An Explication of Existence and Uniqueness Results for a Nonlinear Schrödinger equation

An Introduction to the Shooting Method and Sturm Comparison Theorem

Bachelor's Thesis

at Delft University of Technology, written by

Jasper Sebastiaan Eenhoorn

Student at Faculty for Applied Physics.

This thesis has been approved by

supervisor: dr. M.V. Gnann
dr. M. Blaauboer

The project was previously supervised by dr. F. Genoud.

Keywords: ...

Printed by: ...

Front & Back: ...

An electronic version of this dissertation is available at
`http://repository.tudelft.nl/`.

Contents

1

Physics of NLS

1.1. Derive the wave equation from Maxwell

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \tag{1.1.a}$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t}, \tag{1.1.b}$$

$$\nabla \cdot \vec{\mathcal{D}} = 0, \tag{1.1.c}$$

$$\nabla \cdot \vec{\mathcal{B}} = 0. \tag{1.1.d}$$

The fields are in three-dimensional Cartesian coordinates, for example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents, we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \tag{1.2.a}$$

$$\vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}. \tag{1.2.b}$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [1, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [2]. This reference work also includes an introduction to the necessary vector calculus.

We use vector calculus and Maxwell's laws to rewrite the curl of the curl:

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) \overset{(1.1.a)}{=} \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathcal{B}}\right) \overset{\substack{(1.1.b)\\(1.2.a)}}{=} -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} \overset{(1.2.b)}{=} -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \text{ and}$$

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \nabla^2 \vec{\mathcal{E}} = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \Delta \vec{\mathcal{E}} \overset{\substack{(1.1.c)\\(1.2.b)}}{=} -\Delta \vec{\mathcal{E}}.$$

1

Combining these and using $\mu_0\epsilon_0 = 1/c^2$, we arrive at the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{1.3}$$

1.2. VALIDITY OF PLANE WAVE SOLUTIONS

Stuyding the left and right hand sides of equation (1.3), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{l=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_l^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.4}$$

where k_0 is the wavenumber and ω_0 the frequency. These are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any z-value and the electric field is non-zero in the x and y directions. The wavefronts are spaced by the wavelength λ and the wavenumber k_0 is the reciprocal of the wavelength.

This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane.

We substitute (1.4) in equation (1.3). Note that only Δ_z will be non-zero:

$$\Delta\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation (1.5):

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.5}$$

For a general direction in (x, y, z)-coordinates, define the wavevector

$$\vec{k} = (k_x, k_y, k_z),$$

where $\left|\vec{k}^2\right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$. This satisfies equation (1.3) when $\vec{k}\perp\vec{\mathcal{E}}$ and

$$\mathcal{E}_j = E_c e^{i(\vec{k}\cdot\vec{r}-\omega_0 t)}. \tag{1.6}$$

1.3. DERIVATION OF THE HELMHOLTZ EQUATION

We consider time-harmonic solutions to the scalar wave equation (1.3) of the form

$$\mathcal{E}_j(x, y, z, t) = e^{i\omega_0 t} E(x, y, z) + \text{c.c}, \tag{1.7}$$

which are continuous wave beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, we refer to [3].

Substituting (1.7) in equation (1.3) and taking the derivatives leads to the expression

$$\Delta\left(e^{-i\omega_0 t} E\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(e^{-i\omega_0 t} E\right)$$
$$e^{-i\omega_0 t}\Delta E = \frac{1}{c^2}(-i\omega_0)^2 E e^{-i\omega_0 t},$$

where we can divide by $e^{-i\omega_0 t} \neq 0$ and use the dispersion relation (1.5) to arrive at the scalar linear Helmholtz equation for E

$$\Delta E(x, y, z) + k_0^2 E = 0. \tag{1.8}$$

As an example, equation (1.8) is solved by the general-direction plane waves (1.6), where

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

1.4. DERIVATION OF THE LINEAR SCHRÖDINGER EQUATION

REVISE: We write the incoming field $E_0^{\text{inc}}(x, y)$ as a sum of plane waves. Then the electric field $E(x, y, z)$ for non-zero z-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and it is essential to Fourier optics. We have

$$E_0^{\text{inc}}(x, y) = \frac{1}{2\pi}\int_D E_c(k_x, k_y)\, e^{i(k_x x + k_y y)} \mathrm{d}k_x \mathrm{d}k_y, \quad \text{such that}$$
$$E(x, y, z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} E_c(k_x, k_y)\, e^{i(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z)} \mathrm{d}k_x \mathrm{d}k_y,$$

where D denotes the (circular) laser input beam domain. For laser beams oriented in the z-direction, most of the plane wave modes are nearly parallel to the z-axis, which implies $k_z \approx k_0$. We define $k_\perp^2 = k_x^2 + k_y^2$, such that $k_0^2 = k_\perp^2 + k_z^2$. It is equivalent to $k_0 \approx k_z$ to say that $k_\perp \ll k_z$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x, y, z) \tag{1.9}$$

where $\psi(x, y, z)$ is an envelope (or amplitude) function. The envelope shape may vary over z, in contrast to soliton solutions, see (2.5).

1

Substituting (1.9) into the Helmholtz equation (1.8) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp\psi = 0, \tag{1.10}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that for lasers beams oriented in the z-direction, the wavenumber k_z dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope function $\psi(x,y,z)$ will vary slowly in z and curve even more slowly.

Claim: $|\psi_{zz}| \ll k_0|\psi_z|$ and $|\psi_{zz}| \ll \Delta_\perp\psi$.

REVISE: To see this, we first show that $k_0 - k_z \ll 1$. We factor out k_0^2, take the square root on both sides and linearise the square root term of the right hand side:

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right) \Longrightarrow k_z = k_0\left(1 - \frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right).$$

Finally, we use $k_\perp \ll k_0$ to obtain the intermediate result:

$$k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} \ll 1.$$

For the first statement of the claim, $|\psi_{zz}| \ll k_0|\psi_z|$, it is equivalent to show that the ratio of $|\psi_{zz}|$ over $k_0|\psi_z|$ is much smaller than 1. We calculate the ratio as follows:

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_0(k_0 - k_z)E_c} = \frac{k_0 - k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0}\cdot\frac{1}{k_0} \ll 1.$$

For the other statement of the claim, we calculate:

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0 - k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} \ll \frac{1}{4}\frac{k_\perp^2}{k_0^2} \ll 1.$$

Using the approxations in equation (1.10) yields the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp\psi = 0. \tag{1.11}$$

1.5. POLARISATION FIELD

Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E},0,0),\ \vec{\mathcal{P}} = (\mathcal{P},0,0),\ \vec{\mathcal{D}} = (\mathcal{D},0,0),$$

Furthermore, we assume that $\mathcal{E}$ is the continuous wave electric field from (1.7). We write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.12}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials. The constants c_i are actually a function of the frequency ω_0. We rewrite $c_i = \epsilon_0\chi^{(\mathrm{i})}(\omega_0)$, where $\chi^{(\mathrm{i})}$ is the i-th order susceptibility. Then equation (1.12) reads:

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.13}$$

First we consider linear polarisation:

$$\mathcal{P}_{\mathrm{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}.$$

Then the induction electric field $\mathcal{D}$ is given by:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} = \epsilon_0\mathcal{E} + \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E} = \epsilon_0\mathcal{E}\left(1 + \chi^{(1)}(\omega_0)\right) = \epsilon_0 n_0^2(\omega_0)\mathcal{E},$$

where $n_0^2(\omega_0) \coloneqq 1 + \chi^{(1)}(\omega_0)$ is the linear index of refraction (or refractive index) of the medium.

With this updated induction electric field $\mathcal{D} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}$, we can update the scalar wave equation and Helmholtz equation. Only the dispersion relation is affected by considering linear polarisation:

$$k_0^2 = \frac{\omega_0^2}{c^2} n_0^2(\omega_0). \tag{1.14}$$

We now consider the nonlinear polarisation field $\mathcal{P}_{\mathrm{nl}}$ as the difference between the true polarisation and the linear approximation:

$$\mathcal{P} = \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$\begin{aligned} -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}(-\mathcal{E})^2 + \epsilon_0\chi^{(3)}(-\mathcal{E})^3 + \epsilon_0\chi^{(4)}(-\mathcal{E})^4 + \epsilon_0\chi^{(5)}(-\mathcal{E})^5 + \mathcal{O}\left(\mathcal{P}^6\right) \\ -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\mathcal{E}^2 - \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 - \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right), \end{aligned}$$

where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\mathrm{nl}}$ and we have only the odd terms:

$$\mathcal{P}_{\mathrm{nl}} = \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^7\right) \tag{1.15}$$

The leading-order term is called the Kerr nonlinearity:

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}^3. \tag{1.16}$$

1

1.6. IMPLICATIONS OF NONLINEAR POLARISATION

Substituting the continuous wave electric field (1.7) into equation (1.16) yields

$$\mathcal{P}_{\text{nl}} \approx \epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}^3 = 3\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \chi^{(3)}(\omega_0)E^3 e^{3i\omega_0 t} + \text{c.c.},$$

where the second term has a frequency of $3\omega_0$ (third harmonic). This has almost no contribution due to the phase-mismatch with the first harmonic. Hence, we approximate

$$\mathcal{P}_{\text{nl}} \approx 3\epsilon_0\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \text{c.c.} = 3\epsilon_0\chi^{(3)}(\omega_0)\mathcal{E}.$$

Then we simplify $\mathcal{P}_{\text{nl}}$ by defining

$$n_2 := \frac{3\chi^{(3)}}{4\epsilon_0 n_0},$$

so that we obtain the simplified expression

$$\mathcal{P}_{\text{nl}} = 4\epsilon_0 n_0 n_2 \,|E|^2\,\mathcal{E}.$$

This allows us to write the induction electric field $\mathcal{D}$ as,

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}} = \epsilon_0 n^2 \mathcal{E},$$

where

$$n^2 = n_0^2\left(1 + \frac{4n_2}{n_0}\,|E|^2\right) = n_0^2 + 3\chi^{(3)}(\omega_0)\frac{1}{\epsilon_0}|E|^2.$$

For water, $n_2 \sim 10^{-22}$ which justifies neglecting nonlinear effects. With lasers, the nonlinear effect becomes more relevant, but is still weak. For a typical continuous wave laser with $|E| \sim 10^9$, we still have a weak nonlinearity, as $n_2|E| \sim 10^{-4} \ll n_0 \approx 1.33$.

