An Explication of Existence and Uniqueness Results for a Nonlinear Schrödinger equation

An Introduction to the Shooting Method and Sturm Comparison Theorem

Bachelor's Thesis

at Delft University of Technology, written by

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Physics of NLS

1.1. Derive the wave equation from Maxwell

DONE: Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \quad \nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t},$$
$$\nabla \cdot \vec{\mathcal{D}} = 0, \quad \nabla \cdot \vec{\mathcal{B}} = 0.$$

Unless otherwise specified, these are fields in three-dimensional Cartesian coordinates. For example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \quad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [1, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [2]. This reference work also includes an introduction to the necessary vector calculus.

REVISE: From these relations and the vector identity for the curl of the curl, a wave equation can be derived. We specifically use $\nabla \cdot \vec{\mathcal{D}} = \nabla \cdot \epsilon_0 \vec{\mathcal{E}} = 0$ and $\nabla \times \vec{\mathcal{B}} = \mu_0 \frac{\partial \vec{\mathcal{D}}}{\partial t}$ to simplify the equation:

$$\nabla \times \nabla \times \vec{\mathcal{E}} = \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \vec{\mathcal{B}}) = -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \quad \text{by Maxwell's laws, and}$$
$$\nabla \times \nabla \times \vec{\mathcal{E}} = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \nabla^2 \vec{\mathcal{E}} = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \Delta \vec{\mathcal{E}} = -\Delta \vec{\mathcal{E}}, \quad \text{by vector calculus.}$$

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Combining these and using $\mu_0\epsilon_0 = 1/c^2$ we arrive at the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{1.1}$$

1.2. VALIDITY OF PLANE WAVE SOLUTIONS

DONE: Stuyding the left and right hand sides of equation (1.1), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{j=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_j^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz (educated guess) for the solutions to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.2}$$

which are so called plane wave solutions.

REVISE: Plane waves are of the simplest geometry and mathematical representation. Constant wavefronts at all times. Locations of constant phase are planes. Plane waves satisfy wave equations in homogenous media or free space.

Plane waves need to be extended infinitely, and are not physical. However, real waves can be approximated by them. Since wavefronts are separated by wavelength, many waves fit in a reasonably small volume, justifying the approximation. Fourier optics studies the decomposition into plane waves using spatial Fourier transforms.

REVISE: This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane. Taking the necessary derivatives of 1.2 in equation (1.1)

$$\Delta\mathcal{E}_j = \frac{\partial^2}{\partial x^2}\mathcal{E}_j + \frac{\partial^2}{\partial y^2}\mathcal{E}_j + \frac{\partial^2}{\partial z^2}\mathcal{E}_j = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{E}_j$$

$$\frac{\partial^2}{\partial z^2}\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.3}$$

Dispersion (spreading out) is a result of different frequencies propagating at different speeds. Of course, other plane waves exist. In general, let wavevector $\overrightarrow{k_0} = (k_x, k_y, k_z)$ satisfy the dispersion relation $\left|\overrightarrow{k_0}\right|^2 = \frac{w_0^2}{c^2}$. The wavenumber $\overrightarrow{k} = (k_x, k_y, k_z)$ signifies the direction of propagation. For a plane wave with $\overrightarrow{k} = (0, 0, k_0)$, we say the plane wave travels in the positive z-direction if k_0 is positive.

TODO: Rewrite such that the general wavevector is the endgoal from the start. Not all plane waves are physical (in agreement with Maxwell's laws), for example the wave with electric field $\overrightarrow{\mathcal{E}} = (p, p, p)$ with plane wave component $p = E_c e^{i(k_0 z - \omega_0 t)}$.

Claim: this violates Maxwell's law for the divergence of the electric field: $\nabla \cdot \overrightarrow{\mathcal{E}} = 0$. Substituting the mentioned plane wave yields

$$\begin{aligned}\nabla \cdot \overrightarrow{\mathcal{E}} &= \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial z} \\ &= 0 + 0 + i k_0 E_c e^{i(k_0 z - \omega_0 t)} \neq 0\end{aligned}$$

The nonzero z-component is troublesome in light of the Maxwell divergence law for the electric field.

However, the electric field $\overrightarrow{\mathcal{E}} = (p, p, 0)$ does satisfy Maxwell's law. This field is perpendicular to the wavevector $\overrightarrow{k} = (0, 0, k_0)$. In fact, this relation holds more generally. Plane waves with wavevector $\overrightarrow{k} = (k_x, k_y, k_z)$ are physical when the electric field and wavevector are perpendicular.

