CS166 - Bacteria Growth Model

Minerva University

CS166 - Modeling and Analysis of Complex Systems

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Simulation

The following simulation models the growth of a bacteria and culture medium in a limited space using cellular automata with periodic boundaries and von Neumann's neighbors. The parameters, variables, rules, and assumptions of the simulation are the following:

Parameters

- Grid size: The size of the cell grid with the default value of 100x100.
- Food capacity (**k_f**): The maximum allowance of medium (from now on referred to as "food") at each cell. It has a default value of $k_f = 100$.
- Food growth rate (**g_f**): The rate in which food grows at each cell at each step. $g_f \in [0, \infty]$
- Food reseeding probability (**p_f**): It is the probability that food will spontaneously grow at each cell. It has a default value of $p_{f,} = 0.01$.
- Food diffusion rate (**d_f**): The rate at which food diffuses (or spreads maintaining its total sum) to its von Neumaan's neighboring cells. $d_f \in [0, 1]$.
- Consumption rate (**c_b**): The rate in each bacteria consumes food at each cell. It is the amount of food each bacteria consumes. $c_b \in [0, \infty)$
- Bacteria growth rate (**g_b**): The rate at which bacteria multiply at each cell at each step. $g_b \in [0, 1]$
- Bacteria diffusion rate (**d_b**): The rate at which bacteria diffuses (or spreads) across irs von Neumann's neighboring cells. $d_b \in [0, 1]$

Variables

- Food (**f_t**): The amount of food in a single cell at time step *t*. It is calculated by the following formula. $f_t \in [0, k_f]$

$$f_{t+1} = f_t\left(1 + g_f\left(1 - \frac{f_t}{k_f}\right)\right) \tag{I}$$

- Bacteria (**b_t**): The amount of bacteria in a single cell at time step *t*. It is calculated by the following formula. $b_t \in [0, \infty]$

$$b_{t+1} = b_t(1 + g_b) \tag{II}$$

Rules

The simulation is run according to the following rules:

1) The grid is initialized with food being randomly uniformly scattered across the grid with values between 0 and 100 with probability of 0.2 of selecting each grid.
2) Bacteria is added at the middle cell with a value of 50. The middle will also have food with values of 100 with probability equal to 1.
3) At each step, the following calculations are done for each cell:
    a) Food growth: Food is grown following formula (I)
    b) Food reseeding: Food increases by 1 across the grid with probability p_f.

    $$f = f + 1$$

    c) Food diffusion: A fraction of food moves from the central cell to its neighbors in equal proportions.

    $$f_{central} = f_{central} - d_f \times f_{central}$$
    $$f_{neighbor} = f_{neighbor} + \frac{1}{4} d_f \times f_{central}$$

    d) Bacteria consumption and starvation: If the amount of food in the cell is below than the amount of food the bacteria need to consume, some of them will die by starvation while the others will survive and replicate. Otherwise, the food will decrease according to the total amount required by the bacteria.

    If $f < c_b \times b$:

    $$b = \frac{f}{c_b}$$
    $$f = 0$$

    Else:

    $$f = f - c_b \times b$$

    e) Bacteria reproduction: The bacteria grows according to formula (II).
    f) Bacteria diffusion: The bacteria diffuses in a similar manner as in 3. c) but with a different diffusion rate.

$$b_{central} = b_{central} - d_b \times b_{central}$$
$$b_{neighbor} = b_{neighbor} + \frac{1}{4} d_b \times b_{central}$$

Assumptions

- Periodic boundary: The limits of the grid are periodic; in other words, the limits of the right and leftmost sides are connected while the limits of the upper and lower sides are connected.
- Von Neumann neighborhood: Each cell only diffuses to its north, south, west, and east neighbors, never in the diagonals.
- The food has a maximum bound but none for bacteria at each cell.
- Bacteria growth and diffusion depend on the amount of food.
- Food growth depends on the amount of food in each cell and its neighboring cells; it also depends on random events.

Results

The simulation code is shown in the Appendix A.

The following are the test case for final results showing the amount of bacterial and food growth and their means for the specific set of parameters after 3000 steps:

Test Case 1

```
food_growth_rate = 0.05 # [0, oo]
food_diffusion_rate = 0.01 # [0, 1]
bac_consump_rate = 0.2 # [0, oo]
bac_growth_rate = 0.01 # [0, 1]
bac_diffusion_rate = 0.01 # [0, 1]
```

