $(0, \infty)$ and that any solution that is positive everywhere and has negative derivative everywhere is convergent. By the properties of g and the (3.1), this limit is a zero of g and if κ is a zero of g, then $l \neq \kappa$ by lemma 4.1. Lastly, the shooting method requires that the sets of initial conditions P and N are disjoint, nonempty and open, as proven by lemma 4.2. Then there exist elements with intial condition α not in P or N that have $u(\alpha, r) > 0$ for $r \geqslant 0$ and $u'(\alpha, r) < 0$ for $r > 0$. Such a solution by lemma 4.1 tends to zero for r to infinity and hence is a ground state solution.

5. Uniqueness of ground state

Lemma 5.1. *Suppose that* $V(0) := \lim_{r\to 0} V(r)$ *exists and is finite. Then*

$$0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \implies \alpha \in P.$$

Proof. Remember solutions $u(r;\alpha)$ to 3.1 with $u(0) = \alpha \in P$ are positive everywhere. Do such initial conditions exist? (See Chapter 4: P is nonempty.) What interval of initial conditions belongs to P? To determine such an interval (of initial conditions), define (the Lyapunov or the energy) function $E(r)$ on $(0, z(\alpha))$ as:

$$E(r) := \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1},$$

Then, calculate $E'(r)$, where the IVP can be used to simplify the expression:

$$\begin{aligned} E'(r) &= u''(r)u'(r) - \lambda u(r)u'(r) + V(r)u(r)^p u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1} \\ &= \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right] u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1} \\ &= -\frac{u'(r)^2}{r} + \frac{1}{p+1}V'(r)u(r)^{p+1} \leqslant 0 \text{ for } r > 0. \\ \text{as } &\left[u''(r) - \lambda u(r) + V(r)u(r)^p\right] = -\frac{1}{r}u'(r) \end{aligned}$$

Each of the terms of $E'(r)$ is non-negative: (i) by hypothesis H2, $V'(r) \leqslant 0$; (ii) $u(r)^{p+1} \geqslant 0$ because $u(r)$ is positive on $(0, z(\alpha))$ for any initial condition; (iii) $r \geqslant 0$ and (iv) $u'(r)^2 \geqslant 0$. <u>Result:</u> $E(r)$ is non-increasing $(0, z(\alpha))$.

To conclude about behaviour of solutions by type of initial condition, regard $E(r)$ for large $r > 0$. Suppose $\alpha \in N$, then $u(z(\alpha)) = 0$. Now evaluate:

$$\begin{aligned} \lim_{r\to z(\alpha)} E(r) &= \lim_{r\to z(\alpha)} \left[\frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1}\right] \\ &= u'(z(\alpha))^2 \geqslant 0. \end{aligned}$$

As $E(r)$ is non-increasing, $E(0) \geqslant 0$. Alternatively, suppose $\alpha \in G$. Then $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$. Then $E(r) = 0$ for $r \to \infty$, so again, $E(0) \geqslant 0$. <u>Results:</u> $E(0) \geqslant 0$ and $E(r)$ well-defined on $[0, z(\alpha)]$ for $\alpha \in G \cup N$.

For $\alpha \in P$, require $E(0) < 0$. Now evaluate $E(0)$ and solve for α:

$$\begin{aligned} E(0) = &\frac{1}{2}u'(0)^2 - \frac{\lambda}{2}u(0)^2 + \frac{1}{p+1}V(0)u(0)^{p+1} < 0 \\ &\iff -\frac{\lambda}{2}\alpha^2 + \frac{1}{p+1}V(0)\alpha^{p+1} < 0 \\ &\iff \alpha^{p-1} < \left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)} \\ &\iff \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \end{aligned}$$

<u>Conclusion:</u> $\alpha \in P$ whenever $0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)}$.

$\square$

Lemma 5.2. *Let $\alpha \in G \cup N$, and $u = u(\alpha, r)$. Then $u'(r) < 0$ for all $r \in (0, z(\alpha))$ and $u'(z(\alpha)) < 0$ if $\alpha \in N$.*

Proof. The lemma shows $u(r)$ strictly decreasing on $(0, z(\alpha))$ for $\alpha \in G \cup N$. The argument uses this to conclude that $w(r)$ has a unique zero on $(0, z(\alpha))$. Finally, analysis of solution sets N and G leads to the uniqueness result. Write $z(\alpha) = \infty$ when $\alpha \in G$, since $u(\alpha, r) \to 0$ as $r \to \infty$. Let $\alpha \in G \cup N$. By lemma 5.1, $E(r) \geqslant 0$ on $[0, z(\alpha)]$ and non-increasing.

Needs argument why $u''(0) < 0$. And why $u'(r) < 0$ for r sufficiently small.

