

$$\frac{\frac{P \longrightarrow P}{P, \top \longrightarrow P} \text{ (L}\top\text{)}}{P \wedge \top \longrightarrow P} \text{ (L}\wedge\text{)}$$

$$\frac{P \longrightarrow P \quad \overline{P \longrightarrow \top}}{P \longrightarrow P \wedge \top} \begin{matrix} \text{(R}\top\text{)} \\ \text{(R}\wedge\text{)} \end{matrix}$$

$$\frac{P \longrightarrow P \quad \overline{\perp \longrightarrow P}}{P \vee \perp \longrightarrow P} \begin{matrix} \text{(L}\perp\text{)} \\ \text{(L}\vee\text{)} \end{matrix}$$

$$\frac{\frac{P \longrightarrow P}{P \longrightarrow P, \perp} \text{ (R}\perp\text{)}}{P \longrightarrow P \vee \perp} \text{ (R}\vee\text{)}$$

$$\frac{\overline{\frac{P, \perp \longrightarrow}{P \wedge \perp \longrightarrow} \text{ (L}\perp\text{)}} \text{ (L}\wedge\text{)}}{P \wedge \perp \longrightarrow \perp} \text{ (R}\perp\text{)}$$

$$\overline{\perp \longrightarrow P \wedge \perp} \text{ (L}\perp\text{)}$$

$$\overline{P \vee \top \longrightarrow \top} \text{ (R}\top\text{)}$$

$$\frac{\overline{\longrightarrow P, \top} \text{ (R}\top\text{)}}{\overline{\longrightarrow P \vee \top} \text{ (R}\vee\text{)}} \text{ (L}\top\text{)}$$

$$\frac{P, P \longrightarrow P}{P \wedge P \longrightarrow P} \text{ (L}\wedge\text{)}$$

$$\frac{P \longrightarrow P \quad P \longrightarrow P}{P \longrightarrow P \wedge P} \text{ (R}\wedge\text{)}$$

$$\frac{P \longrightarrow P \quad P \longrightarrow P}{P \vee P \longrightarrow P} \text{ (L}\vee\text{)}$$

$$\frac{P \longrightarrow P, P}{P \longrightarrow P \vee P} \text{ (R}\vee\text{)}$$

$$\frac{P, Q \longrightarrow Q \quad P, Q \longrightarrow P}{\frac{P, Q \longrightarrow Q \wedge P}{P \wedge Q \longrightarrow Q \wedge P} \text{ (L}\wedge\text{)}} \text{ (R}\wedge\text{)}$$

$$\frac{Q, P \longrightarrow P \quad Q, P \longrightarrow Q}{Q, P \longrightarrow P \wedge Q} \text{ (R}\wedge\text{)}$$

$$\frac{Q, P \longrightarrow P \wedge Q}{Q \wedge P \longrightarrow P \wedge Q} \text{ (L}\wedge\text{)}$$

$$\frac{P \longrightarrow Q, P \quad Q \longrightarrow Q, P}{P \vee Q \longrightarrow Q, P} \text{ (L}\vee\text{)}$$

$$\frac{P \vee Q \longrightarrow Q, P}{P \vee Q \longrightarrow Q \vee P} \text{ (R}\vee\text{)}$$

$$\frac{Q \longrightarrow P, Q \quad P \longrightarrow P, Q}{Q \vee P \longrightarrow P, Q} \text{ (L}\vee\text{)}$$

$$\frac{Q \vee P \longrightarrow P, Q}{Q \vee P \longrightarrow P \vee Q} \text{ (R}\vee\text{)}$$

$$\frac{P, Q \longrightarrow P}{P \wedge Q \longrightarrow P} \text{ (L}\wedge\text{)}$$

$$\frac{P, Q \longrightarrow Q}{P \wedge Q \longrightarrow Q} \text{ (L}\wedge\text{)}$$

$$\frac{P \longrightarrow P, Q}{P \longrightarrow P \vee Q} \text{ (R}\vee\text{)}$$

$$\frac{Q \longrightarrow P, Q}{Q \longrightarrow P \vee Q} \text{ (R}\vee\text{)}$$

$$\frac{P, Q, R \longrightarrow P \quad P, Q, R \longrightarrow Q}{P, Q, R \longrightarrow P \wedge Q} \text{ (R}\wedge\text{)}$$

$$\frac{P, Q, R \longrightarrow P \wedge Q \quad P, Q, R \longrightarrow R}{P, Q, R \longrightarrow (P \wedge Q) \wedge R} \text{ (R}\wedge\text{)}$$

$$\frac{P, Q, R \longrightarrow (P \wedge Q) \wedge R}{P, Q \wedge R \longrightarrow (P \wedge Q) \wedge R} \text{ (L}\wedge\text{)}$$

$$\frac{P, Q \wedge R \longrightarrow (P \wedge Q) \wedge R}{P \wedge Q \wedge R \longrightarrow (P \wedge Q) \wedge R} \text{ (L}\wedge\text{)}$$

$$\frac{P, Q, R \longrightarrow P \quad P, Q, R \longrightarrow Q \quad P, Q, R \longrightarrow R}{P, Q, R \longrightarrow Q \wedge R} \text{ (R}\wedge\text{)}$$

$$\frac{P, Q, R \longrightarrow P \wedge Q \wedge R}{P \wedge Q, R \longrightarrow P \wedge Q \wedge R} \text{ (L}\wedge\text{)}$$

$$\frac{P \wedge Q, R \longrightarrow P \wedge Q \wedge R}{(P \wedge Q) \wedge R \longrightarrow P \wedge Q \wedge R} \text{ (L}\wedge\text{)}$$

$$\frac{P \longrightarrow P, Q, R \quad Q \longrightarrow P, Q, R \quad R \longrightarrow P, Q, R}{Q \vee R \longrightarrow P, Q, R} \text{ (L}\vee\text{)}$$

$$\frac{Q \vee R \longrightarrow P, Q, R}{P \vee Q \vee R \longrightarrow P, Q, R} \text{ (L}\vee\text{)}$$

$$\frac{P \vee Q \vee R \longrightarrow P, Q, R}{P \vee Q \vee R \longrightarrow P \vee Q, R} \text{ (R}\vee\text{)}$$

$$\frac{P \vee Q \vee R \longrightarrow P \vee Q, R}{P \vee Q \vee R \longrightarrow (P \vee Q) \vee R} \text{ (R}\vee\text{)}$$

