An Explication of Existence and Uniqueness Results for a Nonlinear Schrödinger equation

An Introduction to the Shooting Method and Sturm Comparison Theorem

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Physics of NLS

1.1. Derive the wave equation from Maxwell

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \qquad (1.1.a) \qquad\qquad \nabla \cdot \vec{\mathcal{D}} = 0, \qquad (1.1.c)$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t}, \qquad (1.1.b) \qquad\qquad \nabla \cdot \vec{\mathcal{B}} = 0. \qquad (1.1.d)$$

The fields are in three-dimensional Cartesian coordinates, for example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents, we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \qquad (1.2.a) \qquad\qquad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}. \qquad (1.2.b)$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [1, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [2]. This reference work also includes an introduction to the necessary vector calculus.

We use vector calculus and Maxwell's laws to rewrite the curl of the curl:

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) \overset{(1.1.a)}{=} \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathcal{B}}\right) \overset{\substack{(1.1.b)\\(1.2.a)}}{=} -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} \overset{(1.2.b)}{=} -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \text{ and}$$

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \nabla^2 \vec{\mathcal{E}} = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \Delta \vec{\mathcal{E}} \overset{\substack{(1.1.c)\\(1.2.b)}}{=} -\Delta \vec{\mathcal{E}}.$$

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Combining these and using $\mu_0\epsilon_0 = 1/c^2$, we arrive at the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{1.3}$$

1.2. VALIDITY OF PLANE WAVE SOLUTIONS

Stuyding the left and right hand sides of equation (1.3), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{l=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_l^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.4}$$

where k_0 is the wavenumber and ω_0 the frequency. These are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any z-value and the electric field is non-zero in the x and y directions. The wavefronts are spaced by the wavelength λ and the wavenumber k_0 is the reciprocal of the wavelength.

This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane.

We substitute (1.4) in equation (1.3). Note that only Δ_z will be non-zero:

$$\Delta\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation (1.5):

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.5}$$

For a general direction in (x, y, z)-coordinates, define the wavevector

$$\vec{k} = (k_x, k_y, k_z),$$

where $\left|\vec{k}^2\right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$. This satisfies equation (1.3) when $\vec{k}\perp\vec{\mathcal{E}}$ and

$$\mathcal{E}_j = E_c e^{i(\vec{k}\cdot\vec{r}-\omega_0 t)}. \tag{1.6}$$

1.3. DERIVATION OF THE HELMHOLTZ EQUATION

We consider time-harmonic solutions to the scalar wave equation (1.3) of the form

$$\mathcal{E}_j(x,y,z,t) = e^{i\omega_0 t}E(x,y,z) + \text{c.c}, \tag{1.7}$$

which are continuous wave beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, we refer to [3].

Substituting (1.7) in equation (1.3) and taking the derivatives leads to the expression

$$\begin{aligned} \Delta\left(e^{-i\omega_0 t}E\right) &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(e^{-i\omega_0 t}E\right) \\ e^{-i\omega_0 t}\Delta E &= \frac{1}{c^2}(-i\omega_0)^2 E e^{-i\omega_0 t}, \end{aligned}$$

where we can divide by $e^{-i\omega_0 t} \neq 0$ and use the dispersion relation (1.5) to arrive at the scalar linear Helmholtz equation for E

$$\Delta E(x,y,z) + k_0^2 E = 0. \tag{1.8}$$

As an example, equation (1.8) is solved by the general-direction plane waves (1.6), where

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

1.4. DERIVATION OF THE LINEAR SCHRÖDINGER EQUATION

REVISE: We write the incoming field $E_0^{\text{inc}}(x,y)$ as a sum of plane waves. Then the electric field $E(x,y,z)$ for non-zero z-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and it is essential to Fourier optics. We have

$$\begin{aligned} E_0^{\text{inc}}(x,y) &= \frac{1}{2\pi}\int_D E_c(k_x,k_y)\, e^{i(k_x x + k_y y)}\mathrm{d}k_x \mathrm{d}k_y, \quad \text{such that} \\ E(x,y,z) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} E_c(k_x,k_y)\, e^{i(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z)}\mathrm{d}k_x \mathrm{d}k_y, \end{aligned}$$

where D denotes the (circular) laser input beam domain. For laser beams oriented in the z-direction, most of the plane wave modes are nearly parallel to the z-axis, which implies $k_z \approx k_0$. We define $k_\perp^2 = k_x^2 + k_y^2$, such that $k_0^2 = k_\perp^2 + k_z^2$. It is equivalent to $k_0 \approx k_z$ to say that $k_\perp \ll k_z$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{1.9}$$

where $\psi(x,y,z)$ is an envelope (or amplitude) function. The envelope shape may vary over z, in contrast to soliton solutions, see (1.21).

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Substituting (1.9) into the Helmholtz equation (1.8) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp\psi = 0, \tag{1.10}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that for lasers beams oriented in the z-direction, the wavenumber k_z dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope function $\psi(x,y,z)$ will vary slowly in z and curve even more slowly.

Claim: $|\psi_{zz}| \ll k_0|\psi_z|$ and $|\psi_{zz}| \ll \Delta_\perp\psi$.

REVISE: To see this, we first show that $k_0 - k_z \ll 1$. We factor out k_0^2, take the square root on both sides and linearise the square root term of the right hand side:

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right) \implies k_z = k_0\left(1 - \frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right).$$

Finally, we use $k_\perp \ll k_0$ to obtain the intermediate result:

$$k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} \ll 1.$$

For the first statement of the claim, $|\psi_{zz}| \ll k_0|\psi_z|$, it is equivalent to show that the ratio of $|\psi_{zz}|$ over $k_0|\psi_z|$ is much smaller than 1. We calculate the ratio as follows:

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_0(k_0 - k_z)E_c} = \frac{k_0 - k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0}\cdot\frac{1}{k_0} \ll 1.$$

For the other statement of the claim, we calculate:

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0 - k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} \ll \frac{1}{4}\frac{k_\perp^2}{k_0^2} \ll 1.$$

Using the approxations in equation (1.10) yields the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp\psi = 0. \tag{1.11}$$

1.5. POLARISATION FIELD

Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E},0,0),\ \vec{\mathcal{P}} = (\mathcal{P},0,0),\ \vec{\mathcal{D}} = (\mathcal{D},0,0),$$

Furthermore, we assume that $\mathcal{E}$ is the continuous wave electric field from (1.7). We write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.12}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials. The constants c_i are actually a function of the frequency ω_0. We rewrite $c_i = \epsilon_0\chi^{(\mathrm{i})}(\omega_0)$, where $\chi^{(\mathrm{i})}$ is the i-th order susceptibility. Then equation (1.12) reads:

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.13}$$

First we consider linear polarisation:

$$\mathcal{P}_{\mathrm{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}.$$

Then the induction electric field $\mathcal{D}$ is given by:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} = \epsilon_0\mathcal{E} + \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E} = \epsilon_0\mathcal{E}\left(1 + \chi^{(1)}(\omega_0)\right) = \epsilon_0 n_0^2(\omega_0)\mathcal{E},$$

where $n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0)$ is the linear index of refraction (or refractive index) of the medium.