These results can be extended to show $u' < 0$ on $(0, z(\alpha))$. Suppose by contradiction $0 < r_0 = \inf(0 < r < z(\alpha), u'(r) = 0)$ exists. Note how $u''(r_0) < 0 \implies u'(r) > 0$ somewhere on $(0, r_0)$. See also figure F1 This contradicts $u'(r) < 0$ on $(0, r_0)$. Again, the combination of $u''(0) = 0$ and $u'(0) = 0$ would imply $u \equiv u(r_0)$. Hence $u''(r_0) > 0$. Invoke 3.1:

$$
\begin{aligned}
& u''(r_0) = \lambda u(r_0) - V(r_0)u(r_0)^p > 0 \\
\implies & u(r_0) < \left[\frac{\lambda}{V(r_0)}\right]^{1/(p-1)} < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(r_0)}\right]^{1/(p-1)} \\
\iff & u(r_0)^{p-1} < \left(\frac{p+1}{2}\right)\frac{\lambda}{V(r_0)} \\
\iff & \frac{1}{p+1}V(r_0)u(r_0)^{p+1} < \frac{\lambda}{2}u(r_0)^2 \\
\iff & -\frac{\lambda}{2}u(r_0)^2 + \frac{1}{p+1}V(r_0)u(r_0)^{p+1} < 0
\end{aligned}
$$

Then using $u'(r_0) = 0$, this yields $E(r_0) < 0$:

$$E(r_0) = -\frac{\lambda}{2}u(r_0)^2 + \frac{1}{p+1}V(r_0)u(r_0)^{p+1} < 0$$

But $E(r_0) < 0$ contradicts $E(r) \geqslant 0$, so $u' < 0$ on $(0, z(\alpha))$.

It remains to show $u'(z(\alpha)) < 0$ whenever $\alpha \in N$. Suppose $u'(z(\alpha)) = 0$ and remember $u(z(\alpha)) = 0$. Then $u \equiv 0$, because $u''(z(\alpha)) = \lambda u(z(\alpha)) - V(z(\alpha))u(z(\alpha))^p = 0$. Conclusion: $u'(z(\alpha)) < 0$.

□

Lemma 5.3. *Let $\alpha \in (G \cup N)$, then w has at least one zero in $(0, z(\alpha))$.*

Proof. The Lagrange identity for 3.1 and 3.2 will yield information about the zeroes of $w(r)$. The cases $\alpha \in N$ and $\alpha \in G$ will be considered seperately. Suppose $\alpha \in N$. The differential equations for u and w can be written as:

$$(ru'(r))' + r\left[-\lambda u(r) + V(r)u(r)^p\right] = 0$$
$$(rw'(r))' + r\left[-\lambda w(r) + pV(r)u(r)^{p-1}w(r)\right] = 0.$$

Multiply by $w(r)$ and $u(r)$ respectively, subtract the equations and integrate from 0 to $z(\alpha)$,

$$\int_0^{z(\alpha)} w(r)(ru'(r))' - u(r)(rw'(r))'dr = \int_0^{z(\alpha)} r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\}dr,$$

and perform partial integration for the left hand side: (remember $u(z(\alpha)) = 0$)

$$rw(r)u'(r)\Big|_0^{z(\alpha)} - ru(r)w'(r)\Big|_0^{z(\alpha)} - \int_0^{z(\alpha)} \underbrace{\{ru'(r)w'(r) - ru'(r)w'(r)\}}_{0} dr$$
$$= (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)dr$$
$$z(\alpha)w(z(\alpha))u'(z(\alpha)) = (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)dr.$$

For $\alpha \in N$, note $r > 0$, $V > 0$, $u^p > 0$ are finite almost everywhere. Suppose $w > 0$ on $(0, z(\alpha))$. Then $z(\alpha)u'(z(\alpha))w(z(\alpha)) < 0$ contradicts $(p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)dr > 0$. (A similar argument holds for $w < 0$.) Hence, w changes sign at least once on $(0, z(\alpha))$.

For $\alpha \in G$, suppose by contradiction that $w > 0$ on $(0, \infty)$. Perform integration over $(0, r)$ and rewrite left hand side using the quotient rule:

$$ru'(r)w(r) - rw'(r)u(r) = r\frac{u'(r)w(r) - w'(r)u(r)}{w(r)^2} = rw(r)^2\left(\frac{u(r)}{w(r)}\right)'$$
$$= (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)dr > 0.$$

<u>Result:</u> $\left(\frac{u}{w}\right)'$ is positive, so $\frac{u}{w}(r)$ is increasing.