$$\begin{array}{c}
\frac{P \longrightarrow P, Q, R \quad Q \longrightarrow P, Q, R}{P \vee Q \longrightarrow P, Q, R} \text{ (L}\vee\text{)} \quad \frac{R \longrightarrow P, Q, R}{(P \vee Q) \vee R \longrightarrow P, Q, R} \text{ (L}\vee\text{)} \\
\frac{(P \vee Q) \vee R \longrightarrow P, Q, R}{(P \vee Q) \vee R \longrightarrow P, Q \vee R} \text{ (R}\vee\text{)} \\
\frac{(P \vee Q) \vee R \longrightarrow P, Q \vee R}{(P \vee Q) \vee R \longrightarrow P \vee Q \vee R} \text{ (R}\vee\text{)} \\
\\
\frac{P, Q \longrightarrow P, P \quad P, R \longrightarrow P, P}{P, Q \vee R \longrightarrow P, P} \text{ (L}\vee\text{)} \quad \frac{P, Q \longrightarrow P, R \quad P, R \longrightarrow P, R}{P, Q \vee R \longrightarrow P, R} \text{ (L}\vee\text{)} \quad \frac{P, Q \longrightarrow Q, P \quad P, R \longrightarrow Q, P}{P, Q \vee R \longrightarrow Q, P} \text{ (L}\vee\text{)} \\
\frac{P, Q \vee R \longrightarrow P, P \quad P, Q \vee R \longrightarrow P, R}{P, Q \vee R \longrightarrow P, P \wedge R} \text{ (R}\wedge\text{)} \quad \frac{P, Q \vee R \longrightarrow Q, P}{P, Q \vee R \longrightarrow P, Q \vee R} \text{ (L}\wedge\text{)} \\
\frac{P, Q \vee R \longrightarrow P, P \wedge R \quad P, Q \vee R \longrightarrow P, Q \vee R}{P, Q \vee R \longrightarrow (P \wedge Q) \vee P \wedge R} \text{ (R}\vee\text{)} \\
\frac{P, Q \vee R \longrightarrow (P \wedge Q) \vee P \wedge R}{P \wedge Q \vee R \longrightarrow (P \wedge Q) \vee P \wedge R} \text{ (L}\wedge\text{)} \\
\\
\frac{P, Q \longrightarrow P}{P \wedge Q \longrightarrow P} \text{ (L}\wedge\text{)} \quad \frac{P, R \longrightarrow P}{P \wedge R \longrightarrow P} \text{ (L}\wedge\text{)} \quad \frac{P, Q \longrightarrow Q, R}{P \wedge Q \longrightarrow Q, R} \text{ (L}\wedge\text{)} \quad \frac{P, R \longrightarrow Q, R}{P \wedge R \longrightarrow Q, R} \text{ (L}\wedge\text{)} \\
\frac{P \wedge Q \longrightarrow P \quad P \wedge R \longrightarrow P}{(P \wedge Q) \vee P \wedge R \longrightarrow P} \text{ (L}\vee\text{)} \quad \frac{P \wedge Q \longrightarrow Q, R \quad P \wedge R \longrightarrow Q, R}{(P \wedge Q) \vee P \wedge R \longrightarrow Q, R} \text{ (L}\vee\text{)} \\
\frac{(P \wedge Q) \vee P \wedge R \longrightarrow P \quad (P \wedge Q) \vee P \wedge R \longrightarrow Q, R}{(P \wedge Q) \vee P \wedge R \longrightarrow Q \vee R} \text{ (R}\vee\text{)} \\
\frac{(P \wedge Q) \vee P \wedge R \longrightarrow Q \vee R}{(P \wedge Q) \vee P \wedge R \longrightarrow P \wedge Q \vee R} \text{ (R}\wedge\text{)} \\
\\
\frac{P \longrightarrow P, Q \quad Q, R \longrightarrow P, Q}{P \vee Q \wedge R \longrightarrow P, Q} \text{ (L}\wedge\text{)} \quad \frac{P \longrightarrow P, R \quad Q, R \longrightarrow P, R}{P \vee Q \wedge R \longrightarrow P, R} \text{ (L}\wedge\text{)} \\
\frac{P \vee Q \wedge R \longrightarrow P, Q \quad P \vee Q \wedge R \longrightarrow P, R}{P \vee Q \wedge R \longrightarrow P \vee Q} \text{ (R}\vee\text{)} \quad \frac{P \vee Q \wedge R \longrightarrow P, R \quad P \vee Q \wedge R \longrightarrow P, R}{P \vee Q \wedge R \longrightarrow P \vee R} \text{ (R}\vee\text{)} \\
\frac{P \vee Q \wedge R \longrightarrow P \vee Q \quad P \vee Q \wedge R \longrightarrow P \vee R}{P \vee Q \wedge R \longrightarrow (P \vee Q) \wedge P \vee R} \text{ (R}\wedge\text{)} \\
\\
\frac{P, P \longrightarrow P, Q \quad P, R \longrightarrow P, Q}{P, P \vee R \longrightarrow P, Q} \text{ (L}\vee\text{)} \quad \frac{Q, P \longrightarrow P, Q \quad Q, R \longrightarrow P, Q}{Q, P \vee R \longrightarrow P, Q} \text{ (L}\vee\text{)} \quad \frac{P, P \longrightarrow P, R \quad P, R \longrightarrow P, R}{P, P \vee R \longrightarrow P, R} \text{ (L}\vee\text{)} \\
\frac{P, P \vee R \longrightarrow P, Q \quad Q, P \vee R \longrightarrow P, Q}{P \vee Q, P \vee R \longrightarrow P, Q} \text{ (L}\vee\text{)} \quad \frac{P, P \vee R \longrightarrow P, R}{P \vee Q, P \vee R \longrightarrow P, R} \text{ (L}\vee\text{)} \\
\frac{P \vee Q, P \vee R \longrightarrow P, Q \quad P \vee Q, P \vee R \longrightarrow P, R}{P \vee Q, P \vee R \longrightarrow P, Q \wedge R} \text{ (R}\vee\text{)} \\
\frac{P \vee Q, P \vee R \longrightarrow P, Q \wedge R}{(P \vee Q) \wedge P \vee R \longrightarrow P \vee Q \wedge R} \text{ (L}\wedge\text{)} \\
\\
\frac{P, Q \longrightarrow P, Q \quad P, R \longrightarrow P, Q}{P, Q \vee R \longrightarrow P, Q} \text{ (L}\vee\text{)} \quad \frac{P, Q \longrightarrow P, R \quad P, R \longrightarrow P, R}{P, Q \vee R \longrightarrow P, R} \text{ (L}\vee\text{)} \\
\frac{P, Q \vee R \longrightarrow P, Q \quad P, Q \vee R \longrightarrow P, R}{P, Q \vee R \longrightarrow P \vee Q} \text{ (R}\vee\text{)} \quad \frac{P, Q \vee R \longrightarrow P, R \quad P, Q \vee R \longrightarrow P, R}{P, Q \vee R \longrightarrow P \vee R} \text{ (R}\vee\text{)} \\
\frac{P, Q \vee R \longrightarrow P \vee Q \quad P, Q \vee R \longrightarrow P \vee R}{P, Q \vee R \longrightarrow (P \vee Q) \wedge P \vee R} \text{ (R}\wedge\text{)} \\
\frac{P, Q \vee R \longrightarrow (P \vee Q) \wedge P \vee R}{P \wedge Q \vee R \longrightarrow (P \vee Q) \wedge P \vee R} \text{ (L}\wedge\text{)} \\
\\
\frac{P, Q \longrightarrow P, Q}{P \wedge Q \longrightarrow P, Q} \text{ (L}\wedge\text{)} \quad \frac{P, R \longrightarrow P, Q}{P \wedge R \longrightarrow P, Q} \text{ (L}\wedge\text{)} \quad \frac{P, Q \longrightarrow P, R}{P \wedge Q \longrightarrow P, R} \text{ (L}\wedge\text{)} \quad \frac{P, R \longrightarrow P, R}{P \wedge R \longrightarrow P, R} \text{ (L}\wedge\text{)} \\
\frac{P \wedge Q \longrightarrow P, Q \quad P \wedge R \longrightarrow P, Q}{(P \wedge Q) \vee P \wedge R \longrightarrow P, Q} \text{ (L}\vee\text{)} \quad \frac{P \wedge Q \longrightarrow P, R \quad P \wedge R \longrightarrow P, R}{(P \wedge Q) \vee P \wedge R \longrightarrow P, R} \text{ (L}\vee\text{)} \\
\frac{(P \wedge Q) \vee P \wedge R \longrightarrow P, Q \quad (P \wedge Q) \vee P \wedge R \longrightarrow P, R}{(P \wedge Q) \vee P \wedge R \longrightarrow P, Q \wedge R} \text{ (R}\wedge\text{)} \\
\frac{(P \wedge Q) \vee P \wedge R \longrightarrow P, Q \wedge R}{(P \wedge Q) \vee P \wedge R \longrightarrow P \vee Q \wedge R} \text{ (R}\vee\text{)}
\end{array}$$

