An Explication of Existence and Uniqueness Results for a Nonlinear Schrödinger equation

An Introduction to the Shooting Method and Sturm Comparison Theorem

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Physics of NLS

1.1. Derive the wave equation from Maxwell

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \quad \nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t},$$
$$\nabla \cdot \vec{\mathcal{D}} = 0, \quad \nabla \cdot \vec{\mathcal{B}} = 0.$$

Unless otherwise specified, these are fields in three-dimensional Cartesian coordinates. For example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \quad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [1, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [2]. This reference work also includes an introduction to the necessary vector calculus.

From these relations and the vector identity for the curl of the curl, a wave equation can be derived. We specifically use $\nabla \cdot \vec{\mathcal{D}} = \nabla \cdot \epsilon_0 \vec{\mathcal{E}} = 0$ and $\nabla \times \vec{\mathcal{B}} = \mu_0 \frac{\partial \vec{\mathcal{D}}}{\partial t}$ to simplify the equation:

$$\nabla \times \nabla \times \vec{\mathcal{E}} = \nabla \times (-\frac{\partial \vec{\mathcal{B}}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \times \vec{\mathcal{B}}) = -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \quad \text{by Maxwell's laws, and}$$
$$\nabla \times \nabla \times \vec{\mathcal{E}} = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \nabla^2 \vec{\mathcal{E}} = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \Delta \vec{\mathcal{E}} = -\Delta \vec{\mathcal{E}}, \quad \text{by vector calculus.}$$

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Combining these and using $\mu_0\epsilon_0 = 1/c^2$ we arrive at the vector wave equation:

$$\Delta\vec{\mathcal{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2}. \tag{1.1}$$

1.2. VALIDITY OF PLANE WAVE SOLUTIONS

Stuyding the left and right hand sides of equation (1.1), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta\vec{\mathcal{E}} = \Delta\begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial x^2}+\frac{\partial^2\mathcal{E}_x}{\partial y^2}+\frac{\partial^2\mathcal{E}_x}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial x^2}+\frac{\partial^2\mathcal{E}_y}{\partial y^2}+\frac{\partial^2\mathcal{E}_y}{\partial z^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial x^2}+\frac{\partial^2\mathcal{E}_z}{\partial y^2}+\frac{\partial^2\mathcal{E}_z}{\partial z^2}\end{bmatrix} = \frac{1}{c^2}\begin{bmatrix}\frac{\partial^2\mathcal{E}_x}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_y}{\partial t^2}\\ \frac{\partial^2\mathcal{E}_z}{\partial t^2}\end{bmatrix}$$

$$\Delta\mathcal{E}_j = \sum_{j=1}^{3}\left[\frac{\partial^2\mathcal{E}_j}{\partial x_j^2}\right] = \frac{1}{c^2}\frac{\partial^2\mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz (educated guess) for the solutions to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z-\omega_0 t)}, \tag{1.2}$$

which are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any z-value and the electric field is non-zero in the x and y directions. The wavefronts are spaced by the wavelength.

REVISE: This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane. Taking the necessary derivatives of 1.2 in equation (1.1)

$$\Delta\mathcal{E}_j = \frac{\partial^2}{\partial x^2}\mathcal{E}_j + \frac{\partial^2}{\partial y^2}\mathcal{E}_j + \frac{\partial^2}{\partial z^2}\mathcal{E}_j = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{E}_j$$

$$\frac{\partial^2}{\partial z^2}\mathcal{E}_j = k_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)} = \frac{1}{c^2}\omega_0^2\cdot E_c e^{i(k_0 z-\omega_0 t)}$$

yields the dispersion relation

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.3}$$

For a general direction in (x, y, z)-coordinates, define the wavevector

$$\vec{k} = (k_x, k_y, k_z),$$

where $\left|\vec{k}^2\right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$, satisfies the scalar wave equation (1.1) when $\vec{k}\perp\vec{\mathcal{E}}$. Still satisfying the same dispersion relation (1.3).

REVISE: Dispersion (spreading out) is a result of different frequencies propagating at different speeds. Of course, other plane waves exist. In general, let wavevector $\vec{k_0} = (k_x, k_y, k_z)$ satisfy the dispersion relation $\left|\vec{k_0}\right|^2 = \frac{w_0^2}{c^2}$.