By Lemma C.1 of [Gen11] there exists two independent solutions that satisfy 3.2 as $r \to \infty$:

$$\xi_0(r) \sim r^{-\frac{1}{2}} \exp^{-\sqrt{\lambda}r} \quad \text{and} \quad \xi_1(r) \sim r^{-\frac{1}{2}} \exp^{\sqrt{\lambda}r}$$

So for $r \to \infty$, $w(r) \sim r^{-\frac{1}{2}}\left[\alpha_0 \exp^{-\sqrt{\lambda}r} + \alpha_1 \exp^{-\sqrt{\lambda}r}\right]$ for some constants α_1, α_0. Since $w > 0$ by hypothesis, and $\lim_{r\to\infty} w(r) \sim \alpha_1$, $\alpha_1 \geqslant 0$. Suppose $\alpha_1 = 0$, then $w(r) \to 0$ exponentially as $r \to \infty$. So w changes sign by Lemma 1.4.9 of [Gen11], a contradiction. On the other hand, suppose $\alpha_1 > 0$. Then by Lemma 1.2.7 and Lemma C.1 of [Gen11], $u(r) \sim r^{-\frac{1}{2}} \exp^{-\sqrt{\lambda}r}$ as $r \to \infty$. <u>Result:</u> for $\alpha_1 > 0$, there exists C such that

$$\lim_{r\to\infty}\frac{u(r)}{w(r)} = \lim_{r\to\infty}\frac{Cr^{-\frac{1}{2}}\exp^{-\sqrt{\lambda}r}}{\alpha_1 r^{-\frac{1}{2}}\exp^{\sqrt{\lambda}r}} = \lim_{r\to\infty}\frac{C}{\alpha_1}\exp^{-2\sqrt{\lambda}r} = 0$$

This contradicts $\frac{u}{w}(r)$ positive and increasing. <u>Conclusion:</u> w changes sign at least once on $(0, z(\alpha))$ for $\alpha \in G \cup N$. $\square$

Intermezzo: some definitions. Let $\theta(r) := -ru'(r)/u(r)$ for $r \in [0, z(\alpha)$ and $\rho := \theta^{-1}$. Then PPProperties. Many auxiliary functions will be introduced now. These functions construct the information needed to prove w has a unique zero. Bear with me as the following functions and variables are introduced: $\theta(r), \beta, \rho(\beta), \phi_\beta(r), \nu_\beta(r)$. In the lemma that follows, even more functions and variables are introduced: $\bar{\beta}, \sigma(\beta), \xi(r), \Xi(r) := \sigma(\beta)^{-1}, \beta_0$.

Define $\theta(r) := -ru'(r)/u(r)$ on $r \in [0, z(\alpha))$. Note $\theta(r)$ has the following properties:

(i) $\theta(0) = 0$,
(ii) $\theta'(r) > 0$ for all $r \in (0, z(\alpha))$, and
(iii) $\lim_{r \to z(\alpha)} \theta(r) = \infty$.

SHOW THETA(R) HAS THESE PROPERTIES.

Define $\rho := \theta^{-1}$. Since $\theta(r)$ is continuous and increasing SHOW.., there exists a unique $r = \rho(\beta) > 0$ such that $\theta(r) = \beta$. SHOW. Then $\rho(r)$ is continuous and increasing, with $\rho(0) = 0$ and $\lim_{\beta \to \infty} \rho(\beta) = z(\alpha)$. SHOW.

Define $\nu_\beta(r) := ru'(r) + \beta u(r) = -u(r)\theta(r) - \beta$. Let $\beta > 0$, then $\nu_\beta(r) > 0$ if $r < \rho(\beta)$ and $\nu_\beta(r) < 0$ if $r > \rho(\beta)$. SHOW.

Define $\phi_\beta(r) := [\beta(p-1) - 2]\, V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r)$. Now observe $\nu_\beta(r)$ satisfies the differential equation

$$\nu''_\beta(r) + \frac{1}{r}\nu_\beta(r)' - \lambda \nu_\beta + pV(r)u(r)^{p-1}\nu_\beta = \phi_\beta(r)$$

SHOW.