$$\frac{\frac{P \longrightarrow P}{\longrightarrow \neg P, P} \text{ (R}\neg\text{)}}{\neg\neg P \longrightarrow P} \text{ (L}\neg\text{)}$$

$$\frac{\frac{P \longrightarrow P}{\neg P, P \longrightarrow} \text{ (L}\neg\text{)}}{P \longrightarrow \neg\neg P} \text{ (R}\neg\text{)}$$

$$\frac{\frac{\frac{P \longrightarrow P}{P, \neg P \longrightarrow} \text{ (L}\neg\text{)}}{P \wedge \neg P \longrightarrow} \text{ (L}\wedge\text{)}}{P \wedge \neg P \longrightarrow \perp} \text{ (R}\perp\text{)}$$

$$\frac{}{\perp \longrightarrow P \wedge \neg P} \text{ (L}\perp\text{)}$$

$$\frac{}{P \vee \neg P \longrightarrow \top} \text{ (R}\top\text{)}$$

$$\frac{\frac{\frac{P \longrightarrow P}{\longrightarrow P, \neg P} \text{ (R}\neg\text{)}}{\longrightarrow P \vee \neg P} \text{ (R}\vee\text{)}}{\top \longrightarrow P \vee \neg P} \text{ (L}\top\text{)}$$

$$\frac{\frac{Q, P \longrightarrow P \quad Q, P \longrightarrow Q}{Q, P \longrightarrow P \wedge Q} \text{ (R}\wedge\text{)}}{\frac{\frac{P \longrightarrow P \wedge Q, \neg Q}{\longrightarrow P \wedge Q, \neg P, \neg Q} \text{ (R}\neg\text{)}}{\longrightarrow P \wedge Q, \neg P \vee \neg Q} \text{ (R}\vee\text{)}} \text{ (L}\neg\text{)}$$

$$\frac{\frac{\frac{P, Q \longrightarrow P}{P, Q, \neg P \longrightarrow} \text{ (L}\neg\text{)}}{P, Q, \neg P \vee \neg Q \longrightarrow} \text{ (L}\wedge\text{)}}{\frac{P \wedge Q, \neg P \vee \neg Q \longrightarrow}{\neg P \vee \neg Q \longrightarrow \neg(P \wedge Q)} \text{ (R}\neg\text{)}} \text{ (L}\vee\text{)}$$

$$\frac{\frac{\frac{P \longrightarrow P, Q}{\longrightarrow P, Q, \neg P} \text{ (R}\neg\text{)}}{\longrightarrow P, Q, \neg P \wedge \neg Q} \text{ (R}\wedge\text{)}}{\frac{\frac{Q \longrightarrow P, Q}{\longrightarrow P, Q, \neg Q} \text{ (R}\neg\text{)}}{\longrightarrow P \vee Q, \neg P \wedge \neg Q} \text{ (R}\vee\text{)}} \text{ (L}\neg\text{)}$$

$$\frac{\frac{P \longrightarrow Q, P \quad Q \longrightarrow Q, P}{P \vee Q \longrightarrow Q, P} \text{ (L}\vee\text{)}}{\frac{\frac{P \vee Q, \neg Q \longrightarrow P}{P \vee Q, \neg P, \neg Q \longrightarrow} \text{ (L}\neg\text{)}}{P \vee Q, \neg P \wedge \neg Q \longrightarrow} \text{ (L}\wedge\text{)}} \text{ (R}\neg\text{)}$$

