On Existence and Uniqueness Results for some Nonlinear Schrödinger equations

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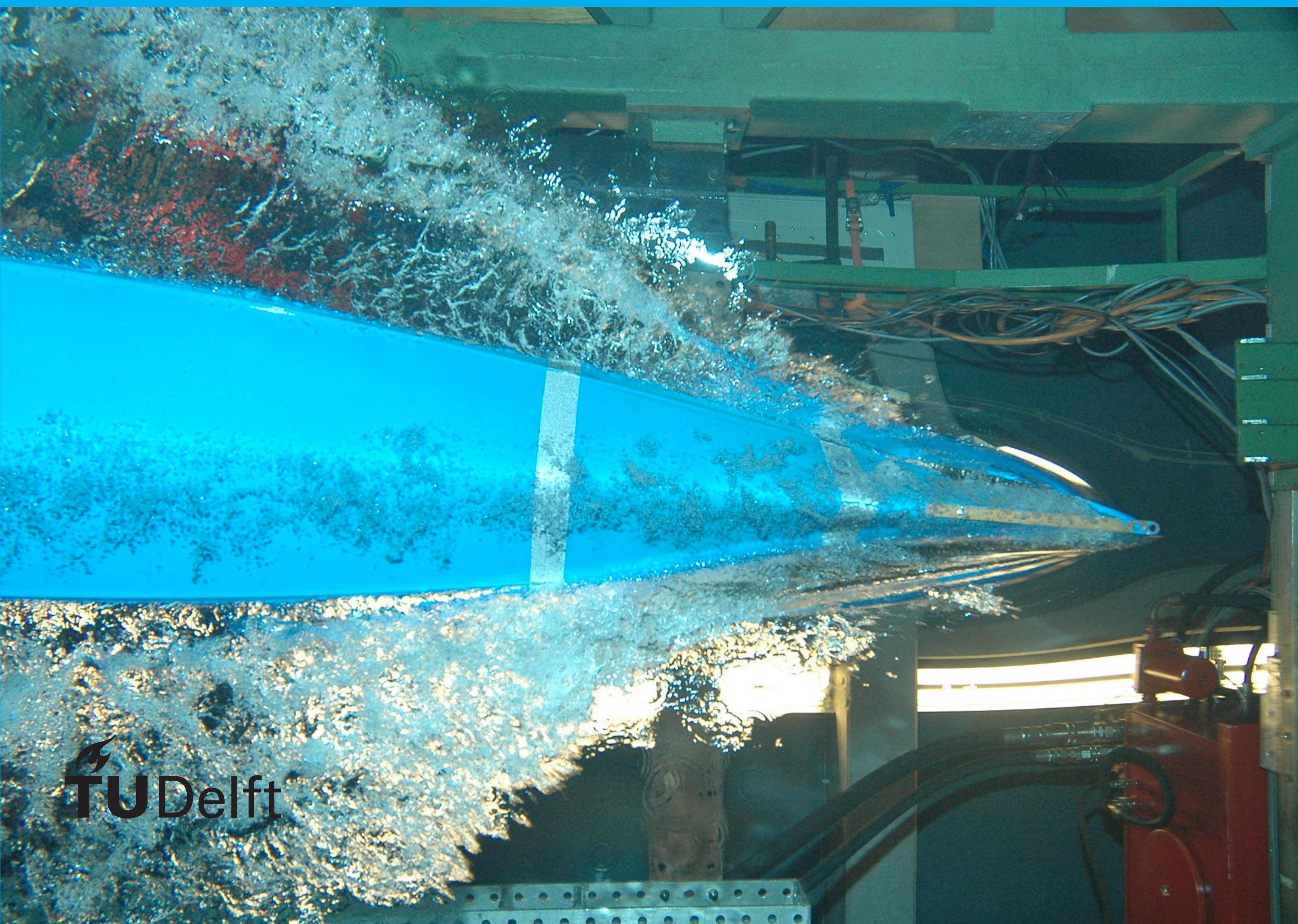

On Existence and Uniqueness Results for some Nonlinear Schrödinger equations

by

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to obtain the degree of Bachelor of Science
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Contents

1

Physics of NLS

1.1. Derive the wave equation from Maxwell

Any electromagnetic wave is governed by Maxwell's laws. In this work, we work in absence of external charges or currents. Then Maxwell's laws for the electric field $\vec{\mathcal{E}}$, magnetic field $\vec{\mathcal{H}}$, induction electric field $\vec{\mathcal{D}}$ and induction magnetic field $\vec{\mathcal{B}}$ are given by:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \qquad (1.1.a) \qquad\qquad \nabla \cdot \vec{\mathcal{D}} = 0, \qquad (1.1.c)$$

$$\nabla \times \vec{\mathcal{H}} = \frac{\partial \vec{\mathcal{D}}}{\partial t}, \qquad (1.1.b) \qquad\qquad \nabla \cdot \vec{\mathcal{B}} = 0. \qquad (1.1.d)$$

The fields are in three-dimensional Cartesian coordinates, for example: $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ in (x, y, z) coordinates. Besides considering no external charges or currents, we consider unitary (relative) permittivities, such that the relation between fields and induction fields (electric or magnetic) is given as:

$$\vec{\mathcal{B}} = \mu_0 \vec{\mathcal{H}}, \qquad (1.2.a) \qquad\qquad \vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}}. \qquad (1.2.b)$$

The notation used here is from "The Nonlinear Schrödinger Equation" by G. Fibich [5, p. 3]. For more background on electrodynamics see "Introduction to Electrodynamics" by D.J. Griffiths [6]. This reference work also includes an introduction to the necessary vector calculus.

We use vector calculus and Maxwell's laws to rewrite the curl of the curl:

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) \overset{(1.1.a)}{=} \nabla \times \left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{\mathcal{B}}\right) \overset{\substack{(1.1.b)\\(1.2.a)}}{=} -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} \overset{(1.2.b)}{=} -\mu_0 \epsilon_0 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \text{ and}$$

$$\nabla \times \left(\nabla \times \vec{\mathcal{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \nabla^2 \vec{\mathcal{E}} = \nabla\left(\nabla \cdot \vec{\mathcal{E}}\right) - \Delta \vec{\mathcal{E}} \overset{\substack{(1.1.c)\\(1.2.b)}}{=} -\Delta \vec{\mathcal{E}}.$$

Combining these and using $\mu_0 \epsilon_0 = 1/c^2$, we arrive at the vector wave equation:

$$\Delta \vec{\mathcal{E}} = \frac{1}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2}. \qquad (1.3)$$

1.2. Validity of plane wave solutions

Stuyding the left and right hand sides of equation (1.3), we see that the vector wave equation is in fact a system of three scalar wave equations.

$$\Delta \vec{\mathcal{E}} = \Delta \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathcal{E}_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} + \frac{\partial^2 \mathcal{E}_x}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2} \end{bmatrix} = \frac{1}{c^2} \begin{bmatrix} \frac{\partial^2 \mathcal{E}_x}{\partial t^2} \\ \frac{\partial^2 \mathcal{E}_y}{\partial t^2} \\ \frac{\partial^2 \mathcal{E}_z}{\partial t^2} \end{bmatrix}$$

$$\Delta \mathcal{E}_j = \sum_{l=1}^{3} \left[\frac{\partial^2 \mathcal{E}_j}{\partial x_l^2} \right] = \frac{1}{c^2} \frac{\partial^2 \mathcal{E}_j}{\partial t^2}.$$

This motivates the following ansatz to such a scalar wave equation:

$$\mathcal{E}_j = E_c e^{i(k_0 z - \omega_0 t)}, \tag{1.4}$$

where k_0 is the wavenumber and ω_0 the frequency. These are so called plane wave solutions. The wavefronts have the simple geometry of an infinite plane at any z-value and the electric field is non-zero in the x and y directions. The wavefronts are spaced by the wavelength λ and the wavenumber k_0 is the reciprocal of the wavelength.

This plane wave travels in the positive z-direction for positive wavenumber k_0 and vice versa. Note that the solution does not depend on x or y. As a result, for a fixed z', the electric field $\mathcal{E}$ is constant in the (x, y, z')-plane.

We substitute (1.4) in equation (1.3). Note that only Δ_z will be non-zero:

$$\Delta \mathcal{E}_j = k_0^2 \cdot E_c e^{i(k_0 z - \omega_0 t)} = \frac{1}{c^2} \omega_0^2 \cdot E_c e^{i(k_0 z - \omega_0 t)}$$

yields the dispersion relation (1.5):

$$k_0^2 = \frac{\omega_0^2}{c^2}. \tag{1.5}$$

For a general direction in (x, y, z)-coordinates, define the wavevector

$$\overrightarrow{k} = (k_x, k_y, k_z),$$

where $\left| \overrightarrow{k}^2 \right| = k_0^2 = k_x^2 + k_y^2 + k_z^2$. This satisfies equation (1.3) when $\overrightarrow{k} \perp \overrightarrow{\mathcal{E}}$ and

$$\mathcal{E}_j = E_c e^{i(\overrightarrow{k} \cdot \overrightarrow{r} - \omega_0 t)}. \tag{1.6}$$

1.3. Derivation of the Helmholtz equation

We consider time-harmonic solutions to the scalar wave equation (1.3) of the form

$$\mathcal{E}_j(x, y, z, t) = e^{i\omega_0 t} E(x, y, z) + \text{c.c}, \tag{1.7}$$

which are continuous wave beam solutions as opposed to pulsed output beams. The continous beam has (approximately) constant power, whereas pulsed beams can reach higher peak powers. For more information on the operating principles of lasers, we refer to [11].