With this updated induction electric field $\mathcal{D} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}$, we can update the scalar wave equation and Helmholtz equation. Only the dispersion relation is affected by considering linear polarisation:

$$k_0^2 = \frac{\omega_0^2}{c^2} n_0^2(\omega_0). \tag{1.14}$$

We now consider the nonlinear polarisation field $\mathcal{P}_{\mathrm{nl}}$ as the difference between the true polarisation and the linear approximation:

$$\mathcal{P} = \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$\begin{aligned} -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\left(-\mathcal{E}\right)^2 + \epsilon_0\chi^{(3)}\left(-\mathcal{E}\right)^3 + \epsilon_0\chi^{(4)}\left(-\mathcal{E}\right)^4 + \epsilon_0\chi^{(5)}\left(-\mathcal{E}\right)^5 + \mathcal{O}\left(\mathcal{P}^6\right) \\ -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\mathcal{E}^2 - \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 - \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right), \end{aligned}$$

where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\mathrm{nl}}$ and we have only the odd terms:

$$\mathcal{P}_{\mathrm{nl}} = \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^7\right) \tag{1.15}$$

The leading-order term is called the Kerr nonlinearity:

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}^3. \tag{1.16}$$

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1.6. IMPLICATIONS OF NONLINEAR POLARISATION

Substituting the continuous wave electric field (1.7) into equation (1.16) yields

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}^3 = 3\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \chi^{(3)}(\omega_0)E^3 e^{3i\omega_0 t} + \text{c.c.},$$

where the second term has a frequency of $3\omega_0$ (third harmonic). This has almost no contribution due to the phase-mismatch with the first harmonic. Hence, we approximate

$$\mathcal{P}_{\mathrm{nl}} \approx 3\epsilon_0\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \text{c.c.} = 3\epsilon_0\chi^{(3)}(\omega_0)\mathcal{E}.$$

Then we simplify $\mathcal{P}_{\mathrm{nl}}$ by defining

$$n_2 := \frac{3\chi^{(3)}}{4\epsilon_0 n_0},$$

so that we obtain the simplified expression

$$\mathcal{P}_{\mathrm{nl}} = 4\epsilon_0 n_0 n_2\,|E|^2\,\mathcal{E}.$$

This allows us to write the induction electric field $\mathcal{D}$ as,

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}} = \epsilon_0 n^2\mathcal{E},$$

where

$$n^2 = n_0^2\left(1 + \frac{4n_2}{n_0}\,|E|^2\right) = n_0^2 + 3\chi^{(3)}(\omega_0)\frac{1}{\epsilon_0}|E|^2.$$

For water, $n_2 \sim 10^{-22}$ which justifies neglecting nonlinear effects. With lasers, the nonlinear effect becomes more relevant, but is still weak. For a typical continuous wave laser with $|E| \sim 10^9$, we still have a weak nonlinearity, as $n_2|E| \sim 10^{-4} \ll n_0 \approx 1.33$.

We update equation (1.8) to the scalar nonlinear Helmholtz equation (NLH):

$$\Delta E(x,y,z) + k^2 E = 0, \quad \text{where } k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right). \tag{1.17}$$

We write $E(x,y,z)$ as the product of the z-propagation and an envelope function $\psi(x,y,z)$:

$$E = e^{ik_0 z}\psi$$

and substitute in (1.17) to obtain:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp\psi + 4k_0^2\frac{n_2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.18}$$

Just as in section 1.4, we apply the paraxial approximation, since for laser beams oriented in the z-direction, we have $\left|\psi_{zz}\right| \ll k_0\left|\psi_z\right|, \left|\psi_{zz}\right| \ll \Delta_\perp\psi$. We finally obtain the nonlinear Schrödinger equation (NLS):

$$2ik_0\psi_z(z,\overline{x}) + \Delta_\perp\psi + k_0^2\frac{4n^2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.19}$$

1.7. SOLITON SOLUTIONS

The NLS equation (1.19) can be written as a dimensionless equation. Starting from equation (1.18), we apply the rescaling of coordinates $(x, y, z) \to (\tilde{x}, \tilde{y}, \tilde{z})$ defined by:

$$\tilde{x} = \frac{x}{r_0} \quad \tilde{y} = \frac{y}{r_0} \quad \tilde{z} = \frac{z}{2L_{\text{diff}}},$$

where r_0 is the input beam width and L_{diff} is the diffraction length. We refer to chapter 2 of [1] for more information on the geometrical optics of lasers. There, we also find that $L_{\text{diff}} = k_0 \cdot r_0^2$. To rescale $\tilde{\psi}$, we define:

$$\tilde{\psi} = \frac{\psi}{E_c}, \quad \text{where } E_c := \max_{x,y} \left|\psi_0(x,y)\right|.$$

Through the rescaling we obtain the dimensionless NLH for $\tilde{\psi}$:

$$\frac{f^2}{4}\tilde{\psi}_{\tilde{z}\tilde{z}}(\tilde{z},\tilde{x},\tilde{y}) + i\tilde{\psi}_{\tilde{z}} + \Delta_\perp \tilde{\psi} + \nu \left|\tilde{\psi}\right|^2 \tilde{\psi} = 0,$$

that depends on a nonparaxiality parameter f and a nonlinearity parameter ν:

$$f = \frac{1}{r_0 k_0} = \frac{r_0}{L_{\text{diff}}}, \quad \nu = r_0^2 k_0^2 \frac{4n_2}{n_0} E_c^2.$$

Here the approximation of paraxiality is valid for small $f \ll 1$ and this leads to the dimensionless NLS equation (1.20), where the tildes have been dropped for brevity.

$$i\psi_z(z,x,y) + \Delta_\perp \psi + \nu \left|\psi\right|^2 \psi = 0. \tag{1.20}$$

Radial solitary-wave solutions to (1.20) were considered in [4] with ψ of the form:

$$\psi_\omega^{\text{solitary}}(r,z) = e^{i\omega z} R_\omega(r), \tag{1.21}$$

where ω is a real number and R_ω is the real solution of

$$-\omega R_\omega + \Delta_\perp R_\omega(r) + R_\omega^3 = 0.$$

This can be solved in general by, for example,

$$R_\omega(r) = \sqrt{\omega} R\left(\sqrt{\omega} r\right).$$

However, taking $\omega = 1$ leads to the simplest soliton equation

$$R''(r) + \frac{1}{r}R' - R + R^3 = 0, \quad 0 < r < \infty, \tag{1.22}$$

subject to initial condition $R'(0) = 0$ and integrability condition $\lim_{r\to\infty} R(r) = 0$. The (numerical) solution is known as the Townes profile, which is positive and monotonically decreasing in r.

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2

Existence of ground state

2.1. Initial value problem and nonlinearity

In this chapter, we will study an existence proof for the initial value problem

$$-u''(r) - \frac{n-1}{r}u'(r) = f(u(r)), \quad \text{on } 0 < r < \infty, \tag{2.1}$$

satisfying initial conditions and an integrability condition

$$\begin{cases} u(0) = \alpha, \\ u'(0) = 0 \\ \lim_{r\to\infty} u(r) = 0. \end{cases} \tag{2.2}$$

The existence proof will be based on [1], which generalises earlier results. One of these is the uniqueness result [2], which was later generalised in [3], which forms the basis for the next chapter.

The proof will be by a shooting method, where we categorise the solutions based on their asymptotic behaviour. Furthermore, solutions to the initial value problem equation (2.1) are also positive radial solutions to the more general problem

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

where $f(u)$ is a given nonlinear function. This partial differential equation is relevant to many areas of mathematical physics.

The solutions $R(r)$ to equation (1.22) are solutions $u(r)$ to (2.1) with $n = 2$ and

$$f(u) = -u + u^3.$$

2.2. DEFINITIONS OF SOLUTION SETS

A **ground state solution** is strictly decreasing everywhere and has no finite zeroes. Yet, the solution should vanish in the limit as $r \to \infty$.

We define the set G of ground state initial conditions as

$$G := \left\{ \alpha > 0 \;\middle|\; u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}. \tag{2.4}$$

We consider two alternatives: either (i) the derivative vanishes, or (ii) the solution vanishes. We define the set P of initial conditions with a vanishing derivative as

$$P := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.5}$$

We define the set N of initial conditions with a vanishing solution as

$$N := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u(r_0,\alpha) = 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.6}$$

REVISE: These solution sets are disjoint by definition and we write the union of initial conditions as $I = P \,\dot\cup\, G \,\dot\cup\, N$.