$$\begin{array}{c}
\frac{P \longrightarrow P, Q \quad P, Q \longrightarrow Q}{P, P \supset Q \longrightarrow Q} \text{ (L } \supset \text{)} \\
\frac{P \supset Q \longrightarrow \neg P, Q}{P \supset Q \longrightarrow \neg P \vee Q} \text{ (R } \neg \text{)} \\
\frac{P \supset Q \longrightarrow \neg P \vee Q}{P \supset Q \longrightarrow \neg P \vee Q} \text{ (R } \vee \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow P, Q}{P, \neg P \longrightarrow Q} \text{ (L } \neg \text{)} \quad \frac{P, Q \longrightarrow Q}{P, \neg P \vee Q \longrightarrow Q} \text{ (L } \vee \text{)} \\
\frac{P, \neg P \vee Q \longrightarrow Q}{\neg P \vee Q \longrightarrow P \supset Q} \text{ (R } \supset \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow P}{P \longrightarrow P, \perp} \text{ (R } \perp \text{)} \\
\frac{P \longrightarrow P, \perp}{\longrightarrow P, P \supset \perp} \text{ (R } \supset \text{)} \\
\frac{\longrightarrow P, P \supset \perp}{\neg P \longrightarrow P \supset \perp} \text{ (L } \neg \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow P \quad \overline{P, \perp \longrightarrow}}{P, P \supset \perp \longrightarrow} \text{ (L } \perp \text{)} \\
\frac{P, P \supset \perp \longrightarrow}{P \supset \perp \longrightarrow \neg P} \text{ (L } \supset \text{)} \\
\frac{P \supset \perp \longrightarrow \neg P}{P \supset \perp \longrightarrow \neg P} \text{ (R } \neg \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow P, Q \quad P, Q \longrightarrow Q}{P, P \supset Q \longrightarrow Q} \text{ (L } \supset \text{)} \\
\frac{P \supset Q \longrightarrow Q, \neg P}{\neg Q, P \supset Q \longrightarrow \neg P} \text{ (R } \neg \text{)} \\
\frac{\neg Q, P \supset Q \longrightarrow \neg P}{P \supset Q \longrightarrow \neg Q \supset \neg P} \text{ (L } \neg \text{)} \\
\frac{P \supset Q \longrightarrow \neg Q \supset \neg P}{P \supset Q \longrightarrow \neg Q \supset \neg P} \text{ (R } \supset \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{Q, P \longrightarrow Q}{P \longrightarrow \neg Q, Q} \text{ (R } \neg \text{)} \quad \frac{P \longrightarrow P, Q}{P, \neg P \longrightarrow Q} \text{ (L } \neg \text{)} \\
\frac{P, \neg Q \supset \neg P \longrightarrow Q}{\neg Q \supset \neg P \longrightarrow P \supset Q} \text{ (L } \supset \text{)} \\
\frac{\neg Q \supset \neg P \longrightarrow P \supset Q}{\neg Q \supset \neg P \longrightarrow P \supset Q} \text{ (R } \supset \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P, Q \longrightarrow P, R \quad \frac{P, Q \longrightarrow Q, R \quad P, Q, R \longrightarrow R}{P, Q, Q \supset R \longrightarrow R} \text{ (L } \supset \text{)}}{P, Q, P \supset Q \supset R \longrightarrow R} \text{ (L } \supset \text{)} \\
\frac{P, Q, P \supset Q \supset R \longrightarrow R}{Q, P \supset Q \supset R \longrightarrow P \supset R} \text{ (R } \supset \text{)} \\
\frac{Q, P \supset Q \supset R \longrightarrow P \supset R}{P \supset Q \supset R \longrightarrow Q \supset P \supset R} \text{ (R } \supset \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{Q, P \longrightarrow Q, R \quad \frac{Q, P \longrightarrow P, R \quad Q, P, R \longrightarrow R}{Q, P, P \supset R \longrightarrow R} \text{ (L } \supset \text{)}}{Q, P, Q \supset P \supset R \longrightarrow R} \text{ (L } \supset \text{)} \\
\frac{Q, P, Q \supset P \supset R \longrightarrow R}{P, Q \supset P \supset R \longrightarrow Q \supset R} \text{ (R } \supset \text{)} \\
\frac{P, Q \supset P \supset R \longrightarrow Q \supset R}{Q \supset P \supset R \longrightarrow P \supset Q \supset R} \text{ (R } \supset \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P, Q \longrightarrow P, R \quad \frac{P, Q \longrightarrow Q, R \quad P, Q, R \longrightarrow R}{P, Q, Q \supset R \longrightarrow R} (L \supset)}{P, Q, P \supset Q \supset R \longrightarrow R} (L \supset) \\
\frac{P, Q, P \supset Q \supset R \longrightarrow R}{P \wedge Q, P \supset Q \supset R \longrightarrow R} (L \wedge) \\
\frac{P \wedge Q, P \supset Q \supset R \longrightarrow R}{P \supset Q \supset R \longrightarrow P \wedge Q \supset R} (R \supset)
\end{array}$$

$$\begin{array}{c}
\frac{Q, P \longrightarrow P, R \quad Q, P \longrightarrow Q, R}{Q, P \longrightarrow P \wedge Q, R} (R \wedge) \\
\frac{Q, P \longrightarrow P \wedge Q, R \quad Q, P, R \longrightarrow R}{Q, P, P \wedge Q \supset R \longrightarrow R} (L \supset) \\
\frac{Q, P, P \wedge Q \supset R \longrightarrow R}{P, P \wedge Q \supset R \longrightarrow Q \supset R} (R \supset) \\
\frac{P, P \wedge Q \supset R \longrightarrow Q \supset R}{P \wedge Q \supset R \longrightarrow P \supset Q \supset R} (R \supset)
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow P, P, Q \quad P, R \longrightarrow P, Q}{P, P \supset R \longrightarrow P, Q} (L \supset) \quad \frac{P, Q \longrightarrow P, Q \quad P, Q, R \longrightarrow Q}{P, Q, P \supset R \longrightarrow Q} (L \supset) \quad \frac{P \longrightarrow P, P, R \quad P, R \longrightarrow P, R}{P, P \supset R \longrightarrow P, R} \\
\frac{P, P \supset R \longrightarrow P, Q \quad P, P \supset R \longrightarrow P, R}{P, P \supset Q, P \supset R \longrightarrow Q} (L \supset) \quad \frac{P, P \supset Q, P \supset R \longrightarrow Q \wedge R}{P \supset Q, P \supset R \longrightarrow P \supset Q \wedge R} (R \supset) \\
\frac{P, P \supset Q, P \supset R \longrightarrow Q \wedge R}{(P \supset Q) \wedge (P \supset R) \longrightarrow P \supset Q \wedge R} (L \wedge)
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow P, Q \quad \frac{P, Q, R \longrightarrow Q}{P, Q \wedge R \longrightarrow Q} (L \wedge)}{P, P \supset Q \wedge R \longrightarrow Q} (L \supset) \quad \frac{P \longrightarrow P, R \quad \frac{P, Q, R \longrightarrow R}{P, Q \wedge R \longrightarrow R} (L \wedge)}{P, P \supset Q \wedge R \longrightarrow R} (L \supset) \\
\frac{P, P \supset Q \wedge R \longrightarrow Q}{P \supset Q \wedge R \longrightarrow P \supset Q} (R \supset) \quad \frac{P, P \supset Q \wedge R \longrightarrow R}{P \supset Q \wedge R \longrightarrow P \supset R} (R \supset) \\
\frac{P \supset Q \wedge R \longrightarrow P \supset Q \quad P \supset Q \wedge R \longrightarrow P \supset R}{P \supset Q \wedge R \longrightarrow (P \supset Q) \wedge (P \supset R)} (R \wedge)
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow Q, P, R \quad P, R \longrightarrow P, R}{P, Q \supset R \longrightarrow P, R} (L \supset) \quad \frac{P, R \longrightarrow Q, R \quad P, R, R \longrightarrow R}{P, R, Q \supset R \longrightarrow R} (L \supset) \quad \frac{Q \longrightarrow Q, P, R \quad Q, R \longrightarrow P, R}{Q, Q \supset R \longrightarrow P, R} \\
\frac{P, Q \supset R \longrightarrow P, R \quad Q, Q \supset R \longrightarrow P, R}{P, P \supset R, Q \supset R \longrightarrow R} (L \supset) \quad \frac{P \vee Q, P \supset R, Q \supset R \longrightarrow R}{P \supset R, Q \supset R \longrightarrow P \vee Q \supset R} (R \supset) \\
\frac{P \vee Q, P \supset R, Q \supset R \longrightarrow R}{(P \supset R) \wedge (Q \supset R) \longrightarrow P \vee Q \supset R} (L \wedge)
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow P, Q, R}{P \longrightarrow P \vee Q, R} (R \vee) \quad \frac{P, R \longrightarrow R}{P, P \vee Q \supset R \longrightarrow R} (L \supset) \quad \frac{Q \longrightarrow P, Q, R}{Q \longrightarrow P \vee Q, R} (R \vee) \quad \frac{Q, R \longrightarrow R}{Q, P \vee Q \supset R \longrightarrow R} (L \supset) \\
\frac{P, P \vee Q \supset R \longrightarrow R}{P \vee Q \supset R \longrightarrow P \supset R} (R \supset) \quad \frac{Q, P \vee Q \supset R \longrightarrow R}{P \vee Q \supset R \longrightarrow Q \supset R} (R \supset) \\
\frac{P \vee Q \supset R \longrightarrow P \supset R \quad P \vee Q \supset R \longrightarrow Q \supset R}{P \vee Q \supset R \longrightarrow (P \supset R) \wedge (Q \supset R)} (R \wedge)
\end{array}$$