1.3. DERIVE THE HELMHOLTZ EQUATION

NEW: Considering time-harmonic solutions to the scalar wave equation (1.1) of the form

$$\mathcal{E}_j(x,y,z,t) = e^{i\omega_0 t} E(x,y,z) + \text{c.c}, \tag{1.4}$$

which are continuous wave (cw) beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, refer to [3].

Substituting (1.4) in equation (1.1) shows that E should satisfy the scalar linear Helmholtz equation

$$\Delta E(x,y,z) + k_0^2 E = 0, \tag{1.5}$$

where k_0 is given by the dispersion relation (1.3).

TODO: Work available, see Note A of May 4th

Helmholtz equation (1.5) is solved, for example, by general direction plane waves when

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

1.4. DERIVE THE LINEAR SCHRÖDINGER

REVISE: Write the incoming field $E_0^{\text{inc}}(x,y)$ as a sum of plane waves, then the electric field for non-zero z-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and is essential to Fourier optics.

$$E_0^{\text{inc}}(x,y) = \frac{1}{2\pi}\int E_c(k_x,k_y)\, e^{i(k_x x + k_y y)} \mathrm{d}k_x \mathrm{d}k_y, \quad \text{such that}$$
$$E(x,y,z) = \frac{1}{2\pi}\int E_c(k_x,k_y)\, e^{i(k_x x + k_y y + \sqrt{k_0^2 - k_x^2 - k_y^2}\, z)} \mathrm{d}k_x \mathrm{d}k_y$$

NEW: For laser beams oriented in the z-direction, most of the plane wave modes are nearly parallel to the z-axis. These paraxial plane waves satisfy

$$k_\perp^2 << k_z^2, \quad k_\perp^2 = k_x^2 + k_y^2.$$

We have $k_0^2 \approx k_z^2$, since $k_0^2 = k_x^2 + k_y^2 + k_z^2 = k_\perp^2 + k_z^2$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{1.6}$$

where $\psi(x,y,z)$ is an envelope (or amplitude) function. Substituting this form into the Helmholtz equation (1.5) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp \psi = 0, \tag{1.7}$$

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where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x, y, z)$.

Remember that the wavenumber k_z dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope will vary slowly in z and curve even more slowly.

Claim: $\psi_{zz} << k_0\psi_z$ and $\psi_{zz} << \Delta_\perp\psi$.

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right)$$

$$\implies k_z = k_0\left(1 - \frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right)$$

$$\implies k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} << 1$$

This helps in the following step

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_0(k_0 - k_z)E_c} = \frac{k_0 - k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0} \cdot \frac{1}{k_0} << 1.$$

Also

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0 - k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0 - k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} << \frac{1}{4}\frac{k_\perp^2}{k_0^2} << 1.$$

Yielding the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp\psi = 0, \tag{1.8}$$

1.5. POLARISATION FIELD

NEW: Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0\vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E}, 0, 0),\ \vec{\mathcal{P}} = (\mathcal{P}, 0, 0),\ \vec{\mathcal{D}} = (\mathcal{D}, 0, 0),$$

Also, we assume that $\mathcal{E}$ is the cw electric field from (1.4) Write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \text{h.o.t.} \tag{1.9}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials TODO: Needs ref. The constants c_i are actually a function of frequency ω_0. Rewrite $c_i = \epsilon_0 \chi^{(\mathrm{i})}(\omega_0)$ where $\chi^{(\mathrm{i})}$ is called the susceptibility.

$$\mathcal{P} = \epsilon_0 \chi^{(1)} \mathcal{E} + \epsilon_0 \chi^{(2)} \mathcal{E}^2 + \epsilon_0 \chi^{(3)} \mathcal{E}^3 + \epsilon_0 \chi^{(4)} \mathcal{E}^4 + \epsilon_0 \chi^{(5)} \mathcal{E}^5 + \text{h.o.t.} \tag{1.10}$$

First we consider linear polarisation. The electric field affects the medium and induces a polarisation proportional to the electric field

$$\mathcal{P} = \mathcal{P}_{\text{lin}} = c\mathcal{E}$$

for some real number c. In fact, we can write

$$\mathcal{P} = \epsilon_0 \chi^{(1)}(\omega_0)\mathcal{E},$$

where $\chi^{(1)}$ is the first-order optical susceptibility, still a function of the frequency ω_0. Then the induction electric field is given by

$$\mathcal{D} = \epsilon_0 \mathcal{E} + \mathcal{P}_{\text{lin}} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}, \quad n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0),$$

where n_0 is the linear index of refraction (or refractive index) of the medium.

