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UNIQUENESS OF GROUND STATE

3.1. INTRODUCTION

In this chapter, we study a paper from 1972 by Charles V. Coffman [1]. The paper proves uniqueness of the positive radially symmetric (ground state) solution $u = \phi_1 \in C^2 \cap L^4$ for the equation

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^3. \tag{3.1}$$

Note that all function spaces consist of real valued functions on $\mathbb{R}^3$. Furthermore, radial symmetry is with respect to the origin only.

The existence of such a function ϕ_1 was shown in [2], where $\phi_1 = v_1(|x|)$ solves (3.1). In fact, there exist functions $v_n(|x|) \in C^2([0,\infty))$, $n = 1,2,\ldots$, such that for each n, v_n has exacly $n-1$ isolated zeroes in $[0,\infty)$, decays exponentially as $r \to \infty$. This was shown in [3, 4]

Moreover, Theorem 3.1 of [1] improves the result of [5], which also studied (3.1) (in the context of variational calculus). In [5], they show that the Lagrangian associated with (3.1) is zero in its first variation and the second variation is positive if $\lambda_1 > 1$. The latter is shown only through approximations. Theorem 3.1 of [1] shows that the Rayleigh quotient J associated with (3.1)

$$J(u) = \frac{\left(\int |\nabla u|^2 + u^2 \, dx\right)^2}{\int u^4 \, dx} \tag{3.2}$$

is indeed minimal for $u = \phi_1$ and for

$$u(x) = k\phi_1(x + x_0) \tag{3.3}$$

for any $k \neq 0$ and $x_0 \in \mathbb{R}^3$.

3.2. PRELIMINARY RESULTS FOR THE INTEGRAL EQUATION

The problem (3.1) subject to $u \in L^4$ is equivalent to the integral equation in L^4

$$\begin{cases} u(x) = \int g(x-y)u^3(y)\,\mathrm{d}y, & \text{where} \\ g(x) = (4\pi)^{-1}|x|^{-1}e^{-|x|}. \end{cases} \tag{3.4}$$

Here $g(x)$ is the Yukawa (screened Coulomb) potential. This potential is associated with the equation

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$$\Delta u - u = 0. \tag{3.5}$$

We consider radially symmetric solutions to (3.5), $r = |x|$, which solve

$$\frac{\mathrm{d}}{\mathrm{d}r^2}(ru) = ru. \tag{3.6}$$

Hence, the Yukawa potential $u(r)$ is of the form $ru = e^{-r} \iff u(r) = r^{-1}e^{-r}$.

The following two subsections discuss (mostly) standard results regarding the Sobolev space H^1 and the convolution operator $\tau : u \to g * u$.

3.2.1. SOME RESULTS REGARDING H^1

First, concerning the space H^1, we have the following results:

a) C_0^∞ is dense in H^1.

b) If $u \in H^1$, then $v = |u| \in H^1$ and
$$|u|_{1,2} = |v|_{1,2}.$$

c) If $u \in H^1$, then $u \in L^4$ and
$$|u|_{0,4} \leqslant 2^{-1/4}|u|_{1,2}. \tag{3.7}$$

d) Let V denote the subspace of H^1 consisting of radially symmetric functions. The embedding $V \to L^4$ is compact.

3.2.2. SOME RESULTS REGARDING THE CONVOLUTION OPERATOR

e) If $u \in L^{4/3}$, then $v = g * u \in H^1 \subseteq L^4$, $\int u\,v\,\mathrm{d}x > 0$ unless $u = 0$, and v is a weak solution of
$$-\Delta v + v = u. \tag{3.8}$$

f) If $u \in L^1 \cap L^\infty$, then $v = g * u$ has bounded continuous first derivatives and
$$\lim_{|x|\to\infty} v(x) = 0$$
.

g) If $u \in L^1 \cap L^\infty \cap C^1$, then $v = g * u \in C^2$ and v satisfies (3.8).

h) Let X and Y denote the subspaces of $L^{4/3}$ and L^4 respectively, consisting of radially symmetric functions. Then $Y = X^*$ and $\tau : X \to Y$ is compact.

3.3. MINIMISATION OF J

This section first states that a solution $u \in L^4$ must belong to H^1. For $u \in L^4$, $u \neq 0$, we define $\sigma(u)$ by

$$\big(\sigma(u)\big)(x) = c\int g(x-t)u^3(t)\,dt \tag{3.9}$$

[...]

Lemma 3.1. *If u is an admissible solution, then $\sigma(u)$ is admissible and*

$$J(\sigma(u)) \leqslant J(u) \tag{3.10}$$

with equality only if $\sigma(u) = u$. Moreover, $\sigma(u) \in L^\infty$ and $v = \sigma^2(u)$ has bounded continuous derivatives and satisfies

$$\lim_{|x|\to\infty} v(x) = 0; \tag{3.11}$$

finally $\sigma^3(u) \in C^2$.

Proof. □

The following two lemmata are corollaries of Lemma 3.1.

