Lemma 5.1. *Suppose that* $V(0) := \lim_{r \to 0} V(r)$ *exists and is finite. Then*

$$0 < \alpha < \left[\left(\frac{p+1}{2} \right) \frac{\lambda}{V(0)} \right]^{1/(p-1)} \implies \alpha \in P.$$

Proof. Remember solutions $u(r;\alpha)$ to 3.1 with $u(0) = \alpha \in P$ are positive everywhere. Do such initial conditions exist? (See Chapter 4: P is nonempty.) What interval of initial conditions belongs to P? To determine such an interval (of initial conditions), define (the Lyapunov or the energy) function $E(r)$ on $(0, z(\alpha))$ as:

$$E(r) := \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1},$$

Then, calculate $E'(r)$, where the IVP can be used to simplify the expression:

$$\begin{aligned} E'(r) &= u''(r)u'(r) - \lambda u(r)u'(r) + V(r)u(r)^p u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1} \\ &= \left[u''(r) - \lambda u(r) + V(r)u(r)^p\right] u'(r) + \frac{1}{p+1}V'(r)u(r)^{p+1} \\ &= -\frac{u'(r)^2}{r} + \frac{1}{p+1}V'(r)u(r)^{p+1} \leqslant 0 \text{ for } r > 0. \\ \text{as } &\left[u''(r) - \lambda u(r) + V(r)u(r)^p\right] = -\frac{1}{r}u'(r) \end{aligned}$$

Each of the terms of $E'(r)$ is non-negative: (i) by hypothesis H2, $V'(r) \leqslant 0$; (ii) $u(r)^{p+1} \geqslant 0$ because $u(r)$ is positive on $(0, z(\alpha))$ for any initial condition; (iii) $r \geqslant 0$ and (iv) $u'(r)^2 \geqslant 0$. Result: $E(r)$ is non-increasing $(0, z(\alpha))$.

To conclude about behaviour of solutions by type of initial condition, regard $E(r)$ for large $r > 0$. Suppose $\alpha \in N$, then $u(z(\alpha)) = 0$. Now evaluate:

$$\begin{aligned} \lim_{r \to z(\alpha)} E(r) &= \lim_{r \to z(\alpha)} \left[\frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{1}{p+1}V(r)u(r)^{p+1} \right] \\ &= u'(z(\alpha))^2 \geqslant 0. \end{aligned}$$

As $E(r)$ is non-increasing, $E(0) \geqslant 0$. Alternatively, suppose $\alpha \in G$. Then $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$. Then $E(r) = 0$ for $r \to \infty$, so again, $E(0) \geqslant 0$. Results: $E(0) \geqslant 0$ and $E(r)$ well-defined on $[0, z(\alpha)]$ for $\alpha \in G \cup N$.

For $\alpha \in P$, require $E(0) < 0$. Now evaluate $E(0)$ and solve for α:

$$\begin{aligned} E(0) =& \frac{1}{2}u'(0)^2 - \frac{\lambda}{2}u(0)^2 + \frac{1}{p+1}V(0)u(0)^{p+1} < 0 \\ \iff& -\frac{\lambda}{2}\alpha^2 + \frac{1}{p+1}V(0)\alpha^{p+1} < 0 \\ \iff& \alpha^{p-1} < \left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)} \\ \iff& \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)} \end{aligned}$$

Conclusion: $\alpha \in P$ whenever $0 < \alpha < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(0)}\right]^{1/(p-1)}$.

$\square$

Lemma 5.2. *Let $\alpha \in G \cup N$, and $u = u(\alpha, r)$. Then $u'(r) < 0$ for all $r \in (0, z(\alpha))$ and $u'(z(\alpha)) < 0$ if $\alpha \in N$.*

Proof. The lemma shows $u(r)$ strictly decreasing on $(0, z(\alpha))$ for $\alpha \in G \cup N$. The argument uses this to conclude that $w(r)$ has a unique zero on $(0, z(\alpha))$. Finally, analysis of solution sets N and G leads to the uniqueness result. Write $z(\alpha) = \infty$ when $\alpha \in G$, since $u(\alpha, r) \to 0$ as $r \to \infty$. Let $\alpha \in G \cup N$. By lemma 5.1, $E(r) \geqslant 0$ on $[0, z(\alpha)]$ and non-increasing.

Needs argument why $u''(0) < 0$. And why $u'(r) < 0$ for r sufficiently small.

These results can be extended to show $u' < 0$ on $(0, z(\alpha))$. Suppose by contradiction $0 < r_0 = \inf(0 < r < z(\alpha), u'(r) = 0)$ exists. Note how $u''(r_0) < 0 \implies u'(r) > 0$ somewhere on $(0, r_0)$. See also figure F1 This contradicts $u'(r) < 0$ on $(0, r_0)$. Again, the combination of $u''(0) = 0$ and $u'(0) = 0$ would imply $u \equiv u(r_0)$. Hence $u''(r_0) > 0$. Invoke 3.1:

$$
\begin{aligned}
& u''(r_0) = \lambda u(r_0) - V(r_0)u(r_0)^p > 0 \\
\implies & u(r_0) < \left[\frac{\lambda}{V(r_0)}\right]^{1/(p-1)} < \left[\left(\frac{p+1}{2}\right)\frac{\lambda}{V(r_0)}\right]^{1/(p-1)} \\
\iff & u(r_0)^{p-1} < \left(\frac{p+1}{2}\right)\frac{\lambda}{V(r_0)} \\
\iff & \frac{1}{p+1}V(r_0)u(r_0)^{p+1} < \frac{\lambda}{2}u(r_0)^2 \\
\iff & -\frac{\lambda}{2}u(r_0)^2 + \frac{1}{p+1}V(r_0)u(r_0)^{p+1} < 0
\end{aligned}
$$

Then using $u'(r_0) = 0$, this yields $E(r_0) < 0$:

$$E(r_0) = -\frac{\lambda}{2}u(r_0)^2 + \frac{1}{p+1}V(r_0)u(r_0)^{p+1} < 0$$

But $E(r_0) < 0$ contradicts $E(r) \geqslant 0$, so $u' < 0$ on $(0, z(\alpha))$.

