An Explication of Existence and Uniqueness Results for a Nonlinear Schrödinger equation

An Introduction to the Shooting Method and Sturm Comparison Theorem

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PHYSICS OF NLS

1.1. DERIVE THE WAVE EQUATION FROM MAXWELL

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \qquad (1.1.a) \qquad\qquad \nabla \cdot \vec{\mathcal{D}} = 0, \qquad (1.1.c)$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t}, \qquad (1.1.b) \qquad\qquad \nabla \cdot \vec{\mathcal{B}} = 0. \qquad (1.1.d)$$

The fields are in three-dimensional Cartesian coordinates, for example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents, we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \qquad (1.2.a) \qquad\qquad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}. \qquad (1.2.b)$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [1, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [2]. This reference work also includes an introduction to the necessary vector calculus.

We use vector calculus and Maxwell's laws to rewrite the curl of the curl:

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) \overset{(1.1.a)}{=} \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathcal{B}}\right) \overset{\substack{(1.1.b)\\(1.2.a)}}{=} -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} \overset{(1.2.b)}{=} -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \text{ and}$$

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \nabla^2 \vec{\mathcal{E}} = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \Delta \vec{\mathcal{E}} \overset{\substack{(1.1.c)\\(1.2.b)}}{=} -\Delta \vec{\mathcal{E}}.$$

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Combining these and using $\mu_0\epsilon_0 = 1/c^2$, we arrive at the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{1.3}$$

1.2. VALIDITY OF PLANE WAVE SOLUTIONS

Stuyding the left and right hand sides of equation (1.3), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{l=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_l^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.4}$$

where k_0 is the wavenumber and ω_0 the frequency. These are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any z-value and the electric field is non-zero in the x and y directions. The wavefronts are spaced by the wavelength λ and the wavenumber k_0 is the reciprocal of the wavelength.

This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane.

We substitute (1.4) in equation (1.3). Note that only Δ_z will be non-zero:

$$\Delta\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation (1.5):

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.5}$$

For a general direction in (x, y, z)-coordinates, define the wavevector

$$\vec{k} = (k_x, k_y, k_z),$$

where $\left|\vec{k}^2\right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$. This satisfies equation (1.3) when $\vec{k}\perp\vec{\mathcal{E}}$ and

$$\mathcal{E}_j = E_c e^{i(\vec{k}\cdot\vec{r}-\omega_0 t)}. \tag{1.6}$$

1.3. DERIVATION OF THE HELMHOLTZ EQUATION

We consider time-harmonic solutions to the scalar wave equation (1.3) of the form

$$\mathcal{E}_j(x,y,z,t) = e^{i\omega_0 t}E(x,y,z) + \text{c.c}, \tag{1.7}$$

which are continuous wave beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, we refer to [3].

Substituting (1.7) in equation (1.3) and taking the derivatives leads to the expression

$$\begin{aligned} \Delta\left(e^{-i\omega_0 t}E\right) &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(e^{-i\omega_0 t}E\right) \\ e^{-i\omega_0 t}\Delta E &= \frac{1}{c^2}(-i\omega_0)^2 E e^{-i\omega_0 t}, \end{aligned}$$

where we can divide by $e^{-i\omega_0 t} \neq 0$ and use the dispersion relation (1.5) to arrive at the scalar linear Helmholtz equation for E

$$\Delta E(x,y,z) + k_0^2 E = 0. \tag{1.8}$$

As an example, equation (1.8) is solved by the general-direction plane waves (1.6), where

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

1.4. DERIVATION OF THE LINEAR SCHRÖDINGER EQUATION

REVISE: We write the incoming field $E_0^{\text{inc}}(x,y)$ as a sum of plane waves. Then the electric field $E(x,y,z)$ for non-zero z-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and it is essential to Fourier optics. We have

$$\begin{aligned} E_0^{\text{inc}}(x,y) &= \frac{1}{2\pi}\int_D E_c(k_x,k_y)\, e^{i(k_x x + k_y y)}\mathrm{d}k_x \mathrm{d}k_y, \quad \text{such that} \\ E(x,y,z) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} E_c(k_x,k_y)\, e^{i\left(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z\right)}\mathrm{d}k_x \mathrm{d}k_y, \end{aligned}$$

where D denotes the (circular) laser input beam domain. For laser beams oriented in the z-direction, most of the plane wave modes are nearly parallel to the z-axis, which implies $k_z \approx k_0$. We define $k_\perp^2 = k_x^2 + k_y^2$, such that $k_0^2 = k_\perp^2 + k_z^2$. It is equivalent to $k_0 \approx k_z$ to say that $k_\perp \ll k_z$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{1.9}$$

where $\psi(x,y,z)$ is an envelope (or amplitude) function. The envelope shape may vary over z, in contrast to soliton solutions, see (1.21).

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Substituting (1.9) into the Helmholtz equation (1.8) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp\psi = 0, \tag{1.10}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that for lasers beams oriented in the z-direction, the wavenumber k_z dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope function $\psi(x,y,z)$ will vary slowly in z and curve even more slowly.

Claim: $|\psi_{zz}| \ll k_0|\psi_z|$ and $|\psi_{zz}| \ll \Delta_\perp\psi$.

REVISE: To see this, we first show that $k_0 - k_z \ll 1$. We factor out k_0^2, take the square root on both sides and linearise the square root term of the right hand side:

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right) \implies k_z = k_0\left(1 - \frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right).$$

Finally, we use $k_\perp \ll k_0$ to obtain the intermediate result:

$$k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} \ll 1.$$

For the first statement of the claim, $|\psi_{zz}| \ll k_0|\psi_z|$, it is equivalent to show that the ratio of $|\psi_{zz}|$ over $k_0|\psi_z|$ is much smaller than 1. We calculate the ratio as follows:

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_0(k_0 - k_z)E_c} = \frac{k_0 - k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0} \cdot \frac{1}{k_0} \ll 1.$$

For the other statement of the claim, we calculate:

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0 - k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} \ll \frac{1}{4}\frac{k_\perp^2}{k_0^2} \ll 1.$$

Using the approxations in equation (1.10) yields the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp\psi = 0. \tag{1.11}$$

1.5. POLARISATION FIELD

Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E},0,0),\ \vec{\mathcal{P}} = (\mathcal{P},0,0),\ \vec{\mathcal{D}} = (\mathcal{D},0,0),$$

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Furthermore, we assume that $\mathcal{E}$ is the continuous wave electric field from (1.7). We write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.12}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials. The constants c_i are actually a function of the frequency ω_0. We rewrite $c_i = \epsilon_0\chi^{(\mathrm{i})}(\omega_0)$, where $\chi^{(\mathrm{i})}$ is the i-th order susceptibility. Then equation (1.12) reads:

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.13}$$

First we consider linear polarisation:

$$\mathcal{P}_{\text{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}.$$

Then the induction electric field $\mathcal{D}$ is given by:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\text{lin}} = \epsilon_0\mathcal{E} + \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E} = \epsilon_0\mathcal{E}\left(1 + \chi^{(1)}(\omega_0)\right) = \epsilon_0 n_0^2(\omega_0)\mathcal{E},$$

where $n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0)$ is the linear index of refraction (or refractive index) of the medium.

With this updated induction electric field $\mathcal{D} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}$, we can update the scalar wave equation and Helmholtz equation. Only the dispersion relation is affected by considering linear polarisation:

$$k_0^2 = \frac{\omega_0^2}{c^2} n_0^2(\omega_0). \tag{1.14}$$

We now consider the nonlinear polarisation field $\mathcal{P}_{\text{nl}}$ as the difference between the true polarisation and the linear approximation:

$$\mathcal{P} = \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$\begin{aligned}-\mathcal{P}_{\text{nl}} &= \epsilon_0\chi^{(2)}(-\mathcal{E})^2 + \epsilon_0\chi^{(3)}(-\mathcal{E})^3 + \epsilon_0\chi^{(4)}(-\mathcal{E})^4 + \epsilon_0\chi^{(5)}(-\mathcal{E})^5 + \mathcal{O}\left(\mathcal{P}^6\right)\\ -\mathcal{P}_{\text{nl}} &= \epsilon_0\chi^{(2)}\mathcal{E}^2 - \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 - \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right),\end{aligned}$$

where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\text{nl}}$ and we have only the odd terms:

$$\mathcal{P}_{\text{nl}} = \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^7\right) \tag{1.15}$$

The leading-order term is called the Kerr nonlinearity:

$$\mathcal{P}_{\text{nl}} \approx \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}^3. \tag{1.16}$$

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1.6. IMPLICATIONS OF NONLINEAR POLARISATION

Substituting the continuous wave electric field (1.7) into equation (1.16) yields

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}^3 = 3\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \chi^{(3)}(\omega_0)E^3 e^{3i\omega_0 t} + \text{c.c.},$$

where the second term has a frequency of $3\omega_0$ (third harmonic). This has almost no contribution due to the phase-mismatch with the first harmonic. Hence, we approximate

$$\mathcal{P}_{\mathrm{nl}} \approx 3\epsilon_0 \chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \text{c.c.} = 3\epsilon_0\chi^{(3)}(\omega_0)\mathcal{E}.$$

Then we simplify $\mathcal{P}_{\mathrm{nl}}$ by defining

$$n_2 := \frac{3\chi^{(3)}}{4\epsilon_0 n_0},$$

so that we obtain the simplified expression

$$\mathcal{P}_{\mathrm{nl}} = 4\epsilon_0 n_0 n_2 \,|E|^2\,\mathcal{E}.$$

This allows us to write the induction electric field $\mathcal{D}$ as,

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}} = \epsilon_0 n^2 \mathcal{E},$$

where

$$n^2 = n_0^2\left(1 + \frac{4n_2}{n_0}\,|E|^2\right) = n_0^2 + 3\chi^{(3)}(\omega_0)\frac{1}{\epsilon_0}|E|^2.$$

For water, $n_2 \sim 10^{-22}$ which justifies neglecting nonlinear effects. With lasers, the nonlinear effect becomes more relevant, but is still weak. For a typical continuous wave laser with $|E| \sim 10^9$, we still have a weak nonlinearity, as $n_2|E| \sim 10^{-4} \ll n_0 \approx 1.33$.

We update equation (1.8) to the scalar nonlinear Helmholtz equation (NLH):

$$\Delta E(x,y,z) + k^2 E = 0, \quad \text{where } k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right). \tag{1.17}$$

We write $E(x,y,z)$ as the product of the z-propagation and an envelope function $\psi(x,y,z)$:

$$E = e^{ik_0 z}\psi$$

and substitute in (1.17) to obtain:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp\psi + 4k_0^2\frac{n_2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.18}$$

Just as in section 1.4, we apply the paraxial approximation, since for laser beams oriented in the z-direction, we have $\left|\psi_{zz}\right| \ll k_0\left|\psi_z\right|, \left|\psi_{zz}\right| \ll \Delta_\perp\psi$. We finally obtain the nonlinear Schrödinger equation (NLS):

$$2ik_0\psi_z(z,\overline{x}) + \Delta_\perp\psi + k_0^2\frac{4n^2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.19}$$

1.7. SOLITON SOLUTIONS

The NLS equation (1.19) can be written as a dimensionless equation. Starting from equation (1.18), we apply the rescaling of coordinates $(x, y, z) \to (\tilde{x}, \tilde{y}, \tilde{z})$ defined by:

$$\tilde{x} = \frac{x}{r_0} \quad \tilde{y} = \frac{y}{r_0} \quad \tilde{z} = \frac{z}{2L_{\text{diff}}},$$

where r_0 is the input beam width and L_{diff} is the diffraction length. We refer to chapter 2 of [1] for more information on the geometrical optics of lasers. There, we also find that $L_{\text{diff}} = k_0 \cdot r_0^2$. To rescale $\tilde{\psi}$, we define:

$$\tilde{\psi} = \frac{\psi}{E_c}, \quad \text{where } E_c := \max_{x,y} \left|\psi_0(x, y)\right|.$$

Through the rescaling we obtain the dimensionless NLH for $\tilde{\psi}$:

$$\frac{f^2}{4}\tilde{\psi}_{\tilde{z}\tilde{z}}(\tilde{z}, \tilde{x}, \tilde{y}) + i\tilde{\psi}_{\tilde{z}} + \Delta_\perp \tilde{\psi} + \nu \left|\tilde{\psi}\right|^2 \tilde{\psi} = 0,$$

that depends on a nonparaxiality parameter f and a nonlinearity parameter ν:

$$f = \frac{1}{r_0 k_0} = \frac{r_0}{L_{\text{diff}}}, \quad \nu = r_0^2 k_0^2 \frac{4n_2}{n_0} E_c^2.$$

Here the approximation of paraxiality is valid for small $f \ll 1$ and this leads to the dimensionless NLS equation (1.20), where the tildes have been dropped for brevity.

$$i\psi_z(z, x, y) + \Delta_\perp \psi + \nu \left|\psi\right|^2 \psi = 0. \tag{1.20}$$

Radial solitary-wave solutions to (1.20) were considered in [4] with ψ of the form:

$$\psi_\omega^{\text{solitary}}(r, z) = e^{i\omega z} R_\omega(r), \tag{1.21}$$

where ω is a real number and R_ω is the real solution of

$$-\omega R_\omega + \Delta_\perp R_\omega(r) + R_\omega^3 = 0.$$

This can be solved in general by, for example,

$$R_\omega(r) = \sqrt{\omega} R\left(\sqrt{\omega} r\right).$$

However, taking $\omega = 1$ leads to the simplest soliton equation

$$R''(r) + \frac{1}{r} R' - R + R^3 = 0, \quad 0 < r < \infty, \tag{1.22}$$

subject to initial condition $R'(0) = 0$ and integrability condition $\lim_{r\to\infty} R(r) = 0$. The (numerical) solution is known as the Townes profile, which is positive and monotonically decreasing in r.