Substituting (1.7) in equation (1.3) and taking the derivatives leads to the expression

$$\begin{aligned} \Delta \left(e^{-i\omega_0 t} E \right) &= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(e^{-i\omega_0 t} E \right) \\ e^{-i\omega_0 t} \Delta E &= \frac{1}{c^2} (-i\omega_0)^2 E e^{-i\omega_0 t}, \end{aligned}$$

where we can divide by $e^{-i\omega_0 t} \neq 0$ and use the dispersion relation (1.5) to arrive at the scalar linear Helmholtz equation for E

$$\Delta E(x, y, z) + k_0^2 E = 0. \tag{1.8}$$

As an example, equation (1.8) is solved by the general-direction plane waves (1.6), where

$$E = E_c e^{i(k_x x + k_y y + k_z z)}.$$

1.4. Derivation of the Linear Schrödinger equation

We write the incoming field $E_0^{\text{inc}}(x,y)$ as a sum of plane waves. Then the electric field $E(x,y,z)$ for non-zero z-value follows from propagation. This is the plane wave spectrum representation of the electromagnetic field and it is essential to Fourier optics. We have

$$\begin{aligned} E_0^{\text{inc}}(x,y) &= \frac{1}{2\pi}\int_D E_c(k_x,k_y)\, e^{i(k_x x+k_y y)}\mathrm{d}k_x\mathrm{d}k_y, \text{ such that} \\ E(x,y,z) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} E_c(k_x,k_y)\, e^{i(k_x x+k_y y+\sqrt{k_0^2-k_x^2-k_y^2}\,z)}\mathrm{d}k_x\mathrm{d}k_y, \end{aligned}$$

where D denotes the (circular) laser input beam domain. For laser beams oriented in the z-direction, most of the plane wave modes are nearly parallel to the z-axis, which implies $k_z \approx k_0$. We define $k_\perp^2 = k_x^2 + k_y^2$, such that $k_0^2 = k_\perp^2 + k_z^2$. It is equivalent to $k_0 \approx k_z$ to say that $k_\perp \ll k_z$.

This motivates studying solutions of the form

$$E = e^{ik_0 z}\psi(x,y,z) \tag{1.9}$$

where $\psi(x,y,z)$ is an envelope (or amplitude) function. The envelope shape may vary over z, in contrast to soliton solutions, see (2.5).

Substituting (1.9) into the Helmholtz equation (1.8) yields

$$\psi_{zz}(x,y,z) + 2ik_0\psi_z + \Delta_\perp\psi = 0, \tag{1.10}$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ such that $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$. Basically, this is the Helmholtz equation for the envelope function $\psi(x,y,z)$. Remember that for lasers beams oriented in the z-direction, the wavenumber k_z dominates over $k_\perp$ such that $k_0 \approx k_z$. The envelope function $\psi(x,y,z)$ will vary slowly in z and curve even more slowly.

Claim: $|\psi_{zz}| \ll k_0|\psi_z|$ and $|\psi_{zz}| \ll \Delta_\perp\psi$.

To see this, we first show that $k_0 - k_z \ll 1$. We factor out k_0^2, take the square root on both sides and linearise the square root term of the right hand side:

$$k_z^2 = k_0^2 + k_\perp^2 = k_0^2\left(1 - \frac{k_\perp^2}{k_0^2}\right) \implies k_z = k_0\left(1-\frac{k_\perp^2}{k_0^2}\right)^{\frac{1}{2}} \approx k_0\left(1 - \frac{1}{2}\frac{k_\perp^2}{k_0^2}\right).$$

Finally, we use $k_\perp \ll k_0$ to obtain the intermediate result:

$$k_0 - k_z \approx k_0 - k_0 + \frac{1}{2}\frac{k_\perp^2}{k_0} = \frac{1}{2}\frac{k_\perp^2}{k_0} \ll 1.$$

For the first statement of the claim, $|\psi_{zz}| \ll k_0|\psi_z|$, it is equivalent to show that the ratio of $|\psi_{zz}|$ over $k_0|\psi_z|$ is much smaller than 1. We calculate the ratio as follows:

$$\frac{[\psi_{zz}]}{[k_0\psi_z]} = \frac{(k_0-k_z)^2 E_c}{k_0(k_0-k_z)E_c} = \frac{k_0-k_z}{k_0} = \frac{k_\perp}{k_0} \approx \frac{1}{2}\frac{k_\perp^2}{k_0}\cdot\frac{1}{k_0} \ll 1.$$

For the other statement of the claim, we calculate:

$$\frac{[\psi_{zz}]}{[\Delta_\perp\psi_z]} = \frac{(k_0-k_z)^2 E_c}{k_\perp^2 E_c} = \frac{(k_0-k_z)^2}{k_\perp^2} \approx \frac{1}{k_\perp^2}\left(\frac{1}{2}\frac{k_\perp^2}{k_0}\right) = \frac{1}{4}\frac{k_\perp^2}{k_0^4} \ll \frac{1}{4}\frac{k_\perp^2}{k_0^2} \ll 1.$$

Using the approxations in equation (1.10) yields the linear Schrödinger equation:

$$2ik_0\psi_z + \Delta_\perp\psi = 0. \tag{1.11}$$

1.5. Polarisation field

Polarisation describes the influence of an electric field on the centers of the electrons of the medium. In our consideration, the medium is isotropic and homogenous. The polarisation field $\vec{P}$ contributes to the induction eletric field

$$\vec{\mathcal{D}} = \epsilon_0 \vec{\mathcal{E}} + \vec{\mathcal{P}}.$$

In the following, we assume that the electric field is linearly polarised, such that

$$\vec{\mathcal{E}} = (\mathcal{E}, 0, 0), \ \vec{\mathcal{P}} = (\mathcal{P}, 0, 0), \ \vec{\mathcal{D}} = (\mathcal{D}, 0, 0),$$

Furthermore, we assume that $\mathcal{E}$ is the continuous wave electric field from (1.7). We write the Taylor expansion of the polarisation field $\mathcal{P} = c\mathcal{E}$ as:

$$\mathcal{P} = c_0 + c_1\mathcal{P} + c_2\mathcal{P}^2 + c_3\mathcal{P}^3 + c_4\mathcal{P}^4 + c_5\mathcal{P}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.12}$$

where the c_i are real for all i. Note that $c_0 = 0$ except in ferro-electric materials. The constants c_i are actually a function of the frequency ω_0. We rewrite $c_i = \epsilon_0\chi^{(\mathrm{i})}(\omega_0)$, where $\chi^{(\mathrm{i})}$ is the i-th order susceptibility. Then equation (1.12) reads:

$$\mathcal{P} = \epsilon_0\chi^{(1)}\mathcal{E} + \epsilon_0\chi^{(2)}\mathcal{E}^2 + \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right) \tag{1.13}$$

First we consider linear polarisation:

$$\mathcal{P}_{\mathrm{lin}} = \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}.$$

Then the induction electric field $\mathcal{D}$ is given by:

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\mathrm{lin}} = \epsilon_0\mathcal{E} + \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E} = \epsilon_0\mathcal{E}\left(1 + \chi^{(1)}(\omega_0)\right) = \epsilon_0 n_0^2(\omega_0)\mathcal{E},$$

where $n_0^2(\omega_0) := 1 + \chi^{(1)}(\omega_0)$ is the linear index of refraction (or refractive index) of the medium.

With this updated induction electric field $\mathcal{D} = \epsilon_0 n_0^2(\omega_0)\mathcal{E}$, we can update the scalar wave equation and Helmholtz equation. Only the dispersion relation is affected by considering linear polarisation:

$$k_0^2 = \frac{\omega_0^2}{c^2} n_0^2(\omega_0). \tag{1.14}$$

We now consider the nonlinear polarisation field $\mathcal{P}_{\mathrm{nl}}$ as the difference between the true polarisation and the linear approximation:

$$\mathcal{P} = \mathcal{P}_{\mathrm{lin}} + \mathcal{P}_{\mathrm{nl}}.$$

In an isotropic medium, the relation between $\mathcal{P}$ and $\mathcal{E}$ should be same in all directions. Replacing $\mathcal{P}$ and $\mathcal{E}$ by $-\mathcal{P}$ and $-\mathcal{E}$ respectively,

$$\begin{aligned} -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\left(-\mathcal{E}\right)^2 + \epsilon_0\chi^{(3)}\left(-\mathcal{E}\right)^3 + \epsilon_0\chi^{(4)}\left(-\mathcal{E}\right)^4 + \epsilon_0\chi^{(5)}\left(-\mathcal{E}\right)^5 + \mathcal{O}\left(\mathcal{P}^6\right) \\ -\mathcal{P}_{\mathrm{nl}} &= \epsilon_0\chi^{(2)}\mathcal{E}^2 - \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(4)}\mathcal{E}^4 - \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^6\right), \end{aligned}$$

where we see that for the even exponents, the negative signs cancel. Hence, the even terms cannot contribute to $\mathcal{P}_{\mathrm{nl}}$ and we have only the odd terms:

$$\mathcal{P}_{\mathrm{nl}} = \epsilon_0\chi^{(3)}\mathcal{E}^3 + \epsilon_0\chi^{(5)}\mathcal{E}^5 + \mathcal{O}\left(\mathcal{P}^7\right) \tag{1.15}$$