We update equation (1.8) to the scalar nonlinear Helmholtz equation (NLH):

$$\Delta E(x,y,z) + k^2 E = 0, \quad \text{where } k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right). \tag{1.17}$$

We write $E(x,y,z)$ as the product of the z-propagation and an envelope function $\psi(x,y,z)$:

$$E = e^{ik_0 z}\psi$$

and substitute in (1.17) to obtain:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp\psi + 4k_0^2\frac{n_2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.18}$$

Just as in section 1.4, we apply the paraxial approximation, since for laser beams oriented in the z-direction, we have $\left|\psi_{zz}\right| \ll k_0\left|\psi_z\right|, \left|\psi_{zz}\right| \ll \Delta_\perp\psi$. We finally obtain the nonlinear Schrödinger equation (NLS):

$$2ik_0\psi_z(z,\overline{x}) + \Delta_\perp\psi + k_0^2\frac{4n^2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.19}$$

1.7. SOLITON SOLUTIONS

The NLS equation (1.19) can be written as a dimensionless equation. Starting from equation (1.18), we apply the rescaling of coordinates $(x, y, z) \to (\tilde{x}, \tilde{y}, \tilde{z})$ defined by:

$$\tilde{x} = \frac{x}{r_0} \quad \tilde{y} = \frac{y}{r_0} \quad \tilde{z} = \frac{z}{2L_{\text{diff}}},$$

where r_0 is the input beam width and L_{diff} is the diffraction length. We refer to chapter 2 of [1] for more information on the geometrical optics of lasers. There, we also find that $L_{\text{diff}} = k_0 \cdot r_0^2$. To rescale $\tilde{\psi}$, we define:

$$\tilde{\psi} = \frac{\psi}{E_c}, \quad \text{where } E_c := \max_{x,y} \left|\psi_0(x, y)\right|.$$

Through the rescaling we obtain the dimensionless NLH for $\tilde{\psi}$:

$$\frac{f^2}{4}\tilde{\psi}_{\tilde{z}\tilde{z}}(\tilde{z}, \tilde{x}, \tilde{y}) + i\tilde{\psi}_{\tilde{z}} + \Delta_\perp\tilde{\psi} + \nu\left|\tilde{\psi}\right|^2\tilde{\psi} = 0,$$

that depends on a nonparaxiality parameter f and a nonlinearity parameter ν:

$$f = \frac{1}{r_0 k_0} = \frac{r_0}{L_{\text{diff}}}, \quad \nu = r_0^2 k_0^2 \frac{4n_2}{n_0} E_c^2.$$

Here the approximation of paraxiality is valid for small $f \ll 1$ and this leads to the dimensionless NLS equation (1.20), where the tildes have been dropped for brevity.

$$i\psi_z(z, x, y) + \Delta_\perp\psi + \nu\left|\psi\right|^2\psi = 0. \tag{1.20}$$

Radial solitary-wave solutions to (1.20) were considered in [4] with ψ of the form:

$$\psi_\omega^{\text{solitary}}(r, z) = e^{i\omega z} R_\omega(r), \tag{1.21}$$

where ω is a real number and R_ω is the real solution of

$$-\omega R_\omega + \Delta_\perp R_\omega(r) + R_\omega^3 = 0.$$

This can be solved in general by, for example,

$$R_\omega(r) = \sqrt{\omega} R\left(\sqrt{\omega} r\right).$$

However, taking $\omega = 1$ leads to the simplest soliton equation

$$R''(r) + \frac{1}{r}R' - R + R^3 = 0, \quad 0 < r < \infty, \tag{1.22}$$

subject to initial condition $R'(0) = 0$ and integrability condition $\lim_{r\to\infty} R(r) = 0$. The (numerical) solution is known as the Townes profile, which is positive and monotonically decreasing in r.

1

REFERENCES

[1] G. Fibich, *The nonlinear Schrödinger equation*, Applied Mathematical Sciences, Vol. 192 (Springer, Cham, 2015) p. 862, singular solutions and optical collapse.

[2] D. Griffiths, *Introduction to Electrodynamics*, Pearson international edition (Prentice Hall, 1999).

[3] A. Siegman, *Lasers* (University Science Books, 1986).

[4] R. Y. Chiao, E. Garmire, and C. H. Townes, *Self-trapping of optical beams,* Phys. Rev. Lett. **13**, 479 (1964).

2

Existence of ground state

2.1. Initial value problem and nonlinearity

In this chapter, we will study an existence proof for the initial value problem

$$-u''(r) - \frac{n-1}{r}u'(r) = f(u(r)), \quad \text{on } 0 < r < \infty, \tag{2.1}$$

satisfying initial conditions and an integrability condition

$$\begin{cases} u(0) = \alpha, \\ u'(0) = 0 \\ \lim_{r\to\infty} u(r) = 0. \end{cases} \tag{2.2}$$

The existence proof will be based on [1], which generalises earlier results. One of these is the uniqueness result [2], which was later generalised in [3], which forms the basis for the next chapter.

The proof will be by a shooting method, where we categorise the solutions based on their asymptotic behaviour. Furthermore, solutions to the initial value problem equation (2.1) are also positive radial solutions to the more general problem

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

where $f(u)$ is a given nonlinear function. This partial differential equation is relevant to many areas of mathematical physics.

The solutions $R(r)$ to equation (1.22) are solutions $u(r)$ to (2.1) with $n = 2$ and

$$f(u) = -u + u^3.$$

2.2. DEFINITIONS OF SOLUTION SETS

A **ground state solution** is strictly decreasing everywhere and has no finite zeroes. Yet, the solution should vanish in the limit as $r \to \infty$.

We define the set G of ground state initial conditions as

$$G := \left\{ \alpha > 0 \;\middle|\; u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}. \tag{2.4}$$

We consider two alternatives: either (i) the derivative vanishes, or (ii) the solution vanishes. We define the set P of initial conditions with a vanishing derivative as

$$P := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.5}$$

We define the set N of initial conditions with a vanishing solution as

$$N := \left\{ \alpha > 0 \;\middle|\; \exists r_0 : u(r_0,\alpha) = 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.6}$$

We note that the sets P and N are disjoint by definition. Either the derivative vanishes first, or the solution vanishes first.

We will show that the sets P and N are non-empty, and open. Then, there exist initial conditions that belong to neither P nor N. Solutions that belong to neither P nor N are *everywhere positive and decreasing*

$$\begin{cases} u(r,\alpha) > 0 & \text{for } r \geqslant 0, \text{ and} \\ u'(r,\alpha) < 0 & \text{for } r > 0. \end{cases} \tag{2.7}$$

Lastly, we will show that under certain assumptions, such an element belongs to G.

2.3. ASSUMPTIONS ON f

We assume that f is locally Lipschitz continuous from $\mathbb{R}_+ \to \mathbb{R}$ and satisfies $f(0) = 0$. Additionally, we assume that hypotheses (H1)–(H5) are satisfied. Firstly,

$$f(\kappa) = 0, \text{ for some } \kappa > 0. \tag{H1}$$

Secondly, defining $F(t)$ as the integral of $f(t)$

$$F(t) := \int_0^t f(s)\,\mathrm{d}s, \tag{2.8}$$

there exists an initial condition $\alpha > 0$ such that $F(\alpha) > 0$. We define

$$\alpha_0 := \inf\left\{ \alpha > 0 \;\middle|\; F(\alpha) > 0 \right\}. \tag{H2}$$

Thirdly, the right-derivative of $f(s)$ at κ is positive

$$f'(\kappa^+) = \lim_{s\downarrow\kappa} \frac{f(s) - f(\kappa)}{s - \kappa} > 0, \tag{H3}$$

and fourthly, we have

$$f(s) > 0 \quad \text{for } s \in \left(\kappa, \alpha_0\right]. \tag{H4}$$

We define

$$\lambda := \inf\left\{ \alpha > \alpha_0 \,\middle|\, f(\alpha) = 0 \right\}, \tag{2.9}$$

and note that $\alpha_0 < \lambda \leqslant \infty$. In the situation where $\lambda = \infty$, we assume

$$\lim_{s\to\infty} \frac{f(s)}{s^l} = 0, \quad \text{with } l < \frac{n+2}{n-2}. \tag{H5}$$

2.4. MAIN THEOREM

Theorem 2.1. *Let f be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0,\infty)$ such that $f(0) = 0$ and f satisfies hypotheses $(H1)-(H5)$. Then there exists a number $\alpha \in (\alpha_0, \lambda)$ such that the solution $u(r,\alpha) \in C^2(\mathbb{R}_+)$ of the initial value problem*

$$\begin{cases} -u''(r) - \dfrac{n-1}{r} u'(r) = f(u(r)), \ \textit{for } r > 0, \\ u(0) = \alpha, \quad u'(0) = 0 \end{cases} \tag{2.10}$$

is an element of solution set G defined in (3.10)

$$G := \left\{ \alpha > 0 \,\middle|\, u(r,\alpha) > 0 \textit{ and } u'(r,\alpha) < 0 \textit{ for all } r > 0 \textit{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}.$$

Proof. We will show in Lemma 2.1-2.3 that solutions to the differential problem (2.10) are defined for $0 < r < \infty$. Furthermore, by Lemma 2.4 solutions with $\alpha \notin \left(P \dot\cup N\right)$ satisfy

$$\lim_{r\to\infty} u(r,\alpha) = 0.$$

Lastly, we will show that solution sets P and N are non-empty and open. In Lemma 2.5 we show that solution set P is non-empty and open. By similar argument, solution set N is open. For the argument that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

In conclusion, G is non-empty. □

2.5. INTERVAL OF DEFINITION

Existence of local unique solutions is guaranteed by the Picard-Lindelöf theorem, see for example [4, Theorem. 2.2].