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2

Existence of ground state

2.1. Initial value problem and nonlinearity

In this chapter, we will study an existence proof for the initial value problem

$$-u''(r) - \frac{n-1}{r}u'(r) = f(u(r)), \quad \text{on } 0 < r < \infty, \tag{2.1}$$

satisfying initial conditions and an integrability condition

$$\begin{cases} u(0) = \alpha, \\ u'(0) = 0 \\ \lim_{r\to\infty} u(r) = 0. \end{cases} \tag{2.2}$$

The existence proof will be based on [1], which generalises earlier results. One of these is the uniqueness result [2], which was later generalised in [3], which forms the basis for the next chapter.

The proof will be by a shooting method, where we categorise the solutions based on their asymptotic behaviour. Furthermore, solutions to the initial value problem equation (2.1) are also positive radial solutions to the more general problem

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

where $f(u)$ is a given nonlinear function. This partial differential equation is relevant to many areas of mathematical physics.

The solutions $R(r)$ to equation (1.22) are solutions $u(r)$ to (2.1) with $n = 2$ and

$$f(u) = -u + u^3.$$

2.2. Definitions of solution sets

A **ground state solution** is strictly decreasing everywhere and has no finite zeroes. Yet, the solution should vanish in the limit as $r \to \infty$.

We define the set G of ground state initial conditions as

$$G := \left\{ \alpha > 0 \,\middle|\, u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}. \tag{2.4}$$

We consider two alternatives: either (i) the derivative vanishes, or (ii) the solution vanishes. We define the set P of initial conditions with a vanishing derivative as

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.5}$$

We define the set N of initial conditions with a vanishing solution as

$$N := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u(r_0,\alpha) = 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.6}$$

REVISE: These solution sets are disjoint by definition and we write the union of initial conditions as $I = P \dot\cup G \dot\cup N$.

2.3. Assumptions on f

We assume that f is locally Lipschitz continuous from $\mathbb{R}_+ \to \mathbb{R}$ and satisfies $f(0) = 0$. Local Lipschitz continuity is an important condition for the Picard-Lindelöf local existence and uniqueness theorem. Additionally, we assume that hypotheses (H1)–(H5) are satisfied. Firstly,

$$f(\kappa) = 0, \text{ for some } \kappa > 0. \tag{H1}$$

Secondly, defining $F(t)$ as the integral of $f(t)$

$$F(t) := \int_0^t f(s)\,\mathrm{d}s, \tag{2.7}$$

there exists an initial condition $\alpha > 0$ such that $F(\alpha) > 0$. We define

$$\alpha_0 := \inf\{\alpha > 0 \mid F(\alpha) > 0\}. \tag{H2}$$

Thirdly, the right-derivative of $f(s)$ at κ is positive

$$f'(\kappa^+) = \lim_{s\downarrow\kappa} \frac{f(s) - f(\kappa)}{s - \kappa} > 0, \tag{H3}$$

and fourthly, we have

$$f(s) > 0 \quad \text{for } s \in (\kappa, \alpha_0]. \tag{H4}$$

We define

$$\lambda := \inf\{\alpha > \alpha_0 \mid f(\alpha) = 0\}, \tag{2.8}$$

and note that $\alpha_0 < \lambda \leqslant \infty$. In the situation where $\lambda = \infty$, we assume

$$\lim_{s\to\infty} \frac{f(s)}{s^l} = 0, \quad \text{with } l < \frac{n+2}{n-2}. \tag{H5}$$

2.4. MAIN THEOREM

Theorem 2.1. *Let f be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0,\infty)$ such that $f(0) = 0$ and f satisfies hypothese $(H1)-(H5)$. Then there exists a number $\alpha \in (\alpha_0, \lambda)$ such that the solution $u(r,\alpha) \in C^2(\mathbb{R}_+)$ of the initial value problem (2.1) has*

$$\begin{cases} u(r,\alpha) > 0 & \text{for all } r \geqslant 0, \\ u'(r,\alpha) < 0 & \text{for all } r > 0, \end{cases} \tag{2.9}$$

and

$$\lim_{r\to\infty} u(r) = 0.$$

Then $\alpha \in G$. Hence G is non-empty.

If in addition, we assume that f satisfies ?? then there exists constants such that etc...

2.5. INTERVAL OF DEFINITION

Existence of local unique solutions is guaranteed by the Picard-Lindelöf theorem, see for example [4, Theorem. 2.2].

In these circumstances, boundedness of the solution $u(r,\alpha)$ is a sufficient condition for the solution to be defined on the maximal interval $[0,\infty)$. This is also called the *blow-up alternative*. Either (i) for some $r_0 > 0$ we have

$$\left|u(r_0,\alpha)\right| > M, \quad \text{for all } M > 0,$$

and the solution is defined on $[0, r_0)$. Or (ii) for some $M > 0$ we have

$$\left|u(r,\alpha)\right| \leqslant M, \quad \text{for all } r \geqslant 0,$$

and the solution is defined for all $r \geqslant 0$.

In this section, we will derive an upper and a lower bound for $u(r,\alpha)$. Since the solution is initially decreasing, possibly the initial condition α is an upper bound.

Lemma 2.1. $u(r,\alpha) \leqslant u(0,\alpha) = \alpha$ *for $r \geqslant 0$.*

Proof. TODO: Seperate into lemma about the quantity (2.11). In this proof, we write $u(r) = u(r,\alpha)$ for brevity. We start with (2.1) and multiply by $u'(r)$. Then we integrate from 0 to r to obtain

$$-\int_0^r \left[u'(s)u''(s)\right] ds - \int_0^r \left[\frac{n-1}{s}[u'(s)]^2\right] \mathrm{d}s = \int_0^r \left[u'(s) f(u(s))\right] \mathrm{d}s. \tag{2.10}$$

We use the chain rule simplify the first term in (2.10) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}[u'(r)^2] = 2u'(r)u''(r) \overset{(2.2)}{\Longleftrightarrow} \frac{1}{2}[u'(r)]^2 = \int_0^r \left[u'(s)u''(s)\right] \mathrm{d}s.$$

2

Then, we rewrite the right-hand side of (2.10) using the fundamental theorem of calculus

$$\int_0^r \left[u'(s) f(u(s))\right] \mathrm{d}s = \int_0^r \left[\tfrac{\mathrm{d}u}{\mathrm{d}s} f(u(s))\right] \mathrm{d}s = \int_{u(0)}^{u(r)} f(u)\, \mathrm{d}u = F(u(r)) - F(u(0)).$$

Finally, using $u(0) = \alpha$, we have rewritten (2.10) as

$$-\frac{1}{2}\left[u'(r)\right]^2 - (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(u(r)) - F(\alpha). \tag{2.11}$$

We suppose by contradiction that

$$u(r_0) > \alpha, \quad \text{for some } r_0 > 0. \tag{2.12}$$

TODO: Be more specific about: in which quantity, with which assumption, do we have which result. By the assumptions on $f(u)$, we have $f(u) > 0$ on (α_0, ∞). As a result, $F(u)$ is increasing on (α_0, ∞). Using assumption (2.12) and $\alpha > \kappa$, we deduce that

$$F(u(r_0)) > F(\alpha) \iff F(u(r_0)) - F(\alpha) > 0.$$

This contradicts (2.11), as the left-hand side is clearly non-positive. □

We will show that $u(r, \alpha)$ has a lower bound for $r < \infty$. Let r_0 be the first zero of $u(r, \alpha)$

$$r_0 := \inf\left\{ r > 0 \,\middle|\, u(r, \alpha) = 0 \right\}. \tag{2.13}$$

If $r_0 = \infty$, then we have $u(r, \alpha) > 0$ for all $r > 0$. Suppose to the contrary that $r_0 < \infty$, then we have the following bound on the derivative $u'(r, \alpha)$.

Lemma 2.2. *Suppose that $r_0 < \infty$. Then for $r \geqslant r_0$, we have*

$$u'(r, \alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0, \alpha) \geqslant u'(r_0, \alpha). \tag{2.14}$$

Proof. We consider the sign of $u'(r_0, \alpha)$. Firstly, if $u'(r_0, \alpha) = 0$ then u and u' vanish simultaneously in r_0. Then from (2.1) we have

$$u'' = 0, \quad \text{with } u(r_0) = u'(r_0) = 0,$$

which is solved by

$$u(r) = c_1 r + c_2,$$

where we must have $c_1 = c_2 = 0$ to satisfy the conditions at r_0, so that $u \equiv 0$. But this contradicts $u(0, \alpha) = \alpha > 0$. Hence, u and u' cannot vanish simultaneously for $\alpha > 0$.

Secondly, if $u'(r_0, \alpha) > 0$ we also reach a contradiction. By (2.1) with $u(r_0, \alpha) = 0$,

$$u''(r_0, \alpha) + \frac{n-1}{r} u'(r_0, \alpha) = -f(0) = 0$$

we see that u'' and u' have opposite signs in r_0. Then either:

$$\begin{cases} \text{(i) } u'' > 0 \quad \text{and} \quad u' < 0 \quad \text{in } r_0, \quad \text{or} \\ \text{(ii) } u'' < 0 \quad \text{and} \quad u' > 0 \quad \text{in } r_0. \end{cases} \tag{2.15}$$

2

The latter case implies that $u(r,\alpha) < 0$ in a left neighborhood of r_0, which contradicts $u(r,\alpha) > 0$ on $[0, r_0)$. Thus, we have $u'(r_0,\alpha) < 0$.

In the following, we extend $f(u) = 0$ for $u \leqslant 0$. Then for $u(r,\alpha) \leqslant 0$ the IVP (2.1) reads

$$-u''(r,\alpha) - \frac{n-1}{r} u'(r,\alpha) = 0, \tag{2.16}$$

We solve (2.16) for $u' = u'(r,\alpha)$ and seperate the variables, resulting in

$$\frac{\mathrm{d}u'}{u'} = -\frac{n-1}{r}\mathrm{d}r.$$

We integrate the expression from r_0 to r and evaluate the limits to get

$$\ln u'\big|_{r_0}^{r} = \left[(n-1)\ln r\right]_{r}^{r_0} \iff \ln u'(r) - \ln u'(r_0) = (n-1)\left[\ln r_0 - \ln r\right].$$

Then, we rewrite the expression to arrive at the desired result

$$\frac{u'(r)}{u'(r_0)} = \left(\frac{r_0}{r}\right)^{n-1} \iff u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha) \geqslant u'(r_0,\alpha). \qquad \square$$

In conclusion, the solution $u(r,\alpha)$ is bounded for bounded r. More specifically, in the case of everywhere positive solutions, we have

$$0 < u(r,\alpha) \leqslant \alpha \quad \text{for all } r > 0.$$

TODO: Explicit expression for $u(r,\alpha)$.Alternatively, for solutions with $u(r_0,\alpha) = 0$ for some $r_0 > 0$, by Lemma 2.2 we have

$$u(r,\alpha) \geqslant \int_{r_0}^{r} \left(\frac{r_0}{s}\right)^{n-1} u'(r_0,\alpha)\,\mathrm{d}s > -\infty \quad \text{for } r_0 < r < \infty. \tag{2.17}$$

2.6. ASYMPTOTICS OF POSITIVE DECREASING SOLUTIONS

We will show that everywhere positive decreasing solutions $u(r,\alpha)$ vanish in the limit as $r \to \infty$. The proof will be in TODO: three steps: (i) we show that the nonlinearity $f(u) \to 0$ in the limit as $r \to \infty$, (ii) we show via a translation $v(r) = u(r) - \kappa$ that $l = \kappa$ does not satisfy the IVP, such that $l = 0$.