$$\begin{array}{c}
\frac{P \longrightarrow Q, P, R \quad P, R \longrightarrow P, R}{P, Q \supset R \longrightarrow P, R} (L \supset) \quad \frac{P, Q \longrightarrow Q, R \quad P, Q, R \longrightarrow R}{P, Q, Q \supset R \longrightarrow R} (L \supset) \\
\frac{P, Q \supset R \longrightarrow P, R \quad P, Q, Q \supset R \longrightarrow R}{P, P \supset Q, Q \supset R \longrightarrow R} (L \supset) \\
\frac{P, P \supset Q, Q \supset R \longrightarrow R}{P \supset Q, Q \supset R \longrightarrow P \supset R} (R \supset)
\end{array}$$

$$\frac{\frac{Q, P \longrightarrow P}{P \longrightarrow Q \supset P} (\text{R} \supset)}{\longrightarrow P \supset Q \supset P} (\text{R} \supset)$$

$$\frac{\frac{P \longrightarrow P, P, R \quad \frac{P \longrightarrow Q, P, R \quad P, R \longrightarrow P, R}{P, Q \supset R \longrightarrow P, R} (\text{L} \supset)}{P, P \supset Q \supset R \longrightarrow P, R} (\text{L} \supset) \quad \frac{P, Q \longrightarrow P, R \quad \frac{P, Q \longrightarrow Q, R \quad P, Q, R \longrightarrow R}{P, Q, Q \supset R \longrightarrow R} (\text{L} \supset)}{P, Q, P \supset Q \supset R \longrightarrow R} (\text{L} \supset)}{\frac{\frac{P, P \supset Q, P \supset Q \supset R \longrightarrow R}{P \supset Q, P \supset Q \supset R \longrightarrow P \supset R} (\text{R} \supset)}{P \supset Q \supset R \longrightarrow (P \supset Q) \supset P \supset R} (\text{R} \supset)} (\text{R} \supset)}$$

$$\frac{\frac{P \longrightarrow P, P \quad \frac{P \longrightarrow Q, P}{P, \neg Q \longrightarrow P} (\text{L} \neg)}{P, P \supset \neg Q \longrightarrow P} (\text{L} \supset) \quad \frac{P, Q \longrightarrow P \quad \frac{P, Q \longrightarrow Q}{P, Q, \neg Q \longrightarrow} (\text{L} \neg)}{P, Q, P \supset \neg Q \longrightarrow} (\text{L} \supset)}{\frac{\frac{P, P \supset Q, P \supset \neg Q \longrightarrow}{P \supset Q, P \supset \neg Q \longrightarrow \neg P} (\text{R} \neg)}{P \supset \neg Q \longrightarrow (P \supset Q) \supset \neg P} (\text{R} \supset)} (\text{R} \supset)}$$

$$\frac{\frac{P, P \longrightarrow P}{P \longrightarrow \neg P, P} (\text{R} \neg) \quad \frac{P \longrightarrow Q, P}{P, \neg Q \longrightarrow P} (\text{L} \neg)}{\frac{P, \neg P \supset \neg Q \longrightarrow P}{\neg P \supset \neg Q \longrightarrow \neg P, P} (\text{R} \neg)} (\text{L} \supset) \quad \frac{P, Q \longrightarrow P}{Q \longrightarrow \neg P, P} (\text{R} \neg) \quad \frac{Q \longrightarrow Q, P}{Q, \neg Q \longrightarrow P} (\text{L} \neg)}{\frac{\neg P \supset Q, \neg P \supset \neg Q \longrightarrow P}{\neg P \supset \neg Q \longrightarrow (\neg P \supset Q) \supset P} (\text{R} \supset)} (\text{R} \supset)}$$

$$\frac{\frac{P \longrightarrow P, P, S \quad P \longrightarrow Q, P, S}{P \longrightarrow P \wedge Q, P, S} (\text{R} \wedge) \quad \frac{P, S \longrightarrow P, S}{P, P \wedge Q \supset S \longrightarrow P, S} (\text{L} \supset) \quad \frac{P, Q \longrightarrow P, S \quad P, Q \longrightarrow Q, S}{P, Q \longrightarrow P \wedge Q, S} (\text{R} \wedge) \quad \frac{P, Q, S \longrightarrow}{P, Q, P \wedge Q \supset S \longrightarrow S} (\text{L} \supset)}{\frac{\frac{P, P \supset Q, P \wedge Q \supset S \longrightarrow S}{P, (P \supset Q) \wedge (P \wedge Q \supset S) \longrightarrow S} (\text{L} \wedge)}{P \wedge (P \supset Q) \wedge (P \wedge Q \supset S) \longrightarrow S} (\text{L} \wedge)} (\text{R} \supset)}$$