The leading-order term is called the Kerr nonlinearity:

$$\mathcal{P}_{\mathrm{nl}} \approx \epsilon_0\chi^{(1)}(\omega_0)\mathcal{E}^3. \tag{1.16}$$

1.6. Implications of nonlinear polarisation

Substituting the continuous wave electric field (1.7) into equation (1.16) yields

$$\mathcal{P}_{\rm nl} \approx \epsilon_0 \chi^{(3)}(\omega_0)\mathcal{E}^3 = 3\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \chi^{(3)}(\omega_0)E^3 e^{3i\omega_0 t} + \text{c.c.},$$

where the second term has a frequency of $3\omega_0$ (third harmonic). This has almost no contribution due to the phase-mismatch with the first harmonic. Hence, we approximate

$$\mathcal{P}_{\rm nl} \approx 3\epsilon_0\chi^{(3)}(\omega_0)\,|E|^2\,Ee^{i\omega_0 t} + \text{c.c.} = 3\epsilon_0\chi^{(3)}(\omega_0)\mathcal{E}.$$

Then we simplify $\mathcal{P}_{\rm nl}$ by defining

$$n_2 := \frac{3\chi^{(3)}}{4\epsilon_0 n_0},$$

so that we obtain the simplified expression

$$\mathcal{P}_{\rm nl} = 4\epsilon_0 n_0 n_2\,|E|^2\,\mathcal{E}.$$

This allows us to write the induction electric field $\mathcal{D}$ as,

$$\mathcal{D} = \epsilon_0\mathcal{E} + \mathcal{P}_{\rm lin} + \mathcal{P}_{\rm nl} = \epsilon_0 n^2\mathcal{E},$$

where

$$n^2 = n_0^2\left(1 + \frac{4n_2}{n_0}\,|E|^2\right) = n_0^2 + 3\chi^{(3)}(\omega_0)\frac{1}{\epsilon_0}|E|^2.$$

For water, $n_2 \sim 10^{-22}$ which justifies neglecting nonlinear effects. With lasers, the nonlinear effect becomes more relevant, but is still weak. For a typical continuous wave laser with $|E| \sim 10^9$, we still have a weak nonlinearity, as $n_2|E| \sim 10^{-4} \ll n_0 \approx 1.33$.

We update equation (1.8) to the scalar nonlinear Helmholtz equation (NLH):

$$\Delta E(x,y,z) + k^2 E = 0, \quad \text{where } k^2 = k_0^2\left(1 + \frac{4n_2}{n_0}|E|^2\right). \tag{1.17}$$

We write $E(x,y,z)$ as the product of the z-propagation and an envelope function $\psi(x,y,z)$:

$$E = e^{ik_0 z}\psi$$

and substitute in (1.17) to obtain:

$$\psi_{zz} + 2ik_0\psi_z + \Delta_\perp\psi + 4k_0^2\frac{n_2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.18}$$

Just as in section 1.4, we apply the paraxial approximation, since for laser beams oriented in the z-direction, we have $\left|\psi_{zz}\right| \ll k_0\left|\psi_z\right|, \left|\psi_{zz}\right| \ll \Delta_\perp\psi$. We finally obtain the nonlinear Schrödinger equation (NLS):

$$2ik_0\psi_z(z,\overline{x}) + \Delta_\perp\psi + k_0^2\frac{4n^2}{n_0}\left|\psi\right|^2\psi = 0. \tag{1.19}$$

1.7. Soliton solutions

The NLS equation (1.19) can be written as a dimensionless equation. Starting from equation (1.18), we apply the rescaling of coordinates $(x,y,z) \to (\tilde{x},\tilde{y},\tilde{z})$ defined by:

$$\tilde{x} = \frac{x}{r_0} \quad \tilde{y} = \frac{y}{r_0} \quad \tilde{z} = \frac{z}{2L_{\rm diff}},$$

where r_0 is the input beam width and $L_{\rm diff}$ is the diffraction length. We refer to chapter 2 of [5] for more information on the geometrical optics of lasers. There, we also find that $L_{\rm diff} = k_0 \cdot r_0^2$. To rescale $\tilde{\psi}$, we define:

$$\tilde{\psi} = \frac{\psi}{E_c}, \quad \text{where } E_c := \max_{x,y}\left|\psi_0(x,y)\right|.$$

Through the rescaling we obtain the dimensionless NLH for $\tilde{\psi}$:

$$\frac{f^2}{4}\tilde{\psi}_{\tilde{z}\tilde{z}}(\tilde{z},\tilde{x},\tilde{y}) + i\tilde{\psi}_{\tilde{z}} + \Delta_{\perp}\tilde{\psi} + \nu\left|\tilde{\psi}\right|^2\tilde{\psi} = 0,$$

that depends on a nonparaxiality parameter f and a nonlinearity parameter ν:

$$f = \frac{1}{r_0 k_0} = \frac{r_0}{L_{\text{diff}}}, \quad \nu = r_0^2 k_0^2 \frac{4n_2}{n_0} E_c^2.$$

Here the approximation of paraxiality is valid for small $f \ll 1$ and this leads to the dimensionless NLS equation (1.20), where the tildes have been dropped for brevity.

$$i\psi_z(z,x,y) + \Delta_{\perp}\psi + \nu\left|\psi\right|^2\psi = 0. \tag{1.20}$$

Radial solitary-wave solutions to (1.20) were considered in [3] with ψ of the form:

$$\psi_{\omega}^{\text{solitary}}(r,z) = e^{i\omega z}R_{\omega}(r), \tag{1.21}$$

where ω is a real number and R_{ω} is the real solution of

$$-\omega R_{\omega} + \Delta_{\perp}R_{\omega}(r) + R_{\omega}^3 = 0.$$

This can be solved in general by, for example,

$$R_{\omega}(r) = \sqrt{\omega}R\left(\sqrt{\omega}r\right).$$

However, taking $\omega = 1$ leads to the simplest soliton equation

$$R''(r) + \frac{1}{r}R' - R + R^3 = 0, \quad 0 < r < \infty, \tag{1.22}$$

subject to initial condition $R'(0) = 0$ and integrability condition $\lim_{r\to\infty} R(r) = 0$. The (numerical) solution is known as the Townes profile, which is positive and monotonically decreasing in r.

2

Existence of ground state

2.1. Initial value problem and nonlinearity

In this chapter, we will study an existence proof for the initial value problem

$$-u''(r) - \frac{n-1}{r}u'(r) = f(u(r)), \quad \text{on } 0 < r < \infty, \tag{2.1}$$

satisfying initial conditions and an integrability condition

$$\begin{cases} u(0) = \alpha, \\ u'(0) = 0 \\ \lim_{r\to\infty} u(r) = 0. \end{cases} \tag{2.2}$$

The existence proof will be based on [1], which generalises earlier results. One of these is the uniqueness result [4], which was later generalised in [7], which forms the basis for the next chapter.

The proof will be by a shooting method, where we categorise the solutions based on their asymptotic behaviour. Furthermore, solutions to the initial value problem equation (2.1) are also positive radial solutions to the more general problem

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

where $f(u)$ is a given nonlinear function. This partial differential equation is relevant to many areas of mathematical physics.

The solutions $R(r)$ to equation (1.22) are solutions $u(r)$ to (2.1) with $n = 2$ and

$$f(u) = -u + u^3.$$

2.2. Definitions of solution sets

A **ground state solution** is strictly decreasing everywhere and has no finite zeroes. Yet, the solution should vanish in the limit as $r \to \infty$.

We define the set G of ground state initial conditions as

$$G := \left\{ \alpha > 0 \,\middle|\, u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r\to\infty} u(r,\alpha) = 0 \right\}. \tag{2.4}$$

We consider two alternatives: either (i) the derivative vanishes, or (ii) the solution vanishes. We define the set P of initial conditions with a vanishing derivative as

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.5}$$

We define the set N of initial conditions with a vanishing solution as

$$N := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u(r_0, \alpha) = 0 \text{ and } u'(r, \alpha) < 0 \text{ for all } r \leqslant r_0 \right\}. \tag{2.6}$$

We note that the sets P and N are disjoint by definition. Either the derivative vanishes first, or the solution vanishes first.

We will show that the sets P and N are non-empty, and open. Then, there exist initial conditions that belong to neither P nor N. Solutions that belong to neither P nor N are everywhere positive and decreasing

$$\begin{cases} u(r,\alpha) > 0 & \text{for } r \geqslant 0, \text{ and} \\ u'(r,\alpha) < 0 & \text{for } r > 0. \end{cases} \tag{2.7}$$

Lastly, we will show that under certain assumptions, such an element belongs to G.