2

Lemma 2.3. *Let $f:\mathbb{R}^+\to\mathbb{R}$ be a locally Lipschitz continuous function such that $f(0)=0$. Let $u(r,\alpha_1)$ be a solution to initial value problem (2.1) with $\alpha_1\in(0,\infty)$ such that*

$$\begin{cases} u(r,\alpha_1)>0 & \text{for all } r\geqslant 0, \quad \text{and} \\ u'(r,\alpha_1)<0 & \text{for all } r>0. \end{cases} \tag{2.18}$$

Then the number $l:=\lim_{r\to\infty}u(r,\alpha_1)$ satisfies $f(l)=0$.

If additionally $f(u)$ satisfies (H3), then $l=0$.

Proof step 1. By assumption (2.18) on $u(r,\alpha_1)$ and the monotone convergence theorem, we have $0\leqslant l<\alpha_1$. Then $f(l)<f(\alpha_1)$. We consider the limit as $r\to\infty$ of the IVP (2.1)

$$\lim_{r\to\infty}\left[-u''(r,\alpha_1)-\frac{n-1}{r}u'(r,\alpha_1)\right]=f(l)<\infty. \tag{2.19}$$

We restate equation (2.11)

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2+(n-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}=F(\alpha_1)-F(u(r,\alpha_1)),$$

and consider the limit as $r\to\infty$

$$\lim_{r\to\infty}\left[\frac{1}{2}\left[u'(r,\alpha_1)\right]^2+(n-1)\int_0^r\left[u'(s,\alpha_1)\right]^2\frac{\mathrm{d}s}{s}\right]=F(\alpha_1)-\lim_{r\to\infty}F(u(r,\alpha_1)),$$

where we use $\lim_{r\to\infty}F(u(r,\alpha_1))=F(l)<\infty$ to write

$$\lim_{r\to\infty}\frac{1}{2}[u'(r,\alpha_1)]^2+(n-1)\int_0^\infty u'(s,\alpha_1)^2\frac{\mathrm{d}s}{s}=F(\alpha_1)-F(l). \tag{2.20}$$

Note that

$$F(\alpha_1)-F(l)<\infty\implies\int_0^\infty u'(s,\alpha_1)^2\frac{\mathrm{d}s}{s}<\infty,$$

such that $u'(r,\alpha_1)^2/r$ converges as $r\to\infty$ by the Levi monotone convergence theorem. Then $u'(r,\alpha_1)^2$ converges, because... and since $u'(r,\alpha_1)<0$ everywhere, we deduce that $u'(r,\alpha_1)$ converges. However, since $0\leqslant u(r,\alpha_1)\leqslant\alpha_1$, we have

$$\lim_{r\to\infty}u'(r,\alpha_1)=0. \tag{2.21}$$

Now, we return to equation (2.19) and use $\lim_{r\to\infty}u'(r,\alpha_1)=0$ to obtain

$$-\lim_{r\to\infty}\left[u''(r,\alpha_1)\right]=f(l).$$

We have (2.21) and hence, we have

$$\lim_{r\to\infty}u''(r,\alpha_1)=0.$$

The desired result follows: $f(l)=0$. □

Proof step 2. The nonlinearity $f(u)$ has more than one zero. Both $f(0) = 0$ and $f(\kappa) = 0$. We will show that only $l = 0$ satisfies the assumptions.

Suppose to the contrary that $l = \kappa$. We will use the substitution

$$v(r) = r^{(1/2)(n-1)}\left[u(r,\alpha_1) - \kappa\right]$$

in equation (2.1) to obtain a differential equation in $v(r)$. In the remainder of the proof, we will abbreviate $u(r,\alpha_1) = u(r)$. We note that $v(r) > 0$ by definition, as the assumption is that $u(r) > \kappa$ for $r > 0$ and $u(r) \downarrow \kappa$.

We proceed to calculate the first derivative

$$v'(r) = \frac{1}{2}(n-1)r^{(n-3)/2}\left[u(r) - \kappa\right] + r^{(n-1)/2}u'(r),$$

and the second derivative, where we gather the terms by $u(r)$, $u'(r)$ and $u''(r)$ as

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-5)/2}\left[u(r) - \kappa\right] + (n-1)r^{(n-3)/2}u'(r) + r^{(n-1)/2}u''(r). \tag{2.22}$$

We multiply the IVP (2.1) by $r^{(n-1)/2}$ to obtain

$$-r^{(n-1)/2}u''(r) - (n-1)r^{(n-1)/2}r^{-1}u'(r) = f(u(r))r^{(n-1)/2}. \tag{2.23}$$

We can use this to simplify (2.22)

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-1)/2}r^{-2}\left[u(r) - \kappa\right] - f(u(r))r^{(n-1)/2}.$$

Now we factor out $v(r) = r^{(n-1)/2}\left[u(r) - \kappa\right]$ to obtain

$$v''(r) = r^{(n-1)/2}\left[u(r) - \kappa\right]\left\{\frac{1}{4}(n-1)(n-3)r^{-2} - \frac{f(u)}{u(r) - \kappa}\right\}.$$

Lastly, we multiply by -1 to obtain the exact expression from [1] as

$$-v''(r) = \left\{\frac{f(u)}{u(r) - \kappa} - \frac{(n-1)(n-3)}{4r^2}\right\}v. \tag{2.24}$$

We can show that there exist $\omega > 0$ and $R_1 > 0$, such that

$$\frac{f(u)}{u(r) - \kappa} - \frac{(n-1)(n-3)}{4r^2} \geqslant \omega \quad \text{for all } r \geqslant R_1. \tag{2.25}$$

This will be done in proof step 3. We will first show how this leads to $l = 0$ to conclude proof step 2.

We have $v''(r) < 0$ for $r \geqslant R_1$, which implies that

$$v'(r) \downarrow L \geqslant -\infty, \quad \text{as } r \to \infty.$$

Suppose that $L < 0$, then $v(r) \to \infty$ as $r \to \infty$, which contradicts $v(r) > 0$. On the other hand, suppose that $L \geqslant 0$, then $v(r) \geqslant v(R_1) > 0$ for $r \geqslant R_1$. Substituting in (2.25), we have

$$-v''(r) \geqslant \omega v(R_1) > 0,$$

such that $v'(r) \to -\infty$ as $r \to \infty$. This contradicts $L \geqslant 0$. Since $l = \kappa$ is contradictory in any case, we have $l = 0$. □

Proof step 3. The first term (2.25) is non-negative and decreasing by (H3). We will write

$$M(r) := \frac{f(u)}{u(r) - \kappa} > 0, \tag{2.26}$$

and rewrite (2.25) to obtain

$$M(r) \geqslant \frac{(n-1)(n-3)}{4r^2} + \omega. \tag{2.27}$$

We choose $2\omega = \max_{r>0} M(r)$ and choose $R_1 > 0$ such that

$$\frac{(n-1)(n-3)}{4r^2} \leqslant \frac{1}{2} M(r) \quad \text{for } r \geqslant R_1.$$ □

2.7. P IS NON-EMPTY AND OPEN

In this section we will show that P is non-empty and open. TODO: Refer to main theorem.The existence of solutions in F also requires that N is non-empty and open. The proof that N is open is similar to the proof given for P. For the proof that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

Lemma 2.4. *Solution set P as defined in* (2.5)

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0, \alpha) = 0 \textit{ and } u(r, \alpha) > 0 \textit{ for all } r \leqslant r_0 \right\}$$

is non-empty and open.

Proof step 1. We will show that solution set P is non-empty. Let $\alpha \in (\kappa, \alpha_0]$. TODO: Refer to definition of α_0. Considering all initial conditions $(0, \infty)$ and the disjoint subsets P and N, if $\alpha \notin N$ and $\alpha \notin \left(P \dot\cup N\right)$, then $\alpha \in P$.

First, we suppose by contradiction that $\alpha \in N$. By the definition of N in (2.6) there exists a number $r_0 > 0$ such that

$$\begin{cases} u(r_0, \alpha) = 0, \\ u'(r, \alpha) < 0 \quad \text{for } r \leqslant r_0. \end{cases} \tag{2.28}$$

TODO: Refer to gint lemma. We restate equation (2.11) for $r = r_0$

$$\frac{1}{2}\left[u'(r_0,\alpha)\right]^2 + (n-1)\int_0^{r_0} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r_0,\alpha)). \tag{2.29}$$

The left hand side of (2.29) is positive. But $F(u(r_0,\alpha)) = F(0) = 0$, by ... Furthermore, for $\alpha \in (\kappa, \alpha_0]$, we have $F(\alpha) < 0$. Hence $\alpha \notin N$.

Next, we suppose that $\alpha \notin (P \dot\cup N)$. TODO: Precise refs Thus, by the definitions of P and N, we have...

$$\begin{cases} u(r,\alpha) > 0 & \text{for } r \geqslant 0, \text{ and} \\ u'(r,\alpha) < 0 & \text{for } r > 0. \end{cases} \tag{2.30}$$

This implies TODO: Be specific about the lemma statement and proof steps I refer to.

$$u(r,\alpha) \downarrow l \geqslant 0 \quad \text{as } r \uparrow \infty, \tag{2.31}$$

and by Lemma (2.3), we know that

$$\lim_{r\to\infty} u'(r,\alpha) = 0, \quad \text{and } l = 0.$$

Thus,

$$(n-1)\int_0^{\infty} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) < 0.$$

By this contradiction we have $\alpha \notin \left(P \dot\cup N\right)$. Hence $(\kappa, \alpha_0] \subset P$. □

Proof step 2. We will show that P is open. TODO: Read Teschl or CodLev to be more precise about the circumstances that I assume, and the type of continuity this implies. We know that the solution $u(r,\alpha)$ and its derivative $u'(r,\alpha)$ depend continuously on α. Let $\alpha \in P$. There exists

$$r_0 := \inf\left\{r > 0 \,\middle|\, u'(r,\alpha) = 0 \text{ and } u(r,\alpha) > 0\right\}$$

such that

$$\begin{cases} u(r,\alpha) > 0 & \text{for all } r \in [0, r_0] \\ u'(r,\alpha) < 0 & \text{for all } r \in (0, r_0). \end{cases} \tag{2.32}$$

Evaluating the IVP (2.1) in r_0 yields

$$u''(r_0,\alpha) = -f(u(r_0,\alpha)).$$

Suppose that $u''(r_0,\alpha) = 0$. Then $-f(u(r_0,\alpha)) = 0$. Therefore $u(r_0,\alpha) = \kappa$ or $u(r_0,\alpha) = 0$. These would imply $u(r,\alpha) \equiv \kappa$ or $u(r,\alpha) \equiv 0$, which are both impossible in light of (2.32). TODO: Mention uniqueness of the IVP, be precise about implication

2

Suppose that $u''(r_0,\alpha)\neq 0$. Since

$$\begin{cases} u'(r_0,\alpha)=0 & \text{and} \\ u'(r,\alpha)<0 & \text{for } r<r_0, \end{cases} \tag{2.33}$$

we have $u''(r_0,\alpha)>0$. Then for $r_1>r_0$ near r_0, we have

$$u(r,\alpha)>u(r_0,\alpha) \quad \text{for all } r\in(r_0,r_1].$$

By the continuous dependence on α, we have

$$\forall\epsilon>0\,\exists\,\delta>0:\ \big|u(r,\alpha)-u(r,\beta)\big|<\epsilon \Longleftarrow \big|\alpha-\beta\big|<\delta.$$

We define

$$\epsilon := \frac{1}{2}\big(u(r_1,\alpha)-u(r_0,\alpha)\big).$$

For $\delta_{r_0}>0$ sufficiently small, we have

$$\big|u(r_0,\alpha)-u(r_0,\beta)\big|<\epsilon,$$

and for $\delta_{r_1}>0$ sufficiently small, we have

$$\big|u(r_1,\alpha)-u(r_1,\beta)\big|<\epsilon.$$

That is, for β near α we have

$$\begin{cases} u(r_1,\beta)>u(r_0,\beta) \\ \beta>u(r,\beta)>0 & \text{for all } r\in(0,r_1]. \end{cases} \tag{2.34}$$

Thus $\beta\in P$ and P is open. □

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3

Uniqueness

The uniqueness of ground state solutions to the initial value problem

$$\#uivp$$

was proven in #Genoud. The central theorem is

Theorem 3.1. *Suppose that hypotheses # to # are satisfied. Then there exists* $\alpha_0 > 0$ *such that the solution sets have the following structure*

$$P = (0, \alpha_0), \quad G = \{\alpha_0\}, \quad N = (\alpha_0, \infty).$$

Proof. # general proof, referring to lemma's □

Central to the proof is the Sturm Comparison theorem

Theorem 3.2.

as stated and proven in for example #Kwong. Furthermore, we define

$$\#wdefinition$$

and from (**??**) we obtain the related initial value problem

$$\#wivp$$

In the end, we wish to show that the function $z(\alpha)$ is monotone decreasing in α. Or actually, that $(\alpha, \alpha+\epsilon) \subset N$ and if $\overline{\alpha} \in N$ then $z(\alpha) : [\overline{\alpha}, \infty) \to (0, \infty)$ is monotone decreasing.

