Continuous Optimization Home Exam

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1

Define $f(x) = (x_1 - x_2)^2 + (2 + x_1 + x_2)^2$

1.1

Prove that f is convex for $x_1 \geq -2$
First, we compute the hessian of the function f.

```
syms x1 x2
dff_sym = hessian(f(x1, x2), [x1, x2]);
```

$$\nabla^2 f = \begin{pmatrix} 4 & 4x_2 - 2 \\ 4x_2 - 2 & 12x_2^2 + 4x_1 + 10 \end{pmatrix}$$

According to Sylvester's criterium for semidefinition:
H is positive semidefinite $\Leftrightarrow$ all determinants of all principal minors are non-negative.

1.1.1 The determinant of the 2×2 matrix

```
d = det(dff_sym);
```

$d = 32x_2^2 + 16x_2 + 16x_1 + 36 \geq 0$
$d = 2x_2^2 + x_2 + x_1 + 2.25 \geq 0$

Assuming $x_1 \geq -2$
$2x_2^2 + x_2 \geq -0.25$

Find a minimum of the function above:
Derivative: $4x_2 + 1$
Minimum at point $x_2 = -\frac{1}{4}$ with value $-\frac{1}{8}$

$-\frac{1}{8} \geq -0.25$

1.1.2 The determinant of the first element of the diagonal

$4 \geq 0$

1.1.3 The determinant of the second element of the diagonal

$12x_2^2 + 4x_1 + 10 \geq 0$
$12x_2^2$ is indeed greater or equal to 0

Which gives $x_1 \geq 2.5$ which satisfies the condition $x_1 \geq -2$

According to Thm 9.5:
Assume continuous $\frac{\delta^2 f}{\delta x_i \delta x_j}$ on convex set X (hence $\nabla^2 f$ symmetric)
$\nabla^2 f$ positive semidefinite on $X \Leftrightarrow f$ is convex on X
R^2 with $x_1 \geq -2$ is a convex set, and $\nabla^2 f$ is positive semidefinite, therefore f is convex for $x_1 \geq -2$.

1.2

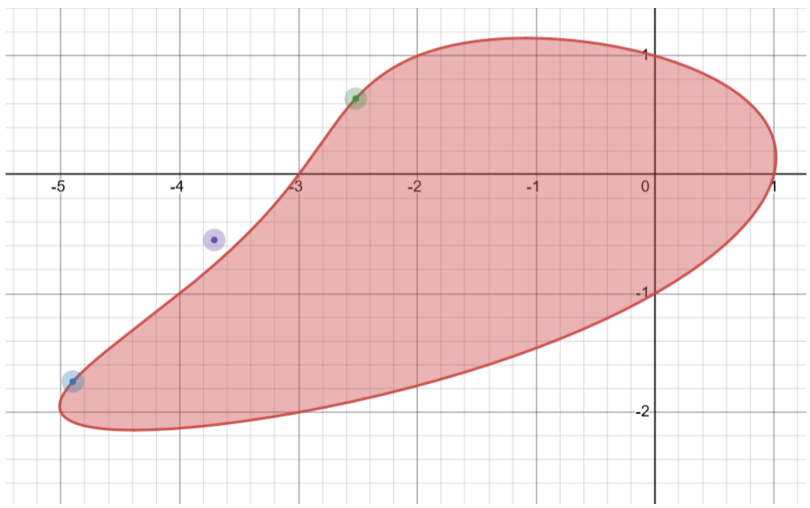
Is the set $X = \{x \in R^2 : f(x) \leq 10\}$ convex? Prove your answer.

No, it isn't.

Let $a = (-4.9, -1.741)$ and $b = (-2.52, 0.64)$. Then $f(a) \approx 9.9965$ and $f(b) \approx 9.9978$ meaning $a, b \in X$.
Let $\lambda = 1/2$, then the midpoint is $m = (-3.71, -0.55)$ and $f(m) \approx 11.9667$, meaning $m \notin X$.

This proves X is not convex.

Set $X = \{x \in R^2 : f(x) \leq 10\}$ with highlighted points a, b and m:



2

The only real stationary point of f is $x^0 = (-\frac{11}{8}, -\frac{1}{2})$ and can be found by taking the gradient of the function f and putting the gradient equal to 0.

```
f = @(x1, x2) (x1 - x2)^2 + (2 + x1 + x2^2)^2;

df_sym = gradient(f(x1,x2),[x1,x2]);
df = matlabFunction(df_sym);
[solx1,solx2] = solve(df(x1, x2) == [0,0]);
```

By running $df(-11/8, -1/2)$, we get a vector $[0; 0]$, therefore x^0 is a stationary point.