In these circumstances, boundedness of the solution $u(r,\alpha)$ is a sufficient condition for the solution to be defined on the maximal interval $[0,\infty)$. This is also called the *blow-up alternative*. Either (i) for some $r_0 > 0$ we have

$$\left|u(r_0,\alpha)\right| > M, \qquad \text{for all } M > 0,$$

and the solution is defined on $[0, r_0)$. Or (ii) for some $M > 0$ we have

$$\left|u(r,\alpha)\right| \leqslant M, \quad \text{for all } r \geqslant 0,$$

and the solution is defined for all $r \geqslant 0$.

Lemma 2.1. *For any initial condition $\alpha > 0$ and $r > 0$, we have the identity*

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)). \tag{2.11}$$

Proof. We multiply the IVP (2.1) by $-u'(r)$. Then we integrate from 0 to r to obtain

$$\int_0^r \left[u'(s)u''(s)\right] ds + \int_0^r \left[\frac{n-1}{s}[u'(s)]^2\right] \mathrm{d}s = -\int_0^r \left[u'(s)f(u(s))\right] \mathrm{d}s. \tag{2.12}$$

We use the chain rule simplify the first term in (2.12) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}[u'(r)^2] = 2u'(r)u''(r) \overset{(2.2)}{\Longleftrightarrow} \frac{1}{2}[u'(r)]^2 = \int_0^r \left[u'(s)u''(s)\right] \mathrm{d}s.$$

Then, we rewrite the right-hand side of (2.12)

$$-\int_0^r \left[u'(s)f(u(s))\right] \mathrm{d}s = \int_r^0 \left[\tfrac{\mathrm{d}u}{\mathrm{d}s} f(u(s))\right] \mathrm{d}s$$

and use the fundamental theorem of calculus

$$\int_{u(r)}^{u(0)} f(u)\,\mathrm{d}u = F(u(0)) - F(u(r)).$$

Finally, using $u(0) = \alpha$, we have rewritten (2.12) as

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)). \qquad \square$$

In this section, we will derive an upper and a lower bound for $u(r,\alpha)$. Since the solution is initially decreasing, possibly the initial condition α is an upper bound.

Lemma 2.2. *Let $\alpha > \kappa$. Then $u(r,\alpha) \leqslant u(0,\alpha) = \alpha \quad$ for $r \geqslant 0$.*

Proof. We suppose by contradiction that

$$\alpha < u(r_0,\alpha) < \lambda, \quad \text{for some } r_0 > 0. \tag{2.13}$$

By (H4) and (2.9), we have F non-decreasing on (κ,λ). Then,

$$F(\kappa) < F(\alpha) < F(u(r_0,\alpha)) < F(\lambda).$$

In particular, we have

$$F(\alpha) - F(u(r_0,\alpha)) < 0.$$

This contradicts Lemma 2.1, as the left-hand side is clearly non-negative. $\square$

We will show that $u(r,\alpha)$ has a lower bound for $r<\infty$. Let r_0 be the first zero of $u(r,\alpha)$

$$r_0 := \inf\{\, r>0 \mid u(r,\alpha)=0\,\}. \tag{2.14}$$

If $r_0=\infty$, then we have $u(r,\alpha)>0$ for all $r>0$. When $r_0<\infty$, we have the following bound on the derivative $u'(r,\alpha)$.

Lemma 2.3. *Suppose that there exists $r_0>0$ such that*

$$\begin{cases} u(r_0,\alpha)=0 \\ u'(r_0,\alpha)<0. \end{cases} \tag{2.15}$$

If we have $f(u)=0$ for $u\leqslant 0$, then for $r\geqslant r_0$ we have

$$u'(r,\alpha)=\left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha)\geqslant u'(r_0,\alpha). \tag{2.16}$$

Proof. For $u(r,\alpha)\leqslant 0$ the IVP (2.1) reads

$$-u''(r,\alpha)-\frac{n-1}{r}u'(r,\alpha)=0, \tag{2.17}$$

We solve (2.17) for $u'=u'(r,\alpha)$ and seperate the variables, resulting in

$$\frac{\mathrm{d}u'}{u'}=-\frac{n-1}{r}\,\mathrm{d}r.$$

We integrate the expression from r_0 to r and evaluate the limits

$$\ln u'\big|_{r_0}^{r}=\left[(n-1)\ln r\right]_r^{r_0} \iff \ln u'(r)-\ln u'(r_0)=(n-1)\left[\ln r_0-\ln r\right].$$

Then, we rewrite the expression to arrive at the desired result

$$\frac{u'(r)}{u'(r_0)}=\left(\frac{r_0}{r}\right)^{n-1} \iff u'(r,\alpha)=\left(\frac{r_0}{r}\right)^{n-1}u'(r_0,\alpha)\geqslant u'(r_0,\alpha). \qquad \square$$

In conclusion, the solution $u(r,\alpha)$ is bounded for bounded r. More specifically, in the case of everywhere positive solutions, we have

$$0<u(r,\alpha)\leqslant\alpha \quad \text{for all } r>0.$$

Alternatively, for solutions with $u(r_0,\alpha)=0$ and $u'(r_0,\alpha)<0$ by Lemma 2.3 we have

$$u(r,\alpha)\geqslant\int_{r_0}^{r}\left(\frac{r_0}{s}\right)^{n-1}u'(r_0,\alpha)\,\mathrm{d}s>-\infty \quad \text{for } r>r_0, \tag{2.18}$$

such that for $n=2$, we have

$$u(r,\alpha)\geqslant r_0\,u'(r_0,\alpha)\left(\ln r-\ln r_0\right) \tag{2.19}$$

and for $n>2$, we have

$$u(r,\alpha)\geqslant\frac{r_0^{n-1}u'(r_0,\alpha)}{2-n}\left(r^{2-n}-r_0^{2-n}\right). \tag{2.20}$$

2.6. ASYMPTOTICS OF POSITIVE DECREASING SOLUTIONS

In this section, we will show that everywhere positive decreasing solutions $u(r,\alpha)$ vanish in the limit as $r \to \infty$.

2

Lemma 2.4. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function such that $f(0) = 0$. Let $u(r,\alpha_1)$ be a solution to initial value problem (2.1) with $\alpha_1 \in (0,\infty)$ such that*

$$\begin{cases} u(r,\alpha_1) > 0 & \text{for all } r \geqslant 0, \quad \text{and} \\ u'(r,\alpha_1) < 0 & \text{for all } r > 0. \end{cases} \tag{2.21}$$

Then the number $l := \lim_{r\to\infty} u(r,\alpha_1)$ satisfies $f(l) = 0$.

If additionally, $f(u)$ satisfies (H3), then $l = 0$.

Proof step 1. By assumption (2.21) on $u(r,\alpha_1)$ and the monotone convergence theorem, we have $0 \leqslant l < \alpha_1$. Then $f(l) < f(\alpha_1)$. We consider the limit as $r \to \infty$ of the IVP (2.1)

$$\lim_{r\to\infty} \left[-u''(r,\alpha_1) - \frac{n-1}{r} u'(r,\alpha_1) \right] = f(l) < \infty. \tag{2.22}$$

We restate equation (2.11)

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2 + (n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha_1) - F(u(r,\alpha_1))$$

and note that the right hand side is finite. We write

$$(n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha_1) - F(u(r,\alpha_1)) - \frac{1}{2}\left[u'(r,\alpha_1)\right]^2$$

and note that the left hand side is increasing and bounded above. Hence,

$$\int_0^\infty u'(s,\alpha_1)^2 \frac{\mathrm{d}s}{s} < \infty.$$

We write

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2 = F(\alpha_1) - F(u(r,\alpha_1)) - (n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s}.$$

Then $\lim_{r\to\infty} u'(r,\alpha_1)^2$ exists. Since $u'(r,\alpha_1) < 0$ and $u(r,\alpha_1)$ is bounded, we have

$$\lim_{r\to\infty} u'(r,\alpha_1) = 0. \tag{2.23}$$

Now, we return to equation (2.22) and use $\lim_{r\to\infty} u'(r,\alpha_1) = 0$ to obtain

$$-\lim_{r\to\infty}\left[u''(r,\alpha_1)\right] = f(l).$$

We have (2.23) and hence, we have

$$\lim_{r\to\infty} u''(r,\alpha_1) = 0.$$

The desired result follows: $f(l) = 0$. □

Proof step 2. The nonlinearity $f(u)$ has more than one zero. Both $f(0) = 0$ and $f(\kappa) = 0$. We will show that under assumption (H3), only $l = 0$ satisfies the IVP (2.1).

Suppose to the contrary that $l = \kappa$. We will use the substitution

$$v(r) = r^{(1/2)(n-1)}\left[u(r,\alpha_1) - \kappa\right] \tag{2.24}$$

in equation (2.1) to obtain a differential equation in $v(r)$. In the remainder of the proof of this lemma, we will abbreviate $u(r,\alpha_1) = u(r)$. We note that $v(r) > 0$ by definition, since we have $u(r) \downarrow \kappa$.