$$\frac{\frac{P \longrightarrow P, Q}{\longrightarrow P, P \supset Q} (\text{R} \supset)}{\longrightarrow P \vee (P \supset Q)} (\text{R} \vee)$$

$$\frac{\frac{P \longrightarrow Q, P}{\longrightarrow P \supset Q, P} (\text{R} \supset) \quad \frac{P \longrightarrow P}{(P \supset Q) \supset P \longrightarrow P} (\text{L} \supset)}{\longrightarrow ((P \supset Q) \supset P) \supset P} (\text{R} \supset)$$

$$\frac{\frac{P(a), \forall x.P(x) \longrightarrow P(a)}{\forall x.P(x) \longrightarrow P(a)} \text{ (L}\forall\text{)} \quad \frac{P(b), \forall x.P(x) \longrightarrow P(b)}{\forall x.P(x) \longrightarrow P(b)} \text{ (L}\forall\text{)}}{\forall x.P(x) \longrightarrow P(a) \wedge P(b)} \text{ (R}\wedge\text{)}$$

$$\frac{\frac{P(a) \longrightarrow P(a), \exists x.P(x)}{P(a) \longrightarrow \exists x.P(x)} \text{ (R}\exists\text{)} \quad \frac{P(b) \longrightarrow P(b), \exists x.P(x)}{P(b) \longrightarrow \exists x.P(x)} \text{ (R}\exists\text{)}}{P(a) \vee P(b) \longrightarrow \exists x.P(x)} \text{ (L}\vee\text{)}$$

$$\frac{\forall x.P(x), \forall x.\forall x.P(x) \longrightarrow \forall x.P(x)}{\forall x.\forall x.P(x) \longrightarrow \forall x.P(x)} \text{ (L}\forall\text{)}$$

$$\frac{\frac{P(z), \forall x.P(x) \longrightarrow P(z)}{\forall x.P(x) \longrightarrow P(z)} \text{ (L}\forall\text{)}}{\frac{\forall x.P(x) \longrightarrow \forall y.P(y)}{\forall x.P(x) \longrightarrow \forall x.\forall x.P(x)}} \text{ (R}\forall\text{)}$$

$$\frac{\frac{P(z) \longrightarrow P(z), \exists x.P(x)}{P(z) \longrightarrow \exists x.P(x)} \text{ (R}\exists\text{)}}{\frac{\exists y.P(y) \longrightarrow \exists x.P(x)}{\exists x.\exists x.P(x) \longrightarrow \exists x.P(x)}} \text{ (L}\exists\text{)}$$

$$\frac{\exists x.P(x) \longrightarrow \exists x.P(x), \exists x.\exists x.P(x)}{\exists x.P(x) \longrightarrow \exists x.\exists x.P(x)} \text{ (R}\exists\text{)}$$

$$\frac{\frac{P(y), Q(y), \forall x.P(x) \wedge Q(x) \longrightarrow P(y)}{P(y) \wedge Q(y), \forall x.P(x) \wedge Q(x) \longrightarrow P(y)} \text{ (L}\wedge\text{)} \quad \frac{P(z), Q(z), \forall x.P(x) \wedge Q(x) \longrightarrow Q(z)}{P(z) \wedge Q(z), \forall x.P(x) \wedge Q(x) \longrightarrow Q(z)} \text{ (L}\wedge\text{)}}{\frac{\frac{\forall x.P(x) \wedge Q(x) \longrightarrow P(y)}{\forall x.P(x) \wedge Q(x) \longrightarrow \forall x.P(x)} \text{ (R}\forall\text{)}}{\forall x.P(x) \wedge Q(x) \longrightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}} \text{ (R}\wedge\text{)}$$

$$\frac{\frac{P(y), \forall x.P(x), \forall x.Q(x) \longrightarrow P(y)}{\forall x.P(x), \forall x.Q(x) \longrightarrow P(y)} \text{ (L}\forall\text{)} \quad \frac{\forall x.P(x), Q(y), \forall x.Q(x) \longrightarrow Q(y)}{\forall x.P(x), \forall x.Q(x) \longrightarrow Q(y)} \text{ (L}\forall\text{)}}{\frac{\frac{\forall x.P(x), \forall x.Q(x) \longrightarrow P(y) \wedge Q(y)}{\forall x.P(x), \forall x.Q(x) \longrightarrow \forall x.P(x) \wedge Q(x)} \text{ (R}\forall\text{)}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \longrightarrow \forall x.P(x) \wedge Q(x)} \text{ (L}\wedge\text{)}$$

$$\frac{\frac{P(y) \longrightarrow P(y), \exists x.P(x), \exists x.Q(x)}{P(y) \longrightarrow \exists x.P(x), \exists x.Q(x)} \text{ (R}\exists\text{)} \quad \frac{Q(y) \longrightarrow \exists x.P(x), Q(y), \exists x.Q(x)}{Q(y) \longrightarrow \exists x.P(x), \exists x.Q(x)} \text{ (R}\exists\text{)}}{\frac{P(y) \vee Q(y) \longrightarrow \exists x.P(x), \exists x.Q(x)}{\exists x.P(x) \vee Q(x) \longrightarrow \exists x.P(x), \exists x.Q(x)} \text{ (L}\exists\text{)}} \text{ (R}\vee\text{)}$$