In order to find the second order approximation $p(x^0)$, we first obtain all the derivatives of f we will need.

```
fx1 = matlabFunction(df_sym(1));
fx2 = matlabFunction(df_sym(2));

dfx1_sym = gradient(fx1(x1, x2), [x1, x2]);
dfx2_sym = gradient(fx2(x1, x2), [x1, x2]);

fx1x1 = matlabFunction(dfx1_sym(1));
fx1x2 = matlabFunction(dfx1_sym(2));
fx2x2 = matlabFunction(dfx2_sym(2));

ptemp = @(a,b) f(a,b) + fx1(a,b) * (x1 - a) + fx2(a,b) * (x2 - b) + 1/2 *
fx1x1() * (x1 - a)^2 + fx1x2(b) * (x1 - a) * (x2 - b) + 1/2 * fx2x2(a,b) *
(x2 - b)^2;
```

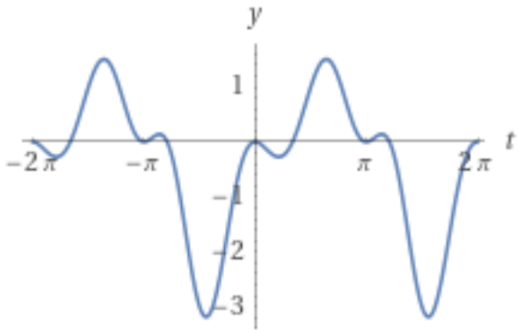
By evaluating ptemp at point $x^0 = (-11/8, -1/2)$ we get the approximation p at point x^0.
Next we create $d(x) = p(x) - f(x)$

```
p = @(x1,x2) (15*(x2 + 1/2)^2)/4 + 2*(x1 + 11/8)^2 - (4*x1 + 11/2)*(x2 + 1/2)  + 49/32;

d_sym = p(x1, x2) - f(x1, x2);
d = matlabFunction(d_sym);
```

The next task is to find a minimum and maximum of d over a circle with a diameter equal to 1 with a centre in x^0.
We parametrize the set by setting $x_1 = cos(t) - \frac{11}{8}$ and $x_2 = sin(t) - \frac{1}{2}$ and get a new function $dparam(t)$, with just one variable, t.

```
x1_param = @(t) cos(t) - 11/8;
x1_param_sym = x1_param(t);
x2_param = @(t) sin(t) - 1/2;
x2_param_sym = x2_param(t);

d_param_sym = d(x1_param_sym,x2_param_sym);
d_param = matlabFunction(d_param_sym);
```

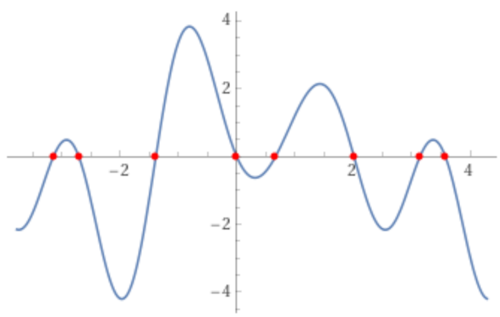
Function dparam looks like this:



Because we are trying to find extremes of a single-variable function, we can simply take the derivative and put it equal to zero to find interesting points.

```
d_param_derivative1 = gradient(d_param(t), t);
```

Looking at the graph of the derivative, we can see that there are 6 roots repeating in the interval of $2\pi k$ where k is a natural number.



From the graph, we read the intervals in which to search for the roots and find six roots accordingly. We use the 2π span of $[-2, 4.3]$

```
root1 = vpasolve(d_param_derivative1,[-2 -1])
root2 = vpasolve(d_param_derivative1,[-1 0.5])
root3 = vpasolve(d_param_derivative1,[0.5 1])
root4 = vpasolve(d_param_derivative1,[1 2.5])
root5 = vpasolve(d_param_derivative1,[2.5 3.25])
root6 = vpasolve(d_param_derivative1,[3.25 4.3])
```

By evaluating dparam at these points, we get values:

```
d_param(root1) = -3.1882718231245494920621811254115
d_param(root2) = 0
d_param(root3) = -0.27433036481422052398729139282714
d_param(root4) = 1.5084790203384482931696202009055
d_param(root5) = 0
d_param(root6) = 0.14088547194003861257865171457888
```

We can see that root1 ≈ -1.385686034 gives minimum and root4 ≈ 2.009897292678 gives maximum. Let's check that we didn't make a mistake in calculations by substituting back into the original equations:

```
x1_min = x1_param(root1)
x2_min = x2_param(root1)
d(x1_min, x2_min)

x1_max = x1_param(root4)
x2_max = x2_param(root4)
d(x1_max, x2_max)
```

The code above confirms that the minimum is at point $(-1.19094505, -1.48291595)$ with value of -3.18827182, and the maximum is at point $(-1.800125890, 0.4051342316)$ with the value of 1.5084790203

These are surely the minimum and maximum points, because we checked all points where the derivative of dparam(t) $= 0$ and then chose the ones with highest and lowest value.