We proceed to calculate the first derivative $v'(r)$

$$v'(r) = \frac{1}{2}(n-1)r^{(n-3)/2}\left[u(r) - \kappa\right] + r^{(n-1)/2}u'(r),$$

and the second derivative $v''(r)$, where we gather the terms by $u(r), u'(r)$ and $u''(r)$

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-5)/2}\left[u(r) - \kappa\right] + (n-1)r^{(n-3)/2}u'(r) + r^{(n-1)/2}u''(r). \tag{2.25}$$

We multiply the IVP (2.1) by $r^{(n-1)/2}$ to obtain

$$-r^{(n-1)/2}u''(r) - (n-1)r^{(n-1)/2}r^{-1}u'(r) = f(u(r))r^{(n-1)/2}. \tag{2.26}$$

We can use this to simplify (2.25) to

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-1)/2}r^{-2}\left[u(r) - \kappa\right] - f(u(r))r^{(n-1)/2}.$$

Now we factor out $v(r) = r^{(n-1)/2}\left[u(r) - \kappa\right]$ to obtain

$$v''(r) = r^{(n-1)/2}\left[u(r) - \kappa\right]\left\{\frac{1}{4}(n-1)(n-3)r^{-2} - \frac{f(u)}{u(r) - \kappa}\right\}.$$

Lastly, we multiply by -1 to obtain the exact expression from [1] as

$$-v''(r) = \left\{\frac{f(u)}{u(r) - \kappa} - \frac{(n-1)(n-3)}{4r^2}\right\}v. \tag{2.27}$$

In proof step 3, we will show that there exist $\omega > 0$ and $R_1 > 0$, such that

$$\frac{f(u)}{u(r)-\kappa} - \frac{(n-1)(n-3)}{4r^2} \geqslant \omega \quad \text{for all } r \geqslant R_1. \tag{2.28}$$

We have $v''(r) < 0$ for $r \geqslant R_1$, which implies by

$$v'(r) = v'(R_1) + \int_{R_1}^{r} v''(s)\,\mathrm{d}s$$

that

$$v'(r) \downarrow L \geqslant -\infty, \quad \text{as } r \to \infty.$$

Suppose that $L < 0$, then $v(r) \to -\infty$ as $r \to \infty$. However, by (2.24) we have $v > 0$.

Then $L \geqslant 0$. This implies $v'(r) \geqslant 0$ for $r \geqslant R_1$. But then $v(r) \geqslant v(R_1) > 0$ for $r \geqslant R_1$. By (2.28) and (2.27), we have

$$-v''(r) \geqslant \omega v(R_1) > 0,$$

such that $v'(r) \to -\infty$ as $r \to \infty$. This contradicts $L \geqslant 0$. Hence, we have $l = 0$. □

Proof step 3. The first term (2.28) is non-negative and decreasing by (H3). We will write

$$M(r) := \frac{f(u)}{u(r)-\kappa} > 0, \tag{2.29}$$

and rewrite (2.28) to obtain

$$M(r) \geqslant \frac{(n-1)(n-3)}{4r^2} + \omega. \tag{2.30}$$

We choose $2\omega = \max_{r>0} M(r)$ and choose $R_1 > 0$ such that

$$\frac{(n-1)(n-3)}{4r^2} \leqslant \frac{1}{2} M(r) \quad \text{for } r \geqslant R_1.$$

□

2.7. P IS NON-EMPTY AND OPEN

In this section we will show that P is non-empty and open. The proof that N is open is similar to the proof given for P. For the proof that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

Lemma 2.5. *Solution set P as defined in* (3.11)

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0, \alpha) = 0 \text{ and } u(r, \alpha) > 0 \text{ for all } r \leqslant r_0 \right\}$$

is non-empty and open.

Proof step 1. We will show that solution set P is non-empty. Let $\alpha \in (\kappa, \alpha_0]$. We refer to (H1) and (H2) for the definitions of κ and α_0.

First, we suppose by contradiction that $\alpha \in N$. By the definition of N in (3.9), there exists a number $r_0 > 0$ such that

$$\begin{cases} u(r_0, \alpha) = 0, \\ u'(r, \alpha) < 0 \quad \text{for } \ r \leqslant r_0. \end{cases} \tag{2.31}$$

We restate equation (2.11) from Lemma 2.1 for $r = r_0$ and use $F(u(r_0, \alpha)) = F(0) = 0$

$$\frac{1}{2}\left[u'(r_0, \alpha)\right]^2 + (n-1)\int_0^{r_0} u'(s, \alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha). \tag{2.32}$$

The left hand side of (2.32) is positive. For $\alpha \in (\kappa, \alpha_0]$, we have $F(\alpha) < 0$. Hence $\alpha \notin N$.

Next, we suppose that $\alpha \notin P$. Thus $\alpha \notin (P \,\dot{\cup}\, N)$. We have the situation of (2.7)

$$\begin{cases} u(r, \alpha) > 0 \quad \text{for } r \geqslant 0, \text{ and} \\ u'(r, \alpha) < 0 \quad \text{for } r > 0, \end{cases}$$

which is the setting of Lemma 2.4. Thus, we have $l = 0$ and by equation (2.23), we have

$$\lim_{r \to \infty} u'(r, \alpha) = 0.$$

Then equation (2.32) evaluates to

$$(n-1)\int_0^{\infty} u'(s, \alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) < 0,$$

but the left hand side is positive. We have $(\kappa, \alpha_0] \subset P$, since α was chosen arbitrarily. □

Proof step 2. We will show that P is open. Let $\alpha \in P$. There exists

$$r_0 := \inf\left\{ r > 0 \,\middle|\, u'(r, \alpha) = 0 \text{ and } u(r, \alpha) > 0 \right\}$$

such that by the definition of P in (3.11)

$$\begin{cases} u(r, \alpha) > 0 \quad \text{for all } r \in [0, r_0], & (2.33a) \\ u'(r, \alpha) < 0 \quad \text{for all } r \in (0, r_0). & (2.33b) \end{cases}$$

Evaluating the IVP (2.1) in r_0 yields

$$u''(r_0, \alpha) = -f(u(r_0, \alpha)).$$

Suppose that $u''(r_0, \alpha) = 0$. Then $-f(u(r_0, \alpha)) = 0$. The zeroes of $f(u)$ are $f(\kappa) = 0$ and $f(0) = 0$. Thus, $u(r_0, \alpha) = \kappa$ by (2.33a).

2

Then, the differential equation (2.1) with

$$\begin{cases} u(r_0,\alpha)=\kappa, \\ u'(r_0,\alpha)=0, \\ u''(r_0,\alpha)=0 \end{cases}$$

is solved by $u\equiv\kappa$, and by uniqueness of solutions this contradicts $u(0,\alpha)=\alpha>\kappa$. Hence $u''(r_0,\alpha)\neq 0$. Since $u'(r,\alpha)<0$ for $r<r_0$ and $u'(r_0,\alpha)=0$, we have

$$u''(r_0,\alpha)>0.$$

Then there exists $r_1>r_0$, such that

$$u(r,\alpha)>u(r_0,\alpha)\quad \text{for all } r\in(r_0,r_1].$$

Since $u(r,\alpha)$ is pointwise continuous in α, we have

$$\forall\epsilon>0\,\exists\,\delta>0:\ |\alpha-\beta|<\delta \implies |u(r,\alpha)-u(r,\beta)|<\epsilon.$$

We define

$$\epsilon:=\frac{1}{2}\left(u(r_1,\alpha)-u(r_0,\alpha)\right).$$

For $\delta_{r_0}>0$ sufficiently small, we have

$$|u(r_0,\alpha)-u(r_0,\beta)|<\epsilon,$$

and for $\delta_{r_1}>0$ sufficiently small, we have

$$|u(r_1,\alpha)-u(r_1,\beta)|<\epsilon.$$

Let $\delta=\min\{\delta_{r_0},\delta_{r_1}\}>0$. Then, for $|\alpha-\beta|<\delta$, we have

$$\begin{cases} u(r_1,\beta)>u(r_0,\beta) \\ \beta>u(r,\beta)>0 \quad \text{for all } r\in(0,r_1]. \end{cases} \tag{2.34}$$

Thus $\beta\in P$ and P is open. □

REFERENCES

[1] H. BERESTYCKI, P. L. LIONS, and L. A. PELETIER, *An ode approach to the existence of positive solutions for semilinear problems in rn,* Indiana University Mathematics Journal **30**, 141 (1981).

[2] C. V. Coffman, *Uniqueness of the ground state solution for* $\Delta u-u+u^3=0$ *and a variational characterization of other solutions,* Arch. Rational Mech. Anal. **46**, 81 (1972).

[3] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbf{R}^n$, Arch. Rational Mech. Anal. **105**, 243 (1989).

[4] G. Teschl, *Ordinary Differential Equations and Dynamical Systems*, Graduate studies in mathematics (American Mathematical Society, 2012).

3

Uniqueness

3.1. Main uniqueness theorem

In the previous chapter, we studied the shooting argument presented in [1] that showed the existence of solutions to initial value problem (2.1). We will study the uniqueness of solutions to a related problem, see (3.2) below. The method is due to Coffman [2] and was later generalised by Kwong [3] for the autonomous problem with arbitrary n.

We note that setting $n = 2$ in (2.1) yields a specific case of (3.2) where

$$\lambda = 0, \ p = 3 \text{ and } V(r) = 1.$$

In relation to the general problem (2.3), we obtain (3.2) by setting $n = 2$ and

$$f(u, r) = \lambda u - V(r)u^p. \tag{3.1}$$

3.1.1. New initial value problem and nonlinearity

In [4], the uniqueness of ground state solutions was shown for

$$\begin{cases} u'' + \dfrac{1}{r}u' - \lambda u + V(r)u^p = 0, \quad r > 0, \quad u = u(r, \alpha) \geqslant 0, \\ u(0, \alpha) = \alpha > 0, \quad \lim\limits_{r \to 0} r u'(r, \alpha) = 0, \end{cases} \tag{3.2}$$

where $\lambda > 0$ and $p > 1$.