Table 1: Parameter values for Test Case 1

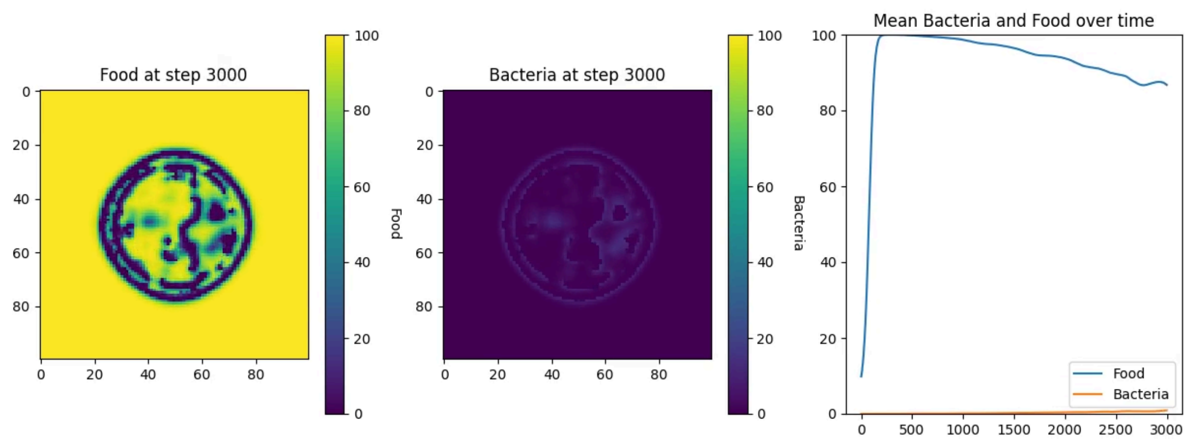

Figure 1: Food and bacteria population with their averages from parameters test case 1

For test case 1, we see that we do not have enough step size to tell if the population will reach a certain equilibrium, since the parameters are too tiny, making the simulation run slow.

Test case 2

```
food_growth_rate = 0.8 # [0, oo]
food_diffusion_rate = 0.1 # [0, 1]
bac_consump_rate = 0.7 # [0, oo]
bac_growth_rate = 0.4 # [0, 1]
bac_diffusion_rate = 0.4 # [0, 1]
```

Table 2: Parameters values for test case 2

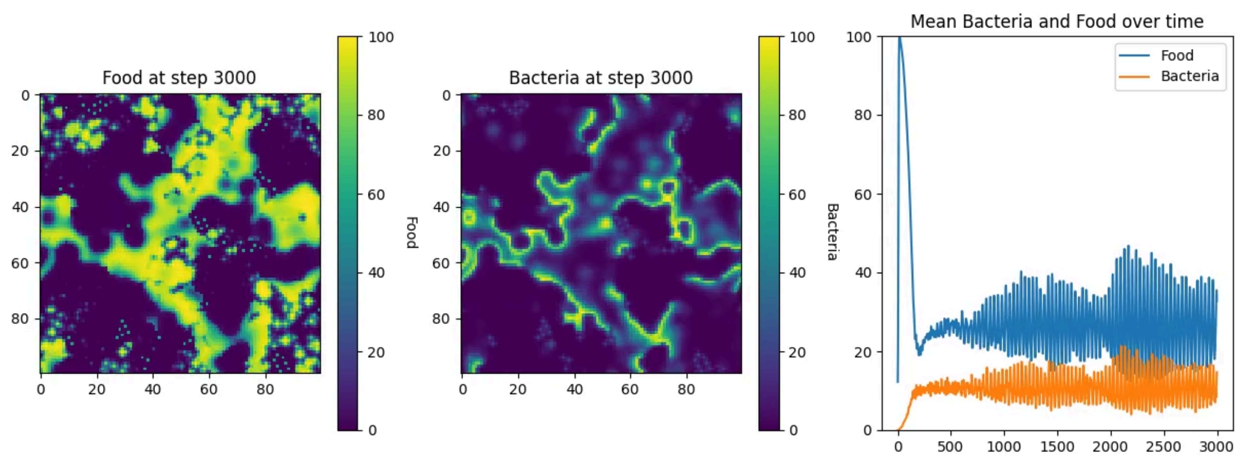

Figure 2: Food and bacteria population with their averages from parameters test case 2

In test case 2, the populations seem to reach an equilibrium after the 500 iterations. After that both populations oscillate around a mean value. Those values seem to be 30 for food and 10 for bacteria as seen in Figure 2.

Test case 3

```
food_growth_rate = 0.8 # [0, oo]
food_diffusion_rate = 0.1 # [0, 1]
bac_consump_rate = 0.7 # [0, oo]
bac_growth_rate = 0.6 # [0, 1]
bac_diffusion_rate = 0.4 # [0, 1]
```