Lemma 3.1. *Suppose that* $V(0) := \lim_{r\to 0} V(r)$ *exists and is finite. Then*

$$0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \implies \alpha \in P.$$

Proof. We define the Lyapunov (or energy) function $E(r)$ on $(0, z(\alpha))$ as

$$E(r) := \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1}.$$

We calculate the derivative $E'(r)$ and gather the terms with $u'(r)$

$$E'(r) = \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right]u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1}.$$

This expression can be simplified by writing the IVP (**??**) as

$$-\frac{1}{r}u'(r) = \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right].$$

Thus, we have

$$E'(r) = -\frac{u'(r)^2}{r} + \frac{1}{p+1}V'(r)u(r)^{p+1} \leqslant 0,$$

where $E'(r) \leqslant 0$ holds because $u(r) > 0$ on $(0, z(\alpha))$ and $V' < 0$ by (**??**).

For $\alpha \in G \cup N$, we have $u(z(\alpha)) = 0$, where $z(\alpha) < \infty$ for $\alpha \in N$ and $z(\alpha) = \infty$ for $\alpha \in G$. We evaluate $E(z(\alpha))$ for $\alpha \in N$

$$E(z(\alpha)) = \frac{1}{2}u'(z(\alpha))^2 \geqslant 0. \tag{3.1}$$

For $\alpha \in G$, we have $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$, such that

$$E(r) \to 0 \text{ as } r \to \infty.$$

Since $E(r)$ is non-increasing, we have $E(0) \geqslant 0$ for $\alpha \in G \cup N$. However, solving $E(0) = 0$ for α we have

$$E(0) = \frac{1}{2}u'(0)^2 - \frac{\lambda}{2}u(0)^2 + \frac{1}{p+1}V(0)u(0)^{p+1} = 0.$$

We use $u'(0) = 0$ and $u(0) = \alpha$ to simplify the expression to

$$E(0) = -\frac{\lambda}{2}\alpha^2 + \frac{1}{p+1}V(0)\alpha^{p+1} = 0.$$

So that we obtain

$$\alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)}.$$

□

Lemma 3.2. *Let $\alpha \in G \cup N$, and $u(r) = u(r, \alpha)$. Then $u'(r) < 0$ for all $r \in (0, z(\alpha))$ and $u'(z(\alpha)) < 0$ if $\alpha \in N$.*

Proof. By equation (3.1) , we have $E(z(\alpha)) \geqslant 0$ and $E(r)$ is non-increasing. Hence $E(0) \geqslant 0$ for $r \geqslant 0$.

By Lemma 3.1 we know that for $\alpha \in G \cup N$, we have $E(r) \geqslant 0$ for all $r \in (0, z(\alpha))$.

By the IVP, we have for $r = 0$

$$u''(0) + \lim_{r \downarrow 0} \frac{1}{r} u'(r) - \lambda u(0) + V(0) u(0)^p = 0.$$

If $V(0) < \infty$ then ...

If $V(0) = \infty$ then ...

Suppose by contradiction that there exists $r_0 \in (0, z(\alpha))$ such that $u'(r_0, \alpha) = 0$. Then

$$u''(r_0, \alpha) - \lambda u(r_0, \alpha) + V(r_0) u(r_0)^p = 0. \tag{3.2}$$

Because ... we have $u''(r_0, \alpha) > 0$. Solving for $u''(r_0, \alpha)$ in (3.2)

$$\lambda u(r_0, \alpha) - V(r_0) u(r_0, \alpha)^p = u''(r_0, \alpha) \geqslant 0,$$

where we can solve for $u(r_0, \alpha)$ to find that

$$u(r_0, \alpha) \leqslant \left[\frac{\lambda}{V(0)} \right]^{\frac{1}{p-1}} < \left[\frac{p+1}{2} \frac{\lambda}{V(0)} \right]^{\frac{1}{p-1}}.$$

Now this implies by Lemma 3.1 that $E(r_0) < 0$, which contradicts $E(r) \geqslant 0$ for $\alpha \in G \cup N$. Therefore, we have $u'(r) < 0$ for all $r \in (0, z(\alpha))$. Lastly, if we have $u'(z(\alpha)) = 0$ for $\alpha \in N$, we have $u \equiv 0$. Hence, we have $u'(z(\alpha)) < 0$.

□

Lemma 3.3. *Let $\alpha \in (G \cup N)$. Then $w(r) = w(r, \alpha)$ has at least one zero in $(0, z(\alpha))$.*

Proof. The cases $\alpha \in N$ and $\alpha \in G$ will be considered seperately. First, we suppose $\alpha \in N$. The differential equations for u and w can be written as:

$$(r u'(r))' + r \left[-\lambda u(r) + V(r) u(r)^p \right] = 0 \tag{3.3}$$

$$(r w'(r))' + r \left[-\lambda w(r) + p V(r) u(r)^{p-1} w(r) \right] = 0. \tag{3.4}$$

We multiply equations (3.3) and (3.4) by $w(r)$ and $u(r)$ respectively and subtract to obtain:

$$z(\alpha) w(r) (r u'(r))' - u(r) (r w'(r))' = r \left\{ p V(r) u(r)^p w(r) - V(r) u(r)^p w(r) \right\},$$

3

Now, we integrate from 0 to $z(\alpha)$ to obtain:

$$\int_0^{z(\alpha)} w(r)(ru'(r))' - u(r)(rw'(r))'\,\mathrm{d}r$$
$$= \int_0^{z(\alpha)} r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\}\mathrm{d}r. \quad (3.5)$$

By partial integration, the left hand side reads

$$w(r)ru'(r)\Big|_0^{z(\alpha)} - \int_0^{z(\alpha)} ru'(r)w'(r)\,\mathrm{d}r - u(r)rw'(r)\Big|_0^{z(\alpha)} + \int_0^{z(\alpha)} rw'(r)u'(r)\,\mathrm{d}r,$$

such that (3.5) can be simplified to

$$w(z(\alpha))z(\alpha)u'('za) = \int_0^{z(\alpha)} r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\}\mathrm{d}r.$$

using $u(z(\alpha)) = 0$ and $r = 0$.

simplify the integral rhs to $(p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)\,\mathrm{d}r$ earlier on

analyse the consequence of $w > 0$ to find contradiction

deal with $\alpha \in G$ case referring to Genouds thesis, which needs to be added to the references.

in the end, the contradiction will be that $\frac{u}{w}$ is everywhere positive, yet by the thesis of Genoud we have lim r to inf of the fraction is zero.

□

Lemma 3.4. *Let $\alpha \in G \cup N$. There exist $\beta_0 > 0$ and a unique function $\sigma : [0, \overline{\beta}] \to [0, \infty)$ with the following properties:*

(a) σ is continuous and decreasing, $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$;

(b) for all $\beta > 0$ we have:

$$\phi_\beta(r) < 0 \text{ if } r < \sigma(\beta), \text{ and } \phi_\beta > 0 \text{ if } r > \sigma(\beta).$$

Proof. We fix $\beta > 0$. Then $\phi_\beta(r)$ is fixed. # Argument why ϕ_β changes sign only once. Then $\sigma(\beta)$ is defined by the zero of ϕ_β. Since β was arbitrary, $\sigma(\beta)$ is defined by ϕ_β for all β and $\sigma(\beta)$ is unique. That $\sigma(\beta)$ is continuous and decreasing is shown below.

To analyse the function $\phi_\beta(r)$ we will write the function more conveniently. We know that $r > 0$ and $V(r) > 0$ and $u(r)^p > 0$ and this will allow us to assert the sign of a part of the function $\phi_\beta(r)$. In $\phi_\beta(r)$, factor out a term of $V(r)u(r)^p > 0$ and gather the other terms into $\xi(r)$ as follows:

$$\begin{aligned}
\phi_\beta(r) &= \left[\beta(p-1)-2\right] V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r) \\
&= V(r)u(r)^p \left[\beta(p-1)-2-\frac{rV'(r)u(r)^p}{V(r)u(r)^p}+\frac{2\lambda u(r)}{V(r)u(r)^p}\right] \\
&= V(r)u(r)^p \left[\beta(p-1)-2-r\frac{V'(r)}{V(r)}+\frac{2\lambda}{V(r)u(r)^{p-1}}\right] \\
&= V(r)u(r)^p \left[\beta(p-1)-2-\xi(r)\right] \\
\text{where } \xi(r) &= r\frac{V'(r)}{V(r)} - \frac{2\lambda}{V(r)u(r)^{p-1}}.
\end{aligned}$$

Hence the sign of $\phi(r)$ depends on β and $\xi(r)$ by the term in brackets. Remember that p is just a constant of the initial value problem resembling a dimension #. Note that $\phi_\beta(r) > 0 \iff [\ldots] > 0$. This allows us to solve for β:

$$\beta(p-1)-2-\xi(r) > 0 \iff \beta > \frac{2+\xi(r)}{p-1} := \Xi(r).$$

That is, $\phi_\beta(r)$ is positive whenever β is greater than this new function $\Xi(r)$. Studying this function $\Xi(r)$ will be our next step.

Before studying the sign of and bounds on $\Xi(r)$, we can show that the function $\xi(r)$ is non-positive and strictly decreasing on the usual interval.. In other words:

$$\xi(r) \leqslant 0 \text{ and } \xi'(r) < 0 \text{ on } (0, z(\alpha)).$$

This implies that the limit approaches negative infinity:

$$\lim_{r\to z(\alpha)} \xi(r) = -\infty.$$

To see this, note how the first fraction in the expression for $\xi(r)$ is nonincreasing:

$$h(r) = r\frac{V'(r)}{V(r)}$$

as the numerator $V'(r)$ is negative everywhere, the denominator $V(r)$ is positive everywhere and of course $r > 0$. By similar reasoning, the second term of $\xi(r)$ is strictly decreasing. We know that $V(r)$ is decreasing and so is $u(r)$ (hence $u(r)^{p-1}$ as well). Since the numerator $\lambda > 0$ is constant and the denominator is strictly decreasing, the fraction is strictly increasing such that the second term of $\xi(r)$ is strictly decreasing. Thus $\xi(r)$ is strictly decreasing. Since $u(z(\alpha)) = 0$, the limit is $-\infty$. Even if $V(r)$ is non-zero, then still, the fraction blows up by the behaviour of $u(r)$ near $z(\alpha)$.