2.3. ASSUMPTIONS ON f

We assume that f is locally Lipschitz continuous from $\mathbb{R}_+ \to \mathbb{R}$ and satisfies $f(0) = 0$. Local Lipschitz continuity is an important condition for the Picard-Lindelöf local existence and uniqueness theorem. Additionally, we assume that hypotheses (H1)–(H5) are satisfied. Firstly,

$$f(\kappa) = 0, \text{ for some } \kappa > 0. \tag{H1}$$

Secondly, defining $F(t)$ as the integral of $f(t)$

$$F(t) := \int_0^t f(s)\,\mathrm{d}s, \tag{2.7}$$

there exists an initial condition $\alpha > 0$ such that $F(\alpha) > 0$. We define

$$\alpha_0 := \inf\{\alpha > 0 \mid F(\alpha) > 0\}. \tag{H2}$$

Thirdly, the right-derivative of $f(s)$ at κ is positive

$$f'(\kappa^+) = \lim_{s\downarrow\kappa} \frac{f(s) - f(\kappa)}{s - \kappa} > 0, \tag{H3}$$

and fourthly, we have

$$f(s) > 0 \quad \text{for } s \in (\kappa, \alpha_0]. \tag{H4}$$

We define

$$\lambda := \inf\{\, \alpha > \alpha_0 \mid f(\alpha) = 0\}, \tag{2.8}$$

and note that $\alpha_0 < \lambda \leqslant \infty$. In the situation where $\lambda = \infty$, we assume

$$\lim_{s\to\infty} \frac{f(s)}{s^l} = 0, \quad \text{with } l < \frac{n+2}{n-2}. \tag{H5}$$

2.4. MAIN THEOREM

Theorem 2.1. *Let f be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0,\infty)$ such that $f(0) = 0$ and f satisfies hypothese $(H1) - (H5)$. Then there exists a number $\alpha \in (\alpha_0, \lambda)$ such that the solution $u(r,\alpha) \in C^2(\mathbb{R}_+)$ of the initial value problem (2.1) has*

$$\begin{cases} u(r,\alpha) > 0 & \text{for all } r \geqslant 0, \\ u'(r,\alpha) < 0 & \text{for all } r > 0, \end{cases} \tag{2.9}$$

and

$$\lim_{r\to\infty} u(r) = 0.$$

Then $\alpha \in G$. Hence G is non-empty.

If in addition, we assume that f satisfies ?? then there exists constants such that etc...

2.5. INTERVAL OF DEFINITION

Existence of local unique solutions is guaranteed by the Picard-Lindelöf theorem, see for example [4, Theorem. 2.2].

In these circumstances, boundedness of the solution $u(r,\alpha)$ is a sufficient condition for the solution to be defined on the maximal interval $[0,\infty)$. This is also called the *blow-up alternative*. Either (i) for some $r_0 > 0$ we have

$$\left|u(r_0,\alpha)\right| > M, \quad \text{for all } M > 0,$$

and the solution is defined on $[0, r_0)$. Or (ii) for some $M > 0$ we have

$$\left|u(r,\alpha)\right| \leqslant M, \quad \text{for all } r \geqslant 0,$$

and the solution is defined for all $r \geqslant 0$.

In this section, we will derive an upper and a lower bound for $u(r,\alpha)$. Since the solution is initially decreasing, possibly the initial condition α is an upper bound.

Lemma 2.1. $u(r,\alpha) \leqslant u(0,\alpha) = \alpha \quad$ *for* $r \geqslant 0$.

Proof. TODO: Seperate into lemma about the quantity (2.11). In this proof, we write $u(r) = u(r,\alpha)$ for brevity. We start with (2.1) and multiply by $u'(r)$. Then we integrate from 0 to r to obtain

$$-\int_0^r \left[u'(s)u''(s)\right] ds - \int_0^r \left[\frac{n-1}{s}[u'(s)]^2\right] \mathrm{d}s = \int_0^r \left[u'(s) f(u(s))\right] \mathrm{d}s. \tag{2.10}$$

We use the chain rule simplify the first term in (2.10) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}[u'(r)^2] = 2u'(r)u''(r) \overset{(2.2)}{\iff} \frac{1}{2}[u'(r)]^2 = \int_0^r \left[u'(s)u''(s)\right] \mathrm{d}s.$$

2

Then, we rewrite the right-hand side of (2.10) using the fundamental theorem of calculus

$$\int_0^r \left[u'(s) f(u(s))\right] \mathrm{d}s = \int_0^r \left[\tfrac{\mathrm{d}u}{\mathrm{d}s} f(u(s))\right] \mathrm{d}s = \int_{u(0)}^{u(r)} f(u)\,\mathrm{d}u = F(u(r)) - F(u(0)).$$

Finally, using $u(0) = \alpha$, we have rewritten (2.10) as

$$-\frac{1}{2}\left[u'(r)\right]^2 - (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(u(r)) - F(\alpha). \tag{2.11}$$

We suppose by contradiction that

$$u(r_0) > \alpha, \quad \text{for some } r_0 > 0. \tag{2.12}$$

TODO: Be more specific about: in which quantity, with which assumption, do we have which result. By the assumptions on $f(u)$, we have $f(u) > 0$ on (α_0, ∞). As a result, $F(u)$ is increasing on (α_0, ∞). Using assumption (2.12) and $\alpha > \kappa$, we deduce that

$$F(u(r_0)) > F(\alpha) \iff F(u(r_0)) - F(\alpha) > 0.$$

This contradicts (2.11), as the left-hand side is clearly non-positive. □

We will show that $u(r,\alpha)$ has a lower bound for $r < \infty$. Let r_0 be the first zero of $u(r,\alpha)$

$$r_0 := \inf\left\{ r > 0 \,\middle|\, u(r,\alpha) = 0 \right\}. \tag{2.13}$$

If $r_0 = \infty$, then we have $u(r,\alpha) > 0$ for all $r > 0$. Suppose to the contrary that $r_0 < \infty$, then we have the following bound on the derivative $u'(r,\alpha)$.

Lemma 2.2. *Suppose that $r_0 < \infty$. Then for $r \geqslant r_0$, we have*

$$u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha) \geqslant u'(r_0,\alpha). \tag{2.14}$$

Proof. We consider the sign of $u'(r_0,\alpha)$. Firstly, if $u'(r_0,\alpha) = 0$ then u and u' vanish simultaneously in r_0. Then from (2.1) we have

$$u'' = 0, \quad \text{with } u(r_0) = u'(r_0) = 0,$$

which is solved by

$$u(r) = c_1 r + c_2,$$

where we must have $c_1 = c_2 = 0$ to satisfy the conditions at r_0, so that $u \equiv 0$. But this contradicts $u(0,\alpha) = \alpha > 0$. Hence, u and u' cannot vanish simultaneously for $\alpha > 0$.

Secondly, if $u'(r_0,\alpha) > 0$ we also reach a contradiction. By (2.1) with $u(r_0,\alpha) = 0$,

$$u''(r_0,\alpha) + \frac{n-1}{r} u'(r_0,\alpha) = -f(0) = 0$$

we see that u'' and u' have opposite signs in r_0. Then either:

$$\begin{cases} \text{(i) } u'' > 0 \quad \text{and} \quad u' < 0 \quad \text{in } r_0, \quad \text{or} \\ \text{(ii) } u'' < 0 \quad \text{and} \quad u' > 0 \quad \text{in } r_0. \end{cases} \tag{2.15}$$

The latter case implies that $u(r,\alpha) < 0$ in a left neighborhood of r_0, which contradicts $u(r,\alpha) > 0$ on $[0, r_0)$. Thus, we have $u'(r_0,\alpha) < 0$.

In the following, we extend $f(u) = 0$ for $u \leqslant 0$. Then for $u(r,\alpha) \leqslant 0$ the IVP (2.1) reads

$$-u''(r,\alpha) - \frac{n-1}{r} u'(r,\alpha) = 0, \tag{2.16}$$

We solve (2.16) for $u' = u'(r,\alpha)$ and seperate the variables, resulting in

$$\frac{\mathrm{d}u'}{u'} = -\frac{n-1}{r}\,\mathrm{d}r.$$

We integrate the expression from r_0 to r and evaluate the limits to get

$$\ln u'\big|_{r_0}^{r} = \big[(n-1)\ln r\big]_{r}^{r_0} \iff \ln u'(r) - \ln u'(r_0) = (n-1)\left[\ln r_0 - \ln r\right].$$

Then, we rewrite the expression to arrive at the desired result

$$\frac{u'(r)}{u'(r_0)} = \left(\frac{r_0}{r}\right)^{n-1} \iff u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha) \geqslant u'(r_0,\alpha). \qquad \square$$

In conclusion, the solution $u(r,\alpha)$ is bounded for bounded r. More specifically, in the case of everywhere positive solutions, we have

$$0 < u(r,\alpha) \leqslant \alpha \quad \text{for all } r > 0.$$

TODO: Explicit expression for $u(r,\alpha)$.Alternatively, for solutions with $u(r_0,\alpha) = 0$ for some $r_0 > 0$, by Lemma 2.2 we have

$$u(r,\alpha) \geqslant \int_{r_0}^{r} \left(\frac{r_0}{s}\right)^{n-1} u'(r_0,\alpha)\,\mathrm{d}s > -\infty \quad \text{for } r_0 < r < \infty. \tag{2.17}$$

2.6. Asymptotics of positive decreasing solutions

We will show that everywhere positive decreasing solutions $u(r,\alpha)$ vanish in the limit as $r \to \infty$. The proof will be in TODO: three steps: (i) we show that the nonlinearity $f(u) \to 0$ in the limit as $r \to \infty$, (ii) we show via a translation $v(r) = u(r) - \kappa$ that $l = \kappa$ does not satisfy the IVP, such that $l = 0$.