$$\begin{array}{c}
\frac{P(y) \longrightarrow P(y), Q(y), \exists x.P(x) \vee Q(x)}{P(y) \longrightarrow P(y) \vee Q(y), \exists x.P(x) \vee Q(x)} \text{ (R}\vee\text{)} \\
\frac{P(y) \longrightarrow P(y) \vee Q(y), \exists x.P(x) \vee Q(x)}{P(y) \longrightarrow \exists x.P(x) \vee Q(x)} \text{ (R}\exists\text{)} \\
\frac{P(y) \longrightarrow \exists x.P(x) \vee Q(x)}{\exists x.P(x) \longrightarrow \exists x.P(x) \vee Q(x)} \text{ (L}\exists\text{)} \\
\frac{\exists x.P(x) \longrightarrow \exists x.P(x) \vee Q(x)}{(\exists x.P(x)) \vee (\exists x.Q(x)) \longrightarrow \exists x.P(x) \vee Q(x)} \text{ (L}\vee\text{)}
\end{array}
\quad
\begin{array}{c}
\frac{Q(z) \longrightarrow P(z), Q(z), \exists x.P(x) \vee Q(x)}{Q(z) \longrightarrow P(z) \vee Q(z), \exists x.P(x) \vee Q(x)} \text{ (R}\vee\text{)} \\
\frac{Q(z) \longrightarrow P(z) \vee Q(z), \exists x.P(x) \vee Q(x)}{Q(z) \longrightarrow \exists x.P(x) \vee Q(x)} \text{ (R}\exists\text{)} \\
\frac{Q(z) \longrightarrow \exists x.P(x) \vee Q(x)}{\exists x.Q(x) \longrightarrow \exists x.P(x) \vee Q(x)} \text{ (L}\exists\text{)} \\
\frac{\exists x.Q(x) \longrightarrow \exists x.P(x) \vee Q(x)}{(\exists x.P(x)) \vee (\exists x.Q(x)) \longrightarrow \exists x.P(x) \vee Q(x)} \text{ (L}\vee\text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y) \longrightarrow P(y), \exists x.\neg P(x)}{\longrightarrow P(y), \neg P(y), \exists x.\neg P(x)} \text{ (R}\neg\text{)} \\
\frac{\longrightarrow P(y), \neg P(y), \exists x.\neg P(x)}{\longrightarrow P(y), \exists x.\neg P(x)} \text{ (R}\exists\text{)} \\
\frac{\longrightarrow P(y), \exists x.\neg P(x)}{\longrightarrow \forall x.P(x), \exists x.\neg P(x)} \text{ (R}\forall\text{)} \\
\frac{\longrightarrow \forall x.P(x), \exists x.\neg P(x)}{\neg(\forall x.P(x)) \longrightarrow \exists x.\neg P(x)} \text{ (L}\neg\text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y), \forall x.P(x) \longrightarrow P(y)}{\forall x.P(x) \longrightarrow P(y)} \text{ (L}\forall\text{)} \\
\frac{\forall x.P(x) \longrightarrow P(y)}{\forall x.P(x), \neg P(y) \longrightarrow} \text{ (L}\neg\text{)} \\
\frac{\forall x.P(x), \neg P(y) \longrightarrow}{\forall x.P(x), \exists x.\neg P(x) \longrightarrow} \text{ (L}\exists\text{)} \\
\frac{\forall x.P(x), \exists x.\neg P(x) \longrightarrow}{\exists x.\neg P(x) \longrightarrow \neg(\forall x.P(x))} \text{ (R}\neg\text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y) \longrightarrow P(y), \exists x.P(x)}{P(y) \longrightarrow \exists x.P(x)} \text{ (R}\exists\text{)} \\
\frac{P(y) \longrightarrow \exists x.P(x)}{\longrightarrow \exists x.P(x), \neg P(y)} \text{ (R}\neg\text{)} \\
\frac{\longrightarrow \exists x.P(x), \neg P(y)}{\longrightarrow \exists x.P(x), \forall x.\neg P(x)} \text{ (R}\forall\text{)} \\
\frac{\longrightarrow \exists x.P(x), \forall x.\neg P(x)}{\neg(\exists x.P(x)) \longrightarrow \forall x.\neg P(x)} \text{ (L}\neg\text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y), \forall x.\neg P(x) \longrightarrow P(y)}{P(y), \neg P(y), \forall x.\neg P(x) \longrightarrow} \text{ (L}\neg\text{)} \\
\frac{P(y), \neg P(y), \forall x.\neg P(x) \longrightarrow}{P(y), \forall x.\neg P(x) \longrightarrow} \text{ (L}\forall\text{)} \\
\frac{P(y), \forall x.\neg P(x) \longrightarrow}{\exists x.P(x), \forall x.\neg P(x) \longrightarrow} \text{ (L}\exists\text{)} \\
\frac{\exists x.P(x), \forall x.\neg P(x) \longrightarrow}{\forall x.\neg P(x) \longrightarrow \neg(\exists x.P(x))} \text{ (R}\neg\text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y) \longrightarrow Q(y), P(y), \exists x.P(x) \wedge \neg Q(x)}{P(y) \longrightarrow Q(y), \neg Q(y), \exists x.P(x) \wedge \neg Q(x)} \text{ (R}\neg\text{)} \\
\frac{P(y) \longrightarrow Q(y), \neg Q(y), \exists x.P(x) \wedge \neg Q(x)}{P(y) \longrightarrow Q(y), P(y) \wedge \neg Q(y), \exists x.P(x) \wedge \neg Q(x)} \text{ (R}\wedge\text{)} \\
\frac{P(y) \longrightarrow Q(y), P(y) \wedge \neg Q(y), \exists x.P(x) \wedge \neg Q(x)}{P(y) \longrightarrow Q(y), \exists x.P(x) \wedge \neg Q(x)} \text{ (R}\exists\text{)} \\
\frac{P(y) \longrightarrow Q(y), \exists x.P(x) \wedge \neg Q(x)}{\longrightarrow P(y) \supset Q(y), \exists x.P(x) \wedge \neg Q(x)} \text{ (R}\supset\text{)} \\
\frac{\longrightarrow P(y) \supset Q(y), \exists x.P(x) \wedge \neg Q(x)}{\longrightarrow \forall x.P(x) \supset Q(x), \exists x.P(x) \wedge \neg Q(x)} \text{ (R}\forall\text{)} \\
\frac{\longrightarrow \forall x.P(x) \supset Q(x), \exists x.P(x) \wedge \neg Q(x)}{\neg(\forall x.P(x) \supset Q(x)) \longrightarrow \exists x.P(x) \wedge \neg Q(x)} \text{ (L}\neg\text{)}
\end{array}$$