TODO: Leads to linear Helmholtz with adjusted k_0...

NEW: Consider the nonlinear polarisation field $\mathcal{P}_{\text{nl}}$ as the difference between the true polarisation and the linear approximation, that is,

$$\mathcal{P} = \mathcal{P}_{\text{lin}} + \mathcal{P}_{\text{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$-\mathcal{P}_{\text{nl}} = \epsilon_0 \chi^{(2)} (-\mathcal{E})^2 + \epsilon_0 \chi^{(3)} (-\mathcal{E})^3 + \epsilon_0 \chi^{(4)} (-\mathcal{E})^4 + \epsilon_0 \chi^{(5)} (-\mathcal{E})^5 + \text{h.o.t.}$$
$$-\mathcal{P}_{\text{nl}} = \epsilon_0 \chi^{(2)} \mathcal{E}^2 - \epsilon_0 \chi^{(3)} \mathcal{E}^3 + \epsilon_0 \chi^{(4)} \mathcal{E}^4 - \epsilon_0 \chi^{(5)} \mathcal{E}^5 + \text{h.o.t.}$$

Where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\text{nl}}$ and we have odd terms only:

$$\mathcal{P}_{\text{nl}} = \epsilon_0 \chi^{(3)} \mathcal{E}^3 + \epsilon_0 \chi^{(5)} \mathcal{E}^5 + \text{h.o.t.} \tag{1.11}$$

The leading-order term is called the Kerr nonlinearity

$$\mathcal{P}_{\text{nl}} \approx \epsilon_0 \chi^{(1)}(\omega_0)\mathcal{E}^3.$$

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1.6. Implications of nonlinear polarisation

TODO: write how this leads to NLH and NLH see notes

1.7. Focusing NLS and solitons

NEW: (General NLS) Substituting $E = e^{ik_0 z}\psi$ in the NLH **??** and applying the paraxial approximation $\psi_{zz} << k_0\psi_z$, we obtain the nonlinear Schrödinger equation (NLS)

$$2ik_0\psi_z(z,\bar{x}) + \Delta_\perp\psi + k_0^2\frac{4n^2}{n_0}|\psi|^2\psi = 0. \tag{1.12}$$

TODO: Instead, use dimensionless NLS (3.4) and rewrite soliton substition to match that. This will yield

$$R'' + \frac{1}{r}R' - R + R^3 = 0$$

as is the entrypoint/simplest form for the next chapters.

REVISE: (Focusing NLS) The previous results lead to the focusing NLS given by

$$i\psi_z(z,\bar{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0. \tag{1.13}$$

Considering envelopes of constant shape (solitons) with

$$\psi_\omega^{\text{soliton}} = e^{i\omega z}R_\omega(\bar{x})$$

leads to an equation in $R_\omega(\bar{x})$ by the following steps

1. $i\psi_z(z,\bar{x}) = i\left(i\omega e^{i\omega z}R_\omega(\bar{x})\right) = -\omega e^{i\omega z}R_\omega(\bar{x})$
2. $\Delta\psi = \left(\Delta e^{i\omega z}\right)R_\omega(\bar{x}) + e^{i\omega z}\left(\Delta R_\omega(\bar{x})\right)$
3. $|\psi|^{2\sigma}\psi = \left|e^{i\omega z}R_\omega(\bar{x})\right|^{2\sigma}e^{i\omega z}R_\omega(\bar{x}) = |R_\omega(\bar{x})|^{2\sigma}e^{i\omega z}R_\omega(\bar{x})$
4. such that
5. $e^{i\omega z}\left[-\omega R_\omega(\bar{x}) + \Delta R_\omega(\bar{x}) + |R_\omega(\bar{x})|^{2\sigma}R_\omega(\bar{x})\right] = 0$
6. and
7. $\Delta R_\omega(\bar{x}) - \omega R_\omega(\bar{x}) + |R_\omega(\bar{x})|^{2\sigma}R_\omega(\bar{x}) = 0$

References

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[3] A. Siegman, *Lasers* (University Science Books, 1986).