2

Lemma 2.3. *Let $f:\mathbb{R}^+\to\mathbb{R}$ be a locally Lipschitz continuous function such that $f(0)=0$. Let $u(r,\alpha_1)$ be a solution to initial value problem (2.1) with $\alpha_1\in(0,\infty)$ such that*

$$\begin{cases} u(r,\alpha_1)>0 & \text{for all } r\geqslant 0, \quad \text{and} \\ u'(r,\alpha_1)<0 & \text{for all } r>0. \end{cases} \tag{2.18}$$

Then the number $l:=\lim_{r\to\infty} u(r,\alpha_1)$ satisfies $f(l)=0$.

If additionally $f(u)$ satisfies (H3), then $l=0$.

Proof step 1. By assumption (2.18) on $u(r,\alpha_1)$ and the monotone convergence theorem, we have $0\leqslant l<\alpha_1$. Then $f(l)<f(\alpha_1)$. We consider the limit as $r\to\infty$ of the IVP (2.1)

$$\lim_{r\to\infty}\left[-u''(r,\alpha_1)-\frac{n-1}{r}u'(r,\alpha_1)\right]=f(l)<\infty. \tag{2.19}$$

We restate equation (2.11)

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2+(n-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}=F(\alpha_1)-F(u(r,\alpha_1)),$$

and consider the limit as $r\to\infty$

$$\lim_{r\to\infty}\left[\frac{1}{2}\left[u'(r,\alpha_1)\right]^2+(n-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}\right]=F(\alpha_1)-\lim_{r\to\infty}F(u(r,\alpha_1)),$$

where we use $\lim_{r\to\infty}F(u(r,\alpha_1))=F(l)<\infty$ to write

$$\lim_{r\to\infty}\frac{1}{2}[u'(r,\alpha_1)]^2+(n-1)\int_0^\infty u'(s,\alpha_1)^2\frac{\mathrm{d}s}{s}=F(\alpha_1)-F(l). \tag{2.20}$$

Note that

$$F(\alpha_1)-F(l)<\infty\implies\int_0^\infty u'(s,\alpha_1)^2\frac{\mathrm{d}s}{s}<\infty,$$

such that $u'(r,\alpha_1)^2/r$ converges as $r\to\infty$ by the Levi monotone convergence theorem. Then $u'(r,\alpha_1)^2$ converges, because... and since $u'(r,\alpha_1)<0$ everywhere, we deduce that $u'(r,\alpha_1)$ converges. However, since $0\leqslant u(r,\alpha_1)\leqslant\alpha_1$, we have

$$\lim_{r\to\infty}u'(r,\alpha_1)=0. \tag{2.21}$$

Now, we return to equation (2.19) and use $\lim_{r\to\infty}u'(r,\alpha_1)=0$ to obtain

$$-\lim_{r\to\infty}\left[u''(r,\alpha_1)\right]=f(l).$$

We have (2.21) and hence, we have

$$\lim_{r\to\infty}u''(r,\alpha_1)=0.$$

The desired result follows: $f(l)=0$. □

Proof step 2. The nonlinearity $f(u)$ has more than one zero. Both $f(0)=0$ and $f(\kappa)=0$. We will show that only $l=0$ satisfies the assumptions.

Suppose to the contrary that $l=\kappa$. We will use the substitution

$$v(r)=r^{(1/2)(n-1)}\left[u(r,\alpha_1)-\kappa\right]$$

in equation (2.1) to obtain a differential equation in $v(r)$. In the remainder of the proof, we will abbreviate $u(r,\alpha_1)=u(r)$. We note that $v(r)>0$ by definition, as the assumption is that $u(r)>\kappa$ for $r>0$ and $u(r)\downarrow\kappa$.

We proceed to calculate the first derivative

$$v'(r)=\frac{1}{2}(n-1)r^{(n-3)/2}\left[u(r)-\kappa\right]+r^{(n-1)/2}u'(r),$$

and the second derivative, where we gather the terms by $u(r)$, $u'(r)$ and $u''(r)$ as

$$v''(r)=\frac{1}{4}(n-1)(n-3)r^{(n-5)/2}\left[u(r)-\kappa\right]+(n-1)r^{(n-3)/2}u'(r)+r^{(n-1)/2}u''(r). \tag{2.22}$$

We multiply the IVP (2.1) by $r^{(n-1)/2}$ to obtain

$$-r^{(n-1)/2}u''(r)-(n-1)r^{(n-1)/2}r^{-1}u'(r)=f(u(r))r^{(n-1)/2}. \tag{2.23}$$

We can use this to simplify (2.22)

$$v''(r)=\frac{1}{4}(n-1)(n-3)r^{(n-1)/2}r^{-2}\left[u(r)-\kappa\right]-f(u(r))r^{(n-1)/2}.$$

Now we factor out $v(r)=r^{(n-1)/2}\left[u(r)-\kappa\right]$ to obtain

$$v''(r)=r^{(n-1)/2}\left[u(r)-\kappa\right]\left\{\frac{1}{4}(n-1)(n-3)r^{-2}-\frac{f(u)}{u(r)-\kappa}\right\}.$$

Lastly, we multiply by -1 to obtain the exact expression from [1] as

$$-v''(r)=\left\{\frac{f(u)}{u(r)-\kappa}-\frac{(n-1)(n-3)}{4r^2}\right\}v. \tag{2.24}$$

We can show that there exist $\omega>0$ and $R_1>0$, such that

$$\frac{f(u)}{u(r)-\kappa}-\frac{(n-1)(n-3)}{4r^2}\geqslant\omega \quad \text{for all } r\geqslant R_1. \tag{2.25}$$

This will be done in proof step 3. We will first show how this leads to $l=0$ to conclude proof step 2.

2

We have $v''(r) < 0$ for $r \geqslant R_1$, which implies that

$$v'(r) \downarrow L \geqslant -\infty, \quad \text{as } r \to \infty.$$

Suppose that $L < 0$, then $v(r) \to \infty$ as $r \to \infty$, which contradicts $v(r) > 0$. On the other hand, suppose that $L \geqslant 0$, then $v(r) \geqslant v(R_1) > 0$ for $r \geqslant R_1$. Substituting in (2.25), we have

$$-v''(r) \geqslant \omega v(R_1) > 0,$$

such that $v'(r) \to -\infty$ as $r \to \infty$. This contradicts $L \geqslant 0$. Since $l = \kappa$ is contradictory in any case, we have $l = 0$. □

Proof step 3. The first term (2.25) is non-negative and decreasing by (H3). We will write

$$M(r) := \frac{f(u)}{u(r) - \kappa} > 0, \tag{2.26}$$

and rewrite (2.25) to obtain

$$M(r) \geqslant \frac{(n-1)(n-3)}{4r^2} + \omega. \tag{2.27}$$

We choose $2\omega = \max_{r>0} M(r)$ and choose $R_1 > 0$ such that

$$\frac{(n-1)(n-3)}{4r^2} \leqslant \frac{1}{2} M(r) \quad \text{for } r \geqslant R_1.$$ □

2.7. P IS NON-EMPTY AND OPEN

In this section we will show that P is non-empty and open. TODO: Refer to main theorem.The existence of solutions in F also requires that N is non-empty and open. The proof that N is open is similar to the proof given for P. For the proof that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

Lemma 2.4. *Solution set P as defined in (2.5)*

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0, \alpha) = 0 \textit{ and } u(r, \alpha) > 0 \textit{ for all } r \leqslant r_0 \right\}$$

is non-empty and open.