Table 3: Parameters values for test case 3

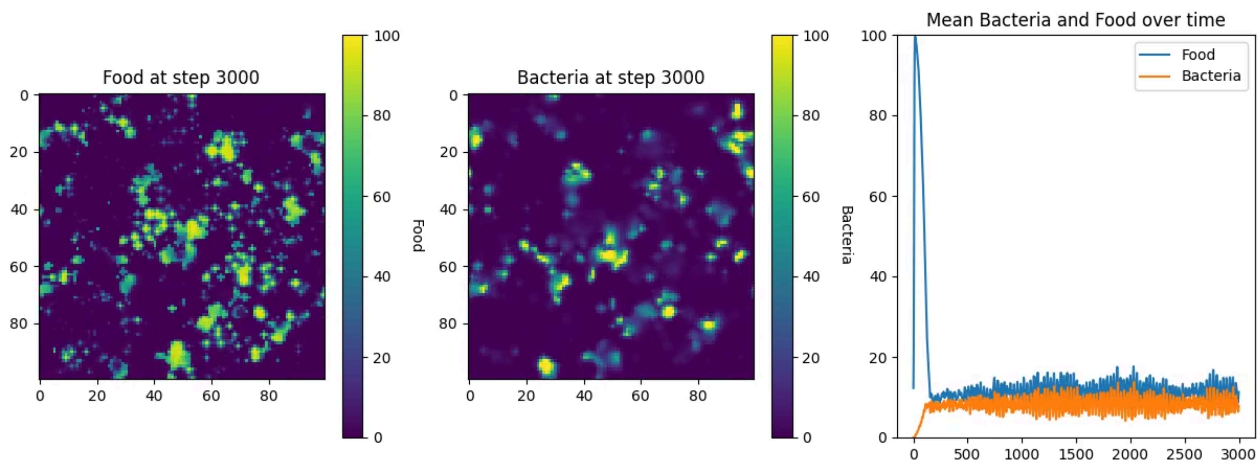

Figure 3: Food and bacteria population with their averages from parameters test case 3

Test case 3 is the same as test case 2, but the bac_growth_rate parameter is 0.6 instead of 0.4. We can see how a tiny parameter change significantly changes the values observed. Here, the populations reach an equilibrium earlier, and the oscillation is lower. The populations' means also oscillate around a lower value, which seems to be around 10 for both food and bacteria, with food's value slightly more than bacteria's. We will see how those parameters affect the population distribution across time for the empirical analysis. For such analysis, we will use the parameters from Test Case 2 as a baseline.

Empirical analysis

For the empirical analysis, we will see how changing the food growth rate, the bacterial growth rate and the consumption rate changes the average amount of food and bacteria at equilibrium. Those values are computed between steps 700 and 1000. For those calculations, the other parameters are set to equal the values in Table 2, given the apparent equilibrium under those values, as observed in Figure 2.

Food Growth Rate

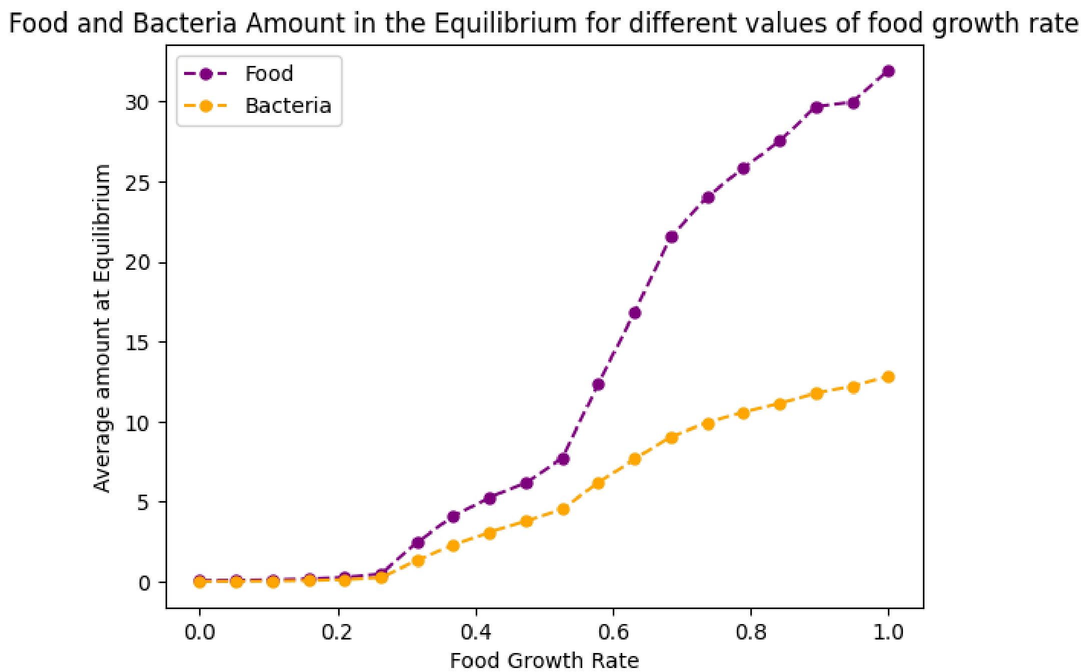

Figure 4: Average amount of food and bacteria at equilibrium for varying values of food growth rate.