Furthermore, we define $w = w(r, \alpha) := \frac{\partial}{\partial \alpha} u(r, \alpha)$, which satisfies the IVP

$$\begin{cases} w'' + \dfrac{1}{r}w' - \lambda w + pV(r)u^{p-1} w = 0, \quad r > 0, \\ w(0, \alpha) = 1, \quad \lim\limits_{r \to 0} r w'(r, \alpha) = 0. \end{cases} \tag{3.3}$$

3

3.1.2. Hypotheses on the non-autonomous term

The nonlinearity (3.1) contains the non-autonomous term $V(r)$. The following hypotheses are made: $V(r)$ is continuously differentiable

$$V \in C^1\left((0,\infty)\right); \tag{3.4}$$

is positive and decreasing

$$V > 0 \quad \text{and} \quad V' \leqslant 0 \quad \text{on } (0,\infty); \tag{3.5}$$

is bounded or at most hyperbolic of quadratic power in r

$$r^k V(r) \in L^\infty(0,\infty) \quad \text{for some } k \in (0,2); \tag{3.6}$$

Lastly, we define

$$h(r) := r\frac{V'(r)}{V(r)} \tag{3.7}$$

and require that $h(r)$ is non-increasing on $(0,\infty)$

$$h'(r) \leqslant 0 \quad \text{on } (0,\infty). \tag{3.8}$$

3.1.3. Definitions of solution sets

In this chapter, the solution sets P, G and N are defined as

$$N := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u(r_0,\alpha) = 0 \right\} \tag{3.9}$$

$$G := \left\{ \alpha > 0 \,\middle|\, u(r,\alpha) > 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\} \tag{3.10}$$

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u(r,\alpha) > 0 \text{ for all } r > 0 \text{ and } \alpha \notin G \right\} \tag{3.11}$$

3.1.4. Main theorem

Theorem 3.1. *Suppose that (3.4)–(3.8) are satisfied. Then there exists $\alpha_0 > 0$ such that the solution sets of (3.2) have the following structure*

$$P = \left(0,\alpha_0\right), \quad G = \left\{\alpha_0\right\}, \quad N = \left(\alpha_0,\infty\right).$$

3.2. Sturm theory and related lemmata

The Sturm comparison stated below is important for the following lemmata is and proven in (for example) [3, p. 246].

Theorem 3.2. *Let U and V be solutions, respectively, of the following equations*

$$U''(x) + f(x)U'(x) + g(x)U(x) = 0, \quad x \in (a,b), \tag{3.12}$$

$$V''(x) + f(x)V'(x) + G(x)V(x) = 0, \quad x \in (a,b), \tag{3.13}$$

where f, g and G are continuous. Let (μ, ν) be a subinterval in which $V(x) \neq 0$ and $U(x) \neq 0$, and in which the comparison condition

$$G(x) \geqslant g(x) \quad \textit{for all } x \in (\mu, \nu) \tag{3.14}$$

holds. Suppose further that

$$\frac{V'(\mu)}{V(\mu)} \leqslant \frac{U'(\mu)}{U(\mu)}. \tag{3.15}$$

Then

$$\frac{V'(x)}{V(x)} \leqslant \frac{U'(x)}{U(x)} \quad \textit{for all } x \in (\mu, \nu). \tag{3.16}$$

Equality in (3.16) can occur only if $U \equiv V$ in $[\mu, x]$. If either μ or ν is a zero of U or V, then the fractions in (3.15) and (3.16) are interpreted as ∞.

3

3.3. BASIC PROPERTIES OF SOLUTIONS

In this section, we derive basic properties of solutions to (3.2). In particular, we will find an initial condition that separates solutions into $\alpha \in P$ and $\alpha \in G \cup N$. Next, we will show that solutions with $\alpha \in G \cup N$ are strictly decreasing everywhere. Lastly, we show that the derivative with respect to the initial condition $w(r)$ has at least one zero in the interval $(0, z(\alpha))$.

Lemma 3.1. *Suppose that $V(0) := \lim_{r \to 0} V(r)$ exists and is finite. Then*

$$0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \implies \alpha \in P.$$

Proof. We define the Lyapunov (or energy) function $E(r)$ as

$$E(r) := \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1} \tag{3.17}$$

for

$$\begin{cases} r \in [0, \infty) & \text{if } \alpha \in P \cup G, \\ r \in [0, z(\alpha)] & \text{if } \alpha \in N. \end{cases}$$

We calculate the derivative $E'(r)$ and gather the terms with $u'(r)$

$$E'(r) = \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right] u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1}.$$

This expression can be simplified by writing the IVP (3.2) as

$$-\frac{1}{r}u'(r) = \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right].$$

Thus, we have

$$E'(r) = -\frac{u'(r)^2}{r} + \frac{1}{p+1} V'(r) u(r)^{p+1} \leqslant 0, \tag{3.18}$$

where $E'(r) \leqslant 0$ holds because $V' \leqslant 0$ by (3.5) and

$$\begin{cases} u(r) > 0 & \text{for } r \in [0,\infty) \quad \text{if } \alpha \in P \cup G, \\ u(r) \geqslant 0 & \text{for } r \in [0, z(\alpha)] \quad \text{if } \alpha \in N. \end{cases} \tag{3.19}$$

For $\alpha \in N$, we have $u(z(\alpha)) = 0$. We evaluate $E(z(\alpha))$

$$E(z(\alpha)) = \frac{1}{2} u'(z(\alpha))^2 \geqslant 0. \tag{3.20}$$

On the other hand, for $\alpha \in G$, we have $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$. Hence, we have $E(r) \to 0$ as $r \to \infty$. Then by (3.18), we have

$$E(r) \geqslant 0 \quad \text{for } \alpha \in G \cup N. \tag{3.21}$$

In particular, we have $E(0) \geqslant 0$ for $\alpha \in G \cup N$. Then the left hand side is non-negative in the inequality (why?)

$$\frac{1}{p+1} V(0) \alpha^{p+1} - \frac{\lambda}{2} \alpha^2 \geqslant -\frac{1}{2} u'(0)^2. \tag{3.22}$$

We solve for α to obtain

$$\alpha \geqslant \left[\left(\frac{p+1}{2} \right) \frac{\lambda}{V(0)} \right]^{\frac{1}{p-1}}. \tag{3.23}$$

These initial conditions must belong to $G \cup N$. Any other initial conditions might belong to $G \cup N$ or to P. □

Lemma 3.2. *Let $\alpha \in G \cup N$, and $u(r) = u(r, \alpha)$. Then $u'(r) < 0$ for all $r \in (0, z(\alpha))$ and $u'(z(\alpha)) < 0$ if $\alpha \in N$.*

Proof. By (3.21) and (**??**), we have $E(r) \geqslant 0$ and $E(0) \geqslant 0$ for $\alpha \in G \cup N$. By the IVP, we have for $r = 0$, where $u(0, \alpha) = \alpha > 0$

$$u''(0) + \lim_{r \downarrow 0} \frac{1}{r} u'(r) - \lambda \alpha + V(0) \alpha^p = 0.$$

If $V(0) < \infty$ then ...

If $V(0) = \infty$ then ...

Then, there exists $\epsilon > 0$ such that $u''(r, \alpha) < 0$ for $r \in (0, \epsilon)$ and

$$u'(\epsilon, \alpha) = \int_0^\epsilon u''(s, \alpha) \, ds < 0.$$

We claim that $u'(r,\alpha) < 0$ for all $r > 0$. Suppose by contradiction that there exists $r_0 \in (0, z(\alpha))$ such that $u'(r_0,\alpha) = 0$. Then by (3.2) we have

$$u''(r_0,\alpha) - \lambda u(r_0,\alpha) + V(r_0)u(r_0,\alpha)^p = 0. \tag{3.24}$$

Since $u'(r,\alpha) < 0$ on $(0, r_0)$, we have $u''(r_0,\alpha) \geqslant 0$. When we separate $u''(r_0,\alpha)$ in (3.24)

$$\lambda u(r_0,\alpha) - V(r_0)u(r_0,\alpha)^p = u''(r_0,\alpha) \geqslant 0,$$

we can solve the left hand side for $u(r_0,\alpha)$ to find that

$$u(r_0,\alpha) \leqslant \left[\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}} < \left[\frac{p+1}{2}\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}}.$$

This implies by (3.17) that $E(r_0) < 0$, which contradicts (3.21). Therefore, $u'(r) < 0$ for all $r \in (0, z(\alpha))$. Lastly, if $u'(z(\alpha)) = 0$ for $\alpha \in N$, then $u \equiv 0$. Hence, $u'(z(\alpha)) < 0$. □

3

Lemma 3.3. *Let $\alpha \in (G \cup N)$. Then $w(r) = w(r,\alpha)$ has at least one zero in $(0, z(\alpha))$.*

Proof. The cases $\alpha \in N$ and $\alpha \in G$ will be considered seperately. First, we suppose $\alpha \in N$. We write $u(r) = u(r,\alpha)$. The differential equations for u and w can be written as:

$$(ru'(r))' + r\left[-\lambda u(r) + V(r)u(r)^p\right] = 0 \tag{3.25}$$

$$(rw'(r))' + r\left[-\lambda w(r) + pV(r)u(r)^{p-1}w(r)\right] = 0. \tag{3.26}$$

We multiply equations (3.25) and (3.26) by $w(r)$ and $u(r)$ respectively and subtract to obtain:

$$w(r)(ru'(r))' - u(r)(rw'(r))' = r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\}. \tag{3.27}$$

Now, we integrate from 0 to $z(\alpha)$ to obtain:

$$\int_0^{z(\alpha)} w(r)(ru'(r))' - u(r)(rw'(r))'\,\mathrm{d}r = \int_0^{z(\alpha)} r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\}\mathrm{d}r. \tag{3.28}$$

By partial integration, the left hand side reads

$$w(r)ru'(r)\Big|_0^{z(\alpha)} - u(r)rw'(r)\Big|_0^{z(\alpha)} + \int_0^{z(\alpha)} rw'(r)u'(r)\,\mathrm{d}r - \int_0^{z(\alpha)} ru'(r)w'(r)\,\mathrm{d}r. \tag{3.29}$$

We use $u(z(\alpha)) = 0$ and $r = 0$, such that (3.29) reads

$$w(z(\alpha))z(\alpha)u'(z(\alpha)). \tag{3.30}$$

Furthermore, the right hand side of (3.28) can be simplified to

$$(p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)\,\mathrm{d}r,$$

such that (3.28) reads

$$w(z(\alpha))z(\alpha)u'(z(\alpha)) = (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)\,\mathrm{d}r. \tag{3.31}$$

3

By Lemma 3.2, we have $u'(z(\alpha)) < 0$. If $w > 0$ holds on $(0, z(\alpha))$, then $w(z(\alpha)) \geqslant 0$. Hence, the left hand side of (3.31) is non-positive. However, the right hand side is positive. By this contradiction, w has at least one zero in $(0, z(\alpha))$.

