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Continuous optimization
Fall 2023
Laboration 2
Disks

“My God, it’s full of disks!”

This laboration is about modelling some charming problems in geometric optimization and trying to solve them with the tools offered by Matlab. Disk packing, as the name suggests, is about covering as large part of a certain area using nonoverlapping disks. In general the problem is notoriously hard to solve and exact solutions exist only for a few special cases.

For example, an old and famous problem concerns how to cover as large part of the 2-dimensional plane as possible by using disks of equal radius. It has been shown that the optimal solution is to arrange them in a honeycomb pattern which results in the disks covering 90.69% ($\frac{\pi}{2\sqrt{3}}$) of the plane.

In this laboration we will try to find optimal or good packings (aka arrangements) of a finite number of disks into a few bounded regions. In the first part we will let all disks have equal radius and in the second part we remove this restriction. Chapter 9.1 in the course book suggests a real-world application and a model formulation for one variant of the problem.

We define the *density* d of a packing as the total area of the disks divided by the area of the region. For a given region, the optimal density for n disks is denoted d_n^*. Recall the fundamental principle of optimization, that any (feasible) packing of n disks with density d gives us a *lower* bound of d_n^*. However, proving optimality is usually quite difficult and all we can hope for is to find a better feasible solution.

For disks with free radii d_n^* is an increasing sequence in n and $d_n^* \to 1$ when $n \to \infty$. (Proof: take an optimal arrangement for n disks and add a new disk with radius ε (small enough to fit), then $d_n^* < d_{n+1}^*$). For disks of equal radius the d_n^*-sequence is not necessarily increasing in n but we still have $d_n^* \leq 0.9069$, as mentioned above. The inverse problem is to ask for the smallest number $N(d)$ of disks needed to obtain a certain density d. Of course, if we can obtain density d using n disks then this only proves that $N(d) \leq n$. We can only hope to obtain good upper bounds of $N(d)$ in this lab.

A model formulation should be stated in the usual form

$$
\begin{aligned}
&\min f(x) \\
&g_1(x) \leq 0, \ldots, g_m(x) \leq 0
\end{aligned}
$$

but we need to find a good choice of $f, g_1, \ldots, g_m$. Some of the constraints may be linear, others nonlinear. There is also the more practical problem of choosing good starting points for a local solver.

The tools we have available are not just the `fmincon`-command — which only does *local* optimization — but also the *global* optimization tools `GlobalSearch` and `MultiStart`. There is also the `ga`-command which is a genetic-algorithm solver. However, it is not clear which is best for these problems. Experiment to see which tool and which choice of command options gets the job done.

Enjoy!

Equality of the radii!

Problem 1: In Figure 1 we show arrangements of 36 (left) and 49 (right) nonoverlapping disks of equal radius packed inside a square. These packings have density $d = \pi/4 \approx 0.785398$. This is optimal for 36 disks because it has been proved that $d^*_{36} = \pi/4$. However, for 49 disks this is not optimal, that is, $d^*_{49} > \pi/4$. Your first problem is to find a better packing of 49 disks. Any solution with $d \geq 0.79$ is accepted. Show it in a figure.

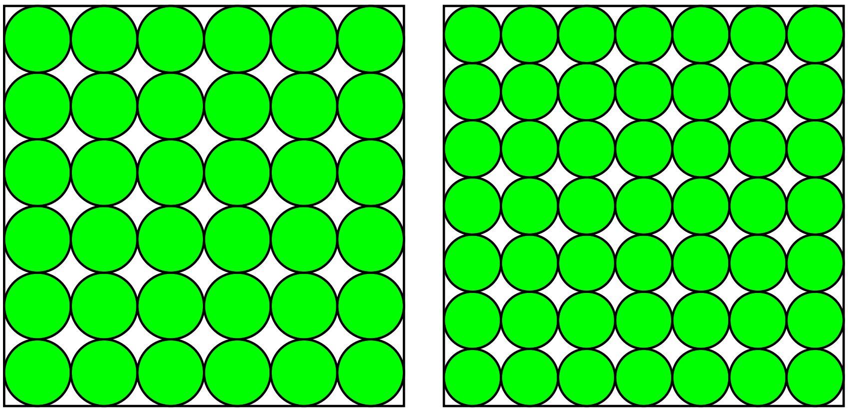

Figure 1: An optimal packing of 36 disks and a nonoptimal packing of 49 disks.

Problem 2: Now we will pack disks of equal radius into a circular quadrant. In Figure 2 we show an arrangement of 12 disks giving density $d = 0.7553$ and this is probably very close to optimal. In fact, 12 is the least number of disks needed to reach a 75% density. The second problem is to determine $N(0.80)$, i.e., the *least* number n of disks needed to obtain a packing with density $d \geq 0.80$. Show the packing in a figure.