2.3. Assumptions on f

We assume that f is locally Lipschitz continuous from $\mathbb{R}_+ \to \mathbb{R}$ and satisfies $f(0) = 0$. Additionally, we assume that hypotheses (H1)–(H5) are satisfied. Firstly,

$$f(\kappa) = 0, \text{ for some } \kappa > 0. \tag{H1}$$

Secondly, defining $F(t)$ as the integral of $f(t)$

$$F(t) := \int_0^t f(s)\,\mathrm{d}s, \tag{2.8}$$

there exists an initial condition $\alpha > 0$ such that $F(\alpha) > 0$. We define

$$\alpha_0 := \inf\{\alpha > 0 \,|\, F(\alpha) > 0\}. \tag{H2}$$

Thirdly, the right-derivative of $f(s)$ at κ is positive

$$f'(\kappa^+) = \lim_{s \downarrow \kappa} \frac{f(s) - f(\kappa)}{s - \kappa} > 0, \tag{H3}$$

and fourthly, we have

$$f(s) > 0 \quad \text{for } s \in (\kappa, \alpha_0]. \tag{H4}$$

We define

$$\lambda := \inf\{\alpha > \alpha_0 \,|\, f(\alpha) = 0\}, \tag{2.9}$$

and note that $\alpha_0 < \lambda \leqslant \infty$. In the situation where $\lambda = \infty$, we assume

$$\lim_{s \to \infty} \frac{f(s)}{s^l} = 0, \quad \text{with } l < \frac{n+2}{n-2}. \tag{H5}$$

2.4. Main theorem

Theorem 2.1. Let f be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0,\infty)$ such that $f(0) = 0$ and f satisfies hypotheses $(H1) - (H5)$. Then there exists a number $\alpha \in (\alpha_0, \lambda)$ such that the solution $u(r,\alpha) \in C^2(\mathbb{R}_+)$ of the initial value problem

$$\begin{cases} -u''(r) - \dfrac{n-1}{r} u'(r) = f(u(r)), \text{ for } r > 0, \\ u(0) = \alpha, \quad u'(0) = 0 \end{cases} \tag{2.10}$$

is an element of solution set G defined in (2.4)

$$G := \left\{ \alpha > 0 \,\middle|\, u(r,\alpha) > 0 \text{ and } u'(r,\alpha) < 0 \text{ for all } r > 0 \text{ and } \lim_{r \to \infty} u(r,\alpha) = 0 \right\}.$$

Proof. We will show in Lemma 2.1-2.3 that solutions to the differential problem (2.10) are defined for $0 < r < \infty$. Furthermore, by Lemma 2.4 solutions with $\alpha \notin \left(P \dot\cup N\right)$ satisfy

$$\lim_{r\to\infty} u(r,\alpha) = 0.$$

Lastly, we will show that solution sets P and N are non-empty and open. In Lemma 2.5 we show that solution set P is non-empty and open. By similar argument, solution set N is open. For the argument that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

In conclusion, G is non-empty. □

2.5. Interval of definition

Existence of local unique solutions is guaranteed by the Picard-Lindelöf theorem, see for example [12, Theorem. 2.2].

In these circumstances, boundedness of the solution $u(r,\alpha)$ is a sufficient condition for the solution to be defined on the maximal interval $[0,\infty)$. This is also called the blow-up alternative. Either (i) for some $r_0 > 0$ we have

$$\left|u(r_0,\alpha)\right| > M, \quad \text{for all } M > 0,$$

and the solution is defined on $[0, r_0)$. Or (ii) for some $M > 0$ we have

$$\left|u(r,\alpha)\right| \leqslant M, \quad \text{for all } r \geqslant 0,$$

and the solution is defined for all $r \geqslant 0$.

Lemma 2.1. For any initial condition $\alpha > 0$ and $r > 0$, we have the identity

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)). \tag{2.11}$$

Proof. We multiply the IVP (2.1) by $-u'(r)$. Then we integrate from 0 to r to obtain

$$\int_0^r \left[u'(s)u''(s)\right] ds + \int_0^r \left[\frac{n-1}{s}[u'(s)]^2\right] \mathrm{d}s = -\int_0^r \left[u'(s) f(u(s))\right] \mathrm{d}s. \tag{2.12}$$

We use the chain rule simplify the first term in (2.12) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}[u'(r)^2] = 2u'(r)u''(r) \overset{(2.2)}{\Longleftrightarrow} \frac{1}{2}[u'(r)]^2 = \int_0^r \left[u'(s)u''(s)\right] \mathrm{d}s.$$

Then, we rewrite the right-hand side of (2.12)

$$-\int_0^r \left[u'(s) f(u(s))\right] \mathrm{d}s = \int_r^0 \left[\tfrac{\mathrm{d}u}{\mathrm{d}s} f(u(s))\right] \mathrm{d}s$$

and use the fundamental theorem of calculus

$$\int_{u(r)}^{u(0)} f(u)\,\mathrm{d}u = F(u(0)) - F(u(r)).$$

Finally, using $u(0) = \alpha$, we have rewritten (2.12) as

$$\frac{1}{2}\left[u'(r)\right]^2 + (n-1)\int_0^r \left[u'(s)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha) - F(u(r)).$$ □

In this section, we will derive an upper and a lower bound for $u(r,\alpha)$. Since the solution is initially decreasing, possibly the initial condition α is an upper bound.

Lemma 2.2. Let $\alpha > \kappa$. Then $u(r,\alpha) \leqslant u(0,\alpha) = \alpha \quad \text{for } r \geqslant 0.$

Proof. We suppose by contradiction that

$$\alpha < u(r_0,\alpha) < \lambda, \quad \text{for some } r_0 > 0. \tag{2.13}$$

By (H4) and (2.9), we have F non-decreasing on (κ,λ). Then,

$$F(\kappa) < F(\alpha) < F(u(r_0,\alpha)) < F(\lambda).$$

In particular, we have

$$F(\alpha) - F(u(r_0,\alpha)) < 0.$$

This contradicts Lemma 2.1, as the left-hand side is clearly non-negative. □

We will show that $u(r,\alpha)$ has a lower bound for $r<\infty$. Let r_0 be the first zero of $u(r,\alpha)$

$$r_0 := \inf\{\, r>0 \mid u(r,\alpha)=0 \,\}. \tag{2.14}$$

If $r_0=\infty$, then we have $u(r,\alpha)>0$ for all $r>0$. When $r_0<\infty$, we have the following bound on the derivative $u'(r,\alpha)$.

Lemma 2.3. Suppose that there exists $r_0>0$ such that

$$\begin{cases} u(r_0,\alpha)=0 \\ u'(r_0,\alpha)<0. \end{cases} \tag{2.15}$$

If we have $f(u)=0$ for $u\leqslant 0$, then for $r\geqslant r_0$ we have

$$u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha) \geqslant u'(r_0,\alpha). \tag{2.16}$$

Proof. For $u(r,\alpha)\leqslant 0$ the IVP (2.1) reads

$$-u''(r,\alpha) - \frac{n-1}{r}u'(r,\alpha) = 0, \tag{2.17}$$

We solve (2.17) for $u'=u'(r,\alpha)$ and seperate the variables, resulting in

$$\frac{\mathrm{d}u'}{u'} = -\frac{n-1}{r}\,\mathrm{d}r.$$

We integrate the expression from r_0 to r and evaluate the limits

$$\ln u'\big|_{r_0}^{r} = \left[(n-1)\ln r\right]_r^{r_0} \iff \ln u'(r) - \ln u'(r_0) = (n-1)\left[\ln r_0 - \ln r\right].$$

Then, we rewrite the expression to arrive at the desired result

$$\frac{u'(r)}{u'(r_0)} = \left(\frac{r_0}{r}\right)^{n-1} \iff u'(r,\alpha) = \left(\frac{r_0}{r}\right)^{n-1} u'(r_0,\alpha) \geqslant u'(r_0,\alpha). \qquad \square$$

In conclusion, the solution $u(r,\alpha)$ is bounded for bounded r. More specifically, in the case of everywhere positive solutions, we have

$$0 < u(r,\alpha) \leqslant \alpha \quad \text{for all } r>0.$$

Alternatively, for solutions with $u(r_0,\alpha)=0$ and $u'(r_0,\alpha)<0$ by Lemma 2.3 we have

$$u(r,\alpha) \geqslant \int_{r_0}^{r} \left(\frac{r_0}{s}\right)^{n-1} u'(r_0,\alpha)\,\mathrm{d}s > -\infty \quad \text{for } r>r_0, \tag{2.18}$$

such that for $n=2$, we have

$$u(r,\alpha) \geqslant r_0 u'(r_0,\alpha)\left(\ln r - \ln r_0\right) \tag{2.19}$$

and for $n>2$, we have

$$u(r,\alpha) \geqslant \frac{r_0^{n-1}u'(r_0,\alpha)}{2-n}\left(r^{2-n} - r_0^{2-n}\right). \tag{2.20}$$

2.6. Asymptotics of positive decreasing solutions

In this section, we will show that everywhere positive decreasing solutions $u(r,\alpha)$ vanish in the limit as $r \to \infty$.

Lemma 2.4. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function such that $f(0) = 0$. Let $u(r,\alpha_1)$ be a solution to initial value problem (2.1) with $\alpha_1 \in (0,\infty)$ such that

$$\begin{cases} u(r,\alpha_1) > 0 & \text{for all } r \geqslant 0, \quad \text{and} \\ u'(r,\alpha_1) < 0 & \text{for all } r > 0. \end{cases} \tag{2.21}$$

Then the number $l := \lim_{r\to\infty} u(r,\alpha_1)$ satisfies $f(l) = 0$.

If additionally, $f(u)$ satisfies (H3), then $l = 0$.