On the other hand, for $\alpha \in G$, we assume by contradiction that $w > 0$ on $(0,\infty)$. We write

$$r\left(u'(r)w(r) - u(r)w'(r)\right) = r\,w(r)^2\,\frac{u'(r)w(r) - u(r)w'(r)}{w(r)^2} = r\,w(r)^2\left(\frac{u}{w}\right)' \tag{3.32}$$

such that (3.27) integrated from 0 to r reads

$$r\,w(r)^2\left(\frac{u}{w}\right)' = (p-1)\int_0^r sV(s)u(s)^p w(s)ds > 0. \tag{3.33}$$

From (3.33), we deduce that

$$\left(\frac{u}{w}\right)' > 0 \quad \text{for } r \in (0,\infty). \tag{3.34}$$

We use $u(0) = \alpha > 0$ and $w(r) = 1$ from (3.2) and (3.3) to conclude that

$$\frac{u(r)}{w(r)} > 0 \quad \text{on } (0,\infty). \tag{3.35}$$

We refer to [4, Proof of Lemma 2, p. 487] for the remaining details of this proof. The proof requires lemmata from the earlier work [5]. In summary, they show that

$$\lim_{r\to\infty}\frac{u(r)}{w(r)} = 0.$$

This contradicts (3.34). Hence, w has at least one zero on $(0,\infty)$. □

3.4. AUXILIARY FUNCTIONS RELATED TO SOLUTIONS

This section is dedicated to introducing functions related to the initial value problems (3.2) and (3.3). We introduce the functions $\theta(r)$, $\rho(\beta)$, $\nu_\beta(r)$ and $\phi_\beta(r)$. Furthermore, in Lemma 3.4, we will define the function $\sigma(\beta)$ as the inverse of $\Xi(r)$, which is a transformation of $\xi(r)$ that relates the zero of $\phi_\beta(r)$ to β. In Lemma 3.5, we conclude that the zero of $v_\beta(r)$ and $\phi_\beta(r)$ coincide in β_0. From then on, we will fix $\beta = \beta_0$. By this construction $v(r) = v_{\beta_0}(r)$ has a unique zero. This is used in Lemma 3.6 to show that $w(r)$ has a unique zero.

First, we define the function $\theta(r)$ as:

$$\theta(r) := -r\frac{u'(r)}{u(r)}, \text{ for } r \in [0, z(\alpha). \tag{3.36}$$

Next, the limit of $\theta(r)$ as $r \to 0$ is calculated as:

$$\theta(0) = \lim_{r\downarrow 0}\theta(r) = \lim_{r\downarrow 0}\left(-r\frac{u'(r)}{u(r)}\right) = \lim_{r\downarrow 0}\frac{-ru'(r)}{\alpha} = 0.$$

From Lemma 4.3 and Remark 4.1 (ii) in [6], we know that $\theta(r)$ is increasing on $(0, z(\alpha))$. Hence,

$$\theta(r) > 0 \text{ on } (0, z(\alpha)).$$

Lastly, let $\alpha \in N$. A similar argument holds for $\alpha \in G$. Then $z(\alpha)$ is the first zero of the solution $u(r,\alpha)$. Also, by Lemma 3.2 the derivative $u'(r,\alpha)$ is negative everywhere. Hence, by the definition of $\theta(r)$ in (3.36), we have

$$\lim_{r\to z(\alpha)}\theta(r) = \infty.$$

We define the function $v_\beta(r)$ that depends on the solution $u(r)$ as

$$v_\beta(r) := ru'(r) + \beta u(r) = -u(r)\left\{\theta(r) - \beta\right\}. \tag{3.37}$$

We define $\rho := \theta^{-1}$, which exists since $\theta(r)$ is continuous and increasing. Then, by (3.37)

$$\rho(\beta) = r \iff \theta(r) = \beta. \tag{3.38}$$

Furthermore, we have $\rho(0) = 0$ and $\lim_{\beta\to\infty}\rho(\beta) = z(\alpha)$. Since $\rho(\beta)$ is continuous and increasing in β, we have $\rho(\beta) > 0$ on $(0,\infty)$.

There is a correspondence between the value of $\beta > 0$ and the sign of $v_\beta(r)$ by the definition of $v_\beta(r)$ in (3.37) and the relation (3.38)

$$v_\beta(r) > 0 \quad \text{if} \quad r < \rho(\beta) \quad \text{and} \quad v_\beta(r) < 0 \quad \text{if} \quad r > \rho(\beta). \tag{3.39}$$

We claim that $v''_\beta(r)$ satisfies the differential equation

$$v''_\beta(r) + \frac{1}{r}v'_\beta(r) - \lambda v_\beta(r) + pV(r)u(r)^{p-1}v_\beta(r) = \phi_\beta(r), \tag{3.40}$$

where the function $\phi_\beta(r)$ is defined as

$$\phi_\beta(r) := \left[\beta(p-1) - 2\right]V(r)u(r)^p - rV'(r)u(r)^p + 2\lambda u(r). \tag{3.41}$$

To verify this, we calculate the first and second derivatives of $v_\beta(r)$ as defined in (3.37)

$$v'_\beta(r) = ru''(r) + u'(r) + \beta u'(r) = \lambda r u(r) - rV(r)u(r)^p + \beta u'(r)$$
$$v''_\beta(r) = \lambda u(r) + \lambda r u'(r) - V(r)u(r)^p - rV'(r)u(r)^p - rpV(r)u(r)^{p-1}u'(r) + \beta u''(r).$$

We calculate the terms of the left-hand side of (3.40) as

$$\begin{aligned} \frac{1}{r}v'_\beta(r) &= u''(r) + \frac{1}{r}u'(r) + \frac{1}{r}\beta u'(r) \\ -\lambda v_\beta(r) &= -\lambda r u'(r) - \lambda\beta u(r) \\ pV(r)u(r)^{p-1}v_\beta(r) &= rpV(r)u(r)^{p-1}u'(r) + \beta pV(r)u(r)^p. \end{aligned}$$

We add all the terms and write the differential equation as

$$\begin{aligned} &v''_\beta(r) + \frac{1}{r}v'_\beta(r) - \lambda v_\beta(r) + pV(r)u(r)^{p-1}v_\beta(r) \\ &\quad = \lambda u(r) + \underline{\lambda r u'(r)} - V(r)u(r)^p - rV'(r)u(r)^p - \underline{rpV(r)u(r)^{p-1}u'(r)} + \beta u''(r) \\ &\qquad + u''(r) + \frac{1}{r}u'(r) + \frac{1}{r}\beta u'(r) \\ &\qquad - \underline{\lambda r u'(r)} - \lambda\beta u(r) \\ &\qquad + \underline{rpV(r)u(r)^{p-1}u'(r)} + \beta pV(r)u(r)^p. \end{aligned} \tag{3.42}$$

The underlined terms cancel out. By IVP (3.2), we can write

$$\begin{cases} \lambda u(r) - V(r)u(r)^p = u''(r) + \frac{1}{r}u'(r) \\ -V(r)u(r)^p = u''(r) + \frac{1}{r}u'(r) - \lambda u(r) \end{cases}$$

and use these substitutions to simplify (3.42) to

$$\begin{aligned} &v''_\beta(r) + \frac{1}{r}v'_\beta(r) - \lambda v_\beta(r) + pV(r)u(r)^{p-1}v_\beta(r) \\ &\quad = V(r)u(r)^p\left[\beta(p-1) - 2\right] - rV'(r)u(r)^p + 2\lambda u(r) = \phi_\beta(r). \end{aligned} \tag{3.43}$$

Similar to how the sign of v_β depends on the continuous increasing function $\rho(\beta)$, the sign of ϕ_β depends on β by a continuous decreasing function $\sigma(\beta)$, see Lemma 3.4 below. In Lemma 3.5 we will conclude that there exists a unique β_0 such that

$$v_{\beta_0}(\rho_0) = \phi_{\beta_0}(\rho_0) = 0 \ \text{ in } \ \rho_0 = \rho(\beta_0).$$

The unique zero of v_β will be used to show that w has a unique zero in Lemma 3.6.