3

Define:

$f(x) = x_1^2 + 3x_2^2 + 5x_3^2 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3 + x_1 - 2x_2 + 3x_3$

$g_1(x) = x_1 + x_2 + x_3 - 1$
$g_2(x) = x_1^2 - x_2 + 1$

First, we check if $f(x)$ is convex, as it will make our task easier.

```
f_hess = hessian(f(x1, x2, x3), [x1, x2, x3])
e = eig(f_hess)
```

The hessian is symmetric:

$$\nabla^2 f = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 6 & -6 \\ 2 & -6 & 10 \end{pmatrix}$$

And all three eigenvalues $e_1 = 2$, $e_2 \approx 1.0718$ and $e_3 \approx 14.9282$ are positive, so according to Thm.9.6, $\nabla^2 f$ is positive definite, and according to Thm.9.5, that means that f is strictly convex on R^3.

Next we check convexity of $g_1(x)$ and $g_2(x)$, and because both functions are linear combination of convex functions, they are also convex.
Therefore setting $g_1(x) \leq b_1$ and $g_2(x) \leq b_2$ defines convex sets (see Lecture Notes)

```
f = @(x1, x2, x3) x1^2 + 3 * x2^2 + 5 * x3^2 - 2 * x1 * x2 + 2 * x1 * x3 - 6 * x2 * x3 + x1 - 2 *

g1 = @(x1, x2, x3) x1 + x2 + x3 - 1;
g2 = @(x1, x2, x3) x1^2 - x2 + 1;

syms x1 x2 x3 v1 v2
```

3.1

a) Minimizing $f(x)$ over $g_1(x) \leq 0$

The Lagrange function for problem a is:

```
L1 = @(x1, x2, x3, v1) f(x1, x2, x3) + v1 * g1(x1, x2, x3);
```

$L(x, v) = f(x) + v \cdot g_1(x)$, with $v \geq 0$
$L(x, v) = x1 - 2x_2 + 3x_3 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3 + v1(x_1 + x_2 + x_3 - 1) + x_1^2 + 3x_2^2 + 5x_3^2$

We define the dual function of problem a as:

$h(v) = min_x L(x, v)$ with $v \geq 0$

Because $L(x, v)$ is convex for a fixed v, we can find $x^*(v)$ by setting the $\nabla L(x, v) = 0$

```
L1_grad = gradient(L1(x1, x2, x3, v1), [x1, x2, x3, v1]);

p1 = solve([L1_grad(1); L1_grad(2); L1_grad(3)] == [0;0;0], [x1,x2,x3]);

x1_p1 = matlabFunction(p1.x1)
x2_p1 = matlabFunction(p1.x2)
x3_p1 = matlabFunction(p1.x3)
```

That gives us:

$x_1(v) = -v - \frac{1}{4}$
$x_2(v) = -v$
$x_3(v) = -\frac{v}{2} - \frac{1}{4}$

Next, we plug in our values for x in terms of v into the dual function, and we get:

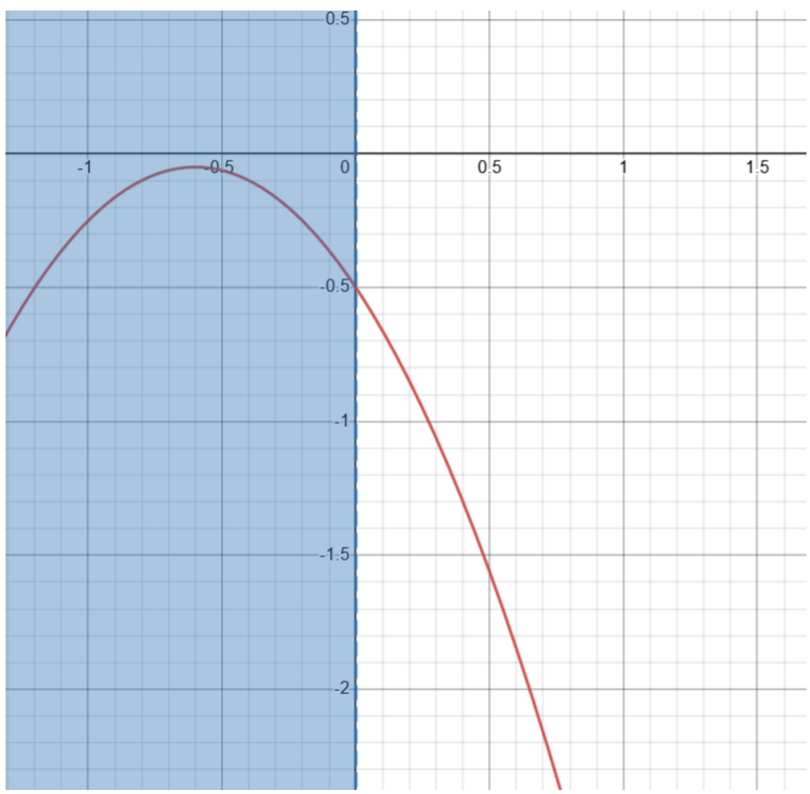
$$h(v) = (v + \tfrac{1}{4})^2 - v(2v + \tfrac{1}{2}) - 6v(\tfrac{v}{2} + \tfrac{1}{4}) - v(\tfrac{5v}{2} + \tfrac{3}{2}) - \tfrac{v}{2} + 5(\tfrac{v}{2} + \tfrac{1}{4})^2 + (2v + \tfrac{1}{2})(\tfrac{v}{2} + \tfrac{1}{4}) + 3v^2 - 1$$

That makes the dual problem of a to maximize $h(v)$ in terms of v.
According to Thm.17.2, dual function is concave, so we set $\nabla h(v) = 0$.

```
h1 = @(v1) L1(x1_p1(v1), x2_p1(v1), x3_p1(v1), v1);

h1_grad = gradient(h1(v1), v1)

v0 = solve(h1_grad == 0)
```

We get $v^* = -3/5$.
Because v has to be ≥ 0, and because $h(v)$ is concave, we choose the closest feasible point $v^* = 0$.

v is shown on the horizontal axis:



This gives us $h(v^*) = -0.5$
$x_1(v^*) = -0.25$, $x_2(v^*) = 0$ and $x_3(v^*) = -0.25$ gives $f(x_1, x_2, x_3) = -0.5$

$h(v^*) = f(x_1^*, x_2^*, x_3^*)$
As proved in section 3, the function f is convex, and the constraint g_1 defines a convex set. According to Thm.17.1, those conditions are sufficient to say that strong duality holds.

3.2

b) Minimizing $f(x)$ over both $g_1(x) \leq 0$ and $g_2(x) \leq 0$

The Lagrange function for problem b is:

```
L2 = @(x1, x2, x3, v1, v2) f(x1, x2, x3) + v1 * g1(x1, x2, x3) + v2 * g2(x1, x2, x3)
```

$L(x,v) = f(x) + v_1 \cdot g_1(x) + v_2 \cdot g_2(x)$, with $v_i \geq 0$
$L(x,v) = x_1 - 2x_2 + 3x_3 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3 + v_1(x_1+x_2+x_3-1) + x_1^2 + 3x_2^2 + 5x_3^2 + v_2(x_1^2 - x_2 + 1)$

We define the dual function of problem b as:

$h(v) = min_x L(x,v)$ with $v \geq 0$

Because $L(x,v)$ is convex for a fixed v, we can find x^* by setting the $\nabla L(x,v) = 0$

```
L2_grad = gradient(L2(x1, x2, x3, v1, v2), [x1, x2, x3, v1, v2])

p2 = solve([L2_grad(1); L2_grad(2); L2_grad(3)] == [0;0;0], [x1, x2, x3])

x1_p2 = matlabFunction(p2.x1)
x2_p2 = matlabFunction(p2.x2)
x3_p2 = matlabFunction(p2.x3)
```

That gives us:

$x_1(v) = -\frac{4v_1 - v_2 + 1}{2(3v_2+2)}$
$x_2(v) = -\frac{8v_1 - 5v_2 + 8v_1v_2 - 5v_2^2}{4(3v_2+2)}$
$x_3(v) = \frac{v_2}{4} - \frac{v_1}{2} - \frac{1}{4}$

Next, we plug in our values for x in terms of v into the dual function, and we get:

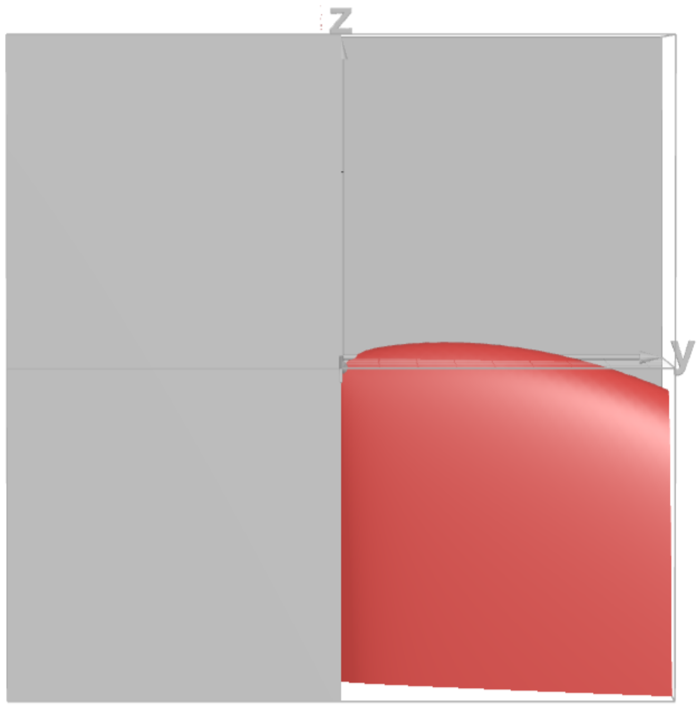
$h(v) = (3*v2)/4 - (3*v1)/2 + v2*((2*v1 - v2/2 + 1/2)^2/(3*v2+2)^2 + (2*v1 - (5*v2)/4 + 2*v1*v2 - (5*v2^2)/4)/(3*v2+2) + 1) - v1*(v1/2 - v2/4 + (2*v1 - (5*v2)/4 + 2*v1*v2 - (5*v2^2)/4)/(3*v2+2) + (2*v1 - v2/2 + 1/2)/(3*v2+2) + 5/4) + (2*v1 - v2/2 + 1/2)^2/(3*v2+2)^2 + 5*(v1/2 - v2/4 + 1/4)^2 + (4*v1 - (5*v2)/2 + 4*v1*v2 - (5*v2^2)/2)/(3*v2+2) + (3*(2*v1 - (5*v2)/4 + 2*v1*v2 - (5*v2^2)/4)^2)/(3*v2+2)^2 - (2*v1 - v2/2 + 1/2)/(3*v2+2) + (2*(2*v1 - v2/2 + 1/2)*(v1/2 - v2/4 + 1/4))/(3*v2+2) - (2*(2*v1 - v2/2 + 1/2)*(2*v1 - (5*v2)/4 + 2*v1*v2 - (5*v2^2)/4))/(3*v2+2)^2 - (6*(v1/2 - v2/4 + 1/4)*(2*v1 - (5*v2)/4 + 2*v1*v2 - (5*v2^2)/4))/(3*v2+2) - 3/4$

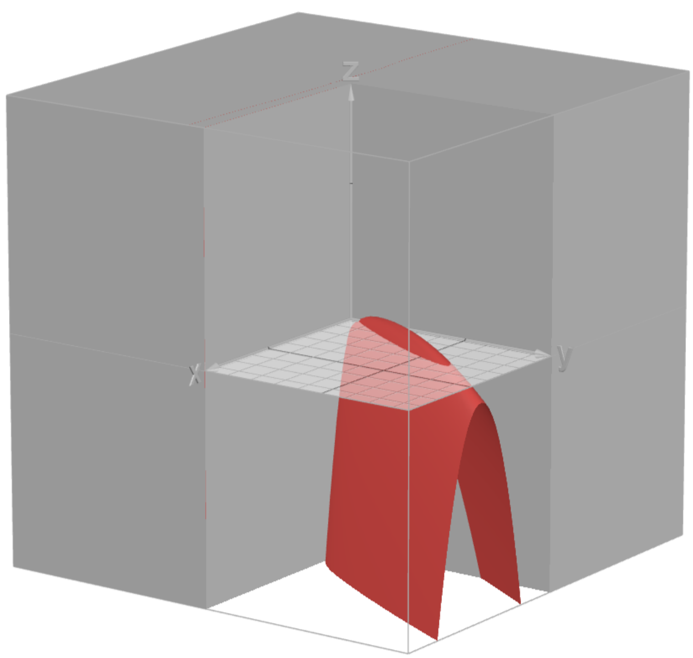
That makes the dual problem of a to maximize $h(v)$ in terms of v.
According to Thm.17.2, dual function is concave, so we set $\nabla h(v) = 0$.

```
h2 = @(v1, v2) L2(x1_p2(v1, v2), x2_p2(v1, v2), x3_p2(v1, v2), v1, v2)

h2_grad = gradient(h2(v1, v2), [v1, v2])

v0 = vpasolve(h2_grad == [0;0], [v1 v2])
```