In Figure 4, we can notice that both food and bacteria only have a non-zero value for values of $g_f > 0.3$. As notice in the plot, increasing the values of g_f increases the values of food and bacteria in equilibrium, where after $g_f = 0.6$ food gets twice as large as the amount of bacteria, which is observed in the plot of Figure 2. The amount of food is always larger or equal to the amount of bacteria.

Bacteria Growth Rate

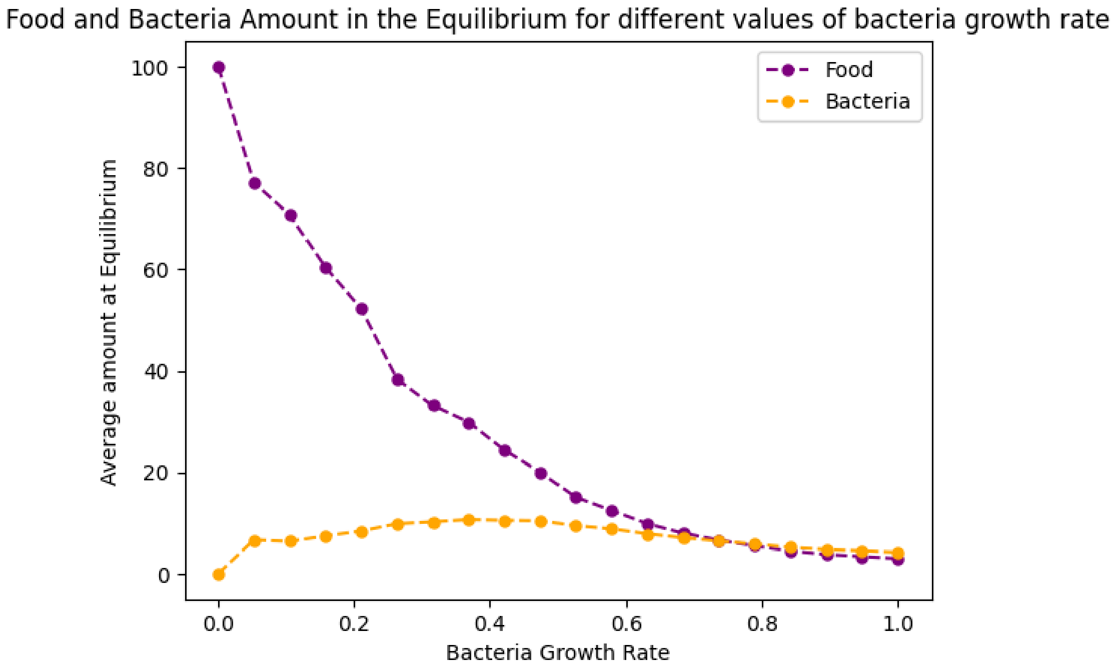

Figure 5: Average amount of food and bacteria in the equilibrium for different values of bacteria growth rate.

In Figure 5, we observe a different trend as in Figure 4. Here, increasing the bacteria growth rate g_b we decrease the amount of food at equilibrium. We can see that it does not matter the value of g_b, the amount of bacteria is always below 20. It has a slight increase before decreasing with a small slope. For higher values of g_b, the amount of food and bacteria equalizes below 10.

Consumption Rate

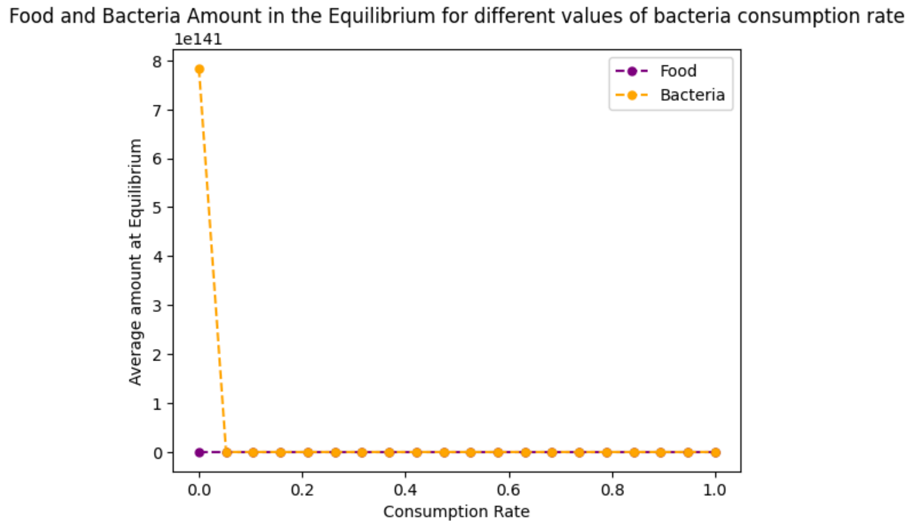

Figure 6-a: Average amount of food and bacteria in the equilibrium for different values of consumption rate.