1.3. DERIVE THE HELMHOLTZ EQUATION

NEW: Considering time-harmonic solutions to the scalar wave equation (1.1) of the form

$$\mathcal{E}_j(x, y, z, t) = e^{i\omega_0 t} E(x, y, z) + \text{c.c}, \tag{1.4}$$

which are continuous wave (cw) beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, refer to [3].

Substituting (1.4) in equation (1.1) shows that E should satisfy the scalar linear Helmholtz equation

$$\Delta E(x, y, z) + k_0^2 E = 0, \tag{1.5}$$

where k_0 is given by the dispersion relation (1.3). TODO: work available in 04052020-1A

The plane waves **??** solve equation (1.5) with

$$E = E_c e^{i(k_x x + k_y y + k_z z)}$$

with $k_x^2 + k_y^2 + k_z^2 = k_0^2$.

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TODO: [1, p. 6]

REVISE: Not necessary? Write the incoming field $E_0^{inc}(x,y)$ as a sum of plane waves, then the electric field for non-zero z-value follows from propagation.

$$E_0^{inc}(x,y) = \frac{1}{2\pi}\int E_c(k_x,k_y)\, e^{i(k_x x + k_y y)}\mathrm{d}k_x \mathrm{d}k_y, \quad \text{such that}$$
$$E(x,y,z) = \frac{1}{2\pi}\int E_c(k_x,k_y)\, e^{i(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z)}\mathrm{d}k_x \mathrm{d}k_y$$

1.4. DERIVE THE LINEAR SCHRÖDINGER

NEW: Most of the plane wave modes in **??** are nearly parallel to the z-axis. These paraxial plane waves satisfy

$$k_\perp^2 << k_z^2, \quad k_\perp^2 = k_x^2 + k_y^2.$$

Since $k_0^2 = k_x^2 + k_y^2 + k_z^2 = k_\perp^2 + k_z^2$, we have $k_0^2 \approx k_z^2$.

NEW: This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{1.6}$$

where the $\psi(x,y,z)$ is an envelope (or amplitude) function. Substituting this form into the Helmholtz equation (1.5) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp \psi = 0, \tag{1.7}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that the wavenumber k_z dominates $k_\perp$ such that $k_0 \approx k_z$.

Claim: $\psi_{zz} << k_0\psi_z$ and $\psi_{zz} << \Delta_\perp\psi$.

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right)$$
$$\implies k_z = k_0\left(1 - \frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right)$$
$$\implies k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} << 1$$

This helps in the following step

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_0(k_0 - k_z)E_c} = \frac{k_0 - k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0}\cdot\frac{1}{k_0} << 1.$$

Also

$$\frac{[\psi_{zz}]}{[\Delta_\perp \psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0 - k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} << \frac{1}{4}\frac{k_\perp^2}{k_0^2} << 1.$$

TODO: Add the expression for Linear Schrödinger

1.5. POLARISATION FIELD

NEW: Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E}, 0, 0),\ \vec{\mathcal{P}} = (\mathcal{P}, 0, 0),\ \vec{\mathcal{D}} = (\mathcal{D}, 0, 0),$$

Also, we assume that $\mathcal{E}$ is the cw electric field from (1.4) Write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \text{h.o.t.} \tag{1.8}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials # ref. The constants c_i are actually a function of frequency ω_0. Rewrite $c_i = \epsilon_0 \chi^{(i)}(\omega_0)$ where $\chi^{(i)}$ is called the susceptibility.

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \text{h.o.t.} \tag{1.9}$$

First we consider linear polarisation involving $\chi^{(1)}$. The electric fields affects the medium and induces a polarisation proportional to the electric field

$$\mathcal{P} = \mathcal{P}_{\text{lin}} = c\mathcal{E}$$

for some real number c. In fact, we can write

$$\mathcal{P} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E},$$

where $\chi^{(1)}$ is the first-order optical susceptibility, whose value depends on the frequency ω_0. Then the induction electric field is given by

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\text{lin}} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}, \quad n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0),$$

where n_0 is the linear index of refraction (or refractive index) of the medium.

TODO: Leads to linear Helmholtz with adjusted k_0...

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1.6. NONLINEAR POLARISATION...