Proof step 1. We will show that solution set P is non-empty. Let $\alpha \in (\kappa, \alpha_0]$. TODO: Refer to definition of α_0. Considering all initial conditions $(0, \infty)$ and the disjoint subsets P and N, if $\alpha \notin N$ and $\alpha \notin \left(P \dot\cup N\right)$, then $\alpha \in P$.

First, we suppose by contradiction that $\alpha \in N$. By the definition of N in (2.6) there exists a number $r_0 > 0$ such that

$$\begin{cases} u(r_0, \alpha) = 0, \\ u'(r, \alpha) < 0 \quad \text{for } \; r \leqslant r_0. \end{cases} \tag{2.28}$$

TODO: Refer to gint lemma. We restate equation (2.11) for $r = r_0$

$$\frac{1}{2}\left[u'(r_0,\alpha)\right]^2 + (n-1)\int_0^{r_0} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r_0,\alpha)). \tag{2.29}$$

The left hand side of (2.29) is positive. But $F(u(r_0,\alpha)) = F(0) = 0$, by ... Furthermore, for $\alpha \in (\kappa, \alpha_0]$, we have $F(\alpha) < 0$. Hence $\alpha \notin N$.

Next, we suppose that $\alpha \notin (P \dot\cup N)$. TODO: Precise refs Thus, by the definitions of P and N, we have...

$$\begin{cases} u(r,\alpha) > 0 & \text{for } r \geqslant 0, \text{ and} \\ u'(r,\alpha) < 0 & \text{for } r > 0. \end{cases} \tag{2.30}$$

This implies TODO: Be specific about the lemma statement and proof steps I refer to.

$$u(r,\alpha) \downarrow l \geqslant 0 \quad \text{as } r \uparrow \infty, \tag{2.31}$$

and by Lemma (2.3), we know that

$$\lim_{r\to\infty} u'(r,\alpha) = 0, \quad \text{and } l = 0.$$

Thus,

$$(n-1)\int_0^{\infty} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) < 0.$$

By this contradiction we have $\alpha \notin \left(P \dot\cup N\right)$. Hence $(\kappa, \alpha_0] \subset P$. □

Proof step 2. We will show that P is open. TODO: Read Teschl or CodLev to be more precise about the circumstances that I assume, and the type of continuity this implies. We know that the solution $u(r,\alpha)$ and its derivative $u'(r,\alpha)$ depend continuously on α. Let $\alpha \in P$. There exists

$$r_0 := \inf\left\{r > 0 \,\middle|\, u'(r,\alpha) = 0 \text{ and } u(r,\alpha) > 0\right\}$$

such that

$$\begin{cases} u(r,\alpha) > 0 & \text{for all } r \in [0, r_0] \\ u'(r,\alpha) < 0 & \text{for all } r \in (0, r_0). \end{cases} \tag{2.32}$$

Evaluating the IVP (2.1) in r_0 yields

$$u''(r_0,\alpha) = -f(u(r_0,\alpha)).$$

Suppose that $u''(r_0,\alpha) = 0$. Then $-f(u(r_0,\alpha)) = 0$. Therefore $u(r_0,\alpha) = \kappa$ or $u(r_0,\alpha) = 0$. These would imply $u(r,\alpha) \equiv \kappa$ or $u(r,\alpha) \equiv 0$, which are both impossible in light of (2.32). TODO: Mention uniqueness of the IVP, be precise about implication

Suppose that $u''(r_0,\alpha) \neq 0$. Since

$$\begin{cases} u'(r_0,\alpha) = 0 & \text{and} \\ u'(r,\alpha) < 0 & \text{for } r < r_0, \end{cases} \tag{2.33}$$

we have $u''(r_0,\alpha) > 0$. Then for $r_1 > r_0$ near r_0, we have

$$u(r,\alpha) > u(r_0,\alpha) \quad \text{for all } r \in (r_0, r_1].$$

By the continuous dependence on α, we have

$$\forall \epsilon > 0 \, \exists \, \delta > 0 : \left| u(r,\alpha) - u(r,\beta) \right| < \epsilon \impliedby \left| \alpha - \beta \right| < \delta.$$

We define

$$\epsilon := \frac{1}{2}\left(u(r_1,\alpha) - u(r_0,\alpha) \right).$$

For $\delta_{r_0} > 0$ sufficiently small, we have

$$\left| u(r_0,\alpha) - u(r_0,\beta) \right| < \epsilon,$$

and for $\delta_{r_1} > 0$ sufficiently small, we have

$$\left| u(r_1,\alpha) - u(r_1,\beta) \right| < \epsilon.$$

That is, for β near α we have

$$\begin{cases} u(r_1,\beta) > u(r_0,\beta) \\ \beta > u(r,\beta) > 0 & \text{for all } r \in (0, r_1]. \end{cases} \tag{2.34}$$

Thus $\beta \in P$ and P is open. □

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3

Uniqueness

3.1. Main uniqueness theorem

This chapter deals with a different initial value problem.

3.1.1. New initial value problem and nonlinearity

The uniqueness of ground state solutions to the initial value problem

$$\begin{cases} u'' + \dfrac{1}{r}u' - \lambda u + V(r)u^p = 0, \quad r > 0, \quad u = u(r,\alpha) \geqslant 0 \\ u(0,\alpha) = \alpha > 0, \quad \lim\limits_{r\to 0} r u'(r,\alpha) = 0 \end{cases} \tag{3.1}$$

was proven in [1]. Here, $\lambda > 0$ and $p > 1$. compare with previous IVP and nonlin.

Furthermore, we define $w = w(r,\alpha) := \frac{\partial}{\partial\alpha} u(r,a) \geqslant 0$, which satisfies the IVP

$$\begin{cases} w'' + \dfrac{1}{r}w' - \lambda w + pV(r)u^{p-1}w = 0, \quad r > 0, \\ w(0,\alpha) = 1, \quad \lim\limits_{r\to 0} r w'(r,\alpha) = 0. \end{cases} \tag{3.2}$$

The main theorem makes certain assumptions on $V(r)$.

3.1.2. Hypotheses on the non-autonomous term

The nonlinearity $f(u,r) = \lambda u - V(r)u^p$ contains the non-autonomous term $V(r)$. The following assumptions will be made on $V(r)$: continuous and differentiable

$$V \in C^1(0,\infty); \tag{3.3}$$

positive and decreasing

$$V > 0 \quad \text{and} \quad V' \leqslant 0 \quad \text{on } (0,\infty); \tag{3.4}$$

3

bounded or at most hyperbolic of quadratic power in r

$$r^k V(r) \in L^\infty(0,\infty) \quad \text{for some } k \in (0,2); \tag{3.5}$$

Lastly, we define

$$h(r) := r\frac{V'(r)}{V(r)} \tag{3.6}$$

and require that $h(r)$ is non-increasing on $(0,\infty)$

$$h'(r) \leqslant 0 \quad \text{on } (0,\infty). \tag{3.7}$$

Theorem 3.1. *Suppose that* (3.3)-(3.7) *are satisfied. Then there exists* $\alpha_0 > 0$ *such that the solution sets have the following structure*

$$P = \left(0,\alpha_0\right), \quad G = \left\{\alpha_0\right\}, \quad N = \left(\alpha_0,\infty\right).$$

3.2. Sturm theory and related lemmata

The paper by Genoud refers to the paper by Kwong, which contains a recap of some Sturmian theory. In particular, the main Sturm comparison theorem and certain corollaries, as well as the concept of a *disconjugay interval.*

Important theory for the following lemmata is Sturm Comparison theorem as stated and proven in for example [2, p. 246].