$$\begin{array}{c}
\frac{\forall x.P(x) \supset Q(x), P(y) \longrightarrow P(y), Q(y) \quad Q(y), \forall x.P(x) \supset Q(x), P(y) \longrightarrow Q(y)}{P(y) \supset Q(y), \forall x.P(x) \supset Q(x), P(y) \longrightarrow Q(y)} \text{ (L } \supset \text{)} \\
\frac{}{\forall x.P(x) \supset Q(x), P(y) \longrightarrow Q(y)} \text{ (L } \forall \text{)} \\
\frac{}{\forall x.P(x) \supset Q(x), P(y), \neg Q(y) \longrightarrow} \text{ (L } \neg \text{)} \\
\frac{}{\forall x.P(x) \supset Q(x), P(y) \wedge \neg Q(y) \longrightarrow} \text{ (L } \wedge \text{)} \\
\frac{}{\forall x.P(x) \supset Q(x), \exists x.P(x) \wedge \neg Q(x) \longrightarrow} \text{ (L } \exists \text{)} \\
\frac{}{\exists x.P(x) \wedge \neg Q(x) \longrightarrow \neg(\forall x.P(x) \supset Q(x))} \text{ (R } \neg \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{Q(y), P(y) \longrightarrow P(y), \exists x.P(x) \wedge Q(x) \quad Q(y), P(y) \longrightarrow Q(y), \exists x.P(x) \wedge Q(x)}{Q(y), P(y) \longrightarrow P(y) \wedge Q(y), \exists x.P(x) \wedge Q(x)} \text{ (R } \wedge \text{)} \\
\frac{}{Q(y), P(y) \longrightarrow \exists x.P(x) \wedge Q(x)} \text{ (R } \exists \text{)} \\
\frac{}{P(y) \longrightarrow \exists x.P(x) \wedge Q(x), \neg Q(y)} \text{ (R } \neg \text{)} \\
\frac{}{\longrightarrow \exists x.P(x) \wedge Q(x), P(y) \supset \neg Q(y)} \text{ (R } \supset \text{)} \\
\frac{}{\longrightarrow \exists x.P(x) \wedge Q(x), \forall x.P(x) \supset \neg Q(x)} \text{ (R } \forall \text{)} \\
\frac{}{\neg(\exists x.P(x) \wedge Q(x)) \longrightarrow \forall x.P(x) \supset \neg Q(x)} \text{ (L } \neg \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y), Q(y), \forall x.P(x) \supset \neg Q(x) \longrightarrow P(y) \quad P(y), Q(y), \neg Q(y), \forall x.P(x) \supset \neg Q(x) \longrightarrow Q(y)}{P(y), Q(y), \forall x.P(x) \supset \neg Q(x) \longrightarrow P(y)} \text{ (L } \neg \text{)} \\
\frac{}{P(y), Q(y), P(y) \supset \neg Q(y), \forall x.P(x) \supset \neg Q(x) \longrightarrow} \text{ (L } \supset \text{)} \\
\frac{}{P(y), Q(y), \forall x.P(x) \supset \neg Q(x) \longrightarrow} \text{ (L } \forall \text{)} \\
\frac{}{P(y) \wedge Q(y), \forall x.P(x) \supset \neg Q(x) \longrightarrow} \text{ (L } \wedge \text{)} \\
\frac{}{\exists x.P(x) \wedge Q(x), \forall x.P(x) \supset \neg Q(x) \longrightarrow} \text{ (L } \exists \text{)} \\
\frac{}{\forall x.P(x) \supset \neg Q(x) \longrightarrow \neg(\exists x.P(x) \wedge Q(x))} \text{ (R } \neg \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y), \forall x.P(x) \longrightarrow P(y), \exists x.P(x)}{P(y), \forall x.P(x) \longrightarrow \exists x.P(x)} \text{ (R } \exists \text{)} \\
\frac{}{\forall x.P(x) \longrightarrow \exists x.P(x)} \text{ (L } \forall \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(y_1), P(x_1) \longrightarrow P(z), P(x_1), \exists x.\forall y.P(y) \supset P(x)}{P(x_1) \longrightarrow P(z), P(y_1) \supset P(x_1), \exists x.\forall y.P(y) \supset P(x)} \text{ (R } \supset \text{)} \\
\frac{}{P(x_1) \longrightarrow P(z), \forall y.P(y) \supset P(x_1), \exists x.\forall y.P(y) \supset P(x)} \text{ (R } \forall \text{)} \\
\frac{}{P(x_1) \longrightarrow P(z), \exists x.\forall y.P(y) \supset P(x)} \text{ (R } \exists \text{)} \\
\frac{}{\longrightarrow P(x_1) \supset P(z), \exists x.\forall y.P(y) \supset P(x)} \text{ (R } \supset \text{)} \\
\frac{}{\longrightarrow \forall y.P(y) \supset P(z), \exists x.\forall y.P(y) \supset P(x)} \text{ (R } \forall \text{)} \\
\frac{}{\longrightarrow \exists x.\forall y.P(y) \supset P(x)} \text{ (R } \exists \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{P(x_1, z), \forall y.P(x_1, y) \longrightarrow P(x_1, z), \exists x.P(x, z)}{P(x_1, z), \forall y.P(x_1, y) \longrightarrow \exists x.P(x, z)} \text{ (R } \exists \text{)} \\
\frac{}{\forall y.P(x_1, y) \longrightarrow \exists x.P(x, z)} \text{ (L } \forall \text{)} \\
\frac{}{\exists x.\forall y.P(x, y) \longrightarrow \exists x.P(x, z)} \text{ (L } \exists \text{)} \\
\frac{}{\exists x.\forall y.P(x, y) \longrightarrow \forall y.\exists x.P(x, y)} \text{ (R } \forall \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{P(a) \longrightarrow P(a), Q, \exists x.P(x) \supset Q}{\longrightarrow P(a), P(a) \supset Q, \exists x.P(x) \supset Q} \text{ (R } \supset \text{)} \quad \frac{\frac{P(b) \longrightarrow P(b), Q, \exists x.P(x) \supset Q}{\longrightarrow P(b), P(b) \supset Q, \exists x.P(x) \supset Q} \text{ (R } \supset \text{)}}{\longrightarrow P(b), \exists x.P(x) \supset Q} \text{ (R } \exists \text{)} \quad \frac{P(y), Q \longrightarrow Q, \exists x.P(x) \supset Q}{Q \longrightarrow P(y) \supset Q, \exists x.P(x) \supset Q} \\
\hline
\frac{\longrightarrow P(a), \exists x.P(x) \supset Q \quad \longrightarrow P(b), \exists x.P(x) \supset Q}{\longrightarrow P(a) \wedge P(b), \exists x.P(x) \supset Q} \text{ (R } \wedge \text{)} \quad \frac{Q \longrightarrow \exists x.P(x) \supset Q}{\longrightarrow \exists x.P(x) \supset Q} \\
\hline
\frac{P(a) \wedge P(b) \supset Q \longrightarrow \exists x.P(x) \supset Q}{\longrightarrow (P(a) \wedge P(b) \supset Q) \supset (\exists x.P(x) \supset Q)} \text{ (R } \supset \text{)}
\end{array}$$

$$\begin{array}{c}
\frac{N(0), N(s(0)), \forall x.N(x) \supset N(s(x)) \longrightarrow N(s(0)), N(s(s(0)))}{N(0), N(s(0)), N(s(0)) \supset N(s(s(0)))} \\
\hline
\frac{N(0), \forall x.N(x) \supset N(s(x)) \longrightarrow N(0), N(s(s(0))) \quad N(0), N(s(0)), \forall x.N(x) \supset N(s(x))}{N(0), N(0) \supset N(s(0)), \forall x.N(x) \supset N(s(x)) \longrightarrow N(s(s(0)))} \text{ (L } \forall \text{)} \\
\hline
N(0), \forall x.N(x) \supset N(s(x)) \longrightarrow N(s(s(0)))
\end{array}$$