- After these three lemmata, the following results are obtained:
  - $\alpha \in P$ for $\alpha \in (0, \alpha_0)$ with $\alpha_0 = \left[\frac{\lambda}{V(0)} \frac{p+1}{2}\right]^{\frac{1}{p-1}}$, we have $\alpha \in P$.
  - For $\alpha \in G$ we have $u'(r) < 0$ on $(0, z(\alpha))$ and for $\alpha \in N$ we have $u'(r) < 0$ on $(0, z(\alpha)]$.
  - For $\alpha \in G \cup N$ at least one zero for $w(r)$ in $(0, z(\alpha))$. In other words, w oscillates faster.
- To obtain results about the zeroes of u we will need more information about the zeroes of w.
- In fact, it will be shown that w has a unique zero in $(0, z(\alpha))$.
- Then, by the difference function $z := u - \tilde{u}$ and Sturm theory:
  - $\alpha \in N \implies \tilde{\alpha} \in N$ for $\tilde{\alpha} \in (\alpha, \alpha + \epsilon)$.
- Lastly, $\alpha \in N \implies \tilde{\alpha} \in N$ for $\tilde{\alpha} \in [\alpha, \infty)$.
  - So all initial conditions greater than or equal to initial conditions with solution in N are also in N.
  - Furthermore, the difference between u and $\tilde{u}$ is decreasing with respect to increasing initial condition.
- This is sufficient to show that any initial condition below $\alpha_0 \in G$ cannot be in N as that would contradict $\alpha_0 \in G$ by this property of initial conditions in N. As G is nonempty by chapter 4 and G and N are now disjoint... also noting P is disjoint with G and N and nonempty by lemma 5.1... $\alpha_0 \in G$ is unique!
  - Most work remains in explaining using lemmata 7 and 8 that G contains at most one point.

- In the lemmata following, many related properties are used. This section is dedicated to properly introducing those functions and their properties.
- First, let $\theta(r) := -r\frac{u'(r)}{u(r)}$ for $r \in [0, z(\alpha)]$.
  - Note that by IVP: $\theta(0) = \lim_{r\downarrow 0}\theta(r) = \lim_{r\downarrow 0} - r\frac{u'(r)}{u(r)} = \lim_{r\downarrow 0}\frac{-ru'(r)}{\alpha} = 0$
  - By properties $r > 0$, $u'(r) < 0$ on $(0, z(\alpha))$ and $u(r) > 0$ on $(0, z(\alpha))$: $\theta(r) > 0$ on $(0, z(\alpha))$.
  - Also, $\theta(r)$ is increasing, that is $\theta(r)' > 0$.
    - To see this... SHOW
  - The limit $\lim_{r\to z(\alpha)}\theta(r) = \infty$.
    - To see this for $\alpha \in N$ SHOW
    - To see this for $\alpha \in G$ SHOW
- As $\theta(r)$ is continuous and increasing, there exists $\rho := \theta^{-1}$ as there exists a unique $r = \rho(\beta) > 0$ s.t. $\theta(r) = \beta$.
  - Note that $\rho(0) = 0$ and $\lim_{\beta\to\infty}\rho(\beta) = z(\alpha)$.
  - Note this $\rho(\beta)$ is continuous and increasing with respect to β.
  - Note that $\rho(r) > 0$ on $(0, \infty)$.
- Now define $\nu_\beta := ru'(r) + \beta u(r) = -u(r)\left[\theta(r) - \beta\right]$
- Let $\beta > 0$ then $\nu_\beta(r) > 0$ if $r < \rho(\beta)$ and $\nu_\beta < 0$ if $r > \rho(\beta)$.
  - Note that $u(r) > 0$ so $\nu_\beta > 0$ if $\theta(r) - \beta < 0$. That is, $0 > \theta(r) - \beta \iff \beta > \theta(r) \iff \rho(\beta) > r$. Similarly, $\nu_\beta < 0$ if $\beta < \theta(r) \iff \rho(\beta) < r$.
- All of these properties can also be illustrated by beautiful figures. SHOW
- *Complicated:* $\nu_\beta(r)$ satisfies the following differential equation:
  - $\nu_\beta''(r) + \frac{1}{r}\nu_\beta'(r) - \lambda\nu_\beta(r) + pV(r)u(r)^{p-1}\nu_\beta(r) = \phi_\beta(r)$
  - Where $\phi_\beta(r) = \left[\beta(p-1) - 2\right]V(r)u(r)^p - V'(r)u(r)^p + 2\lambda u(r)$.
  - SHOWWWWWWWW
- In lemma 4 this will be used to show that the sign of ϕ_β is related to a continuous decreasing function $\sigma(\beta)$ and this function crosses $\rho(\beta)$ in a unique β_0 (lemma 5.5). The specific ν_{β_0} and $\rho_0 = \rho(\beta_0)$ then are used in strong version of Sturm comparison theorem in lemma 6 to show that w has a unique zero on $(0, z(\alpha))$! COOL! Hooray! Useful.

Lemma 5.4. *Let $\alpha \in G \cup N$. There exist $\beta_0 > 0$ and a unique function $\sigma : [0, \bar{\beta}] \to [0, \infty)$ with the following properties:*

(a) σ is continuous and decreasing, $\sigma(0) > 0$ and $\sigma(\bar{\beta}) = 0$;
(b) for all $\beta > 0$ we have: $\phi_\beta(r) < 0$ if $r < \sigma(\beta)$, and $\phi_\beta > 0$ if $r > \sigma(\beta)$.