Theorem 3.2. *Let U and V be solutions, respectively, of the following equations*

$$U''(x) + f(x)U'(x) + g(x)U(x) = 0, \quad x \in (a,b), \tag{3.8}$$

$$V''(x) + f(x)V'(x) + G(x)V(x) = 0, \quad x \in (a,b), \tag{3.9}$$

where f, g and G are continuos. Let (μ,ν) be a subinterval in which $V(x) \neq 0$ and $U(x) \neq 0$, and in which the comparison condition

$$G(x) \geqslant g(x) \quad \textit{for all } x \in (\mu,\nu) \tag{3.10}$$

holds. Suppose further that

$$\frac{V'(\mu)}{V(\mu)} \leqslant \frac{U'(\mu)}{U(\mu)}. \tag{3.11}$$

Then

$$\frac{V'(x)}{V(x)} \leqslant \frac{U'(x)}{U(x)} \quad \textit{for all } x \in (\mu,\nu). \tag{3.12}$$

Equality in (3.12) *can occur only if $U \equiv V$ in $[\mu,x]$. If either μ or ν is a zero of U or V, then the fractions in* (3.11) *and* (3.12) *are interpreted as ∞.*

3.3. BASIC PROPERTIES OF SOLUTIONS

In this section, we will derive several more basic properties of solutions to (3.1). In particular, we will find an initial condition α_p that separates solutions into $\alpha \in P$ and $\alpha \in G \cup N$. Next, we will find that solutions with $\alpha \in G \cup N$ are strictly decreasing everywhere. Lastly, we will find that the derivative with respect to the initial condition $w(r)$ has at least one zero in the interval $(0, z(\alpha))$.

Lemma 3.1. *Suppose that* $V(0) := \lim_{r\to 0} V(r)$ *exists and is finite. Then*

$$0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \implies \alpha \in P.$$

Proof. We define the Lyapunov (or energy) function $E(r)$ as

$$E(r) := \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1} \tag{3.13}$$

for

$$\begin{cases} r \in [0, \infty) & \text{if } \alpha \in P \cup G, \\ r \in [0, z(\alpha)] & \text{if } \alpha \in N. \end{cases}$$

We calculate the derivative $E'(r)$ and gather the terms with $u'(r)$

$$E'(r) = \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right] u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1}.$$

This expression can be simplified by writing the IVP (3.1) as

$$-\frac{1}{r}u'(r) = \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right].$$

Thus, we have

$$E'(r) = -\frac{u'(r)^2}{r} + \frac{1}{p+1}V'(r)u(r)^{p+1} \leqslant 0, \tag{3.14}$$

where $E'(r) \leqslant 0$ holds because $V' \leqslant 0$ by (3.4) and

$$\begin{cases} u(r) > 0 & \text{on } [0, \infty) & \text{if } \alpha \in P \cup G \\ u(r) \geqslant 0 & \text{on} [0, z(\alpha)] & \text{if } \alpha \in N. \end{cases} \tag{3.15}$$

For $\alpha \in N$, we have $u(z(\alpha)) = 0$. We evaluate $E(z(\alpha))$

$$E(z(\alpha)) = \frac{1}{2}u'(z(\alpha))^2 \geqslant 0. \tag{3.16}$$

For $\alpha \in G$, we have

$$u(r) \to 0 \quad \text{and} \quad u'(r) \to 0 \quad \text{as } r \to \infty.$$

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Hence, we have $E(r) \to 0$ as $r \to \infty$. Then by (3.14), we have

$$E(r) \geqslant 0 \quad \text{for } \alpha \in G \cup N. \tag{3.17}$$

In particular, we have

$$\begin{cases} E(0) \geqslant 0 & \text{for } \alpha \in G \cup N \\ E(0) < 0 & \text{for } \alpha \in P. \end{cases} \tag{3.18}$$

We will solve $E(0) < 0$ for α, where we remember that $u(0,\alpha) = \alpha > 0$

$$E(0) = \frac{1}{2}u'(0)^2 - \frac{\lambda}{2}\alpha^2 + \frac{1}{p+1}V(0)\alpha^{p+1} < 0.$$

actually, lim $(ru') = 0$ for r down to 0.

$$E(0) = -\frac{\lambda}{2}\alpha^2 + \frac{1}{p+1}V(0)\alpha^{p+1} = 0.$$

So that we obtain

$$\alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}}.$$

□

Lemma 3.2. *Let $\alpha \in G \cup N$, and $u(r) = u(r,\alpha)$. Then $u'(r) < 0$ for all $r \in (0, z(\alpha))$ and $u'(z(\alpha)) < 0$ if $\alpha \in N$.*

Proof. By (3.17) and (3.18), we have $E(r) \geqslant 0$ and $E(0) \geqslant 0$ for $\alpha \in G \cup N$. By the IVP, we have for $r = 0$, where $u(0,\alpha) = \alpha > 0$

$$u''(0) + \lim_{r \downarrow 0} \frac{1}{r}u'(r) - \lambda\alpha + V(0)\alpha^p = 0.$$

If $V(0) < \infty$ then ...

If $V(0) = \infty$ then ...

Then there exists $\epsilon > 0$ such that $u''(r,\alpha) < 0$ for $r \in (0,\epsilon)$ and we have

$$u'(\epsilon,\alpha) = \int_0^\epsilon u''(s,\alpha)\,ds < 0.$$

We claim that $u'(r,\alpha) < 0$ for all $r > 0$. Suppose by contradiction that there exists $r_0 \in (0, z(\alpha))$ such that $u'(r_0,\alpha) = 0$. Then by (3.1) we have

$$u''(r_0,\alpha) - \lambda u(r_0,\alpha) + V(r_0)u(r_0)^p = 0. \tag{3.19}$$

Since $u'(r,\alpha) < 0$ on $(0, r_0)$, we have $u''(r_0,\alpha) \geqslant 0$. Solving for $u''(r_0,\alpha)$ in (3.19)

$$\lambda u(r_0,\alpha) - V(r_0)u(r_0,\alpha)^p = u''(r_0,\alpha) \geqslant 0,$$

where we can solve for $u(r_0,\alpha)$ to find that

$$u(r_0,\alpha) \leqslant \left[\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}} < \left[\frac{p+1}{2}\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}}.$$

Now this implies by or (3.13) that $E(r_0) < 0$, which contradicts (3.17). Therefore, we have $u'(r) < 0$ for all $r \in (0, z(\alpha))$. Lastly, if we have $u'(z(\alpha)) = 0$ for $\alpha \in N$, we have $u \equiv 0$. Hence, we have $u'(z(\alpha)) < 0$. □

Lemma 3.3. *Let $\alpha \in (G \cup N)$. Then $w(r) = w(r,\alpha)$ has at least one zero in $(0, z(\alpha))$.*

Proof. The cases $\alpha \in N$ and $\alpha \in G$ will be considered seperately. First, we suppose $\alpha \in N$. We write $u(r) = u(r,\alpha)$. The differential equations for u and w can be written as:

$$(ru'(r))' + r\left[-\lambda u(r) + V(r)u(r)^p\right] = 0 \tag{3.20}$$

$$(rw'(r))' + r\left[-\lambda w(r) + pV(r)u(r)^{p-1}w(r)\right] = 0. \tag{3.21}$$

We multiply equations (3.20) and (3.21) by $w(r)$ and $u(r)$ respectively and subtract to obtain:

$$w(r)(ru'(r))' - u(r)(rw'(r))' = r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\}. \tag{3.22}$$

Now, we integrate from 0 to $z(\alpha)$ to obtain:

$$\int_0^{z(\alpha)} w(r)(ru'(r))' - u(r)(rw'(r))' \, \mathrm{d}r$$
$$= \int_0^{z(\alpha)} r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\} \mathrm{d}r. \tag{3.23}$$

By partial integration, the left hand side reads

$$w(r)ru'(r)\Big|_0^{z(\alpha)} - u(r)rw'(r)\Big|_0^{z(\alpha)} + \int_0^{z(\alpha)} rw'(r)u'(r)\,\mathrm{d}r - \int_0^{z(\alpha)} ru'(r)w'(r)\,\mathrm{d}r,$$

and using $u(z(\alpha)) = 0$ and $r = 0$, the only term that remains is

$$w(z(\alpha))z(\alpha)u'(z(\alpha)).$$

Furthermore, the right hand side of (3.23) can be simplified to

$$(p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)\,\mathrm{d}r,$$

such that (3.23) can be simplified to

$$w(z(\alpha))z(\alpha)u'(z(\alpha)) = (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)\,\mathrm{d}r. \tag{3.24}$$

By Lemma 3.2, we have $u'(z(\alpha)) < 0$. If $w > 0$ holds, then $w(z(\alpha)) > 0$ and the left hand side of (3.24) is negative. However, the right hand side is positive. Hence, $w(z(\alpha)) < 0$ and w has at least one zero in $(0, z(\alpha))$.