It remains to show $u'(z(\alpha)) < 0$ whenever $\alpha \in N$. Suppose $u'(z(\alpha)) = 0$ and remember $u(z(\alpha)) = 0$. Then $u \equiv 0$, because $u''(z(\alpha)) = \lambda u(z(\alpha)) - V(z(\alpha))u(z(\alpha))^p = 0$. Conclusion: $u'(z(\alpha)) < 0$.

□

Lemma 5.3. *Let $\alpha \in (G \cup N)$, then w has at least one zero in $(0, z(\alpha))$.*

Proof. The Lagrange identity for 3.1 and 3.2 will yield information about the zeroes of $w(r)$. The cases $\alpha \in N$ and $\alpha \in G$ will be considered seperately. Suppose $\alpha \in N$. The differential equations for u and w can be written as:

$$(ru'(r))' + r\left[-\lambda u(r) + V(r)u(r)^p\right] = 0$$
$$(rw'(r))' + r\left[-\lambda w(r) + pV(r)u(r)^{p-1}w(r)\right] = 0.$$

Multiply by $w(r)$ and $u(r)$ respectively, subtract the equations and integrate from 0 to $z(\alpha)$,

$$\int_0^{z(\alpha)} w(r)(ru'(r))' - u(r)(rw'(r))' dr = \int_0^{z(\alpha)} r\left\{pV(r)u(r)^p w(r) - V(r)u(r)^p w(r)\right\} dr,$$

and perform partial integration for the left hand side: (remember $u(z(\alpha)) = 0$)

$$rw(r)u'(r)\Big|_0^{z(\alpha)} - ru(r)w'(r)\Big|_0^{z(\alpha)} - \int_0^{z(\alpha)} \underbrace{\left\{ru'(r)w'(r) - ru'(r)w'(r)\right\}}_{0} dr$$
$$= (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r) dr$$
$$z(\alpha)w(z(\alpha))u'(z(\alpha)) = (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r) dr.$$

For $\alpha \in N$, note $r > 0,\ V > 0,\ u^p > 0$ are finite almost everywhere. Suppose $w > 0$ on $(0, z(\alpha))$. Then $z(\alpha)u'(z(\alpha))w(z(\alpha)) < 0$ contradicts $(p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r)dr > 0$. (A similar argument holds for $w < 0$.) Hence, w changes sign at least once on $(0, z(\alpha))$.

For $\alpha \in G$, suppose by contradiction that $w > 0$ on $(0, \infty)$. Perform integration over $(0, r)$ and rewrite left hand side using the quotient rule:

$$ru'(r)w(r) - rw'(r)u(r) = r\frac{u'(r)w(r) - w'(r)u(r)}{w(r)^2} = rw(r)^2\left(\frac{u(r)}{w(r)}\right)'$$
$$= (p-1)\int_0^{z(\alpha)} rV(r)u(r)^p w(r) dr > 0.$$

<u>Result:</u> $\left(\frac{u}{w}\right)'$ is positive, so $\frac{u}{w}(r)$ is increasing.

By Lemma C.1 of [Gen11] there exists two independent solutions that satisfy 3.2 as $r \to \infty$:

$$\xi_0(r) \sim r^{-\frac{1}{2}} \exp^{-\sqrt{\lambda} r} \quad \text{and} \quad \xi_1(r) \sim r^{-\frac{1}{2}} \exp^{\sqrt{\lambda} r}$$

So for $r \to \infty$, $w(r) \sim r^{-\frac{1}{2}}\left[\alpha_0 \exp^{-\sqrt{\lambda} r} + \alpha_1 \exp^{-\sqrt{\lambda} r}\right]$ for some constants α_1, α_0. Since $w > 0$ by hypothesis, and $\lim_{r\to\infty} w(r) \sim \alpha_1$, $\alpha_1 \geqslant 0$. Suppose $\alpha_1 = 0$, then $w(r) \to 0$ exponentially as $r \to \infty$. So w changes sign by Lemma 1.4.9 of [Gen11], a contradiction. On the other hand, suppose $\alpha_1 > 0$. Then by Lemma 1.2.7 and Lemma C.1 of [Gen11], $u(r) \sim r^{-\frac{1}{2}} \exp^{-\sqrt{\lambda} r}$ as $r \to \infty$. <u>Result:</u> for $\alpha_1 > 0$, there exists C such that

$$\lim_{r\to\infty} \frac{u(r)}{w(r)} = \lim_{r\to\infty} \frac{Cr^{-\frac{1}{2}} \exp^{-\sqrt{\lambda} r}}{\alpha_1 r^{-\frac{1}{2}} \exp^{\sqrt{\lambda} r}} = \lim_{r\to\infty} \frac{C}{\alpha_1} \exp^{-2\sqrt{\lambda} r} = 0$$

This contradicts $\frac{u}{w}(r)$ positive and increasing. <u>Conclusion:</u> w changes sign at least once on $(0, z(\alpha))$ for $\alpha \in G \cup N$. $\square$