Next, we can show that $\Xi(0) > 0$ and $\Xi(r)$ is continuous and strictly decreasing. First $\Xi(r)$ is strictly decreasing because $\xi(r)$ is strictly decreasing and $\Xi(r)$ is a monotonous transformation # ?. By similar argument, $\Xi(r)$ is continous. $\Xi(r) = \left[2+\xi(r)\right]/(p-1)$ satisfies $\Xi(0) > 0$. To see that $\Xi(0) > 0$, evaluate $\xi(r)$ in $r = 0$ and compare:

$$\Xi(0) > 0 \iff \xi(0) > -2.$$

3

Evaluating $\xi(0)$ for infinite $V(0)$ relies on #. When $V(0) = \infty$ then the second term in $\xi(0)$ is zero, since $u(0,\alpha)^{p-1} = \alpha^{p-1}$. Then $\xi(0) = h_0$. And by # $h_0 + k \geqslant 0$ so $h_0 > -k$. Since $k \in (0,2)$ the lowest bound for h_0 is $-k > -2$. Alternatively, when $V(0)$ is finite, then $h_0 = 0$ by #. Then

$$\xi(0) = -\frac{2\lambda}{\alpha^{p-1}V(0)}.$$

Now we can solve for α to find the values of α for which $\xi(0) > -2$.

$$\begin{aligned}
-\frac{2\lambda}{\alpha^{p-1}V(0)} &> -2 \\
\frac{\lambda}{\alpha^{p-1}V(0)} &< 1 \\
\frac{\lambda}{V(0)} &< \alpha^{p-1} \\
\alpha &> \left[\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}}
\end{aligned}$$

which is confirmed by the assumption that $\alpha \in G \cup N$ # is it?.

Trailing back our steps, we see that $\xi(0) > -2$ which implies $\Xi(0) > 0$. For any $\beta \in (0, \overline{\beta})$ we have

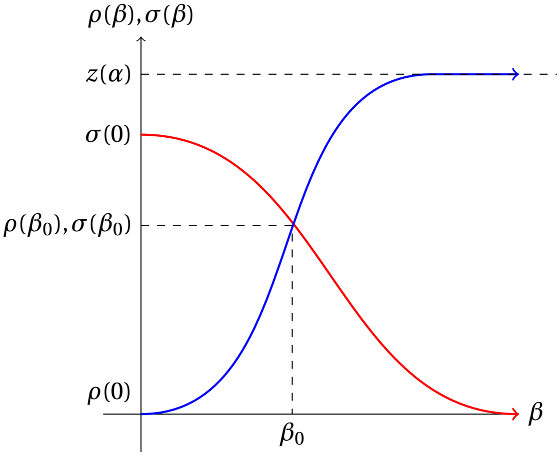
$$\beta < \Xi(r) \iff \beta(p-1) - 2 - \xi(r) < 0 \quad \text{and vice versa.}$$

conclude that this verifies the existence of the unique function $\sigma(\beta)$ with the aforementioned properties. that is $\sigma(\beta)$ is continuous and decreasing and $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$ as well as $\sigma(\beta)$ determing the sign of $\phi_\beta(r)$ for all $\beta > 0$.

□

Lemma 3.5. *Let $\alpha \in G \cup N$. There exists a unique $\beta_0 > 0$ such that $\rho\left(\beta_0\right) = \sigma\left(\beta_0\right)$.*

Proof. This follows immediately from the aforementioned properties of $\rho(\beta)$ and $\sigma(\beta)$. Sketching the two graphs, there is a unique intersection.



3

□

We fix $v = v_{\beta_0}$ and $\rho_0 = \rho\left(\beta_0\right)$. We will apply the Sturm comparison theorem to equations (**??**) and (**??**) in the following lemma.

Lemma 3.6. *For $\alpha \in G \cup N$, $w(r,\alpha)$ has a unique zero $r_0 \in (0, z(\alpha))$. Furthermore, $w(z(\alpha)) < 0$ for $\alpha \in N$ and if $\alpha \in G$, we have*

$$\lim_{r\to\infty} w(r) = -\infty.$$

Proof. v diff eq neg and $v > 0$ on $(0, \rho_0)$

v diff eq pos and $v < 0$ on $(\rho_0, z(\alpha))$.

Moreover, $v(0) = \beta_0\alpha > 0$ and $\lim_{r\to 0} r v'(r) = 0$. Furthermore, if $\alpha \in N$, ten

$$v(z(\alpha)) = z(\alpha)u'(z(\alpha)) < 0$$

by Lemma # and so v has a unique zero ρ_0 in $[0, z(\alpha)]$.

Let $\tau \in (0, z(\alpha))$ be the first zero of w, which exists by Lemma #. Since w satisfies

w ivp with initial data $w(0) = 1$ and $\lim_{r\to 0} r w'(r) = 0$

by Sturm comparison theorem, see Theorem #, we know that v oscillates faster than w. Then $\rho_0 \in (0, \tau)$ and thus

v ivp on $(\tau, z(\alpha))$

Since $w(\tau) = 0$ and v has no zero larger than ρ_0, by Sturm we have that w has no further zero in $(\tau, z(\alpha))$ and we can set $r_0 = \tau$. If $z(\alpha) < \infty$ then w has no zero on $(\tau, z(\alpha)]$ and we have $w(z(\alpha)) < 0$.

However, if $z(\alpha) = \infty$, we can apply Lemma 6 of [**?** , kwong] The disconjugacy interal (d, ∞) is such that

$$d < \rho_0 < r_0.$$

Indeed, if we had $\rho_0 \leqslant d$, we could find a solution $\tilde{w}$ linearly independent of w such that $\tilde{w}(\tilde{r}) = 0$ for some $\tilde{r} > d \geqslant \rho_0$. But then $w + \tilde{w}$ is a solution of # w ivp with two zeroes in (ρ_0, ∞) and thus, v should have another zero in that interval. By this contradiction, we have $w(r_0) = 0$ with $r_0 \in (d, \infty)$ and Lemma 6 of [**?** , kwong]mplies that

$$\lim_{r\to\infty} w(r) = -\infty.$$

□

3

Lemma 3.7. *Let* $\alpha \in G$. *There exists* $\epsilon > 0$ *such that* $(\alpha, \alpha+\epsilon) \subset N$.

Proof. define r_1 and r_2 left and right of r_0

use w to conclude that for $\tilde{\alpha} \in (\alpha, \alpha+\epsilon)$ we have

$$\tilde{u}(r_1) > u(r_1) \quad \text{and} \quad \tilde{u}(r_2) < u(r_2)$$

then the graph of $\tilde{u}$ intersects the graph of u for some $r_3 \in (r_1, r_2)$. we will prove that there exists $\tilde{r} \in (r_3, \infty)$ such that $\tilde{u}(\tilde{r}) = 0$, such that $\tilde{\alpha} \in N$.

By contradiction, we suppose that $\tilde{u}(r) > 0$ for all $r > r_3$.

We will show that $\tilde{u}(r) < u(r)$ for all $r > r_3$.

By contradiction, we suppose that $\tilde{u}(r_4) = u(r_4)$ for some $r_4 > r_3$ and $u - \tilde{u} > 0$ on (r_3, r_4).

The function $z := u - \tilde{u}$ satisfies

$$z'' + \frac{1}{r} z' + \left[V(r) \frac{u^p - \tilde{u}^p}{u - \tilde{u}} - \lambda \right] z = 0 \quad \text{on } (r_3, r_4).$$

We shall compare this differential equation with # w ivp.

Since $z(r_3) = z(r_4)$ by assumption and

$$\frac{u^p - \tilde{u}^p}{u - \tilde{u}} < p u^{p-1}$$

on (r_3, r_4), we can apply Sturm's theorem # ref. Who oscillates faster than who? Let y be any solution of w ivp linearly independent of w. ... There is no positive solution of w ivp on (d, ∞), contradicting ...

Hence $z(r) > 0$ for all $r > r_3$. Then w oscillates faster than z ???

Sturm comparing fractions

Integrating sturm comparands

Hence $z(r) \equiv u(r) - \tilde{u}(r) \to \infty$ as $r \to \infty$. This is impossible, since $0 < \tilde{u}(r) < u(r)$ on (r_3, ∞) and $u(r) \to 0$ as $r \to \infty$. Therefore, $\tilde{u}(r)$ must vanish at some point $\tilde{r} \in (r_3, \infty)$ and the proof is complete.

□

Lemma 3.8. *Let $\alpha^* \in N$. Then $[\alpha^*, \infty) \subset N$ and $z : [\alpha^*, \infty) \to (0, \infty)$ is monotone decreasing.*

Proof. N is open subset of $(0, \infty)$ ...

z is continuous ...

By Lemma 3.6, we have $w(z(\alpha^*)) < 0$. Then, there exists $\epsilon > 0$ such that

$$(\alpha^*, \alpha^* + \epsilon) \subset N \quad \text{and} \quad u(z(\alpha^*), \alpha) < 0 \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon).$$

3

The intermediate value theorem implies existence of an $r \in (0, z(\alpha^*))$ so that $u(r, \alpha) = 0$. We note that

$$z(\alpha) \leqslant r < z(\alpha^*) \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon)$$

and z is decreasing, since ...

We define

$$\overline{\alpha} := \sup\left\{\alpha > \alpha^* : [\alpha^*, \alpha) \subset N \text{ and } z : [\alpha^*, \alpha) \to (0, \infty) \text{ is decreasing}\right\}.$$

By contradiction, we suppose that $\overline{\alpha} < \infty$. Then there exists ...

$\square$

3.1. INITIAL VALUE PROBLEM AND SOLUTION SETS

We are interested in the uniqueness of ground state solutions to the soliton equation (**??**). The existence of solutions was treated in chapter **??**. We study the proof by Genoud in [**?**]. Several lemmata will lead to the conclusion that the set of initial conditions has the following structure

$$P = (0, \alpha_0) \quad G = \{\alpha_0\} \quad N = (\alpha_0, \infty)$$

Broadly speaking, we wish to study zeroes of the solution $u(r, \alpha)$ as a function of the initial condition α. We define

$$z(\alpha).$$

By studying the related differential equations in $w(r)$ and $v(r)$, and applying the Sturm comparison theorem, we will show that $z(\alpha)$ is monotone decreasing. Furthermore G is open and contiguous.

Through the chapter, we assume $n = 2$. There are more assumptions on $f(u)$

$$f(u) = \lambda V(r) u(r) - u(r)^p$$

Compare with $f(u) = u - u^3$.

3.2. Lemma 1

By # solution set P is non-empty. That is, solutions to # that are positive for all r exist. What initial conditions yield such solutions? By studying the sign of a Lyapunov function $E(r)$ for our problem, we can show that $(0,\alpha^*) \subset P$ for certain α^*. # Expand on Lyapunov. By the initial value problem, we can show that $E(r)$ is non-increasing for all r. This will separate the initial conditions into $\alpha \in P$ and $\alpha \in G \cup N$. After this lemma, we will study the implications of the latter case.

3

The Lyapunov (or energy) function can yield information about the solutions. Let the Lyapunov function be defined as

$$E(r) := \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1}$$

on $(0, z(\alpha))$. Evaluating the Lyapunov function in $r=0$, we see that the sign in $r=0$ depends on the initial condition. Solving for α in

$$E(0) = 0 \iff -\frac{\lambda}{2}\alpha^2 + \frac{1}{p+1}V(0)\alpha^{p+1} = 0$$

yields the critical initial condition $\alpha^* = \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)}$.

Additionally, the Lyapunov function $E(r)$ is non-increasing everywhere. Calculating the derivative as

$$E'(r) = -\frac{u'(r)^2}{r} + \frac{1}{p+1}V'(r)u(r)^{p+1}$$

and using the initial value problem, we obtain $E'(r) \leqslant 0$ for $r > 0$. Now, initial conditions below α^* are in solution set P. This relies on #...

Let $\alpha \in G \cup N$. The Lyapunov function $E(r)$ evaluated in $r=0$ will now be greater than or equal to zero. Consider the case for $\alpha \in N$ and $\alpha \in G$ seperately. Evaluate the Lyapunov function $E(r)$ in $r = z(\alpha)$. Clearly, $E(z(\alpha)) > 0$ #... Now, evaluate the Lyapunov function $E(r)$ in the limit for $r \to \infty$. Denote this as # lim r etc.. $E(\infty)$. Clearly $E(\infty) = 0$. # weave in ... To see this, note that both $u(r)$ and $u'(r)$ vanish for $r \to \infty$ and calculate the limit $\lim_{r\to\infty} E(r) = 0$. But $E(r)$ is non-increasing, so $E(0) \geqslant 0$.