Figure 6-a seems to be an odd plot, but it happens because when the consumption rate $c_b = 0$, the code calculates $b = \lim_{c_b \to 0} \frac{f}{c_b} = \infty$, which disproportionalities the other values. A way to fix this is by plotting food and bacteria for values of (0, 1].

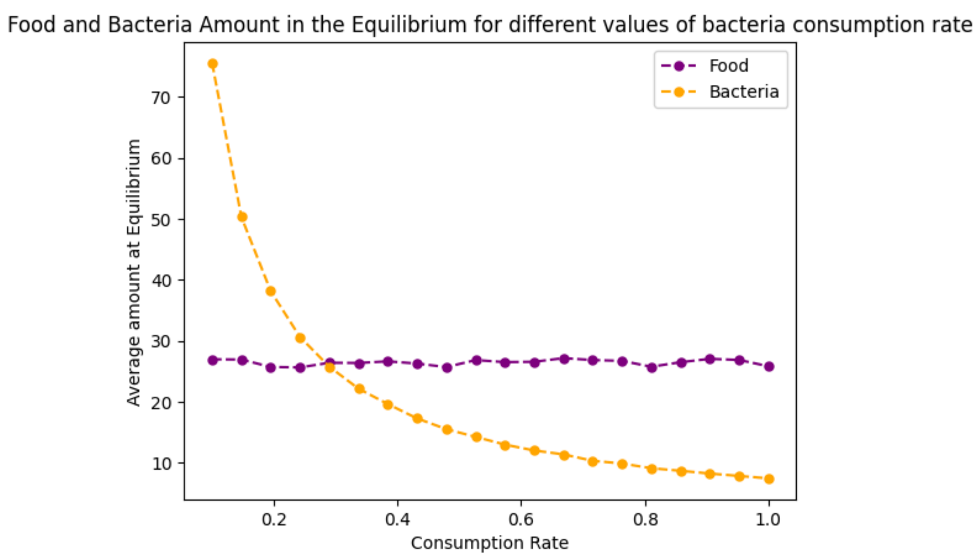

Figure 6-b: Average amount of food and bacteria in the equilibrium for different values of consumption rate in (0, 1]

We can observe in Figure 6-b how the food amount in the equilibrium gets constant around value 30 while the bacteria amount has an exponential decay for increased consumption rate.

Theoretical Analysis

Considering the formula (I) and (II) for food and bacteria growth, we can derive one that implements the other rules of simulation. The following is a Mean Field Approximation (MFA) for the food and bacterial amounts in the equilibrium. For the MFA we will compute in relation to the average amount of bacteria (b) and food (f), the mean-field variables.

Bacteria

Assuming the MFA, diffusion will not be used in this calculation because diffusion does not change the average amount of bacteria in the grid. Therefore, bacterial growth will only rely on starvation caused by consumption and its natural growth. We have that the average amount of bacteria that will grow is the minimum between those that survive starvation or all the bacteria if the food was enough for all of them.

$$b_{t+1} = min\left(\frac{f}{c_b}, b_t\right)\left(1 + g_b\right)$$

Where f is the amount of food in the equilibrium.

In the equilibrium the amount of bacteria does not change, indicating that the number of bacteria who die is equal to the number who is born. We have that $b_{t+1} = b_t = b$.

$$b = min\left(\frac{f}{c_b}, b\right)\left(1 + g_b\right) \tag{III}$$

If $b < \frac{f}{c_b}$:

$$b = b\left(1 + g_b\right)$$

Assuming there is no starvation the bacteria only reaches the equilibrium when $g_b = 0$.

If $\frac{f}{c_b} < b$:

$$b = \frac{f}{c_b}\left(1 + g_b\right)$$

(IV)

Figure 7 is a cobweb plot for function (III). We see that the bacteria amount in the equilibrium is equal to 50 from the parameters in Table 2 and the assumption that $f = 25$. Here we have a stable equilibrium. This result is a bit counterintuitive since as observed in the Figure 2, $b < f$.