Lemma 3.4. *Let $\alpha \in G \cup N$. There exist $\beta_0 > 0$ and a function $\sigma : [0, \overline{\beta}] \to [0, \infty)$ with the following properties:*

(a) σ is continuous and decreasing, $\sigma(0) > 0$ and $\sigma(\overline{\beta}) = 0$;

(b) for all $\beta > 0$ we have:

$$\phi_\beta(r) < 0 \quad \textit{if} \quad r < \sigma(\beta), \quad \textit{and} \quad \phi_\beta > 0 \quad \textit{if} \quad r > \sigma(\beta).$$

Proof. We fix $\beta > 0$. We write (3.41) as

$$\phi_\beta(r) = V(r)u(r)^p \left[\beta(p-1) - 2 - \xi(r)\right], \tag{3.44}$$

where

$$\xi(r) := r\frac{V'(r)}{V(r)} - \frac{2\lambda}{V(r)u(r)^{p-1}}. \tag{3.45}$$

The function $\xi(r)$ is negative and strictly decreasing on $(0, z(\alpha))$ with

$$\lim_{r\to z(\alpha)} \xi(r) = -\infty.$$

The sign of (3.44) depends on the term in brackets, since $V > 0$ by (3.5) and $u(r)^p > 0$ by (3.19). We are interested in the zero of ϕ_β. We define the function $\Xi(r)$ such that

$$\Xi(r) := \frac{2+\xi(r)}{p-1} = \beta \iff \beta(p-1) - 2 - \xi(r) = 0.$$

Since $\Xi(r)$ is strictly decreasing and $\beta > 0$, there are no solutions unless $\Xi(0) > 0$. We set $\Xi(0) = \overline{\beta}$. We claim that $\Xi(0) > 0$, which is equivalent to $\xi(0) > -2$. If $V(0) = \infty$, then

$$\xi(0) = \lim_{r\downarrow 0} r\frac{V'}{V} = h_0.$$

Thus, $h_0 \geqslant -k > -2$ by [4, Remark 2.1]. On the other hand, if $V(0) < \infty$, then

$$\xi(0) = -\frac{2\lambda}{\alpha^{p-1}V(0)}. \tag{3.46}$$

We solve (3.46) for α

$$\alpha > \left[\frac{\lambda}{V(0)}\right]^{\frac{1}{p-1}},$$

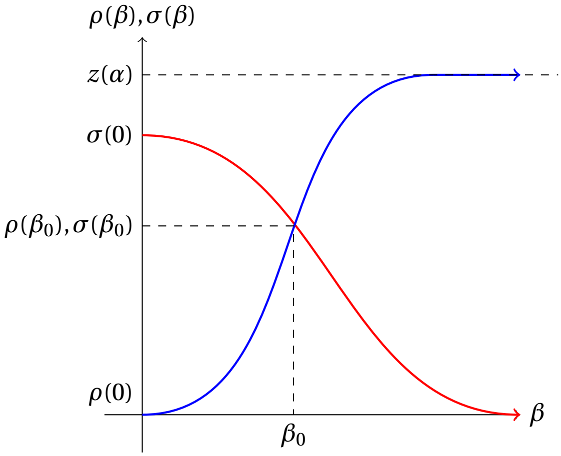
to conclude that $\xi(0) > -2$ for $\alpha \in G \cup N$ by Lemma 3.1. Thus, for $\beta \in [0, \overline{\beta}]$

$$\phi_\beta(r) > 0 \iff \Xi(r) > \beta \;\text{ and }\; \phi_\beta(r) < 0 \iff \Xi(r) < \beta.$$

In conclusion, $\sigma(\beta) := \Xi(r)^{-1}\Big|_{[0,\overline{\beta}]}$ satisfies the properties of the lemma. $\square$

Lemma 3.5. *Let $\alpha \in G \cup N$. There exists a unique $\beta_0 > 0$ such that $\rho\left(\beta_0\right) = \sigma\left(\beta_0\right)$.*

Proof. This follows immediately from property (b) of Lemma 3.4 and correspondence (3.39). This unique intersection can also be seen from a sketch of the two graphs.



□

3.5. More advanced properties of solutions

We fix $\beta = \beta_0$. This fixes at least $v = v_{\beta_0}$ and $\rho_0 = \rho_{\beta_0}$, which will be used in the following lemma. We will apply the Sturm comparison theorem to equations (3.3) and (2.27) in the following lemma.

Lemma 3.6. *For $\alpha \in G \cup N$, $w(r, \alpha)$ has a unique zero $r_0 \in (0, z(\alpha))$. Furthermore, $w(z(\alpha)) < 0$ for $\alpha \in N$ and if $\alpha \in G$, we have*

$$\lim_{r \to \infty} w(r) = -\infty.$$

Proof. REVISE: With the simplifications from before, we have

$$\begin{cases} v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v < 0 \quad \text{and } v > 0 \quad \text{on } (0, \rho_0) \quad \text{and} \\ v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v > 0 \quad \text{and } v < 0 \quad \text{on } (\rho_0, z(\alpha)). \end{cases} \tag{3.47}$$

DONE: Thus v has a unique zero ρ_0 on $(0, z(\alpha))$. Moreover, $v(0) = \beta_0 \alpha > 0$ and $\lim_{r \to 0} r v'(r) = 0$. Furthermore, if $\alpha \in N$, then $v(z(\alpha)) = z(\alpha)u'(z(\alpha)) < 0$ by Lemma 3.2. Thus, ρ_0 is a unique zero of v on $[0, z(\alpha)]$.

Let $\tau \in (0, z(\alpha))$ be the first zero of w, which exists by Lemma 3.3. We remember that w satisfies

$$w'' + \frac{1}{r}w' + \left[pV(r)u^{p-1} - \lambda w\right]w = 0 \quad \text{for } r \in (0, z(\alpha)), \tag{3.48}$$

with initial data $w(0) = 1$ and $\lim_{r\to 0} r\,w'(r) = 0$. Because of this, by # sturm, we know that v oscillates faster than w.

Then $\rho_0 \in (0,\tau)$ and thus

$$v'' + \frac{1}{r}v' + \left[pV(r)u(r)^{p-1} - \lambda\right]v < 0 \quad \text{and } v < 0 \quad \text{on } (\tau, z(\alpha)). \tag{3.49}$$

Since $w(\tau) = 0$ and v has no zero larger than ρ_0, by Sturm we have that w has no further zero in $(\tau, z(\alpha))$ and we can set $r_0 = \tau$. If $z(\alpha) < \infty$, then w has no zero on $(\tau, z(\alpha)]$ and we have $w(z(\alpha)) < 0$.

3

However, if $z(\alpha) = \infty$, we can apply Lemma 6 of [3, p. 249]. The disconjugacy interal (d,∞) is such that

$$d < \rho_0 < r_0.$$

DONE: The disconjugacy interval of a differential equation is the largest left-neighbourhood (c,b) of the right most b on which there exists a solution to the differential equation without zeroes. From Sturmian theory, no non-trivial solution can have more than one zero in (c,b). On the other hand, unless $c = a$, with a the left most, any solution of the differential equation with a zero before c must have another zero in (c,b).

Consider the same setting: equations (3.12) and (3.13) satisfying the comparison condition (3.14). In addition, we assume that $U \neq V$ in *any* neighborhood of b. If there exists a solution V of (3.13) with a largest zero at the point ρ, then the disconjugacy interval of (3.12) is a strict superset of (ρ, b).

One way to interpret the above is to remember that V oscillates faster than U and note that any solution U with a zero in (ρ, b)

Indeed, if we had $\rho_0 \leqslant d$, we could find a solution $\widetilde{w}$ linearly independent of w such that $\widetilde{w}(\widetilde{r}) = 0$ for some $\widetilde{r} > d \geqslant \rho_0$. But then $w + \widetilde{w}$ is a solution of # w ivp with two zeroes in (ρ_0, ∞) and thus, v should have another zero in that interval. By this contradiction, we have $w(r_0) = 0$ with $r_0 \in (d,\infty)$ and Lemma 6 of [3, p. 249] implies that

$$\lim_{r\to\infty} w(r) = -\infty.$$

□

Lemma 3.7. *Let $\alpha \in G$. There exists $\epsilon > 0$ such that $(\alpha, \alpha+\epsilon) \subset N$.*

Proof. DONE: We remember that $w(r)$ has a unique zero r_0 by Lemma 3.6 and since $\alpha \in G$, we have $w(r) \to -\infty$ as $r \to \infty$. This implies by Lemma 6 of [3, p. 249] that $r_0 \in (d,\infty)$, where (d,∞) is the disconjugacy interval of initial value problem (3.3). We refer to # for the definition of disconjugacy interval. We can choose r_1 and r_2

$$d < r_1 < r_0 < r_2,$$

such that by (??), there exists $\epsilon > 0$ such that for all $\tilde{\alpha} \in (\alpha, \alpha + \epsilon)$, we have

$$\tilde{u}(r_1) > u(r_1) \quad \text{and} \quad \tilde{u}(r_2) < u(r_2),$$

where $\tilde{u}(r) = u(r, \tilde{\alpha})$. Hence, the graphs of u and $\tilde{u}$ intersect in some $r_3 \in (r_1, r_2)$, where r_3 may depend on the choice of $\tilde{\alpha}$. Next, we will show that there exists $\tilde{r} \in (r_3, \infty)$ such that $\tilde{u}(\tilde{r}) = 0$. Then, we would have $\tilde{\alpha} \in N$.

Proof step 2. To see that $\tilde{u}(\tilde{r}) = 0$ for some $\tilde{r} \in (r_3, \infty)$, we suppose by contradiction that $\tilde{u}(r) > 0$ for all $r > r_3$. We will show that $\tilde{u}(r) < u(r)$ for all $r > r_3$. □

Proof step 3. To see that $\tilde{u}(r) < u(r)$ for $r > r_3$, we suppose by contradiction that $\tilde{u}(r_4) = u(r_4)$ for some $r_4 > r_3$ and $u - \tilde{u} > 0$ on (r_3, r_4). □

The function $z := u - \tilde{u}$ satisfies

$$z'' + \frac{1}{r} z' + \left[V(r) \frac{u^p - \tilde{u}^p}{u - \tilde{u}} - \lambda \right] z = 0 \quad \text{on } (r_3, r_4). \tag{3.50}$$

Since $z(r_3) = z(r_4) = 0$ by assumption and

$$\frac{u^p - \tilde{u}^p}{u - \tilde{u}} < p u^{p-1} \tag{3.51}$$

on (r_3, r_4), we can apply Sturm's theorem # ref. w oscillates faster than z Let y be any solution of w ivp linearly independent of w. ... There is no positive solution of w ivp on (d, ∞), contradicting ...

Hence $z(r) > 0$ for all $r > r_3$.