Proof step 1. By assumption (2.21) on $u(r,\alpha_1)$ and the monotone convergence theorem, we have $0 \leqslant l < \alpha_1$. Then $f(l) < f(\alpha_1)$. We consider the limit as $r \to \infty$ of the IVP (2.1)

$$\lim_{r\to\infty} \left[-u''(r,\alpha_1) - \frac{n-1}{r} u'(r,\alpha_1) \right] = f(l) < \infty. \tag{2.22}$$

We restate equation (2.11)

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2 + (n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha_1) - F(u(r,\alpha_1))$$

and note that the right hand side is finite. We write

$$(n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s} = F(\alpha_1) - F(u(r,\alpha_1)) - \frac{1}{2}\left[u'(r,\alpha_1)\right]^2$$

and note that the left hand side is increasing and bounded above. Hence,

$$\int_0^\infty u'(s,\alpha_1)^2 \frac{\mathrm{d}s}{s} < \infty.$$

We write

$$\frac{1}{2}\left[u'(r,\alpha_1)\right]^2 = F(\alpha_1) - F(u(r,\alpha_1)) - (n-1)\int_0^r \left[u'(s,\alpha_1)\right]^2 \frac{\mathrm{d}s}{s}.$$

Then $\lim_{r\to\infty} u'(r,\alpha_1)^2$ exists. Since $u'(r,\alpha_1) < 0$ and $u(r,\alpha_1)$ is bounded, we have

$$\lim_{r\to\infty} u'(r,\alpha_1) = 0. \tag{2.23}$$

Now, we return to equation (2.22) and use $\lim_{r\to\infty} u'(r,\alpha_1) = 0$ to obtain

$$-\lim_{r\to\infty} \left[u''(r,\alpha_1)\right] = f(l).$$

We have (2.23) and hence, we have

$$\lim_{r\to\infty} u''(r,\alpha_1) = 0.$$

The desired result follows: $f(l) = 0$. □

Proof step 2. The nonlinearity $f(u)$ has more than one zero. Both $f(0) = 0$ and $f(\kappa) = 0$. We will show that under assumption (H3), only $l = 0$ satisfies the IVP (2.1).

Suppose to the contrary that $l = \kappa$. We will use the substitution

$$v(r) = r^{(1/2)(n-1)}\left[u(r,\alpha_1) - \kappa\right] \tag{2.24}$$

in equation (2.1) to obtain a differential equation in $v(r)$. In the remainder of the proof of this lemma, we will abbreviate $u(r,\alpha_1) = u(r)$. We note that $v(r) > 0$ by definition, since we have $u(r) \downarrow \kappa$.

We proceed to calculate the first derivative $v'(r)$

$$v'(r) = \frac{1}{2}(n-1)r^{(n-3)/2}\left[u(r)-\kappa\right] + r^{(n-1)/2}u'(r),$$

and the second derivative $v''(r)$, where we gather the terms by $u(r), u'(r)$ and $u''(r)$

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-5)/2}\left[u(r)-\kappa\right] + (n-1)r^{(n-3)/2}u'(r) + r^{(n-1)/2}u''(r). \tag{2.25}$$

We multiply the IVP (2.1) by $r^{(n-1)/2}$ to obtain

$$-r^{(n-1)/2}u''(r) - (n-1)r^{(n-1)/2}r^{-1}u'(r) = f(u(r))r^{(n-1)/2}. \tag{2.26}$$

We can use this to simplify (2.25) to

$$v''(r) = \frac{1}{4}(n-1)(n-3)r^{(n-1)/2}r^{-2}\left[u(r)-\kappa\right] - f(u(r))r^{(n-1)/2}.$$

Now we factor out $v(r) = r^{(n-1)/2}\left[u(r)-\kappa\right]$ to obtain

$$v''(r) = r^{(n-1)/2}\left[u(r)-\kappa\right]\left\{\frac{1}{4}(n-1)(n-3)r^{-2} - \frac{f(u)}{u(r)-\kappa}\right\}.$$

Lastly, we multiply by -1 to obtain the exact expression from [1] as

$$-v''(r) = \left\{\frac{f(u)}{u(r)-\kappa} - \frac{(n-1)(n-3)}{4r^2}\right\}v. \tag{2.27}$$

In proof step 3, we will show that there exist $\omega > 0$ and $R_1 > 0$, such that

$$\frac{f(u)}{u(r)-\kappa} - \frac{(n-1)(n-3)}{4r^2} \geqslant \omega \quad \text{for all } r \geqslant R_1. \tag{2.28}$$

We have $v''(r) < 0$ for $r \geqslant R_1$, which implies by

$$v'(r) = v'(R_1) + \int_{R_1}^{r} v''(s)\,\mathrm{d}s$$

that

$$v'(r) \downarrow L \geqslant -\infty, \quad \text{as } r \to \infty.$$

Suppose that $L < 0$, then $v(r) \to -\infty$ as $r \to \infty$. However, by (2.24) we have $v > 0$.

Then $L \geqslant 0$. This implies $v'(r) \geqslant 0$ for $r \geqslant R_1$. But then $v(r) \geqslant v(R_1) > 0$ for $r \geqslant R_1$. By (2.28) and (2.27), we have

$$-v''(r) \geqslant \omega v(R_1) > 0,$$

such that $v'(r) \to -\infty$ as $r \to \infty$. This contradicts $L \geqslant 0$. Hence, we have $l = 0$. □

Proof step 3. The first term (2.28) is non-negative and decreasing by (H3). We will write

$$M(r) := \frac{f(u)}{u(r)-\kappa} > 0, \tag{2.29}$$

and rewrite (2.28) to obtain

$$M(r) \geqslant \frac{(n-1)(n-3)}{4r^2} + \omega. \tag{2.30}$$

We choose $2\omega = \max_{r>0} M(r)$ and choose $R_1 > 0$ such that

$$\frac{(n-1)(n-3)}{4r^2} \leqslant \frac{1}{2}M(r) \quad \text{for } r \geqslant R_1.$$ □

2.7. P is non-empty and open

In this section we will show that P is non-empty and open. The proof that N is open is similar to the proof given for P. For the proof that N is non-empty, we refer to "I_- is non-empty" in [1, p. 147].

Lemma 2.5. Solution set P as defined in (2.5)

$$P := \left\{ \alpha > 0 \,\middle|\, \exists r_0 : u'(r_0,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \text{ for all } r \leqslant r_0 \right\}$$

is non-empty and open.

Proof step 1. We will show that solution set P is non-empty. Let $\alpha \in (\kappa, \alpha_0]$. We refer to (H1) and (H2) for the definitions of κ and α_0.

First, we suppose by contradiction that $\alpha \in N$. By the definition of N in (2.6), there exists a number $r_0 > 0$ such that

$$\begin{cases} u(r_0,\alpha) = 0, \\ u'(r,\alpha) < 0 \quad \text{for } \; r \leqslant r_0. \end{cases} \tag{2.31}$$

We restate equation (2.11) from Lemma 2.1 for $r = r_0$ and use $F(u(r_0,\alpha)) = F(0) = 0$

$$\frac{1}{2}\left[u'(r_0,\alpha)\right]^2 + (n-1)\int_0^{r_0} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha). \tag{2.32}$$

The left hand side of (2.32) is positive. For $\alpha \in (\kappa, \alpha_0]$, we have $F(\alpha) < 0$. Hence $\alpha \notin N$.

Next, we suppose that $\alpha \notin P$. Thus $\alpha \notin (P \,\dot\cup\, N)$. We have the situation of (2.7)

$$\begin{cases} u(r,\alpha) > 0 \quad \text{for } r \geqslant 0, \text{ and} \\ u'(r,\alpha) < 0 \quad \text{for } r > 0, \end{cases}$$

which is the setting of Lemma 2.4. Thus, we have $l = 0$ and by equation (2.23), we have

$$\lim_{r\to\infty} u'(r,\alpha) = 0.$$

Then equation (2.32) evaluates to

$$(n-1)\int_0^{\infty} u'(s,\alpha)^2 \frac{\mathrm{d}s}{s} = F(\alpha) < 0,$$

but the left hand side is positive. We have $(\kappa, \alpha_0] \subset P$, since α was chosen arbitrarily. □

Proof step 2. We will show that P is open. Let $\alpha \in P$. There exists

$$r_0 := \inf\left\{ r > 0 \,\middle|\, u'(r,\alpha) = 0 \text{ and } u(r,\alpha) > 0 \right\}$$

such that by the definition of P in (2.5)

$$\begin{cases} u(r,\alpha) > 0 \quad \text{for all } r \in [0, r_0], & \text{(2.33a)} \\ u'(r,\alpha) < 0 \quad \text{for all } r \in (0, r_0). & \text{(2.33b)} \end{cases}$$

Evaluating the IVP (2.1) in r_0 yields

$$u''(r_0,\alpha) = -f(u(r_0,\alpha)).$$

Suppose that $u''(r_0,\alpha) = 0$. Then $-f(u(r_0,\alpha)) = 0$. The zeroes of $f(u)$ are $f(\kappa) = 0$ and $f(0) = 0$. Thus, $u(r_0,\alpha) = \kappa$ by (2.33a).