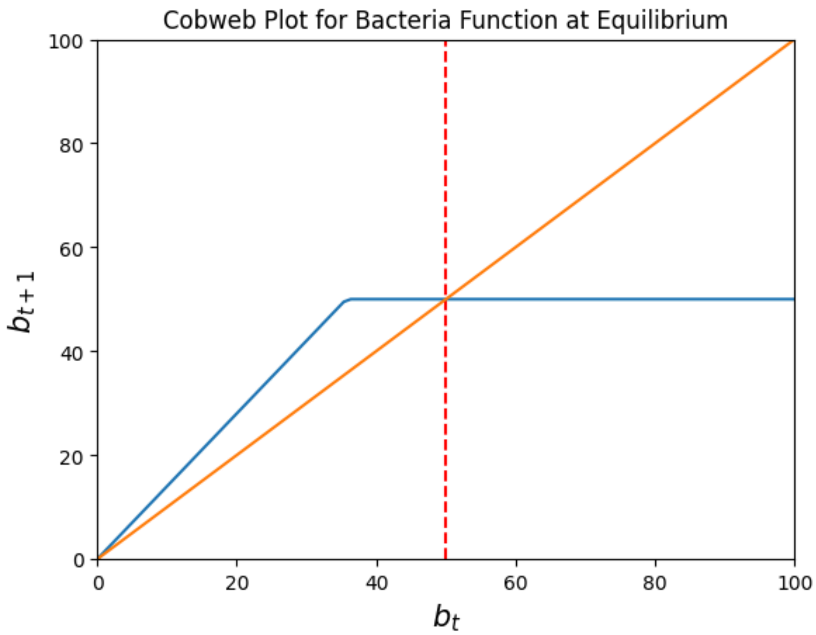

Figure 7: Cobweb plot for Bacteria Function. We see that the equilibrium is reached when $b = 50$.

Food

From the MFA, we assume a cell will behave like the entire grid. We are calculating the average number of foods in equilibrium. Therefore, diffusion does not make sense here because diffusion does not change the average number of food inside the grid. The calculations show that the food increases following (I), but also a factor of $c_b b_t$ decrease it. From the reseeding, we also assume that the food increases by 0.01 at each iteration. Food can never have a negative value. Therefore, we create a bound to limit the food by taking the maximum of the calculation and zero. Then, the food formula is:

$$f_{t+1} = max\left(f_t\left(1 + g_f\left(1 - \frac{f_t}{k_f}\right)\right) - c_b b + 0.01,\ 0\right)$$

In the equilibrium, we do not have a change in the amount of food. Then $f_{t+1} = f_t$:

$$f_t = max\left(f_t\left(1 + g_f\left(1 - \frac{f_t}{k_f}\right)\right) - c_b b + 0.01,\ 0\right)$$

For sake of simplicity, we will write $f_t = f$. Substituting (III) in the previous equation we have:

$$f = max\left(f\left(1 + g_f\left(1 - \frac{f}{k_f}\right)\right) - c_b * min\left(\frac{f}{c_b},\ b\right)\left(1 + g_b\right) + 0.01,\ 0\right) \tag{V}$$

In Figure 8, we plot the previous function in the cobweb plot for values of $b = 15$, assumed from Figure 2 and with parameters values from Table 2.

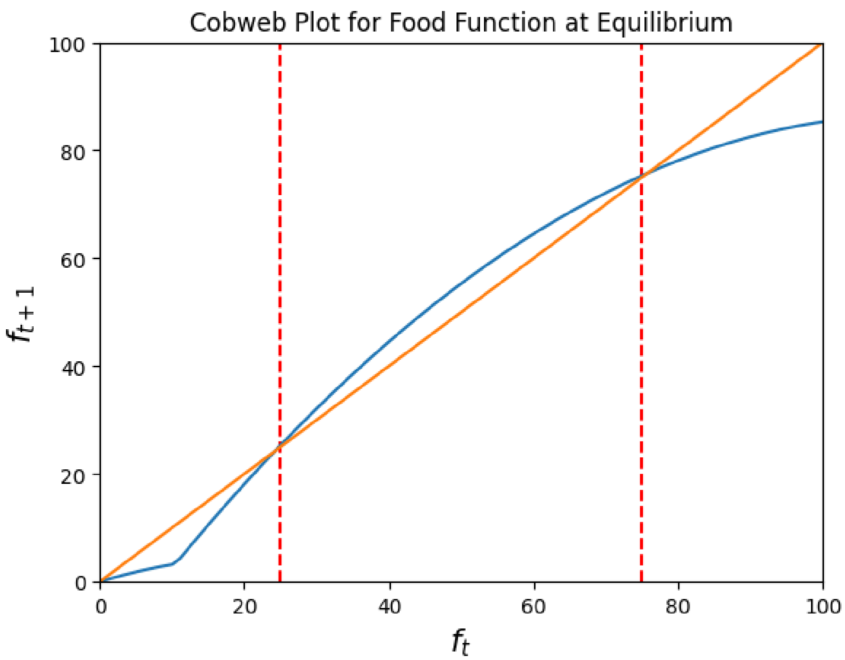

Figure 8: Cobweb Plot for food function showing two equilibrium values around $f = 25,\ 75$

We can see from Figure 8 that there is a unstable equilibrium when $f = 25$, which seems to be pictured in the plot of Figure 2, but also a stable equilibrium when $f = 75$, which was not observed.