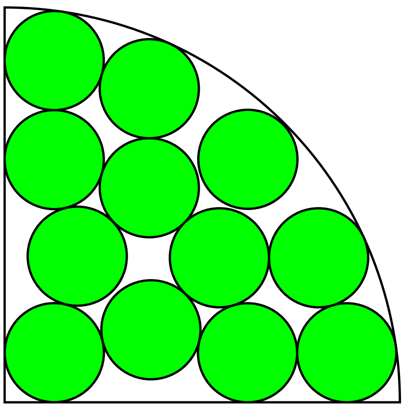

Figure 2: Optimal packing of 12 disks in a circular quadrant.

Free the radii!

Problem 3: We pack disks into a unit square with three forbidden zones. Figure 3 shows 11 disks packed into this region (forbidden zones in gray) and this particular arrangement gives density $d = 0.7990$. Your third problem is to determine good upper bounds on **a)** $N(0.85)$ and **b)** $N(0.90)$. Make figures of these packings.

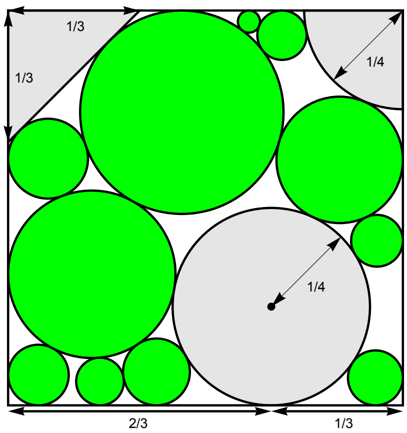

Figure 3: Packing of 11 disks with $d = 0.799$

Problem 4: Now we pack the disks into three-quadrants of a circle. Figure 4 shows 12 disks packed into the region and this particular arrangement gives density $d = 0.84926$. Can you get $d \geq 0.85$ with 12 disks? The fourth problem is to find **a)** $N(0.85)$ and **b)** $N(0.88)$. Make figures of these packings.

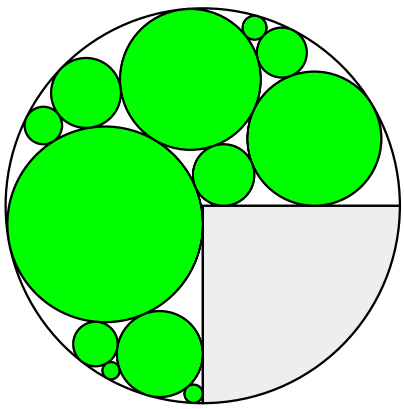

Figure 4: Is this an optimal packing of 12 disks?

1 The Report

The report should be kept as short and succinct as possible, like a technical report that your reviewer can read and appreciate, written in Proper English. Give the answers to the questions and illustrate with nice figures with clear captions. **The models must be stated explicitly with each constraint clearly explained**. Please comment on any difficulties or unforeseen complexities you encountered during the laboration. If you also want to investigate a variant of the problems that you found interesting, feel free to do so but place this in separate chapters. Using LaTeX almost guarantees an approved report and is thus very much recommended but is of course not absolutely necessary.

The code should be written in Matlab. Put the relevant code (no need to include the plot commands) at the end of the report as an appendix. Split the code into convenient functions instead of using the monolithic style.

You may work alone or in teams up to three, but two is probably best. One person from each group must send me an email with the names of the members (including one-member teams). **If you are in the same group as before then you don't have to send me anything. Otherwise send me an email and say which group you leave and which group you join.** To get good reports we will use peer-review. We will pair you with another group well before the deadline, saying which group reviews which. The review is a written one-page document containing critique, suggestions for improvement and comments on any errors in the text or code.

As reviewer your job is to check that all tasks have been completed, all questions answered and all figures included, all presented in good style. Remember that nitpicking is always appreciated, especially by the nitpicked. The reviews are then returned at least a few days before the deadline so that each group can take the critique into account and update the report, unless they think the critique is misguided. Write a response to the review as a separate one-page document. The final version of the report is then submitted to Canvas before the deadline. What you upload to Canvas is **one (1) pdf-file** (all else is ignored) containing the following:

1. a cover-sheet with the names of the team members
2. the review by the other team
3. your response to their critique
4. the updated and now (probably) perfect report with code in the appendix

Only *one* person from each group submits the file. The deadline is **17.00 on October 30**. Only very minor corrections may be handed in after we have evaluated the report but everything should be completed no later than November 10. After that you will be assigned a *new* laboration and its report should be handed in no later than November 30.