Then, the differential equation (2.1) with

$$\begin{cases} u(r_0,\alpha)=\kappa, \\ u'(r_0,\alpha)=0, \\ u''(r_0,\alpha)=0 \end{cases}$$

is solved by $u \equiv \kappa$, and by uniqueness of solutions this contradicts $u(0,\alpha)=\alpha>\kappa$. Hence $u''(r_0,\alpha)\neq 0$. Since $u'(r,\alpha)<0$ for $r<r_0$ and $u'(r_0,\alpha)=0$, we have

$$u''(r_0,\alpha)>0.$$

Then there exists $r_1>r_0$, such that

$$u(r,\alpha)>u(r_0,\alpha) \quad \text{for all } r\in(r_0,r_1].$$

Since $u(r,\alpha)$ is pointwise continuous in α, we have

$$\forall \epsilon>0\,\exists\,\delta>0:\ \left|\alpha-\beta\right|<\delta \implies \left|u(r,\alpha)-u(r,\beta)\right|<\epsilon.$$

We define

$$\epsilon := \frac{1}{2}\left(u(r_1,\alpha)-u(r_0,\alpha)\right).$$

For $\delta_{r_0}>0$ sufficiently small, we have

$$\left|u(r_0,\alpha)-u(r_0,\beta)\right|<\epsilon,$$

and for $\delta_{r_1}>0$ sufficiently small, we have

$$\left|u(r_1,\alpha)-u(r_1,\beta)\right|<\epsilon.$$

Let $\delta=\min\left\{\delta_{r_0},\delta_{r_1}\right\}>0$. Then, for $\left|\alpha-\beta\right|<\delta$, we have

$$\begin{cases} u(r_1,\beta)>u(r_0,\beta) \\ \beta>u(r,\beta)>0 \quad \text{for all } r\in(0,r_1]. \end{cases} \tag{2.34}$$

Thus $\beta\in P$ and P is open. □

3

Uniqueness of ground state

3.1. Introduction

In Chapter 2, we studied the existence result [1] for ground state solutions to the equation

$$\Delta u - u = f(u) \quad \text{in } \mathbb{R}^3, \tag{3.1}$$

where $f(u)$ satisfies the general conditions described in Section 2.3. Motivated by Chapter 1, we are interested in the specific case $f(u) = u - u^3$. In this chapter, we study the uniqueness of positive radially symmetric solutions for the equation

$$\Delta u - u + u^3 = 0 \quad \text{in } \mathbb{R}^3. \tag{3.2}$$

The chapter is based on [4], which proves uniqueness of the positive radially symmetric (ground state) solution $u = \phi_1 \in C^2 \cap L^4$ for (3.2). All function spaces considered in this chapter consist of real valued functions on $\mathbb{R}^3$. Furthermore, radial symmetry is with respect to the origin only.

The existence of such a solution ϕ_1 was shown in [8], where $\phi_1 = v_1(|x|)$ solves $f(u) = u - u^k$ with $1 < k \leqslant 4$. In addition, [4] refers to the existence of functions $v_n(|x|) \in C^2([0,\infty))$, where for each $n \geqslant 1$, the solution v_n has exactly $n-1$ isolated zeroes in $[0,\infty)$ and decays exponentially as $r \to \infty$. This was shown in [10], which considered $f(u) = u - ug(u^2)$, as well as [2], which considered

$$i\frac{\partial u}{\partial t} = \Delta u + f(|x|,|u|^2)u$$

under the assumption that $f(|x|,|u|^2) = k(|x|)|u|^\sigma$ for $0 < \sigma < 4$ and $k(|x|)$ a Lipschitz continuous positive bounded function. We note that (3.2) is the specific case when $k = 3$, $g(u^2) = u^2$ and $f(|x|,|u|^2) = u^2$ respectively.

Furthermore, Theorem 3.1 of [4] improves (vague) the result of [9], which studied (3.2) in the context of variational calculus. In [9], they show that the Lagrangian associated with (3.2) is zero in its first variation and the second variation is positive if $\lambda_1 > 1$. The latter is shown only through approximations. Theorem 3.1 of [4] shows that the Rayleigh quotient J associated with (3.2)

$$J(u) = \frac{\left(\int |\nabla u|^2 + u^2 \, dx\right)^2}{\int u^4 \, dx} \tag{3.3}$$

is indeed minimal for

$$u(x) = k\phi_1(x + x_0), \quad \text{where } k \neq 0 \text{ and } x_0 \in \mathbb{R}^3 \tag{3.4}$$

The right hand side of (3.3) is meaningful for admissible functions.

Definition 3.1. A function u is admissible if $u \in H^1$ and $u \neq 0$.

3.2. Preliminary results for the integral equation

The problem (3.2) subject to $u \in L^4$ is equivalent to the integral equation in L^4

$$\begin{cases} u(x) = \int g(x-y)u^3(y)\,\mathrm{d}y, \quad \text{where} \\ g(x) = (4\pi)^{-1}|x|^{-1}e^{-|x|}. \end{cases} \tag{3.5}$$

Here $g(x)$ is the Yukawa (screened Coulomb) potential, see ... below.

The following two subsections discuss (mostly) standard results regarding the Sobolev space H^1 and the convolution operator $\tau : u \to g * u$.

3.2.1. Some results regarding H^1

First, concerning the space H^1, we have the following results:

a) C_0^∞ is dense in H^1.

b) If $u \in H^1$, then $v = |u| \in H^1$ and
$$|u|_{1,2} = |v|_{1,2}.$$

c) If $u \in H^1$, then $u \in L^4$ and
$$|u|_{0,4} \leqslant 2^{-1/4}|u|_{1,2}. \tag{3.6}$$

d) Let V denote the subspace of H^1 consisting of radially symmetric functions. The embedding $V \to L^4$ is compact.

Proof of (a). C_0^∞ smooth functions with compact support. $H^1 = W^{1,2}$. Proof not fully given by Evans 5.3.2 Theorem 2, since we consider $\mathbb{R}^3$ rather than U bounded. □

3.2.2. Some results regarding the convolution operator

e) If $u \in L^{4/3}$, then $v = g * u \in H^1 \subseteq L^4$, $\int u\ v\,\mathrm{d}x > 0$ unless $u = 0$, and v is a weak solution of
$$-\Delta v + v = u. \tag{3.7}$$

f) If $u \in L^1 \cap L^\infty$, then $v = g * u$ has bounded continuous first derivatives and
$$\lim_{|x|\to\infty} v(x) = 0$$
.

g) If $u \in L^1 \cap L^\infty \cap C^1$, then $v = g * u \in C^2$ and v satisfies (3.7).

h) Let X and Y denote the subspaces of $L^{4/3}$ and L^4 respectively, consisting of radially symmetric functions. Then $Y = X^*$ and $\tau : X \to Y$ is compact.

3.3. Minimisation of J

This section first states that a solution $u \in L^4$ must belong to H^1. For $u \in L^4$, $u \neq 0$, we define $\sigma(u)$ by

$$\big(\sigma(u)\big)(x) = c\int g(x-t)u^3(t)\,\mathrm{d}t, \tag{3.8}$$

where $c > 0$ is chosen to normalise $v = \sigma(u)$. That is,

$$\int v^4\,\mathrm{d}x = \int \Big(\sigma^4(u)\Big)(x)\,\mathrm{d}x = c^4 \int \left(\int g(x-t)u^3(t)\,\mathrm{d}t\right)^4 \mathrm{d}x = 1. \tag{3.9}$$

Further details:

- $u \in L^4 \implies u^3 \in L^{4/3}$

- $g*(u^3)\in L^4$ and $g*(u^3)\neq 0$
- Fixed points of σ are non-trivial solutions of (3.5)
- $\sigma : L^4 \to H^1$
- σ is an operator in $H^1\setminus\{0\}$
- L^4 solution of (3.5) must belong to H^1

Lemma 3.1. Let u be an admissible solution with

$$\int u^4\,\mathrm{d}x = 1. \tag{3.10}$$

Then $\sigma(u)$ is admissible and

$$J(\sigma(u)) \leqslant J(u) \tag{3.11}$$

with equality only if $\sigma(u) = u$. Moreover, $\sigma(u)\in L^\infty$ and $v=\sigma^2(u)$ has bounded continuous derivatives and satisfies

$$\lim_{|x|\to\infty} v(x) = 0; \tag{3.12}$$

finally $\sigma^3(u)\in C^2$.

Proof. Proof steps

- $\sigma(u)$ admissible follows from e) of Section 3.2
- By e), (3.8) and (3.10), $w=\sigma(u)$ satisfies

$$c = c\int u^4\,\mathrm{d}x = \int (\nabla w\cdot\nabla u + wu)\,\mathrm{d}x \leqslant |u|_{1,2}|w|_{1,2}.$$

- By e), (3.9) and (3.10)

$$\int\left(|\nabla w|^2 + w^2\right)\mathrm{d}x = c\int wu^3\,\mathrm{d}x \leqslant c|w|_{0,4}|u|_{0,4}^3 = c. \tag{3.13}$$

- Combining the inequalities yields

$$|w|_{1,2}\leqslant |u|_{1,2},$$

- which imply (3.11) in view of (3.8) and (3.10).
- Equality can hold only if w and u are proportional, from the application the Schwarz inequality.
- The constant of proportionality must be 1, by (3.9), (3.10) and (3.13).
- The boundedness of $\sigma(u)$ follows from applying the Schwarz inequality to (3.8) and then using the inequality

$$\int u^6\,\mathrm{d}x \leqslant 48\left(\int|\nabla u|^2\,\mathrm{d}x\right)^3;$$

  from 14, p.12 or 11, Theorem 9.3, p.24.
- The remaining assertions follow from f) and g).

□

The following two lemmata are corollaries of Lemma 3.1.

Lemma 3.2. If $v\in L^4$ is a solution of (3.5) then $v\in C^2$, v has bounded first derivatives, and v satisfies (3.12).