Analytically we can solve the first part of the maximum function by developing it further:

$$f = f\left(1 + g_f\left(1 - \frac{f}{k_f}\right)\right) - c_b b_t + 0.01$$

$$f \ = f \ + f \ g_f - \ f \ g_f \frac{f}{k_f} - c_b b_t + 0.01$$

$$0 = -\frac{f^{\ 2} g_f}{k_f} + \ f \ \ g_f - c_b b_t + 0.01$$

By substituting b_t by (IV) instead of (III), we have:

$$0 = -\frac{f^{\ 2} g_f}{k_f} + \ f g_f - c_b \left(\frac{f}{c_b} \left(1 + g_b \right) \right) + 0.01$$

$$0 = -\frac{f^{\ 2} g_f}{k_f} + \ f g_f - f \left(1 + g_b \right) + 0.01 \tag{VI}$$

$$0 = -\frac{f^{\ 2} g_f}{k_f} + \ f \left(g_f - \left(1 + g_b \right) \right) + 0.01$$

$$0 = -\frac{f^2 g_f}{k_f} + \ f \ \left(g_f - g_b - 1 \right) + 0.01$$

For simplification, we will write $g_f - g_b - 1 = \delta$

By solving this equation we have:

$$f = \frac{-\delta \pm \sqrt{\delta^2 - 4 \left(-\frac{g_f}{k_f} \right) 0.01}}{2 \left(-\frac{g_f}{k_f} \right)}$$

$$f = \frac{\left(\delta \pm \sqrt{\delta^2 + 0.04 \left(\frac{g_f}{k_f} \right)} \right) k_f}{2 g_f}$$

For the parameters in Table 2:

$g_f = 0.8$

$g_b = 0.4$

$k_f = 100$

$c_b = 0.7$

$f_1 \approx 0.01666$
$f_2 \approx -75$

The positive value of $f = 0.0166$ is smaller than the expected given the assumption of $b = \frac{f}{c_b}(1 + g_b)$, instead of $b = min\left(\frac{f}{c_b}, b\right)(1 + g_b)$. Figure 8 shows a more accurate solution for f.

Parameters

We can also use functions (IV) and (V) to compute the theoretical values of the food and bacteria in the equilibrium for varying parameters. For comparison sake, we will compute as well the impact that the food growth rate, bacteria growth rate, and consumption rate have on those variables. Here $f = 25$ and $b = 10$ were used alongside the values in Table 2.

Food Growth Rate

Figure 9 shows that increasing the food growth rate g_f there is no impact in the bacteria growth which is not observed in Figure 4. This observation comes from the assumption that f is constant at equilibrium, which is not true as seen in Figure 2. However, Figure 9 does a good job in showing the increase of amount of food when increasing g_f as observed in Figure 4. In both figures the food ranges up to around 32. Figure 4 has a somewhat linear behavior for $g_f > 0.3$.

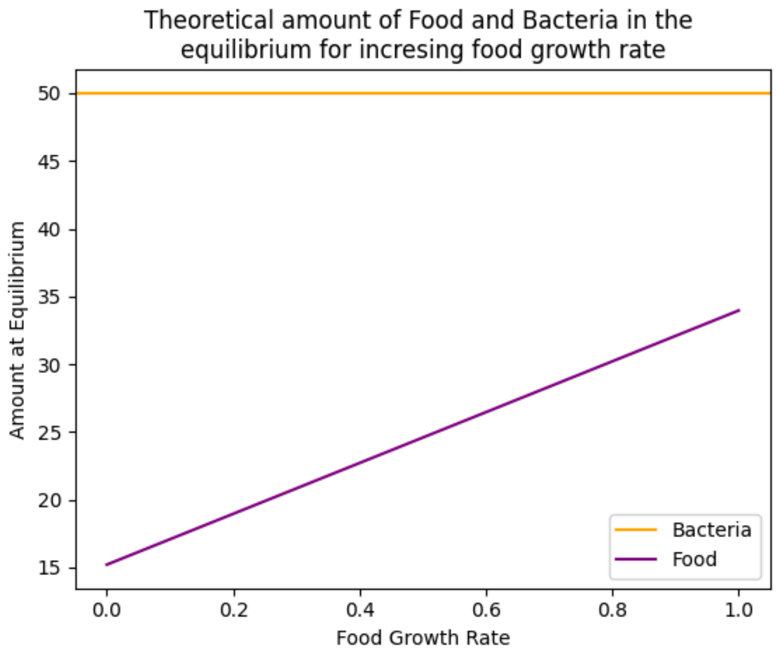

Figure 9: Theoretical amount of food and bacteria in equilibrium for increasing food growth rate

Bacteria Growth Rate

Figure 10 does not depict well the trend of the bacteria growth as shown in Figure 5. As in Figure 9, a cause for such behavior could be the assumption that f is constant and therefore g_f does not impact the growth of the bacteria. However, Figure 10 shows the observed decrease in the amount of food for increasing bacteria growth rate as observed in Figure 5, although the scale does not match.