Some remarks: Why is this σ unique? There are infinitely many functions that agree with $\sigma(0) > 0$ and $\sigma(\bar{\beta}) = 0$. Clearly, the other property of σ uniquely defines the function. Draw the following: x-axis is the β-axis and goes from 0 to $\bar{\beta}$. The y-axis is the r-axis and goes from 0 (in $\bar{\beta}$) to some finite number $\sigma(0) > 0$. For any β between 0 and $\bar{\beta}$ there is a corresponding $\sigma(\beta)$. As this value needs to agree with the property of $\phi_\beta(r)$–$\phi_\beta(r) < 0$ for $r < \sigma(\beta)$ and $\phi_\beta(r) > 0$ for $r > \sigma\beta$– for all β then σ is uniquely defined!

Proof. Let $\beta > 0$ and $r \in [0, z(\alpha))$. Then

$$\begin{aligned}
\phi_\beta(r) &= [\beta(p-1) - 2]\, V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r) \\
&= V(r)u(r)^p \left[\beta(p-1) - 2 - r\frac{V'(r)}{V(r)} + \frac{2\lambda}{V(r)u(r)^{p-1}}\right] \\
&= V(r)u(r)^p \left[\beta(p-1) - 2 - \xi(r)\right]
\end{aligned}$$

$$\text{where } \xi(r) = r\frac{V'(r)}{V(r)} - \frac{2\lambda}{V(r)u(r)^{p-1}}.$$

To conclude about the sign of $\phi(r)$, note that $V(r) \geqslant 0$ and $u(r) \geqslant 0$. Hence the sign of $\phi(r)$ will vary with β and r as dictated by the term in brackets. Write $\beta(p-1) - 2 - \xi(r) > 0 \iff \beta > \frac{2+\xi(r)}{p-1} := \Xi(r)$.

The function $\xi(r) \leqslant 0$ is strictly decreasing on $(0, z(\alpha))$ with $\lim_{r\to z(\alpha)} \xi(r) = -\infty$. To see this, note $h(r) = r\frac{V'(r)}{V(r)}$ is nonincreasing, $V'(r) \leqslant 0$ and $u'(r) < 0$ hence the second term of $\xi(r)$ is strictly decreasing. Thus $\xi(r)$ is strictly decreasing. Since $u(z(\alpha)) = 0$, the limit is $-\infty$.

It remains to show $\Xi(r) = [2 + \xi(r)]\,/(p-1)$ satisfies $\Xi(0) > 0$. Remember that $\xi(r)$ is continuous and decreasing, so $\Xi(r)$ is too.

With this information, $\Xi(r)$ is also continuous and strictly decreasing.

Indeed, let $\Xi(0) = \bar{\beta}$, LLLLLL, for all $\beta \in [0, \bar{\beta}]$. Note $\Xi(r)$ is continuous and decreasing, so $\sigma(\beta) := \Xi(r)^{-1}\big|_{[0,\bar{\beta}]}$ has properties (a) and (b). SHOW.

Also, note $\Xi(0) > 0 \iff \xi(0) > -2$. SHOW.

TODO: Show that there is an implication from the property of $V(0)$ to the property of $\xi(0)$. By definition? $\xi(0) = 0 * V'(0)/V(0) + 2\lambda/(V(0) * u(0)^{p-1}\ldots$ What do I know about $V'(0)$? And about $V(0)$? By lemma 5.1 ?? Also, as $u(0) = \alpha > 0$, we have $u(0)^{p-1} > 0$ and $V(0) > 0$ (finite or infinite) implies the second term of $\xi > 0\ldots$ Yes, definitely some Lyapunov/lemma 5.1 property involved here.

Consider the following cases ......$V(0) < \infty$ .... $V(0) = \infty$ ... hence $\xi(0) > -2$ for $\alpha \in (G \cup N)$.

$\square$

Lemma 5.5. *Let* $\alpha \in G \cup N$. *There exists a unique* $\beta_0 > 0$ *such that* $\rho(\beta_0) = \sigma(\beta_0)$. *This follows immediately from the aforementioned properties of* $\rho(\beta)$ *and* $\sigma(\beta)$. *Sketching the two graphs, there is a unique intersection.*

Lemma 5.6. *For $\alpha \in G \cup N, w$ has a unique zero $r_0 \in (0, z(\alpha))$. Furthermore, $w(z(\alpha)) < 0$ if $\alpha \in N$ and $\lim_{r\to\infty} w(r) = -\infty$ if $\alpha \in G$.*

Some remarks: the functions $u(r)$ and $w(r)$ will be compared ''in the rest of the uniqueness proof'. This lemma derives that $w(r)$ has a **unique** zero r_0 to the left of $z(\alpha)$. Remember $z(\alpha)$ is the zero of $u(r)$. On the other hand, if $u(r)$ is positive and decreasing everywhere–has no finite zero $z(\alpha)$ such that $u(z(\alpha)) = 0$–then still $w(r)$ has a finite zero r_0 and $w(r)$remains negative to the right of the zero. Even more, the function $w(r)$ tends to negative infinity for $r \to \infty$, $\lim_{r\to\infty} w(r) = -\infty$.