In conclusion, the Lyapunov function $E(r)$ defined for $r > 0$ splits the initial conditions around α^*. That is, $\alpha_P < \alpha^* < \alpha_{G\cup N}$. By the derivative $E'(r)$ and the initial value problem #, we know that $E(r)$ is non-increasing for $r > 0$. Lastly, for $\alpha \in P$ we have $E(0) < 0$. And for $\alpha \in G$ we have $E(0) \geqslant 0$ and for $\alpha \in N$, we have $E(0) > 0$.

Note: # Similarly, for $\alpha \in N$, the solution $u(r)$ vanishes in some $z(\alpha)$. Evaluating $\lim_{r\to z(\alpha)} E(r) = \frac{1}{2}u'(z(\alpha))^2$ and noting that by definition of N we have $u'(z(\alpha)) \neq 0$, we find that $E(0) \geqslant 0$.

Note: # This begs the question, is $E(0)$ non-negative for all initial conditions? Solving $E(0) < 0$ yields initial conditions that must belong to P, not to contradict the results that for $\alpha \in G \cup N$ we have $E(0) \geqslant 0$.

This lemma states an interval of initial conditions $0 < \alpha < \dots$ for which the corresponding solution belongs to solution set P. The proof of the lemma involves a Lyapunov (or energy) functions $E(r)$ and its derivative $E'(r)$, which are related to # the IVP. Enough is known about the terms in the derivative $E'(r)$ to conclude that the function $E(r)$ is non-increasing for all r. For initial conditions in solution sets G or N, the defining properties of $u(r)$ and $u'(r)$ yield that $E(0) \geqslant 0$. Furthermore, the function $E(r)$ is non-increasing, so that for any $r \geqslant 0$ we have $E(r) \geqslant 0$. This splits the initial conditions in two sets, as we will see, for $\alpha \in G \cup N$ we have $E(0) \geqslant 0$. Suppose that there are initial conditions which have $E(0) < 0$, the the assumption that these belong to G or N leads to a contradiction since that would imply $E(0) \geqslant 0$. Then these initial conditions must belong to solution set P. The actual upper bound in $0 < \alpha < \dots$ arises from solving $E(0) < 0$.

3

Proof. Define the Lyapunov (or energy) function $E(r)$ on $(0, z(\alpha))$ as:

$$E(r) := \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1},$$

We directly calculate the derivative $E'(r)$ and use the equation (2.1) to simplify.

$$\left[u''(r) - \lambda u(r) + V(r)u(r)^p\right] = -\frac{1}{r}u'(r) \quad \text{from the IVP, and}$$

$$\begin{aligned} E'(r) &= u''(r)u'(r) - \lambda u(r)u'(r) + V(r)u(r)^p u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1} \\ &= \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right]u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1} \\ &= -\frac{u'(r)^2}{r} + \frac{1}{p+1}V'(r)u(r)^{p+1} \leqslant 0 \text{ for } r > 0. \end{aligned}$$

We know that $E'(r)$ is non-positive for $r > 0$, because:

- $V'(r) \leqslant 0$, by hypothesis H2;
- $u(r)^{p+1} \geqslant 0$ because $u(r)$ is positive on $(0, z(\alpha))$ for any initial condition α;
- $r \geqslant 0$;
- $u'(r)^2 \geqslant 0$.

Then $E(r)$ is non-increasing $(0, z(\alpha))$.

Suppose $\alpha \in G$. Then $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$. Then $E(r) = 0$ for $r \to \infty$. In conclusion $E(0) \geqslant 0$. Alternatively, suppose $\alpha \in N$, then $u(z(\alpha)) = 0$. Now evaluate:

$$\begin{aligned} \lim_{r \to z(\alpha)} E(r) &= \lim_{r \to z(\alpha)} \left[\frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1}\right] \\ &= u'(z(\alpha))^2 \geqslant 0. \end{aligned}$$

As $E(r)$ is non-increasing, $E(0) \geqslant 0$. Result: $E(0) \geqslant 0$ and $E(r)$ well-defined on $[0, z(\alpha)]$ for $\alpha \in G \cup N$.

Proof step We solve $E(0) = 0$ to find an upper bound $0 < \alpha < \ldots$ such that $E(0) < 0$):

$$\begin{aligned} E(0) = \frac{1}{2}u'(0)^2 - \frac{\lambda}{2}u(0)^2 + \frac{1}{p+1}V(0)u(0)^{p+1} < 0 \\ \iff -\frac{\lambda}{2}\alpha^2 + \frac{1}{p+1}V(0)\alpha^{p+1} < 0 \\ \iff \alpha^{p-1} < \left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)} \\ \iff \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \end{aligned}$$

In conclusion, we have $\alpha \in P$ whenever $0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)}$. □

3.3. Lemma 2

To separate the solutions corrsponding to initial conditions belonging to G or N, we need to study the zero (or zeroes) of $u(r, \alpha)$. This can not be done directly. However, it can be shown directly that all solutions with initial condition in G or N are decreasing everywhere. That is, $u'(r, \alpha) < 0$ where the derivative is with respect to r.

This will be done again by studying the Lyapunov function $E(r)$. Note that the assumption "$V(0)$ is finite" is no longer present. From the previous lemma and the IVP it follows, in either case, that $u''(0) < 0$ and $u'(0^+) < 0$. If $V(0)$ is finite, then #. If $V(0)$ is infinite, then #.

Suppose now that the derivative of the solution with respect to r vanishes in r_0, that is $u'(r_0) = 0$ for some r_0. Then the equation (2.1) yields an equation in $u(r_0)$ and $u''(r_0)$. By concavity of the graph of the solution $u(r)$, we have $u''(r_0) > 0$. Expand concavity argument #. Then using the Lyapunov function evaluated in r_0, we find $E(r_0) < 0$ which contradicts the assumption that $\alpha \in G \cup N$. Give evaluation #. Hence, $u'(r) < 0$ everywhere.

In conclusion, the solution sets G and N have $u'(r) < 0$ everywhere on $(0, z(\alpha))$ for $\alpha \in G$ and $(0, z(\alpha)]$ for $\alpha \in N$. The lemma shows $u(r)$ strictly decreasing on $(0, z(\alpha))$ for $\alpha \in G \cup N$. The rest of the proof uses this to conclude that $w(r)$ has a unique zero on $(0, z(\alpha))$. Finally, analysis of solution sets N and G leads to the uniqueness result. Write $z(\alpha) = \infty$ when $\alpha \in G$, since $u(r, \alpha) \to 0$ as $r \to \infty$. Let $\alpha \in G \cup N$. By lemma # 4 (1?), $E(r) \geqslant 0$ on $[0, z(\alpha)]$ and non-increasing.

Question: what is an example of infinite $V(0)$ and is it relevant for my thesis? Problem with $E(0) = \ldots V(0)$, might blow up.

Proof. By equation (2.1), we have $E(r) \geqslant 0$ in the limit that r goes to $z(\alpha)$. Since $E(r)$ is non-increasing, $E(r) \geqslant 0$ for all $r \in (0, z(\alpha)]$. Note that $E(0)$ is no longer well-defined as the assumption that $V(0)$ is finite is no longer present.

Lemma 1 and the # equation (2.1) yield $u''(0) < 0$ and $u'(r) < 0$ for $r > 0$ and small. No explicit argument is presented in # Genoud. The argument that $u''(0) < 0$ is not expanded upon here. To see why $u''(0) < 0$ implies $u'(r) < 0$ for $r > 0$ and small, reason by concavity. Since $u''(0) < 0$, the graph is concave down. This implies that $u'(r) < u'(0)$ for $r > 0$ and small.

This result $u'(r) < 0$ for r small, can be extended for larger r, even up to (or including) $z(\alpha)$. Suppose by contradiction that $r_0 = \inf(r < z(\alpha), u'(r) = 0)$ exists. That is, the derivative $u'(r)$ vanishes in r_0 or $u'(r_0) = 0$. This implies that $u''(r)$ was positive on (r, r_0) for some $r > 0$. Both $u''(r_0) = 0$ and $u''(r_0) > 0$ lead to a contradiction. The combination of $u''(r_0)$ and $u'(r_0) = 0$ would imply $u \equiv u(r_0)$, a contradiction. To see how $u''(r_0) > 0$ leads to a contradiction, invoke the # equation (2.1):

$$u''(r_0) = \lambda u(r_0) - V(r_0) u(r_0)^p > 0$$

$$\Longrightarrow u(r_0) < \left[\frac{\lambda}{V(r_0)}\right]^{1/(p-1)} < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(r_0)}\right]^{1/(p-1)}$$

suppose that $u(r_0) \neq 0$

$$\Longleftrightarrow u(r_0) < \left[\frac{\lambda}{V(r_0)}\right] < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(r_0)}\right]^{1/(p-1)}$$

since $\dfrac{p+1}{2} > 1$

$$\Longleftrightarrow u(r_0)^{p-1} < \left(\frac{p+1}{2}\right)\frac{\lambda}{V(r_0)}$$

$$\Longleftrightarrow \frac{1}{p+1} V(r_0) u(r_0)^{p+1} < \frac{\lambda}{2} u(r_0)^2$$

$$\Longleftrightarrow -\frac{\lambda}{2} u(r_0)^2 + \frac{1}{p+1} V(r_0) u(r_0)^{p+1} < 0$$

Then evaluating the Lyapunov function $E(r)$ with the assumption $u'(r_0) = 0$, this yields $E(r_0) < 0$:

$$E(r_0) = -\frac{\lambda}{2} u(r_0)^2 + \frac{1}{p+1} V(r_0) u(r_0)^{p+1} < 0$$

But $E(r_0) < 0$ contradicts $E(r) \geqslant 0$, so $u' < 0$ on $(0, z(\alpha))$.

To see that $u'(z(\alpha)) < 0$ when $\alpha \in N$, suppose by contradiction that $u'(z(\alpha)) = 0$ and remember that by definition $u(z(\alpha)) = 0$. Then $u \equiv 0$, because by the # equation (2.1) we have that $u''(z(\alpha)) = \lambda u(z(\alpha)) - V(z(\alpha)) u(z(\alpha))^p = 0$. # Contradiction: the solution $u =\equiv$ is not in N.

Conclusion: $u'(z(\alpha)) < 0$ for $r \in (0, z(\alpha))$ if $\alpha \in G$ and $r \in (0, z(\alpha)]$ if $\alpha \in N$. $\square$

3.4. LEMMA 3

Even though solutions in sets G and N have been shown to be decreasing everywhere, there is no information about the zeroes. In this lemma, through the Lagrange identity, information about zeroes of the derivative of the solution with respect to the initial condition can be achieved. Define

$$w(r,\alpha) = \frac{\partial}{\partial\alpha}u(r,\alpha)$$

3

and evaluate the Lagrange identity for # and # in 0 and $z(\alpha)$. The conclusion will be that $w(r,\alpha)$ changes sign at least once on $(0, z(\alpha))$. Later, this result will be improved, the zero of $w(r,\alpha)$ is unique and this ultimately leads to the uniqueness of the ground state solution. But for now, enough work to do!

Proof. The Lagrange identity for equation (2.1) and **??** will yield information about the zeroes of $w(r)$. The cases $\alpha \in N$ and $\alpha \in G$ will be considered seperately. First, suppose $\alpha \in N$. The differential equations for u and w can be written as:

$$(ru'(r))' + r\left[-\lambda u(r) + V(r)u(r)^p\right] = 0$$
$$(rw'(r))' + r\left[-\lambda w(r) + pV(r)u(r)^{p-1}w(r)\right] = 0.$$

Multiply these equations by $w(r)$ and $u(r)$ respectively and subtract to obtain:

$$z(\alpha)w(r)(ru'(r))' - u(r)(rw'(r))' = r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\},$$

Now integrate from 0 to $z(\alpha)$ to obtain:

$$\int_0^{z(\alpha)} w(r)(ru'(r))' - u(r)(rw'(r))'dr = \int_0^{z(\alpha)} r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\}dr.$$

Now perform partial integration for the left hand side and remember $u(z(\alpha)) = 0$:

For $\alpha \in N$, note $r > 0,\ V > 0,\ u^p > 0$ are finite almost everywhere. Suppose $w > 0$ on $(0, z(\alpha))$. Then the left hand side and right hand side disagree,

$$z(\alpha)u'(z(\alpha))w(z(\alpha)) < 0, \text{ while } (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)dr > 0.$$

A similar argument holds for $w < 0$. Conclusion: w changes sign at least once on $(0, z(\alpha))$.