Lemma 3.2. *If $v \in L^4$ is a solution of (3.4) then $v \in C^2$, v has bounded first derivatives, and v satisfies (3.11).*

Lemma 3.3. *If u is any (radially symmetric) admissible function, then there is a (raaially symmetric) admissible function $v \in C^2$ which is positive, has bounded first derivatives and satisfies (3.11) and*

$$J(v) \leqslant J(u). \tag{3.12}$$

Moreover, unless u itself has the same properties and is a solution of (3.4) (to within a positive factor), then v can be chosen so that inequality (3.12) is strict.

Proof. □

Theorem 3.1. *Let*

$$\lambda_1 = \inf\{J(u) : u \text{ admissible}\}.$$

There exists a $\phi_1 \in V$ with

$$J(\phi_1) = \lambda_1.$$

For $u \in H^1$, $J(u) > \lambda_1$ unless u is of the form (3.3).

The proof of Theorem 3.1 shows the desired results of the paper except for the last statement. This requires 3.2 of the next section.

3.4. UNIQUENESS THEOREM

The radially symmetric solutions of (3.1) are of the form

$$u(x) = |x|^{-1} w(|x|),$$

where $w(r)$ $(r = |x|)$ solves

$$w'' - w + r^{-2} w^3 = 0. \tag{3.13}$$

We refer to 3.4.1 for the details.

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To prove the uniqueness of ground state solution ϕ_1 for (3.1), it suffices to prove that (3.13) has at most one positive solution satisfying the following boundary conditions

$$0 < \lim_{r\to 0} r^{-1} w(r) < \infty, \quad \lim_{r\to\infty} w(r) = 0. \tag{3.14}$$

The problem (3.13) is transformed to an initial value problem where

$$\lim_{r\to 0} r^{-1} w(r) = a > 0. \tag{3.15}$$

The basic facts regarding the problem (3.13), (3.15) are summarised in Lemma 4.1. The proofs are omitted in [1].

Lemma 3.4. *For each $a > 0$ the equation (3.13) has a unique solution $w = w(r,a)$ which is of class C^2 on $\left(0,\infty\right)$ and satisfies (3.15). The partial derivatives $\partial w(r,a)/\partial a$ and $\partial w'(r,a)/\partial a$ exist for all positive r and a. Furthermore, $\partial w(r,a)/\partial a$ coincides on $(0,\infty)$ with the solution $\delta = \delta(r,a)$ of the regular initial value problem*

$$\begin{cases} \delta'' - \delta + 3r^{-2} w^2 \delta = 0, \\ \delta(0) = 0, \quad \delta'(0) = 1, \end{cases} \tag{3.16}$$

with $w = w(r,a)$; $\partial w'(r,a)/\partial a = \delta'(r,a)$.

It is clear that a solution of (3.13) which satisfies (3.14) belongs to the one-parameter family $w = w(r,a)$, $a > 0$; we therefore formulate our uniqueness result as follows.

Theorem 3.2. *There is at most one positive value of a for which*

$$w(r,a) > 0, \quad 0 < r < \infty \tag{3.17}$$

and

$$\lim_{r\to\infty} w(r,a) = 0. \tag{3.18}$$

Theorem 4.1 is implied by the following lemma.

Lemma 3.5. *(i) If $a > 0$ and $w(r,a) > 0$ on $(0, z_1)$ with $w(z_1,a) = 0$, then $\delta(z_1,a) < 0$.*

(ii) If $a > 0$ and $w(r,a)$ satisfies (4.5) and (4.6) then

$$\lim_{r\to\infty} e^{-r} \delta(r,a) < 0. \tag{3.19}$$

Proof. □

By studying the zeroes of $w(r,a)$, we can show that A (the set of $a>0$ such that $w(r,a)$ has at least one zero in $(0,\infty)$) has a left endpoint.

3.4.1. DERIVATION OF EQUATION FOR RADIALLY SYMMETRIC SOLUTIONS

Consider radially symmetric solutions to (3.1). Then $u(x)=u(|x|)=u(r)$. This transforms (3.1) to the ODE (2.1), restated here for $N=3$

$$u'' + \frac{2}{r}u' - u + u^3 = 0 \tag{3.20}$$

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Furthermore, substituting $u(r) = r^{-1}w(r)$, we calculate the derivatives of u as

1. $u'(r) = -r^{-2}w(r) + r^{-1}w'(r)$
2. $u''(r) = 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r)$.

We substitute in (3.20) to obtain

$$\begin{aligned} u''(r) + \frac{2}{r} - u(r) + u^3(r) \\ &= 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r) + \frac{2}{r}\left(-r^{-2}w(r) + r^{-1}w'(r)\right) \\ &\quad - r^{-1}w(r) + r^{-3}w^3(r) = 0, \end{aligned} \tag{3.21}$$

which is simplified to

$$r^{-1}\left(w'' - w + r^{-2}w^3\right) = 0.$$

In conclusion, since $r \neq 0$, we obtain

$$w'' - w + r^{-2}w^3 = 0. \tag{3.22}$$

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