TODO: merge with previous section NEW: The linear polarisation field is an approximation and we wish to study the nonlinear effects too. Consider the nonlinear polarisation field $\mathcal{P}_{\mathrm{nl}}$ as

$$\mathcal{P} = \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}}.$$

NEW: Since the medium is isotropic, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be the same in all directions. Specifically, when considering $-\mathcal{P}$ the expression should be equivalent, with $-\mathcal{E}$.

$$-\mathcal{P}_{\mathrm{nl}} = \epsilon_0 \chi^{(2)} (-\mathcal{E})^2 + \epsilon_0 \chi^{(3)} (-\mathcal{E})^3 + \epsilon_0 \chi^{(4)} (-\mathcal{E})^4 + \epsilon_0 \chi^{(5)} (-\mathcal{E})^5 + \text{h.o.t.}$$
$$-\mathcal{P}_{\mathrm{nl}} = \epsilon_0 \chi^{(2)} \mathcal{E}^2 - \epsilon_0 \chi^{(3)} \mathcal{E}^3 + \epsilon_0 \chi^{(4)} \mathcal{E}^4 - \epsilon_0 \chi^{(5)} \mathcal{E}^5 + \text{h.o.t.}$$

Where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\mathrm{nl}}$ and we have

$$\mathcal{P}_{\mathrm{nl}} = \epsilon_0 \chi^{(3)} \mathcal{E}^3 + \epsilon_0 \chi^{(5)} \mathcal{E}^5 + \text{h.o.t.} \tag{1.10}$$

The leading-order term is called the Kerr nonlinearity

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0 \chi^{(1)} (\omega_0) \mathcal{E}^3.$$

TODO: Check in notes if the ϵ_0 is actually there. Continue writing about nonlinear (Kerr) polarisation, e.g. about Kerr mediums and about the nonlinear Helmholz that follows. And that this Kerr nonlinearity is weak.

TODO: write how this leads to NLH and NLH see notes

1.7. FOCUSING NLS AND SOLITONS

NEW: (General NLS) Substituting $E = e^{ik_0 z}\psi$ in the NLH **??** and applying the paraxial approximation $\psi_{zz} << k_0 \psi_z$, we obtain the nonlinear Schrödinger equation (NLS)

$$2ik_0\psi_z(z, \bar{x}) + \Delta_\perp \psi + k_0^2 \frac{4n^2}{n_0} |\psi|^2 \psi = 0. \tag{1.11}$$

TODO: Instead, use dimensionless NLS (3.4)

REVISE: (Focusing NLS) The previous results lead to the focusing NLS given by

$$i\psi_z(z, \bar{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0. \tag{1.12}$$

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Considering envelopes of constant shape (solitons) with

$$\psi_{\omega}^{\text{soliton}} = e^{i\omega z} R_{\omega}(\bar{x})$$

leads to an equation in $R_{\omega}(\bar{x})$ by the following steps

1. $i\psi_z(z,\bar{x}) = i\left(i\omega e^{i\omega z} R_{\omega}(\bar{x})\right) = -\omega e^{i\omega z} R_{\omega}(\bar{x})$
2. $\Delta\psi = \left(\Delta e^{i\omega z}\right) R_{\omega}(\bar{x}) + e^{i\omega z}\left(\Delta R_{\omega}(\bar{x})\right)$
3. $|\psi|^{2\sigma}\psi = \left|e^{i\omega z} R_{\omega}(\bar{x})\right|^{2\sigma} e^{i\omega z} R_{\omega}(\bar{x}) = |R_{\omega}(\bar{x})|^{2\sigma} e^{i\omega z} R_{\omega}(\bar{x})$
4. such that
5. $e^{i\omega z}\left[-\omega R_{\omega}(\bar{x}) + \Delta R_{\omega}(\bar{x}) + |R_{\omega}(\bar{x})|^{2\sigma} R_{\omega}(\bar{x})\right] = 0$
6. and
7. $\Delta R_{\omega}(\bar{x}) - \omega R_{\omega}(\bar{x}) + |R_{\omega}(\bar{x})|^{2\sigma} R_{\omega}(\bar{x}) = 0$

REFERENCES

[1] G. Fibich, *The nonlinear Schrödinger equation*, Applied Mathematical Sciences, Vol. 192 (Springer, Cham, 2015) p. 862, singular solutions and optical collapse.

[2] D. Griffiths, *Introduction to Electrodynamics*, Pearson international edition (Prentice Hall, 1999).

[3] A. Siegman, *Lasers* (University Science Books, 1986).