For $\alpha \in G$, suppose by contradiction that $w > 0$ on $(0,\infty)$. Integrate # equation (2.1) from 0 to r and rewrite the left hand side using the quotient rule:

$$ru'(r)w(r) - rw'(r)u(r) = r\frac{u'(r)w(r) - w'(r)u(r)}{w(r)^2} = rw(r)^2\left(\frac{u(r)}{w(r)}\right)'$$
$$rw(r)^2\left(\frac{u(r)}{w(r)}\right)' = (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)dr > 0.$$

By similar reasoning as before, the right hand side is still positive. Hence, the left hand side is also positive. This implies that the term quotient of u and w is increasing. Result: $\left(\frac{u}{w}\right)'$ is positive, so $\frac{u}{w}(r)$ is increasing.

Before, the left hand side was contradictory by simple reasons. Unfortunately, this is not the case now. In summary, by Lemma C.1 of [**?**], for $\alpha > 0$ the quantity $\frac{u(r)}{w(r)}$ vanishes in the limit that $r \to \infty$. This contradicts the earlier assumption that $\frac{u}{w}(r)$ is increasing. # Is it really this simple? What about Theoreme 1.4.9. And the subtlety in having asymptotic behaviour of both w and u?

In conclusion, for $\alpha \in G$ the assumption that $w > 0$ is again contradictory. # and for $w < 0$... Hence, w changes sign at least once on $(0, z(\alpha))$ for $\alpha \in G \cup N$. □

3.5. LEMMATA A

The previous result can be improved upon. In fact, w can be shown to have a unique zero. With the Sturm comparison theorem, this can be used to construct more lemmata and finally to show that G contains at most one element, i.e. the ground state solution is unique.

In the lemmata following, many related properties are used. This section is dedicated to properly introducing those functions and their properties. A *shopping list*: we will introduce $\theta(r)$, $\rho(\beta)$, $\nu_\beta(r)$, $\phi_\beta(r)$. Then lemma 4 introduces $\sigma(\beta)$, $\xi(r)$, $\Xi(r)$. All of these functions are related to the original initial value problem in $u(r)$, to the problem in $w(r)$ or to the problem $\nu(r)$ which is a transformation of the problem in $u(r)$ including the β parameter. This β will be fixed in lemma 5 to construct a zero ρ_0 for the $\nu(r)$ problem. In lemma 6 the problem in $\nu(r)$ and $w(r)$ are Sturm compared to show that the zero of $w(r)$ is unique. The uniqueness of the zero of $w(r)$ will translate to the monotone decreasing $z(\alpha)$ which translates to the structure of $I = P \cup G \cup N$ and the required uniqueness result.

First, define the function $\theta(r)$ as:

$$\theta(r) := -r\frac{u'(r)}{u(r)}, \text{ for } r \in [0, z(\alpha).$$

The limit for r to zero is calculated as:

$$\theta(0) = \lim_{r\downarrow 0}\theta(r) = \lim_{r\downarrow 0}\left(-r\frac{u'(r)}{u(r)}\right) = \lim_{r\downarrow 0}\frac{-ru'(r)}{\alpha} = 0.$$

Combined with the fact that $\theta(r)$ is increasing ($\theta'(r) > 0$), we have $\theta(r) > 0$ on $(0, z(\alpha))$, # argument that θ is increasing. Lastly, the function is unbounded as $\lim_{r\to z(\alpha)} \theta(r) = \infty$. To see this, let $\alpha \in N$ then $z(\alpha)$ is the first zero of the solution $u(r, \alpha)$. Also, by # the derivative is negative everywhere. Hence by definition of $\theta(r)$ the limit for $r \to z(\alpha)$ is infinity. A similar argument holds for $\alpha \in G$.

The function $\theta(r)$ is continuous and increasing, so there exists an inverse function. Define $\rho := \theta^{-1}$. Then

$$\rho(\beta) = r \iff \theta(r) = \beta.$$

But what are the properties of this inverse function $\rho(\beta)$? First, we must have $\rho(0) = 0$ and $\lim_{\beta\to\infty} \rho(\beta) = z(\alpha)$. Note how $\rho(\beta)$ too is continuous and increasing, but with respect to β. Note that $\rho(\beta)$ is positive, that is, $\rho(\beta) > 0$ on $(0,\infty)$. # image of $\theta(r)$ and inverse $\rho(\beta)$.

Now define a transformed version of the #

$$v_\beta(r) := r u'(r) + \beta u(r) = -u(r)\theta(r) - \beta.$$

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While the solution $u(r,\alpha)$ was positive everywhere on $(0, z(\alpha))$ this is not necessarily true for v_β. Note that $u(r) > 0$ so $v_\beta > 0$ if $\theta(r) - \beta < 0$. That is, $0 > \theta(r) - \beta \iff \beta > \theta(r) \iff \rho(\beta) > r$. Let $\beta > 0$, then $v_\beta(r) > 0$ if $r < \rho(\beta)$ and $v_\beta(r) < 0$ if $r > \rho(\beta)$. Similarly, $v_\beta < 0$ if $\beta < \theta(r) \iff \rho(\beta) < r$. All of these properties can also be illustrated #.

To see that $v_\beta(r)$ satisfies the differential equation...

$$v_\beta''(r) + \frac{1}{r} v_\beta(r)' - \lambda v_\beta + pV(r)u(r)^{p-1} v_\beta = \phi_\beta(r)$$

where the function $\phi_\beta(r)$ on the right hand side is defined as

$$\phi_\beta(r) := \left[\beta(p-1) - 2\right] V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r).$$

derivation must be somewhere

In lemma 4 the above w # argument will be used to show that the sign of $\phi_\beta(r)$ is related to a continuous decreasing function $\sigma(\beta)(r)$ and this function intersects $\rho(\beta)$ in a unique β_0 (lemma 5). The specific v_{β_0} and $\rho_0 = \rho(\beta_0)$ are used in lemma 6 with the Sturm comparison theorem to show that w has a unique zero on $(0, z(\alpha))$.

3.6. LEMMA 4

Proof. Why is this σ unique? Maybe because inverse of uniquely defined $\rho(\beta)$? There are infinitely many functions that agree with $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$. Clearly, the other property of σ uniquely defines the function. Let β be arbitrary. This makes $\phi_\beta(r)$ fixed. # Argument why ϕ_β changes sign only once. Then $\sigma(\beta)$ is defined by the zero of ϕ_β. Since β was arbitrary, $\sigma(\beta)$ is defined by ϕ_β for all β and $\sigma(\beta)$ is unique. That $\sigma(\beta)$ is continuous and decreasing is shown below.

To analyse the function $\phi_\beta(r)$ we will write the function more conveniently. We know that $r > 0$ and $V(r) > 0$ and $u(r)^p > 0$ and this will allow us to assert the sign of a part of the function $\phi_\beta(r)$. In $\phi_\beta(r)$, factor out a term of $V(r)u(r)^p > 0$ and gather the other terms into $\xi(r)$ as follows:

$$\begin{aligned}
\phi_\beta(r) &= \left[\beta(p-1) - 2\right] V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r) \\
&= V(r)u(r)^p \left[\beta(p-1) - 2 - r\frac{V'(r)}{V(r)} + \frac{2\lambda}{V(r)u(r)^{p-1}}\right] \\
&= V(r)u(r)^p \left[\beta(p-1) - 2 - \xi(r)\right]
\end{aligned}$$

where $\xi(r) = r\dfrac{V'(r)}{V(r)} - \dfrac{2\lambda}{V(r)u(r)^{p-1}}$.

Hence the sign of $\phi(r)$ depends on β and $\xi(r)$ by the term in brackets. Remember that p is just a constant of the initial value problem resembling a dimension #. Note that $\phi_\beta(r) > 0 \iff [\ldots] > 0$. This allows us to solve for β:

$$\beta(p-1) - 2 - \xi(r) > 0 \iff \beta > \frac{2+\xi(r)}{p-1} := \Xi(r).$$

That is, $\phi_\beta(r)$ is positive whenever β is greater than this new function $\Xi(r)$. Studying this function $\Xi(r)$ will be our next step.

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Before studying the sign of and bounds on $\Xi(r)$, we can show that the function $\xi(r)$ is non-positive and strictly decreasing on the usual interval.. In other words:

$$\xi(r) \leqslant 0 \text{ and } \xi'(r) < 0 \text{ on } (0, z(\alpha)).$$

This implies that the limit approaches negative infinity:

$$\lim_{r \to z(\alpha)} \xi(r) = -\infty.$$

To see this, note how the first fraction in the expression for $\xi(r)$ is nonincreasing:

$$h(r) = r\frac{V'(r)}{V(r)}$$

as the numerator $V'(r)$ is negative everywhere, the denominator $V(r)$ is positive everywhere and of course $r > 0$. By similar reasoning, the second term of $\xi(r)$ is strictly decreasing. We know that $V(r)$ is decreasing and so is $u(r)$ (hence $u(r)^{p-1}$ as well). Since the numerator $\lambda > 0$ is constant and the denominator is strictly decreasing, the fraction is strictly increasing such that the second term of $\xi(r)$ is strictly decreasing. Thus $\xi(r)$ is strictly decreasing. Since $u(z(\alpha)) = 0$, the limit is $-\infty$. Even if $V(r)$ is non-zero, then still, the fraction blows up by the behaviour of $u(r)$ near $z(\alpha)$.

Next, we can show that $\Xi(0) > 0$ and $\Xi(r)$ is continuous and strictly decreasing. First $\Xi(r)$ is strictly decreasing because $\xi(r)$ is strictly decreasing and $\Xi(r)$ is a monotonous transformation # ?. By similar argument, $\Xi(r)$ is continous. $\Xi(r) = \left[2+\xi(r)\right]/(p-1)$ satisfies $\Xi(0) > 0$. To see that $\Xi(0) > 0$, evaluate $\xi(r)$ in $r = 0$ and compare:

$$\Xi(0) > 0 \iff \xi(0) > -2.$$

Evaluating $\xi(0)$ for infinite $V(0)$ relies on #. When $V(0) = \infty$ then the second term in $\xi(0)$ is zero, since $u(0,\alpha)^{p-1} = \alpha^{p-1}$. Then $\xi(0) = h_0$. And by # $h_0 + k \geqslant 0$ so $h_0 > -k$. Since $k \in (0,2)$ the lowest bound for h_0 is $-k > -2$. Alternatively, when $V(0)$ is finite, then $h_0 = 0$ by #. Then

$$\xi(0) = -\frac{2\lambda}{\alpha^{p-1}V(0)}.$$

Now we can solve for α to find the values of α for which $\xi(0) > -2$.

$$
\begin{aligned}
-\frac{2\lambda}{\alpha^{p-1}V(0)} &> -2 \\
\frac{\lambda}{\alpha^{p-1}V(0)} &< 1 \\
\frac{\lambda}{V(0)} &< \alpha^{p-1} \\
\alpha &> \left[\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}}
\end{aligned}
$$

which is confirmed by the assumption that $\alpha \in G \cup N$ # is it?.

Trailing back our steps, we see that $\xi(0) > -2$ which implies $\Xi(0) > 0$. For any $\beta \in (0, \overline{\beta})$ we have

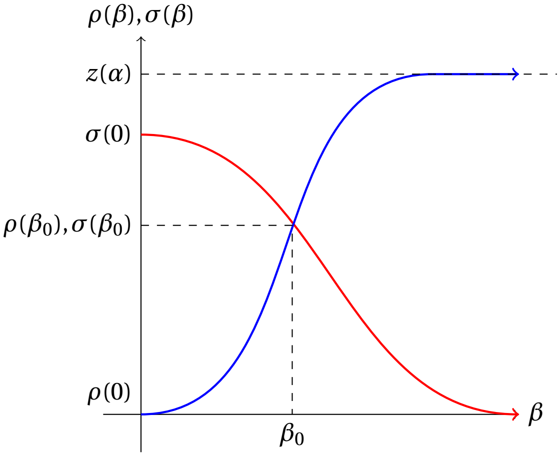
$$\beta < \Xi(r) \iff \beta(p-1) - 2 - \xi(r) < 0 \quad \text{and vice versa.}$$

conclude that this verifies the existence of the unique function $\sigma(\beta)$ with the aforementioned properties. that is $\sigma(\beta)$ is continuous and decreasing and $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$ as well as $\sigma(\beta)$ determing the sign of $\phi_\beta(r)$ for all $\beta > 0$.