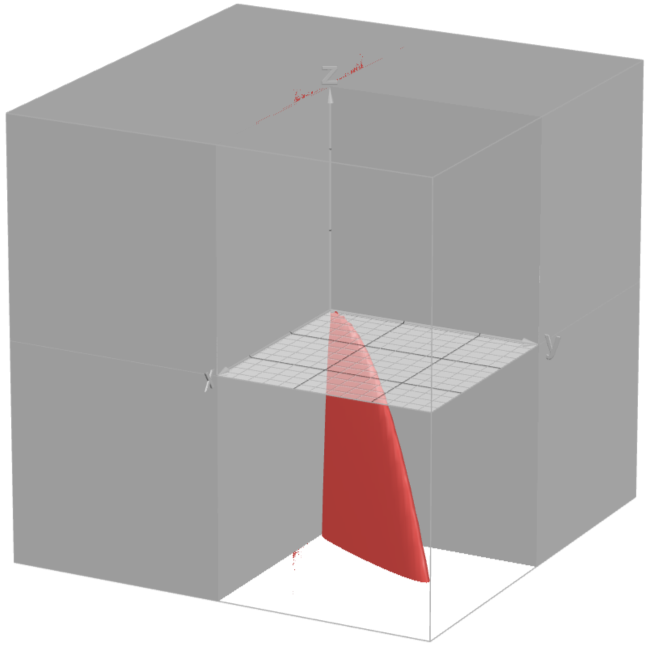
We get a real point $v^* \approx (2.3301335, 6.0404474)$
The point satisfies the condition of $v_i \geq 0$, so this solution is feasible.

x-axis represents v_1, y-axis represents v_2, z-axis represents $h(v_1, v_2)$





The same $h(v_1, v_2)$ function as above, plotted from distance.



The dual function gives us $h(v^*) \approx 0.899070241$
$x_1(v^*) \approx -0.1063568$, $x_2(v^*) \approx 1.01131178$ and $x_3(v^*) \approx 0.09504509$ gives $f(x_1, x_2, x_3) \approx 0.899070241$

$h(v^*) = f(x_1^*, x_2^*, x_3^*)$
As proved in section 3, the function f is convex, and the constraints g_1 and g_2 define convex sets. According to Thm.17.1, those conditions are sufficient to say that strong duality holds.

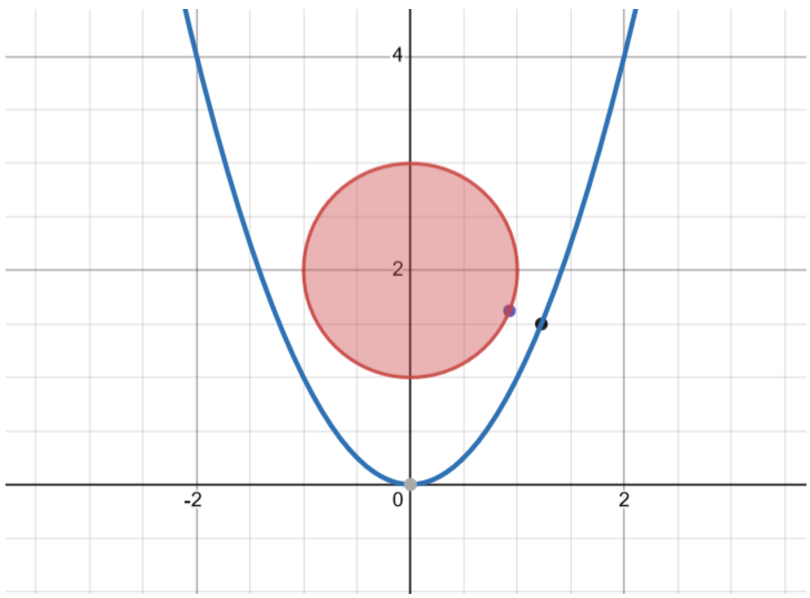
4

4.1

First, we compute the distance between the disk D and $p(t)$ using Matlab.

```
d = @(x) (x(1)-x(3))^2 + (x(2)-x(4))^2;

g1 = @(x) x(1)^2 + (x(2) - 2)^2 - 1;
g2 = @(x) - x(3)^2 + x(4);

lb = -inf(4,1);
ub = inf(4,1);

syms x

nlc = @(x) deal([g1(x)], [g2(x)]);

x0 = fmincon(d, [0,2,1,1], [], [], [], [], lb, ub, nlc)

sqrt(d(x0))
```

We get $x^0 = (0.9258, 1.6220, 1.2247, 1.5)$, which gives us a point $c = (0.9258, 1.6220)$ and $t = 1.2247$, where $c \in D$ and $t^2 = 1.4999 \approx 1.5$



Let's check that x^0 is a feasible point:

$$g1(x^0) = -1.2389e - 06 \approx 0 \text{ and } g2(x^0) = -1.1806e - 12 \approx 0$$

4.2

Now let's look at the dual problem.