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On the other hand, for $\alpha \in G$, we assume by contradiction that $w > 0$ on $(0,\infty)$. We write

$$r\left(u'(r)w(r) - u(r)w'(r)\right) = rw(r)^2 \frac{u'(r)w(r) - u(r)w'(r)}{w(r)^2} = rw(r)^2 \left(\frac{u}{w}\right)' \tag{3.25}$$

such that equation (3.22) integrated from 0 to r reads

$$rw(r)^2\left(\frac{u}{w}\right)' = (p-1)\int_0^r sV(s)u(s)^p w(s)ds > 0. \tag{3.26}$$

From (3.26), we deduce that

$$\left(\frac{u}{w}\right)' > 0 \quad \text{for } r \in (0,\infty), \tag{3.27}$$

and combined with $u(0) = \alpha > 0$ and $w(r) = 1$ from (3.1) and (3.2), we conclude that

$$\frac{u(r)}{w(r)} > 0 \quad \text{on } (0,\infty). \tag{3.28}$$

By several theorems and lemmata in [3], we know that the limit as r to infinity of (3.28) is 0, contradicting (3.27).

deal with $\alpha \in G$ case referring to Genouds thesis, which needs to be added to the references. EXPAND EXPAND □

3.4. Auxiliary functions related to solutions

θ and ρ and ν and ϕ and ξ mostly

3.5. More advanced properties of solutions

Lemma 3.4. *Let $\alpha \in G \cup N$. There exist $\beta_0 > 0$ and a unique function $\sigma : [0, \overline{\beta}] \to [0,\infty)$ with the following properties:*

(a) σ is continuous and decreasing, $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$;

(b) for all $\beta > 0$ we have:

$$\phi_\beta(r) < 0 \quad \text{if } r < \sigma(\beta), \quad \text{and} \quad \phi_\beta > 0 \quad \text{if } r > \sigma(\beta).$$

Proof. We fix $\beta > 0$. Then $\phi_\beta(r)$ is fixed, as we can see from (**??**)

$$\phi_\beta(r) = V(r)u(r)^p \left[\beta(p-1) - 2 - \xi(r)\right].$$

Since $V > 0$ by (3.4) and $u(r)^p > 0$ by (3.15), the sign of ϕ_β depends only on the term in brackets. To study how the zero of this term depends on the choice of β as we will define and analyse the function $\Xi(r)$

$$\beta(p-1) - 2 - \xi(r) = 0 \iff \beta = \frac{2+\xi(r)}{p-1} := \Xi(r).$$

3

why does ξ have only one zero? Argument why ϕ_β changes sign only once.

Then $\sigma(\beta)$ is defined by the zero of ϕ_β. Since β was arbitrary, $\sigma(\beta)$ is defined by ϕ_β for all β and $\sigma(\beta)$ is unique. That $\sigma(\beta)$ is continuous and decreasing is shown below.

REVISE: In $\phi_\beta(r)$, factor out a term of $V(r)u(r)^p > 0$ and gather the other terms into $\xi(r)$ as follows:

$$\begin{aligned}
\phi_\beta(r) &= \left[\beta(p-1)-2\right] V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r) \\
&= V(r)u(r)^p \left[\beta(p-1) - 2 - \frac{rV'(r)u(r)^p}{V(r)u(r)^p} + \frac{2\lambda u(r)}{V(r)u(r)^p}\right] \\
&= V(r)u(r)^p \left[\beta(p-1) - 2 - r\frac{V'(r)}{V(r)} + \frac{2\lambda}{V(r)u(r)^{p-1}}\right] \\
&= V(r)u(r)^p \left[\beta(p-1) - 2 - \xi(r)\right] \\
\text{where } \xi(r) &= r\frac{V'(r)}{V(r)} - \frac{2\lambda}{V(r)u(r)^{p-1}}.
\end{aligned}$$

We note that by (**??**)

$$\phi_\beta(r) > 0 \iff \beta > \Xi(r) \quad \text{and} \quad \phi_\beta(r) < 0 \iff \beta < \Xi(r).$$

choose between two representations We claim that $\xi(r)$ is non-positive and strictly decreasing on (,)

$$\begin{cases} \xi(r) \leqslant 0 & \text{on } (,) \\ \xi'(r) < 0 & \text{on } (,) \end{cases} \tag{3.29}$$

In other words:

$$\xi(r) \leqslant 0 \text{ and } \xi'(r) < 0 \text{ on } (0, z(\alpha)).$$

By (3.7), we have $h'(r) \leqslant 0$ on $(0,\infty)$. Since $u'(r,\alpha) < 0$ by Lemma 3.2 and $V' \leqslant 0$ by (3.4), the second term in $\xi(r)$ is strictly decreasing. Hence, $\xi'(r) < 0$.

3

Since $u(z(\alpha)) = 0$ for $\alpha \in N$ and $u(r,\alpha) \to 0$ as $r \to \infty$ for $\alpha \in G$, we conclude that

$$\lim_{r \to z(\alpha)} \xi(r) = -\infty, \tag{3.30}$$

maybe clean up notation: introduce somewhere the notation $z(\alpha) = \infty$ for $\alpha \in G$ to unite the cases for $\alpha \in G$ and $\alpha \in N$.

REVISE: Even if $V(r)$ is non-zero, then still, the fraction blows up by the behaviour of $u(r)$ near $z(\alpha)$.

Next, we can show that $\Xi(0) > 0$ and $\Xi(r)$ is continuous and strictly decreasing. First $\Xi(r)$ is strictly decreasing because $\xi(r)$ is strictly decreasing and $\Xi(r)$ is a monotonous transformation # ?. By similar argument, $\Xi(r)$ is continous. $\Xi(r) = \left[2 + \xi(r)\right]/(p-1)$ satisfies $\Xi(0) > 0$. To see that $\Xi(0) > 0$, evaluate $\xi(r)$ in $r = 0$ and compare:

$$\Xi(0) > 0 \iff \xi(0) > -2.$$

Evaluating $\xi(0)$ for infinite $V(0)$ relies on #. When $V(0) = \infty$ then the second term in $\xi(0)$ is zero, since $u(0,\alpha)^{p-1} = \alpha^{p-1}$. Then $\xi(0) = h_0$. And by # $h_0 + k \geqslant 0$ so $h_0 > -k$. Since $k \in (0,2)$ the lowest bound for h_0 is $-k > -2$. Alternatively, when $V(0)$ is finite, then $h_0 = 0$ by #. Then

$$\xi(0) = -\frac{2\lambda}{\alpha^{p-1} V(0)}.$$

Now we can solve for α to find the values of α for which $\xi(0) > -2$.

$$\begin{aligned} -\frac{2\lambda}{\alpha^{p-1} V(0)} &> -2 \\ \frac{\lambda}{\alpha^{p-1} V(0)} &< 1 \\ \frac{\lambda}{V(0)} &< \alpha^{p-1} \\ \alpha &> \left[\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}} \end{aligned}$$

which is confirmed by the assumption that $\alpha \in G \cup N$ # is it?.

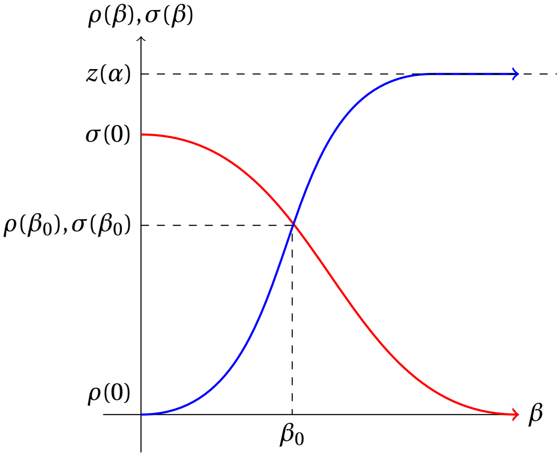
Trailing back our steps, we see that $\xi(0) > -2$ which implies $\Xi(0) > 0$. For any $\beta \in (0, \overline{\beta})$ we have

$$\beta < \Xi(r) \iff \beta(p-1) - 2 - \xi(r) < 0 \quad \text{and vice versa.}$$

conclude that this verifies the existence of the unique function $\sigma(\beta)$ with the aforementioned properties. that is $\sigma(\beta)$ is continuous and decreasing and $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$ as well as $\sigma(\beta)$ determing the sign of $\phi_\beta(r)$ for all $\beta > 0$. □

Lemma 3.5. *Let $\alpha \in G \cup N$. There exists a unique $\beta_0 > 0$ such that $\rho\left(\beta_0\right) = \sigma\left(\beta_0\right)$.*

Proof. This follows immediately from the aforementioned properties of $\rho(\beta)$ and $\sigma(\beta)$. Sketching the two graphs, there is a unique intersection.