Proof. The solution $\nu(r)$ changes sign in $\rho_0 \in [0, z(\alpha)]$. As $\nu(z(\alpha))z(\alpha)u'(z(\alpha)) < 0$, the solution $\nu(r)$ changes sign once. Let τ be the first zero of $w(r)$. Then by Sturm comparison, $\rho_0 \in (0, \tau)$, i.e. ν oscillates faster than w.

Also by Sturm comparison, w can have no further zero and τ is the unique zero of w.

Sign of v changes in ρ_0 ''By previous lemmata' the sign of ν changes in ρ_0. Refer to lemma ... and/or figure ... Also, remember ...

ν changes sign only once The argument that ν changes sign only once is given in lemma ... Remember that ...

w has a first zero Suppose $w(r)$ does not have a zero. Then ...

Comparing w and v, conclusion: w has a unique zero One can compare ν and w with Sturm's comparison theorem, which is given as ... In this theorem, take ... as ... and ... as ... to see that ...

Also if $\alpha \in G$ When $\alpha \in G$ then $u(r)$ has no zero. The solution depends on α and the change of $u(r)$ with respect to α is given by the function $w(r) = \frac{\partial}{\partial\alpha}u(r)$. ''Since there are solutions that have a zero and some that don't', the function $w(r)$ must have a zero. Even more, the zero of w will be to the left of the zero of $u(r)$. CIRKELREDENERING? Then $u(r)$ does not have a zero, the limit of $w(r)$ for r to infinity is negative infinity. To see this, note that for increasing α, one will obtain a solution that *has* a zero. Hence in that r-value, the function $u(r)$ decreases with increasing α, ergo, the function $w(r)$ is increasingly ?? negative for r-values where $u(r)$ is positive.

$\square$

Lemma 5.7. *Let $\alpha^* \in N$. Then $[\alpha^*, \infty) \subset N$ and $z : [\alpha^*, \infty) \to (0, \infty)$ is monotone decreasing.*

Proof. Let $\hat{\alpha} \in N$. By definition of N, there exists a $\hat{r} > 0$ such that $u(\hat{r}; \hat{\alpha}) < 0$. By continuous dependence on the initial data [[Cod. Lev.]], $u(\hat{r}; \alpha) < 0$ for all α sufficiently close to $\hat{\alpha}$.

$\underline{z \text{ is continuous}}$ None of these solutions can be tangent to the r-axis, _SHOW hence the function $z : N \to (0, \infty)$ is continuous. That is...

$\underline{z \text{ is decreasing}}$ Let $\alpha^* \in N$. Then by **??**, $w(z(\alpha^*)) < 0$ and for $\epsilon > 0$ sufficiently small, $(\alpha^*, \alpha^* + \epsilon) \subset N$ and $u(z(\alpha^*), \alpha) < 0$ for all $\alpha \in (\alpha^*, \alpha^* + \epsilon)$.

Remember that w is the derivative of u with respect to the initial condition. Since $w(z(\alpha^*)) < 0$, for initial conditions upward of α^* ($\alpha \in (\alpha^*, \alpha^* + \epsilon)$): $u(\alpha, z(\alpha^*)) < u(\alpha^*, z(\alpha^*)) = 0$. By the intermediate value theorem [[Cod. Lev.]], there exists a $r \in (0, z(\alpha^*))$ such that $u(\alpha, r) = 0$. Then $z(\alpha) \leqslant r \leqslant z(\alpha^*)$ for all $\alpha \in (\alpha^*, \alpha^* + \epsilon)$. Conclusion: z is decreasing on $(\alpha^*, \alpha^* + \epsilon)$.