Sturm comparing fractions

Integrating sturm comparands

Hence $z(r) \equiv u(r) - \tilde{u}(r) \to \infty$ as $r \to \infty$. This is impossible, since $0 < \tilde{u}(r) < u(r)$ on (r_3, ∞) and $u(r) \to 0$ as $r \to \infty$. Therefore, $\tilde{u}(r)$ must vanish at some point $\tilde{r} \in (r_3, \infty)$ and the proof is complete.

□

Lemma 3.8. *Let $\alpha^* \in N$. Then $[\alpha^*, \infty) \subset N$ and $z : [\alpha^*, \infty) \to (0, \infty)$ is monotone decreasing.*

Proof. N is open subset of $(0, \infty)$ ... The set N is an open subset of $(0, \infty)$ by continuous dependence on the initial data as given by #. For $\hat{\alpha}$ close to α, we have $u(\hat{r}, \hat{\alpha}) = 0$ for some $\hat{r}$ near r.

By Lemma 3.2, we have $u'(r) < 0$ for all $r > 0$ if $\alpha \in N$. Hence, the graph of a solution $u(r, \alpha)$ with $\alpha \in N$ cannot be tangent to the r-axis and so the function $z(\alpha) : N \to (0, \infty)$ is continuous.

By Lemma 3.6, we have $w(z(\alpha^*)) < 0$. Then, there exists $\epsilon > 0$ such that

$$(\alpha^*, \alpha^* + \epsilon) \subset N \quad \text{and} \quad u(z(\alpha^*), \alpha) < 0 \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon).$$

The intermediate value theorem implies existence of an $r \in (0, z(\alpha^*))$ so that $u(r, \alpha) = 0$. We note that

$$z(\alpha) \leqslant r < z(\alpha^*) \quad \text{for all } \alpha \in (\alpha^*, \alpha^* + \epsilon)$$

and z is decreasing, since ...

We define

$$\overline{\alpha} := \sup\left\{\alpha > \alpha^* : \left[\alpha^*, \alpha\right) \subset N \text{ and } z : \left[\alpha^*, \alpha\right) \to (0, \infty) \text{ is decreasing}\right\}.$$

By contradiction, we suppose that $\overline{\alpha} < \infty$. Then there exists ... □

3

3.6. PROOF OF MAIN THEOREM

Proof of Theorem 3.1. # general proof, referring to lemma's □

REFERENCES

[1] H. BERESTYCKI, P. L. LIONS, and L. A. PELETIER, *An ode approach to the existence of positive solutions for semilinear problems in rn,* Indiana University Mathematics Journal **30**, 141 (1981).

[2] C. V. Coffman, *Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions,* Arch. Rational Mech. Anal. **46**, 81 (1972).

[3] M. K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$,* Arch. Rational Mech. Anal. **105**, 243 (1989).

[4] F. Genoud, *A uniqueness result for $\Delta u - \lambda u + V(|x|)u^p = 0$ on $\mathbb{R}^2$,* Adv. Nonlinear Stud. **11**, 483 (2011).

[5] F. Genoud, *Théorie de bifurcation et de stabilité pour une équation de Schrödinger avec une non-linéarité compacte* (EPFL, Lausanne, 2008).

[6] E. Yanagida, *Uniqueness of positive radial solutions of $\Delta u + f(u, |x|) = 0$,* Nonlinear Anal. **19**, 1143 (1992).

4

UNIQUENESS

From [?], we know that

$$\Delta u - u + u^3 = 0 \tag{4.1}$$

in $\mathbb{R}^3$ has a positive radially symmetric solution u. in which space This positive radially symmetric solution is unique.

Moreover, $u \in H^1$ and $u \neq 0$ implies $J(\phi_1) < J(u)$ unless $u(x) = \lambda\phi_1\left(x + x_0\right)$ for some non-zero, real λ and $x_0 \in \mathbb{R}^3$. Here J is the Rayleigh quotient associated with (4.1),

$$J(u) = \frac{\left(\int_{\mathbb{R}^3} |\mathrm{grad} u|^2 + u^2 dx\right)^2}{\int_{R^3} u^4 \mathrm{d}x} \tag{4.2}$$

The right hand side of expression (4.2) is meaningful for $u \in H^1$ and $u \neq 0$. Such functions will be referred to as *admissible* functions.

In [?] the equation (4.1) was considered.

Lemma 4.1. *There exist $v_n(r) \in C^2\left(\left[0,\infty\right)\right)$ where $n = 1,2,\ldots$, such that for each n, v_n has exactly $n-1$ isolated zeroes in $\left[0,\infty\right)$, decays exponentially as $r \to \infty$ and $\phi_n(x) = v_n(|x|)$ is a solution of* (4.1).

The proofs were given in [?] and [?]. The same results were obtained in [?] using Lyusternik-Schnirelman theory.

This paper proves the main result below to answer the questions raised in [?].

We seek solutions to (4.1) subject to the boundary condition at infinity given by

$$u \in L^4. \tag{4.3}$$

The problem (4.1), (4.3) is equivalent to the integral equation

$$u(x) = \int_{\mathbb{R}^3} g(x-t)u^3(t)\,\mathrm{d}t, \tag{4.4}$$

in L^4, where

$$g(x) = (4\pi)^{-1}|x|^{-1}e^{-|x|}. \tag{4.5}$$

The details to the results below (as well as a more complete bibliography concering (4.1)) can be found in [**?**].

- C_0^∞ is dense in H^1.
- If $u \in H^1$ then $v = |u| \in H^1$ and

$$|u|_{1,2} = |v|_{1,2}. \tag{4.6}$$

- If $u \in H^1$ then $u \in L^4$ and

$$|u|_{0,4} \leqslant 2^{-\frac{1}{2}}|u|_{1,2}. \tag{4.7}$$

- Let V denote the subspace of H^1 consisting of radially symmetric functions. The embedding $V \to L^4$ is compact.

Except for the constant, (4.7) follows from [**?**] or from the more general inequality in [**?**] or [**?**]. The constant can be obtained from the representation $u = g * w$ where $w = -\Delta u + u$. It suffices to prove (4.7) for $u \in C_0^\infty$. The assertion d) follows in a straightforward way from the Sobolev imbedding theorem and the inequality

$$4\pi \int_{|x|\geqslant\rho} |v(x)|^4\,\mathrm{d}x \leqslant 2\rho^{-1}|v|_{1,2}^4 \tag{4.8}$$

for $v \in V$, $\rho > 0$.

Concerning the convolution operator $\tau : u \to g * u$, where

$$(g*u)(x) = \int_{\mathbb{R}^3} g(x-t)u(t)\,\mathrm{d}t \tag{4.9}$$

and g is given by (4.5), we have the following results:

- $u \in L^{4/3}$ then $v = g * u \in H^1 \subseteq L^4$, $\int_{\mathbb{R}^3} uv\,\mathrm{d}x > 0$ unless $u = 0$, and v is a weak solution of

$$-\Delta v + v = u. \tag{4.10}$$

- If $u \in L^1 \cap L^\infty$ then $v = g * u$ has bounded continuous first derivatives and

$$\lim_{|x|\to\infty} v(x) = 0. \tag{4.11}$$

- If $u \in L^1 \cap L^\infty \cap C^1$ then $v = g * u \in C^2$ and v satisfies (4.10).

- Let X and Y denote the subspaces of $L^{4/3}$ and L^4 respectively, consisting of radially symmetric functions. Then $Y = X^*$ and $\tau : X \to Y$ is compact.

The first assertion of h) is obvious, the second follows immediately from d) and e).

Remark For consideration of the equation

$$\Delta u - u + |u|^{p-2} u = 0, \tag{4.12}$$

one replaces L^4 by L^p and $L^{4/3}$ by L^q where $p^{-1} + q^{-1} = 1$. If $2 < p < 6$, then c), d), e) and h) remain valid in this more general case (except for a change in the constant in (4.7); e) and h) of course fail for $p = 2$.

4.1. MINIMIZATION OF J

4

For $u \in L^4$, $u \neq 0$, we define $\sigma(u)$ by

$$\big(\sigma(u)\big)(x) = c \int_{\mathbb{R}^3} g(x-t) u^3(t)\, \mathrm{d}t \tag{4.13}$$

where $c > 0$ is chosen so that $v = \sigma(u)$ satisfies

$$\int_{\mathbb{R}^3} v^4\, \mathrm{d}x = 1. \tag{4.14}$$

This is possible since $u \in L^4$ implies $u^3 \in L^{4/3}$; thus by e), $g * (u^3) \in L^4$ and is non-zero. It is clear that up to positive factors the fixed points of σ are precisely the non-trivial solutions of (4.2). From e) above it follows that σ actually maps L^4 into H^1 and thus by c), σ can also be regarded as an operator in H^4
$\{0\}$. In particular it follows that an L^4 solution of (4.2) must belong to H^1.

Lemma 4.2. *Let u be an admissible function with*

$$\int_{\mathbb{R}^3} u^4\, \mathrm{d}x = 1. \tag{4.15}$$

Then $\sigma(u)$ is admissible and

$$J\big(\sigma(u)\big) \leqslant J(u) \tag{4.16}$$

with equality only if $\sigma(u) = u$. Moreover, $\sigma(u) \in L^\infty$, and $v = \sigma^2(u)$ has bounded continuous derivatives and satisfies

$$\lim_{|x|\to\infty} v(x) = 0; \tag{4.17}$$

finally $\sigma^3(u) \in C^2$.

Proof. The admissibility of $\sigma(u)$ follows from e). By e), (4.13) and (4.15), $w = \sigma(u)$ satisfies

$$c = c\int_{\mathbb{R}^3} u^4\, \mathrm{d}x = \int_{\mathbb{R}^3} \big(\operatorname{grad} w \cdot \operatorname{grad} u + wu\big)\, \mathrm{d}x \leqslant |u|_{1,2} |w|_{1,2}. \tag{4.18}$$

□

REFERENCES