We are trying to minimize the distance between a point inside of the circle, and a point $p(t)$.
We can write the function we want to minimize as $d(x_1, x_2, t) = (x_1 - t)^2 + (x_2 - t^2)^2$.
With this as our function to minimize, we just need a constraint for the points x_1 and x_2. That will be $g(x_1, x_2) = x_1^2 + (x_2 - 2)^2 - 1$.

```
d = @(x1, x2, t) (x1 - t)^2 + (x2 - t^2)^2;
```

```
g = @(x1, x2) x1^2 + (x2 - 2)^2 - 1;
```

With these functions, we can make a Lagrangian function for this problem:

$$L(x_1, x_2, t, v) = d(x_1, x_2, t) + v \cdot g(x_1, x_2)$$

Setting the gradient in x_1 and x_2 equal to 0 gives us the explicit functions for x_1 and x_2 in regards of v and parameter t.

```
L = @(x1, x2, t, v) d(x1, x2, t) + v * g(x1, x2);

L_grad = gradient(L(x1, x2, t, v), [x1, x2])

p = solve(L_grad == [0;0], [x1, x2])

x1_p = matlabFunction(p.x1);
x2_p = matlabFunction(p.x2);
```

$x_1(v,t) = \frac{t}{v+1}$
$x_2(v,t) = \frac{t^2+2v}{v+1}$

Now, by inputing the values of x_i in terms of v and parameter t into the Lagrangian, we get the dual function $h(v,t)$.

```
h = @(v, t) L(x1_p(t, v), x2_p(v, t), t, v)
```

$$h(v,t) = (t - \frac{t}{v+1})^2 + (\frac{v^2+2t}{t+1} - t^2)^2 + v((\frac{v^2+2t}{t+1} - 2)^2 + \frac{t^2}{(v+1)^2} - 1)$$

Setting the derivative in v equal to 0 finds a v^* dependent on parameter t.

```
h_grad = gradient(h(v, t), v)

v0 = vpasolve(h_grad == 0, v)
```

$$h_{grad} = ((v^2+2*t)/(t+1)-2)^2 + t^2/(v+1)^2 - v*((2*t^2)/(v+1)^3 - (4*v*((v^2+2*t)/(t+1)-2))/(t+1)) + (4*v*((v^2+2*t)/(t+1)-t^2))/(t+1) + (2*t*(t-t/(v+1)))/(v+1)^2 - 1 = 0$$

Solving a high degree equation like this produces from Matlab a result in a form "root(*result*, z, *number of solution*)". I tried many different ways to get a solution of just $v =$ *equation using t as a parameter* but I didn't succeed.
After trying to separate v from the equation by hand and failing after two pages of computations, I decided to just give the v^* in the same form Matlab gave.

$$v^*(t) = root(z^6 + (14*z^5)/5 + z^4/5 - (z^3*(-8*t+4*t^2+4*t^3+20))/5 - (z^2*(-14*t+9*t^2+8*t^3+9))/5 + (z*(4*t-6*t^2-4*t^3+6))/5 - (2*t)/5 + t^4/5 + (2*t^3)/5 + 3/5, z, 1)$$

By back-substituting v^* dependent on parameter t back into the dual function $h(v,t)$, we get a dual function dependent solely on the parameter t.

```
h(v0, t)
```