$\underline{\text{Domain of } z \text{ extends to infinity}}$ In fact, z is decreasing on $[\alpha^*, \infty)$. That is, let

$$\bar{\alpha} \coloneqq \sup\{\alpha > \alpha^* \subset N \text{ and } z : [\alpha^*, \alpha) \to (0, \infty) \text{ is decreasing}\}.$$

Then the lemma requires $\bar{\alpha} = \infty$. Suppose by contradiction $\bar{\alpha} < \infty$. Then there exists $z(\bar{\alpha}) \coloneqq \lim_{\alpha \to \bar{\alpha}} z(\alpha) \in [0, \infty)$. Clearly, $\bar{\alpha} \in N$, since $u(\bar{\alpha}, z(\bar{\alpha}) = 0$ by continuity of z. But then $[\bar{\alpha}, \bar{\alpha} + \epsilon) \in N$ for $\epsilon > 0$ sufficiently small. This contradicts the definition of $\bar{\alpha}$ as the supremum. Then $\bar{\alpha} = \infty$. Conclusion: for $\alpha^* \in N$, $[\alpha^*, \infty) \subset N$ and $z : [\alpha^*, \infty) \to (0, \infty)$ is decreasing.

□

Lemma 5.8. *Let $\alpha \in G$. There exists $\epsilon > 0$ such that $(\alpha, \alpha + \epsilon) \subset N$.*

Proof.
- By lemma 5.6, for $\alpha \in G$, w is unbounded and there exists unique r_0 in $(0, z(\alpha))$ such that $w(r_0) = 0$.
- Then by Lemma 6 of Kwong, $r_0 \in (d, \infty)$, that is, the unique zero r_0 of $w(r)$ on $(0, z(\alpha))$ is in the disconjugacy interval (d, ∞) of **??** w-IVP. USED WHERE?
- Let r_1, r_2 be such that: $d < r_1 < r_0 < r_2 < \infty$. Since $w(r_1) > 0$ and $w(r_2) < 0$, there exists $\epsilon > 0$ such that for all $\tilde{\alpha} \in (\alpha, \alpha + \epsilon)$,
$$\tilde{u}(r_1) > u(r_1) \text{ and } \tilde{u}(r_2) < u(r_2)$$
where $\tilde{u} = u(r; \tilde{\alpha})$.
- Hence, for given $\tilde{\alpha} \in (\alpha, \alpha + \epsilon)$, the graph of $\tilde{u}$ intersects the graph of u at some point $r_3 \in (r_1, r_2)$.
- Claim: there exists $\tilde{r} \in (r_3, \infty)$ such that $\tilde{u}(\tilde{r}) = 0$, thus $\tilde{\alpha} \in N$.
  - Suppose by contradiction that $\tilde{u}(r) > 0$ for all $r > r_3$.
  - Then $\tilde{u}(r) < u(r)$ for all $r > r_3$ by the following argument:
    * Suppose by contradiction that $\tilde{u}(r_4) = u(r_4)$ for some $r_4 > r_3$ and $u - \tilde{u} > 0$ on (r_3, r_4).
    * Define on (r_3, r_4) function $z := u - \tilde{u}$ that satisfies SHOW:
$$z'' + \frac{1}{r} z' - \lambda z + V(r) \frac{u^p - \tilde{u}^p}{u - \tilde{u}} z = 0$$
    * Before applying Sturm comparison between z and w, let y be a solution to **??** w-IVP linearly independent of w. Then y must vanish somewhere in (r_3, r_4) BY DISCONJUGACY? WHY? STURM COMPARISON WITH Z? QUESTION: DOES ANY SOLUTION TO w-IVP NEED TO YIELD THE SUPPOSED PROPERTIES OF u AND $\tilde{u}$? PROBABLY NOT.. PROBABLY BY STURM COMPARISON, AS THE COMPARISON BETWEEN $\frac{u^p - \tilde{u}^p}{u - \tilde{u}}$ AND pu^p IS MADE!Thus $\{w, y\}$ is a basis of solutions of **??** w-IVP and it is impossible to find a positive solution on (d, ∞), a contradiction with the supposed property of z !! CHECK !!
  - Anyway.... The contradiction shows that $z(r) > 0$ ($\tilde{u}(r) < u(r)$) for all $r > r_3$. And w-IVP **??** is a Sturm majorant of z-IVP **??**
  - Let $\tilde{w}$ be a solution to w-IVP **??** such that $\tilde{w}(r_3) = 0$.
    * Note that $\tilde{w}(r_3) = 0$ is a completely different solution than the aforementioned $w(r)$.
  - Since $r_3 \in (d, \infty)$ by Lemma 6 in Kwong our $\tilde{w}$ is unbounded and we can assume that $\tilde{w}(r) \to +\infty$. Why PLUS infinity? Our w goes to $-\infty$... Does this have to do with linear independence? SHOW
  - Also $\tilde{w} > 0$ on (r_3, ∞). No solution can vanish more than once in the disconjugacy interval.. SHOW/USE
  - Then by the strong version of Sturm comparison:
$$\frac{w'(r)}{w(r)} \leqslant \frac{z'(r)}{z(r)} \text{ for all } r \in (r_3, r_4)$$
  - Which by integration from r_4 to r yields:
$$\ln \tilde{w}(r) \leqslant \ln \frac{\tilde{w}(r_4)}{z(r_4)} + \ln z(r)$$