Lemma 3.3. If u is any (radially symmetric) admissible function, then there is a (radially symmetric) admissible function $v \in C^2$ which is positive, has bounded first derivatives and satisfies (3.12) and

$$J(v) \leqslant J(u). \tag{3.14}$$

Moreover, unless u itself has the same properties and is a solution of (3.5) (to within a positive factor), then v can be chosen so that inequality (3.14) is strict.

Proof.
- Except for the positivity and the assertion about radial symmetry, the result follows immediately from lemma 3.1.
- It suffices to prove the positivity assertion for continuous u; we replace such a u by $|u|$, whence by b)

$$J(|u|) = J(u).$$

- By continuity u and $|u|$ must vanish at some point of $\mathbb{R}^3$ unless u is already of one sign.
- If the latter is not the case, then since g is positive, $\sigma(|u|)$ will be positive; we cannot then have $\sigma(|u|) = |u|$, and therefore $J(\sigma(|u|)) < J(|u|) = J(u)$.
- The assertion concerning radial symmetry follows from the observation that σ preserves radial symmetry.

□

Theorem 3.1. Let

$$\lambda_1 = \inf\{J(u) : u \text{ admissible}\}.$$

There exists a $\phi_1 \in V$ with

$$J(\phi_1) = \lambda_1.$$

For $u \in H^1$, $J(u) > \lambda_1$ unless u is of the form (3.4).

Proof. Proof steps

- That J attains an infimum in the class of radially symmetric admissible functions was shown in [8]. This also follows from assertion d) of Section 3.2.
- Suppose that $u \in H^1$ but no translate of u is essentially radially symmetric.
- For the purpose of showing that $J(u) > \lambda$, we can suppose by Lemma 3.3 that u is positive and of class C^2, and that $u(x) \to 0$ as $|x| \to \infty$.
- By the Schwarz symmetrization procedure (see 19 or 16, Section 8) we can produce a radially symmetric admissible function v with $J(v) < J(u)$.
- This shows that the infimum over all admissible functions is the same as its infimum over the radially symmetric functions and therefore that this infimum is attained.
- The final assertion of the theorem follows from Theorem 3.2 below.

□

The proof of Theorem 3.1 shows the desired results of the paper except for the last statement. This requires Theorem 3.2 of the next section.

3.4. Uniqueness theorem

The radially symmetric solutions of (3.2) are of the form

$$u(x) = |x|^{-1} w(|x|),$$

where $w(|x|) = w(r)$ solves

$$w'' - w + r^{-2} w^3 = 0. \tag{3.15}$$

The derivation of (3.15) is included in Section 3.4.1 below. To prove the uniqueness of the ground state solution ϕ_1 for (3.2), it suffices to prove that (3.15) has at most one positive solution satisfying the boundary conditions

$$0 < \lim_{r\to 0} r^{-1} w(r) < \infty, \quad \lim_{r\to\infty} w(r) = 0. \tag{3.16}$$

Similar to the shooting method in Chapter 2, we consider the initial condition

$$\lim_{r\to 0} r^{-1} w(r) = \alpha > 0, \tag{3.17}$$

rather than the boundary conditions (3.16). We write $w = w(r,\alpha)$ to denote the solution $w(r)$ with initial condition $\alpha > 0$. The notation in this chapter differs from the source [4] to agree with Chapter 2. The initial conditions $\alpha > 0$ are categorised into three sets P, G and N, similar to (2.5), (2.4) and (2.6) in Section 2.2.

Furthermore, $z(\alpha) < \infty$ is the first zero of the solution $u(r)$ if $\alpha \in N$ and we write $z(\alpha) = \infty$ if $\alpha \in G \cup N$. Formally, let

$$z(\alpha) := \sup\left\{z^* > 0 \;\middle|\; w(r,\alpha) > 0 \text{ for all } r \in [0, z^*)\right\}. \tag{3.18}$$

We formulate the uniqueness of the ground state solution in terms of this initial condition $\alpha > 0$ in Theorem 3.2 below. The proof method is to show that $z(\alpha)$ is monotonically decreasing and that an initial condition α_2 in a right neighborhood of $\alpha_1 \in G$ must belong to N. The basic facts regarding the problem (3.15), (3.17) are summarised in Lemma 3.4. The proof of Lemma 3.4 is omitted in the original paper [4].

Lemma 3.4. For each $\alpha > 0$ the equation (3.15) has a unique solution $w = w(r,\alpha)$ which is of class $C^2\left((0,\infty)\right)$ and satisfies (3.17). The partial derivatives $w_\alpha(r,\alpha) = \partial_\alpha w(r,\alpha) = \frac{\partial w(r,\alpha)}{\partial \alpha}$ and $w'_\alpha(r,\alpha) = \partial_\alpha w'(r,\alpha) = \frac{\partial w'(r,\alpha)}{\partial \alpha}$ exist for all $r > 0$ and $\alpha > 0$. Furthermore, $w_\alpha = w_\alpha(r,\alpha)$ solves the regular initial value problem

$$\begin{cases} w''_\alpha - w_\alpha + 3r^{-2} w^2 w_\alpha = 0, & \text{on } (0,\infty) \\ w_\alpha(0,\alpha) = 0, \quad w'_\alpha(0,\alpha) = 1, \end{cases} \tag{3.19}$$

Theorem 3.2. There is at most one $\alpha \in G$. That is, at most one $\alpha > 0$ for which

$$w(r,\alpha) > 0 \quad \text{on } (0,\infty) \tag{3.20}$$

and

$$\lim_{r\to\infty} w(r,\alpha) = 0. \tag{3.21}$$

(Expand introduction of lemma) Theorem 3.2 is implied by the following lemma.

Lemma 3.5. (i) If $\alpha > 0$ and $w(r,\alpha) > 0$ on $(0, z(\alpha))$ with $w(z(\alpha),\alpha) = 0$, then $w_\alpha(z(\alpha),\alpha) < 0$.

(ii) If $\alpha > 0$ and $w(r,\alpha)$ satisfies (3.20) and (3.21) then

$$\lim_{r\to\infty} e^{-r} w_\alpha(r,\alpha) < 0. \tag{3.22}$$

Proof of Theorem 3.2. We assume Lemma 3.5. The proof will be in two steps. We first show that $z(\alpha)$ is monotonically decreasing in $\alpha > 0$. Then, we show that if $\alpha_1 > 0$ satisfies the conditions (3.20), (3.21) of Theorem 3.2, then any $\alpha_2 > \alpha_1$ with $\alpha_2 - \alpha_1 > 0$ arbitrarily small has $z(\alpha_2) < \infty$. That is, $\alpha_2 \in N$.

If $\alpha \in N$, then $z(\alpha) < \infty$ and we have $w(z(\alpha),\alpha) = 0$ and $w'(z(\alpha),\alpha) < 0$. Then, by the implicit function theorem $z(\alpha)$ is differentiable with respect to α on N and we have

$$w_\alpha(z(\alpha),\alpha) + w'(z(\alpha),\alpha)\frac{\mathrm{d}z(\alpha)}{\mathrm{d}\alpha} = 0.$$

The term $w_\alpha(z(\alpha),\alpha) < 0$ by lemma 3.4(i) and $w'(z(\alpha),\alpha) < 0$ by the definition of $z(\alpha)$. Hence

$$\frac{\mathrm{d}z(\alpha)}{\mathrm{d}\alpha} < 0 \quad \text{on } N.$$

Therefore, $z(\alpha)$ is monotonically decreasing in $\alpha > 0$. Hence, if N is non-empty, it is a semi-infinite interval.

In the second proof step, we assume that (3.20) and (3.21) hold for $\alpha_1 > 0$, or equivalently, that $\alpha_1 \in G$. Under this assumption, we deduce that α_1 is the left endpoint of N. This will clearly imply Theorem 3.2.

Let $\alpha_1 \in G$ and let $\alpha_2 > \alpha_1$. We show that if $\alpha_2 - \alpha_1$ is sufficiently small, then the assumption

$$w(r,\alpha_2) > 0 \quad \text{on } (0,\infty) \tag{3.23}$$

leads to a contradiction. Note that assumption (3.23) implies $\alpha_2 \notin N$. We put $w_i = w_i(r,\alpha_i)$ for $i = 1,2$[1].

By our assumption on α_1 and ii) of lemma 3.4, we can choose $r_0 > 0$ such that

$$3r^{-2}w_1^2 < \frac{1}{2}, \quad \text{for } r \geqslant r_0 \tag{3.24}$$

and

$$w_\alpha(r_0,\alpha_1) < 0, \quad \text{and } w'_\alpha(r_0,\alpha_1) < 0. \tag{3.25}$$

From lemma 3.4 and (3.25) it follows that if $\alpha_2 > \alpha_1$ and $\alpha_2 - \alpha_1$ is sufficiently small, then

$$w_2(r_0,\alpha_1) < w_1(r_0,\alpha_1), \quad w'_2(r_0,\alpha_1) < w'_1(r_0,\alpha_1). \tag{3.26}$$

We put $v = w_1 - w_2$, so that v satisfies[2]

$$v'' - v + r^{-2}\left(w_1^2 + w_1w_2 + w_2^2\right)v = 0. \tag{3.27}$$

We suppose that $w_2(r,\alpha_2)$ satisfies (3.23) and that for some $r_1 > r_0$

$$0 < w_2 < w_1 \quad \text{on } \left[r_0, r_1\right). \tag{3.28}$$

Such an r_1 exists by (3.26).