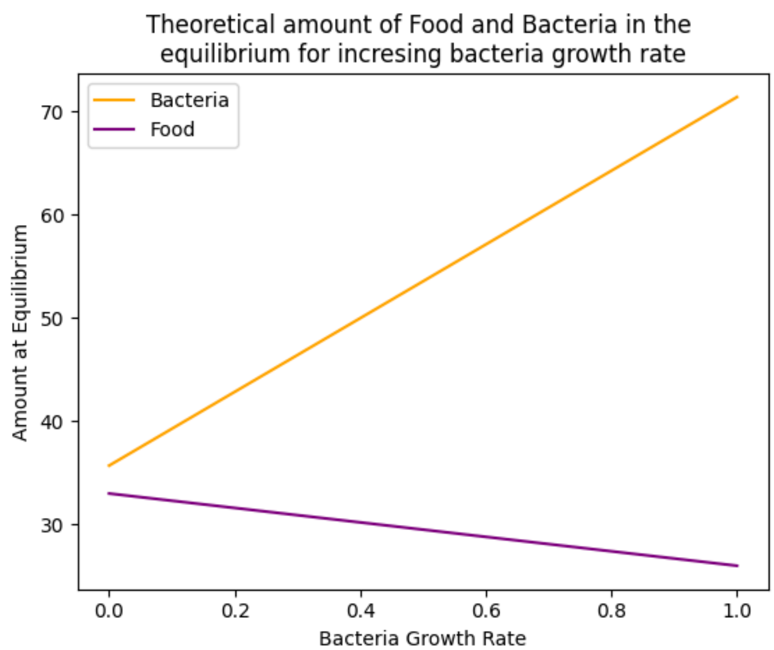

Figure 10: Theoretical amount of food and bacteria in equilibrium for increasing bacteria growth rate

Consumption Rate

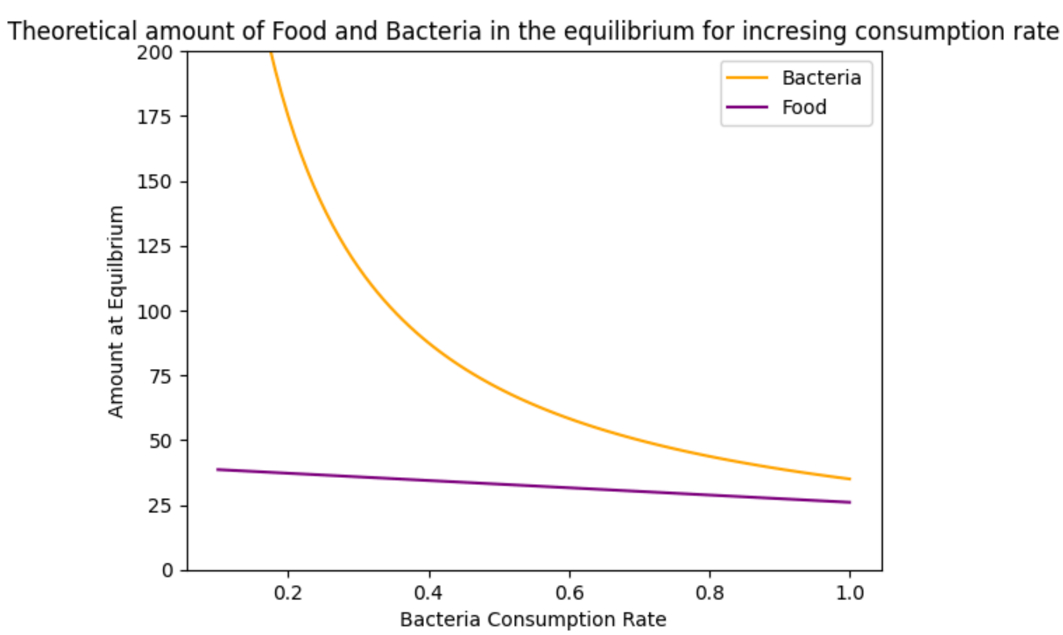

Figure 11: Theoretical amount of food and bacteria in equilibrium for increasing bacteria consumption rate

Figure 11 shows a function of type $y \propto \frac{1}{x}$ for the number of bacteria for different consumption rates while the food has a small negative slope. By comparing both functions, the food plot almost looks like a constant for changing the consumption rate. As we can see, the consumption rate significantly affects the amount of bacteria at equilibrium. The consumption rate can assume values higher than 1. However, here we decided to bound it for consistency. However, we shall see that:

$$b = \lim_{c_b \to \infty} \frac{f}{c_b} = 0$$

We can see how well Figure 11 matches Figure 6-b both in the amount of food and bacteria! From both plots, we can conclude that the consumption rate does not play a role in the amount of food at equilibrium as observed in the function (VI). We also conclude how the bacteria amount exponentially decreases when increasing the consumption rate which is also pictured by function (IV). For higher consumption rates, there is not much difference in the decrease of bacteria population, which indicates that:

$$\frac{f}{c_b}(1 + g_b) = b(1 + g_b)$$

$$\frac{f}{c_b} = b.$$

Appendix

The code that generated the simulation and the plots observed can be found here:
Code for CS166 Assignment 2.ipynb

AI Statement

Grammarly was used for grammar proofreading and Chat GPT was used to debug the code and add docstring.