  - Now, remember $z \equiv u(r) - \tilde{u}(r) \to +\infty$ as $r \to \infty$, which is impossible since $0 < \tilde{u}(r) < u(r)$ on (r_3, ∞) and $u(r) \to \infty$! (How could $z \equiv u(r) - \tilde{u}(r)$ go to infinity if the majorant u goes to zero and the subtrahend is positive everywhere!)
  - Then the contradiction is reached and $\tilde{u}(r)$ has a zero for some $r \in (r_3, \infty)$.
- Conclusion: $\tilde{u}$ has a zero so $\tilde{\alpha} \in N$.
- Since $\tilde{\alpha}$ was chosen arbitrarily in $(\alpha, \alpha + \epsilon)$, all $\alpha' \in (\alpha, \alpha + \epsilon)$ have $\alpha' \in N$.

By chapter 4, the solution set G is non-empty. Let $\alpha \in G$ and let $u(r; \alpha)$ be the corresponding solution. The function $w(\alpha, r) = \frac{\partial}{\partial \alpha} u(\alpha; r)$ satisfies **??**

By lemma 5.6, the function w is unbounded. By lemma **??**, w has a unique zero $r_0 \in (d, \infty)$. Kwong discusses the disconjugacy interval (d, ∞) of **??** in more detail. Let r_1, r_2 be such that $d < r_1 < r_0 < r_2$ and note that $w(r_1) > 0$ and $w(r_2) < 0$. There exists $\epsilon > 0$ such that, for all $\tilde{\alpha} \in (\alpha, \alpha + \epsilon)$,

$$\tilde{u}(r_1) > u(r_1) \text{ and } \tilde{u}(r_2) < u(r_2),$$

where $\tilde{u} = u(\tilde{\alpha}, r)$. Hence there exists $r_3 \in (r_1, r_2)$ such that the graphs of u and $\tilde{u}$ intersect, i.e. $\tilde{u}(r_3) = u(r_3)$. See also figure **??**.

To conclude $\tilde{\alpha} \in N$ requires existence of a $\tilde{r}$ such that $\tilde{u}(\tilde{r}) = 0$. By contradiction, suppose that $\tilde{u}(r) > 0$ for all $r > r_3$. (Note that $\tilde{u}(r) \geqslant u(r) > 0$ on $r \leqslant r_3$, since $u(r) > 0$ for all r.)

Now, for $r > r_3$ and small, $\tilde{u}(r) < u(r)$. Claim: $u(r) > \tilde{u}(r)$ for all $r > r_3$. Define the difference between u and $\tilde{u}$ as $z := u - \tilde{u}$. Note $u(r) > \tilde{u}(r) \iff z(r) > 0$. By contradiction, suppose there exists $r_4 > r_3$ such that $\tilde{u}(r_4) = u(r_4)$ (equivalent to $z(r_4) = 0$). On (r_3, r_4) the function z satisfies:

$$\begin{aligned}
& z'' + \frac{1}{r} z' + \left[V(r) \frac{u^p - \tilde{u}^p}{u - \tilde{u}} \right] z = 0 \quad \text{because} \\
(1): \quad & u'' + \frac{1}{r} u' - \lambda u + V u^p = 0 \\
(2): \quad & \tilde{u}'' + \frac{1}{r} \tilde{u}' - \lambda \tilde{u} + V \tilde{u}^p = 0 \\
(1) - (2): \quad & u'' - \tilde{u}'' + \frac{1}{r} u' - \frac{1}{r} \tilde{u}' - \lambda u + \lambda \tilde{u} + V u^p - V \tilde{u}^p = 0 \\
& z'' + \frac{1}{r} z' - \lambda z + [V u^p - V \tilde{u}^p] \frac{u - \tilde{u}}{u - \tilde{u}} = 0 \\
& z'' + \frac{1}{r} z' + \left[V(r) \frac{u^p - \tilde{u}^p}{u - \tilde{u}} - \lambda \right] z = 0
\end{aligned}$$

Also by Sturm comparison of z and w, the latter oscillates faster. Let $\tilde{w}$ be a solution of (**??**) such that $\tilde{w}(r_3) = 0$.

By integration of the strong version of Sturm, $z(r) \to \infty$ as $r \to \infty$, but this is impossible as $0 < \tilde{u}(r) < u(r)$ on (r_3, ∞) and $u(r) \to 0$ as $r \to \infty$. Therefore, $\tilde{u}$ vanishes at some point $\tilde{r} \in (r_3, \infty)$ and the proof is complete.

□

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