□

3.7. LEMMA 5

Lemma 3.9. *Let $\alpha \in G \cup N$. There exists a unique $\beta_0 > 0$ such that $\rho(\beta_0) = \sigma(\beta_0)$. This follows immediately from the aforementioned properties of $\rho(\beta)$ and $\sigma(\beta)$. Sketching the two graphs, there is a unique intersection.*



3.8. LEMMA 6

Lemma 3.10. *For $\alpha \in G \cup N$ the derivative of the solution with respect to the initial condition $w(r,\alpha)$ has a unique zero $r_0 \in (0, z(\alpha))$. Furthermore,*

$$\begin{aligned} w(z(\alpha)) &< 0, \quad && \text{if } \alpha \in N \\ \lim_{r\to\infty} w(r) &= -\infty, \quad && \text{if } \alpha \in G. \end{aligned}$$

Proof. Interpreting the statements in this lemma, we will see that $w(r,\alpha)$ has a **unique** zero r_0 to the left of $z(\alpha)$, the zero of the solution $u(r,\alpha)$. When concerned with a ground state solution, $u(r,\alpha)$ has no finite zero, signified by $z(\alpha) = \infty$. Then still $w(r,\alpha)$ has a finite zero $r_0 \in (0,\infty)$ and $w(r,\alpha) < 0$ for $r > r_0$ and after r_0 the function $w(r,\alpha)$ tends to negative infinity for $r \to \infty$, that is $\lim_{r\to\infty} w(r) = -\infty$.

Remember that the differential equation in $v_\beta(r)$ changes sign when equal to a value of $\phi_\beta(r)$. In turn, this function $\phi_\beta(r)$ changes sign in the r-value $\sigma(\beta)$. By fixing $\beta = \beta_0$ such that $\rho(\beta)$ and $\sigma(\beta)$ intersect in β_0 ($\sigma(\beta_0) = \rho(\beta_0) = \rho_0$) we have achieved a crucial step. Certain zeroes related to $v_\beta(r)$ now coincide because of this choice $\beta = \beta_0$. Remember that ϕ_{β_0} changes sign in $r = \rho_0$. Which implies that the differential equation in $v_{\beta_0}(r)$ changes sign in ρ_0. But by the definition of $v_\beta(r)$, we can see that the zero of $v_\beta(r)$ now coincides with the sign change of the differential equation: ρ_0 is the value in which $v_{\beta_0}(r)$ changes sign, since

$$v_{\beta_0}(\rho_0) = -u(\rho_0)\left\{\theta(\rho_0) - \beta_0\right\} = 0.$$

The last equality follows from $\theta(\rho_0) = \beta_0 \iff \rho(\beta_0) = \rho_0$ such that $\theta(\rho_0) - \beta_0 = 0$.

$$v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v < 0 \quad \text{on} \quad (0,\rho_0) \quad \text{where} \quad v > 0$$
$$v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v > 0 \quad \text{on} \quad (\rho_0,\infty) \quad \text{where} \quad v < 0$$

By # we know that the solution $v(r)$ changes sign in $\rho_0 \in [0, z(\alpha)]$.

3

$$v(z(\alpha))z(\alpha)u'(z(\alpha)) < 0,$$

the solution $v(r)$ changes sign once. # expand Since by # $w(r,\alpha)$ has at least one zero for $\alpha \in G \cup N$, let τ be the first zero of $w(r)$. # setup up the Sturm comparison functions for w and v and explain Since $v(r)$ oscillates faster than $w(r,\alpha)$, there will be a zero to the left of τ. That is, $v(\rho_0) = 0$ with $\rho_0 \in (0,\tau)$.

Also by Sturm comparison, w can have no further zero and τ is the unique zero of w. Suppose that w has a zero with $r_1 > r_0$. Then v must have a zero between r_0 and r_1. Yet, v has a unique zero $\rho_)$, a contradiction.

''By previous lemmata' the sign of v changes in ρ_0. Refer to lemma ... and/or figure ... Also, remember ... v changes sign only once The argument that v changes sign only once is given in lemma ... Remember that ... w has a first zero Suppose $w(r)$ does not have a zero. Then ... Comparing w and v, conclusion: w has a unique zero One can compare v and w with
Sturm's comparison theorem, which is given as ... In this theorem, take ... as ... and ... as ... to see that ... Also if $\alpha \in G$ When $\alpha \in G$ then $u(r)$ has no zero. The solution depends on α and the change of $u(r)$ with respect to α is given by the function $w(r) = \frac{\partial}{\partial\alpha}u(r)$. ''Since there are solutions that have a zero and some that don't', the function $w(r)$ must have a zero. Even more, the zero of w will be to the left of the zero of $u(r)$. CIRKELREDENERING? Then $u(r)$ does not have a zero, the limit of $w(r)$ for r to infinity is negative infinity. To see this, note that for increasing α, one will obtain a solution that *has* a zero. Hence in that r-value, the function $u(r)$ decreases with increasing α, ergo, the function $w(r)$ is increasingly ?? negative for r-values where $u(r)$ is positive. □

3.9. Lemma 7

Lemma 3.11. *Let $\alpha^* \in N$. Then $[\alpha^*,\infty) \subset N$ and $z : [\alpha^*,\infty) \to (0,\infty)$ is monotone decreasing.*

Proof. Let $\hat{\alpha} \in N$. By definition of N, there exists a $\hat{r} > 0$ such that $u(\hat{r};\hat{\alpha}) < 0$. By continuous dependence on the initial data [[Cod. Lev.]], for all α sufficiently close to $\hat{\alpha}$, $u(\hat{r};\alpha) < 0$ as well.

By definition of N and continuous dependence on the initial data, none of the solutions in N can be tangent to the r-axis. Thus $z : N \to (0,\infty)$ is continuous.

Let $\alpha^* \in N$. Then by **??**, $w(z(\alpha^*)) < 0$ and for $\epsilon > 0$ sufficiently small, $(\alpha^*, \alpha^* + \epsilon) \subset N$ and $u(z(\alpha^*),\alpha) < 0$ for all $\alpha \in (\alpha^*, \alpha^* + \epsilon)$.

Remember that w is the derivative of u with respect to the initial condition. Since $w(z(\alpha^*)) < 0$, for initial conditions upward of α^* ($\alpha \in (\alpha^*, \alpha^* + \epsilon)$): $u(\alpha, z(\alpha^*)) < u(\alpha^*, z(\alpha^*)) = 0$. By the intermediate value theorem [[Cod. Lev.]], there exists a $r \in (0, z(\alpha^*))$ such that $u(\alpha, r) = 0$. Then $z(\alpha) \leqslant r \leqslant z(\alpha^*)$ for all $\alpha \in (\alpha^*, \alpha^* + \epsilon)$. Conclusion: z is decreasing on $(\alpha^*, \alpha^* + \epsilon)$.

Domain of z extends to infinity In fact, z is decreasing on $[\alpha^*, \infty)$. That is, let

$$\overline{\alpha} := \sup\{\alpha > \alpha^* \subset N \text{ and } z : [\alpha^*, \alpha) \to (0, \infty) \text{ is decreasing}\}.$$

Then the lemma requires $\overline{\alpha} = \infty$. Suppose by contradiction $\overline{\alpha} < \infty$. Then there exists $z(\overline{\alpha}) := \lim_{\alpha \to \overline{\alpha}} z(\alpha) \in [0, \infty)$. Clearly, $\overline{\alpha} \in N$, since $u(\overline{\alpha}, z(\overline{\alpha}) = 0$ by continuity of z. But then $[\overline{\alpha}, \overline{\alpha} + \epsilon) \in N$ for $\epsilon > 0$ sufficiently small. This contradicts the definition of $\overline{\alpha}$ as the supremum. Then $\overline{\alpha} = \infty$. Conclusion: for $\alpha^* \in N$, $[\alpha^*, \infty) \subset N$ and $z : [\alpha^*, \infty) \to (0, \infty)$ is decreasing.

□

3.10. LEMMA 8

Lemma 3.12. *Let $\alpha \in G$. There exists $\epsilon > 0$ such that $(\alpha, \alpha + \epsilon) \subset N$.*

Proof. By chapter 4, the solution set G is non-empty. Let $\alpha \in G$ and let $u(r; \alpha)$ be the corresponding solution. The function $w(\alpha, r) = \frac{\partial}{\partial \alpha} u(\alpha; r)$ satisfies **??**

By lemma 3.10, the function w is unbounded. By lemma **??**, w has a unique zero $r_0 \in (d, \infty)$. Kwong discusses the disconjugacy interval (d, ∞) of **??** in more detail. Let r_1, r_2 be such that $d < r_1 < r_0 < r_2$ and note that $w(r_1) > 0$ and $w(r_2) < 0$. There exists $\epsilon > 0$ such that, for all $\tilde{\alpha} \in (\alpha, \alpha + \epsilon)$,

$$\tilde{u}(r_1) > u(r_1) \text{ and } \tilde{u}(r_2) < u(r_2),$$

where $\tilde{u} = u(\tilde{\alpha}, r)$. Hence there exists $r_3 \in (r_1, r_2)$ such that the graphs of u and $\tilde{u}$ intersect, i.e. $\tilde{u}(r_3) = u(r_3)$. See also figure **??**.

To conclude $\tilde{\alpha} \in N$ requires existence of a $\tilde{r}$ such that $\tilde{u}(\tilde{r}) = 0$. By contradiction, suppose that $\tilde{u}(r) > 0$ for all $r > r_3$. (Note that $\tilde{u}(r) \geqslant u(r) > 0$ on $r \leqslant r_3$, since $u(r) > 0$ for all r.)

Now, for $r > r_3$ and small, $\tilde{u}(r) < u(r)$. Claim: $u(r) > \tilde{u}(r)$ for all $r > r_3$. Define the difference between u and $\tilde{u}$ as $z := u - \tilde{u}$. Note $u(r) > \tilde{u}(r) \iff z(r) > 0$. By contradiction, suppose there exists $r_4 > r_3$ such that $\tilde{u}(r_4) = u(r_4)$ (equivalent to $z(r_4) = 0$). On (r_3, r_4)

the function z satisfies:

$$z'' + \frac{1}{r}z' + \left[V(r)\frac{u^p - \tilde{u}^p}{u - \tilde{u}}\right] z = 0 \quad \text{because}$$

$$(1): \quad u'' + \frac{1}{r}u' - \lambda u + V u^p = 0$$

$$(2): \quad \tilde{u}'' + \frac{1}{r}\tilde{u}' - \lambda \tilde{u} + V \tilde{u}^p = 0$$

$$(1)-(2): \quad u'' - \tilde{u}'' + \frac{1}{r}u' - \frac{1}{r}\tilde{u}' - \lambda u + \lambda \tilde{u} + V u^p - V \tilde{u}^p = 0$$

$$z'' + \frac{1}{r}z' - \lambda z + \left[V u^p - V \tilde{u}^p\right]\frac{u - \tilde{u}}{u - \tilde{u}} = 0$$

$$z'' + \frac{1}{r}z' + \left[V(r)\frac{u^p - \tilde{u}^p}{u - \tilde{u}} - \lambda\right] z = 0$$

Also by Sturm comparison of z and w, the latter oscillates faster. Let $\tilde{w}$ be a solution of (**??**) such that $\tilde{w}(r_3) = 0$.

By integration of the strong version of Sturm, $z(r) \to \infty$ as $r \to \infty$, but this is impossible as $0 < \tilde{u}(r) < u(r)$ on (r_3, ∞) and $u(r) \to 0$ as $r \to \infty$. Therefore, $\tilde{u}$ vanishes at some point $\tilde{r} \in (r_3, \infty)$ and the proof is complete.

□

REFERENCES