- From (3.26), (3.30) and (3.24) it follows that v is positive and convex on $\left[r_0, r_1\right)$;
- moreover, from (3.26), $v'(r_0) > 0$ so that v is increasing on $\left[r_0, r_1\right)$.
- Thus (3.28) holds at $r = r_1$;
- hency by a standard argument we conclude that (3.28) holds on $\left[r_0, \infty\right)$ and that v is increasing there.
- The inequality (3.28) on $\left[r_0,\infty\right)$ implies that

$$\lim_{r\to\infty}\left(w_1^2 + w_1w_2 + w_2^2\right) = 0. \tag{3.29}$$

- Using this fact and the monotone character of v, we conclude from asymptotic integration of (3.30) that v grows exponentially as $r \to \infty$.
- From the definition of v, this is clearly a contradiction of (3.21) for $\alpha = \alpha_1$ and (3.23).

[1] Correcting a typo in the definition of w_i given in [4].
[2] Correcting a typo in (4.12) of [4].

- We conclude therefore that (3.23) cannot hold for $\alpha_2 > \alpha_1$ and $\alpha_2 - \alpha_1$ arbitrarily small;
- therefore $\alpha_2 \in N$ for all $\alpha_2 > \alpha_1$ with $\alpha_2 - \alpha_1$ sufficiently small.
- This completes the proof of Theorem 3.2.

□

By studying the zeroes of $w(r,\alpha)$, we can show that N has a left endpoint.

Lemma 3.6. Let $\alpha > 0$ and let $w = w(r,\alpha)$ either vanish at least once in $(0,\infty)$ or satisfy (3.21); then $\alpha > \sqrt{2}$, $w(r,\alpha) = r$ for precisely one value $r = r_0$ in $(0, z(\alpha))$, and $w'(r_0,\alpha) < 0$.

Proof. Proof steps. The function $v(r,a) = r^{-1}w(r,\alpha)$ satisfies

$$v'' + 2r^{-1}v' - v + v^3 = 0 \tag{3.30}$$

and

$$\lim_{r\to 0} v(r,\alpha) = \alpha, \quad \lim_{r\to 0} v'(r,\alpha) = 0. \tag{3.31}$$

We define the function

$$\Phi(r) = (v')^2 + \frac{1}{2}v^4 - v^2 \quad \big(v = v(r,a)\big)$$

and differentiate with respect to r

$$\Phi'(r) = 2v'v'' + 2v^3v' - 2vv'.$$

By rewriting (3.30) as

$$v'' - v + v^3 = -2r^{-1}v',$$

we calculate $\Phi'(r)$ as

$$\Phi'(r) = 2v'\left(v'' - v + v^3\right) = -4r^{-1}(v')^2.$$

Hence, $\Phi(r)$ is a strictly decreasing function of r.

- Since $-v^2 < \Phi(r)$, it follows from the monotone character of Φ that if $\Phi(r_0) \leqslant 0$ for some r_0 in $[0,\infty)$, then v does not vanish in (r_0,∞) and $\liminf_{r\to\infty} v^2(r) > 0$.
- If $0 < \alpha \leqslant \sqrt{2}$, then from (3.31) it follows that $\Phi(0) \leqslant 0$ and that w neither vanishes on $(0,\infty)$ nor satisfies (3.21), since (3.21) clearly implies $\lim_{r\to\infty} v(r) = 0$.
- Suppose now that $w_\alpha(r,\alpha)$ were to satisfy the hypothesis of lemma 3.6 but that for $r_0 \in (0, z(\alpha))$, $w(r_0) = r_0$ while $w'(r_0) \geqslant 0$.
- Since w is convex as long as $0 < w < r$, this assumption would imply the existence of an r_1 with $r_0 < r_1 < z(\alpha)$ such that $w(r_1) = r_1$, so that the assertion concerning the sign of the slope where w crosses the 45° line reduces to the assertion that there is single such crossing in $(0, z(\alpha))$.
- Suppose that there were two such crossings, r_0, r_1. We would then have $v(r_0) = v(r_1) = 1$, and we can assume that $0 < v < 1$ on (r_0, r_1).
- There is then an $r_3 \in (r_0, r_1)$ with $v'(r_3) = 0$, but then $\Phi(r_3) < 0$.
- This implies, as indicated above, that w cannot vanish on (r_3,∞) nor satisfy (3.21).
- Thus the assumption that $w(r) - r$ can vanish twice in $(0, z(\alpha))$ has led to a contradiction and the proof of the lemma is complete.

□

For $w = w_\alpha(r,\alpha)$ as in Lemma 3.6, it follows from that result that there will exist positive numbers a, b, c which are, respectively, the least positive values of r for which

$$w'(r) = 1, \quad w'(r) = 0, \quad w(r) = r.$$

Moreover, by Lemma 3.6, $0 < a < b < c < z(\alpha)$, $r < w(r)$ on $(0,c)$ and $0 < w(r) < r$ on $(c, z(\alpha))$. Finally, w is concave on $(0,c)$ and convex on $(c, z(\alpha))$.

We shall require the following identities which are valid for $w = w_\alpha(r,\alpha)$, $w_\alpha = w_\alpha(r,\alpha)$:

$$(w' w_\alpha - w'_\alpha w)' = 2r^{-2} w^3 w_\alpha, \tag{3.32}$$

$$(w' w'_\alpha - w'' w_\alpha)' = -2r^{-3} w^3 w_\alpha, \tag{3.33}$$

$$(typo)\big(r(w' w'_\alpha - w'' w_\alpha)' = -2w w_\alpha, \tag{3.34}$$

$$\big((w'-1)w'_\alpha - w'' w_\alpha\big)' = -r^{-3} w_\alpha (w-r)^2(2w+r), \tag{3.35}$$

$$(typo)\Big[r\left((w'-1)w'_\alpha - w'' w_\alpha\right) - (w'-1)w_\alpha\big)\Big]' = r^{-1} w_\alpha (w-r)(3w+r). \tag{3.36}$$

Let y_1 denote the least positive zero of $w_\alpha = w_\alpha(r,\alpha)$.

Lemma 3.7. $\alpha < y_1 < \beta$.

Proof.
- Suppose first that
$$y_1 \leqslant \alpha$$
and integrate (4.20) between 0 and y_1. The expression $q(w-r)(3w+3)$ is positive on $(0, y_1)$, which implies
$$y_1(w'(y_1) - 1)q'(y_1) > 0.$$
- Because of the definition of α the assumption (4.21) implies that $w'(y_1) \geqslant 1$, while clearly $q'(y_1) < 0$, so that the assumption (4.21) has led to a contradiction.
- Suppose next that
$$y_1 \geqslant \beta$$
and integrate (4.18) between 0 and β. This give
$$-w''(\beta)q(\beta) < 0;$$
but (4.22) implies $q(\beta) \geqslant 0$ and clearly $w''(\beta) < 0$, so (4.22) has also led to a contradiction and the lemma is proved.

□

- Since w is concave on $(0,\gamma)$, we have $w'(r) < 1$ on (α,γ). In particular by Lemma 3.4
$$w'(y_1) < 1.$$

Remainder of proof of Lemma 3.5.
- Suppose that q has a zero, say y_2, in $(y_1, z_1]$ and integrate (4.19) between y_1 and y_2. This gives
$$\ldots \mathrm{d}r. \tag{3.37}$$
- We assume (as we obviously can) that $q < 0$ on (y_1, y_2). Then $q'(y_1) < 0$ and by (4.23), $(w'(y_1) - 1) < 0$, so that the right side of (4.24) is positive. Since $q'(y_2) > 0$, (4.24) implies that $w'(y_2) > 1$. This is clearly a contradiction since $w' < 1$ on (α,γ) and, since w is convex on $(\gamma, z_1]$, $w' < 0$ on that interval. Thus $q < 0$ on $(y_1, z_2]$ and i) of Lemma 3.5 is proved.

□

If w satisfies (4.5) and (4.6), then $w'(r) < 0$ on (β,∞) and $q < 0$ on (y_1,∞). Integration of (4.19) from y_1 to γ gives $q'(\gamma) < 0$, and integration of (4.17) from γ then gives

$$\lim_{r\to\infty} (w'(r)q'(r) - w''(r)q(r)) > 0,$$

and this implies that $-q$ grows exponentially. This comples the proof of Lemma 3.5.

3.4.1. Derivation of equation for radially symmetric solutions

Consider radially symmetric solutions to (3.2). Then $u(x) = u(|x|) = u(r)$. This transforms (3.2) to the ODE (2.1), restated here for $N = 3$

$$u'' + \frac{2}{r}u' - u + u^3 = 0 \tag{3.38}$$

Furthermore, substituting $u(r) = r^{-1}w(r)$, we calculate the derivatives of u as

1. $u'(r) = -r^{-2}w(r) + r^{-1}w'(r)$
2. $u''(r) = 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r)$.

We substitute in (3.38) to obtain

$$\begin{aligned} u''(r) + \frac{2}{r} - u(r) + u^3(r) \\ &= 2r^{-3}w(r) - 2r^{-2}w'(r) + r^{-1}w''(r) + \frac{2}{r}\left(-r^{-2}w(r) + r^{-1}w'(r)\right) \\ &\quad - r^{-1}w(r) + r^{-3}w^3(r) = 0, \end{aligned} \tag{3.39}$$

which is simplified to

$$r^{-1}\left(w'' - w + r^{-2}w^3\right) = 0.$$

In conclusion, since $r \neq 0$, we obtain

$$w'' - w + r^{-2}w^3 = 0. \tag{3.40}$$

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