□

3

3.6. MORE ADVANCED PROPERTIES OF SOLUTIONS

We fix $v = v_{\beta_0}$ and $\rho_0 = \rho\left(\beta_0\right)$. We will apply the Sturm comparison theorem to equations (**??**) and (**??**) in the following lemma.

Lemma 3.6. *For $\alpha \in G \cup N$, $w(r, \alpha)$ has a unique zero $r_0 \in (0, z(\alpha))$. Furthermore, $w(z(\alpha)) < 0$ for $\alpha \in N$ and if $\alpha \in G$, we have*

$$\lim_{r\to\infty} w(r) = -\infty.$$

Proof. With simplifications from before, we have

$$v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v < 0 \quad \text{and } v < 0 \quad \text{on } (0, \rho_0) \tag{3.31}$$

whereas

$$v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v < 0 \quad \text{and } v < 0 \quad \text{on } (\rho_0, z(\alpha)). \tag{3.32}$$

Moreover, $v(0) = \beta_0\alpha > 0$ and $\lim_{r\to 0} rv'(r) = 0$. Furthermore, if $\alpha \in N$, then

$$v(z(\alpha)) = z(\alpha)u'(z(\alpha)) < 0$$

by Lemma 3.2 and so v has a unique zero ρ_0 in $[0, z(\alpha)]$.

Let $\tau \in (0, z(\alpha))$ be the first zero of w, which exists by Lemma 3.3. We remember that w satisfies

$$w'' + \frac{1}{r}w' + \left[pV(r)u^{p-1} - \lambda w\right]w = 0 \quad \text{for } r \in (0, z(\alpha)), \tag{3.33}$$

with initial data $w(0) = 1$ and $\lim_{r\to 0} r\,w'(r) = 0$. Because of this, by # sturm, we know that v oscillates faster than w.

Then $\rho_0 \in (0,\tau)$ and thus

$$v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v < 0 \quad \text{and } v < 0 \quad \text{on } (\tau, z(\alpha)). \tag{3.34}$$

Since $w(\tau) = 0$ and v has no zero larger than ρ_0, by Sturm we have that w has no further zero in $(\tau, z(\alpha))$ and we can set $r_0 = \tau$. If $z(\alpha) < \infty$, then w has no zero on $(\tau, z(\alpha)]$ and we have $w(z(\alpha)) < 0$.

However, if $z(\alpha) = \infty$, we can apply Lemma 6 of [2, p. 249]. The disconjugacy interal (d,∞) is such that

$$d < \rho_0 < r_0.$$

*T*he disconjugacy interval of a differential equation is the largest left-neighbourhood (c,b) of the right most b on which there exists a solution to the differential equation without zeroes. From Sturmian theory, no non-trivial solution can have more than one zero in (c,b). On the other hand, unless $c = a$, with a the left most, any solution of the differential equation with a zero before c must have another zero in (c,b).

Consider the same setting: equations (3.8) and (3.9) satisfying the comparison condition (3.10). In addition, we assume that $U \neq V$ in *any* neighborhood of b. If there exists a solution V of (3.9) with a largest zero at the point ρ, then the disconjugacy interval of (3.8) is a strict superset of (ρ, b).

One way to interpret the above is to remember that V oscillates faster than U and note that any solution U with a zero in (ρ, b)

Indeed, if we had $\rho_0 \leqslant d$, we could find a solution $\tilde{w}$ linearly independent of w such that $\tilde{w}(\tilde{r}) = 0$ for some $\tilde{r} > d \geqslant \rho_0$. But then $w + \tilde{w}$ is a solution of # w ivp with two zeroes in (ρ_0,∞) and thus, v should have another zero in that interval. By this contradiction, we have $w(r_0) = 0$ with $r_0 \in (d,\infty)$ and Lemma 6 of [2, p. 249] implies that

$$\lim_{r\to\infty} w(r) = -\infty.$$

□

Lemma 3.7. *Let $\alpha \in G$. There exists $\epsilon > 0$ such that $(\alpha, \alpha+\epsilon) \subset N$.*

Proof. We remember that $w(r)$ has a unique zero r_0. Furthermore, by Lemma 6 of [2, p. 249] we know $r_0 \in (d,\infty)$, where (d,∞) is the disconjugacy interval of (3.2). We refer to # for the definition of disconjugacy interval. We can choose r_1 and r_2 around r_0

$$d < r_1 < r_0 < r_2,$$

such that by (**??**), there exists $\epsilon > 0$ such that for all $\widetilde{\alpha} \in (\alpha, \alpha + \epsilon)$, we have

$$\widetilde{u}(r_1) > u(r_1) \quad \text{and} \quad \widetilde{u}(r_2) < u(r_2).$$

Hence the graphs of u and $\widetilde{u}$ intersect in some $r_3 \in (r_1, r_2)$, where r_3 depends on the choice of $\widetilde{\alpha}$. Next, we will show that there exists $\widetilde{r} \in (r_3, \infty)$ such that $\widetilde{u}(\widetilde{r}) = 0$. LATER Hence, $\widetilde{\alpha} \in N$.

3

By contradiction, we suppose that $\widetilde{u}(r) > 0$ for all $r > r_3$.

We will show that $\widetilde{u}(r) < u(r)$ for all $r > r_3$.

By contradiction, we suppose that $\widetilde{u}(r_4) = u(r_4)$ for some $r_4 > r_3$ and $u - \widetilde{u} > 0$ on (r_3, r_4).

The function $z := u - \widetilde{u}$ satisfies

$$z'' + \frac{1}{r} z' + \left[V(r) \frac{u^p - \widetilde{u}^p}{u - \widetilde{u}} - \lambda \right] z = 0 \quad \text{on } (r_3, r_4). \tag{3.35}$$

Since $z(r_3) = z(r_4) = 0$ by assumption and

$$\frac{u^p - \widetilde{u}^p}{u - \widetilde{u}} < p u^{p-1} \tag{3.36}$$

on (r_3, r_4), we can apply Sturm's theorem # ref. w oscillates faster than z Let y be any solution of w ivp linearly independent of w. ... There is no positive solution of w ivp on (d, ∞), contradicting ...

Hence $z(r) > 0$ for all $r > r_3$.

Sturm comparing fractions

Integrating sturm comparands

Hence $z(r) \equiv u(r) - \widetilde{u}(r) \to \infty$ as $r \to \infty$. This is impossible, since $0 < \widetilde{u}(r) < u(r)$ on (r_3, ∞) and $u(r) \to 0$ as $r \to \infty$. Therefore, $\widetilde{u}(r)$ must vanish at some point $\widetilde{r} \in (r_3, \infty)$ and the proof is complete.

□

Lemma 3.8. *Let $\alpha^* \in N$. Then $[\alpha^*, \infty) \subset N$ and $z : [\alpha^*, \infty) \to (0, \infty)$ is monotone decreasing.*

Proof. N is open subset of $(0, \infty)$ ...

z is continuous ...

By Lemma 3.6, we have $w(z(\alpha^*)) < 0$. Then, there exists $\epsilon > 0$ such that

$$(\alpha^*, \alpha^* + \epsilon) \subset N \quad \text{and} \quad u(z(\alpha^*), \alpha) < 0 \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon).$$

The intermediate value theorem implies existence of an $r \in (0, z(\alpha^*))$ so that $u(r,\alpha) = 0$. We note that

$$z(\alpha) \leqslant r < z(\alpha^*) \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon)$$

and z is decreasing, since ...

We define

$$\overline{\alpha} := \sup \left\{ \alpha > \alpha^* : \left[\alpha^*, \alpha\right) \subset N \text{ and } z : \left[\alpha^*, \alpha\right) \to (0,\infty) \text{ is decreasing} \right\}.$$

3

By contradiction, we suppose that $\overline{\alpha} < \infty$. Then there exists ... □

Proof of Theorem 3.1. # general proof, referring to lemma's □

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