$$h(v^*(t)) = (t^2 - (2*t + root(z^6+(14*z^5)/5+z^4/5-(z^3*(-8*t+4*t^2+4*t^3+20))/5-(z^2*(-14*t+9*t^2+8*t^3+9))/5+(z*(4*t-6*t^2-4*t^3+6))/5-(2*t)/5+t^4/5+(2*t^3)/5+3/5, z, 1)^2)/(t+1))^2 + (t - t/(root(z^6+(14*z^5)/5+z^4/5-(z^3*(-8*t+4*t^2+4*t^3+20))/5-(z^2*(-14*t+9*t^2+8*t^3+9))/5+(z*(4*t-6*t^2-4*t^3+6))/5-(2*t)/5+t^4/5+(2*t^3)/5+3/5, z, 1)+1))^2 + (((2*t+root(z^6+(14*z^5)/5+z^4/5-(z^3*(-8*t+4*t^2+4*t^3+20))/5-(z^2*(-14*t+9*t^2+8*t^3+9))/5+(z*(4*t-6*t^2-4*t^3+6))/5-(2*t)/5+t^4/5+(2*t^3)/5+3/5, z, 1)^2)/(t+1)-2)^2 + t^2/(root(z^6+(14*z^5)/5+z^4/5-(z^3*(-8*t+4*t^2+4*t^3+20))/5-(z^2*(-14*t+9*t^2+8*t^3+9))/5+(z*(4*t-6*t^2-4*t^3+6))/5-(2*t)/5+t^4/5+(2*t^3)/5+3/5, z, 1)+1)^2-1)*root(z^6+(14*z^5)/5+z^4/5-(z^3*(-8*t+4*t^2+4*t^3+20))/5-$$

$(z^2*(-14*t+9*t^2+8*t^3+9))/5+(z*(4*t-6*t^2-4*t^3+6))/5-(2*t)/5+t^4/5+(2*t^3)/5+3/5, z, 1)$

Due to the format of the results, I couldn't verify that strong duality holds by checking the value of $d(x_1(v^*(t)), x_2(v^*(t)), t) = h(v^*(t), t)$

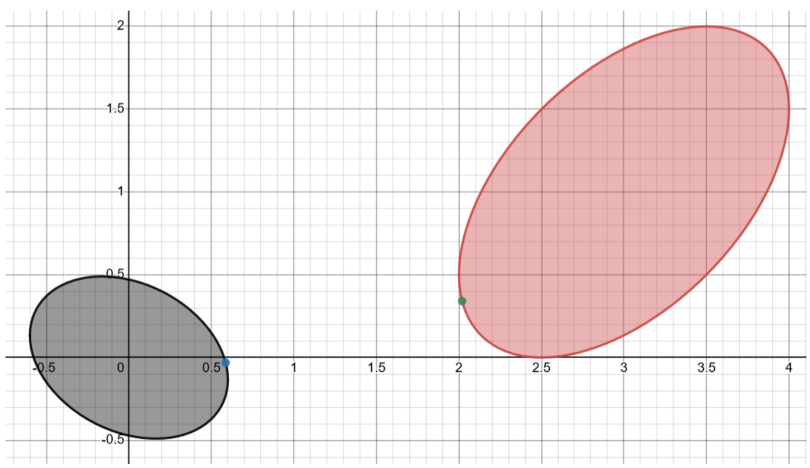
5

We compute the x^0 and the distance d using Matlab.

```
d = @(x) sqrt((x(1)-x(3))^2 + (x(2)-x(4))^2);

g1 = @(x) 6 * x(1)^2 + 9 * x(2)^2 + 4 * x(1) * x(2) - 2;
g2 = @(x) 4 * x(3)^2 + 4 * x(4)^2 - 4 * x(3) * x(4) - 20 * x(3) + 4 * x(4) + 25;

lb = -inf(4,1);
ub = inf(4,1);

syms x

nlc = @(x) deal([g1(x),g2(x)], []);

x0 = fmincon(d, [0,0,3,1], [], [], [], [], lb, ub, nlc)

d(x0)
```

Gives $x^0 = (0.5865, -0.0307, 2.0192, 0.3408)$ and the distance between the points $d(x^0) = 1.4801$.

Sets X_1 and X_2 with highlighted points (x_1, x_2) and (x_3, x_4):



First, let's prove that x^0 is a feasible solution.

$g1(x^0) = -1.4289e-07 \approx 0$ and $g2(x^0) = -1.0764e-07 \approx 0$ which is ≤ 0 and close enough to 0 to be rounding mistake of matlab.

Prove that x^0 is a KKT point:

1. $\nabla f(x^0) = \sum_{i=1}^{m} v_i \nabla g_i(x^0)$
2. $v_i \leq 0$
3. $g_i(x^0) \leq b_i$
4. $v_i(b_i - g_i(x^0)) = 0$

```
d = @(x1, x2, x3, x4) (x1-x3)^2 + (x2-x4)^2;
g1 = @(x1, x2, x3, x4) 6 * x1^2 + 9 * x2^2 + 4 * x1 * x2 - 2;
g2 = @(x1, x2, x3, x4) 4 * x3^2 + 4 * x4^2 - 4 * x3 * x4 - 20 * x3 + 4 * x4 + 25;

syms x1 x2 x3 x4

grad_d = matlabFunction(gradient(d(x1, x2, x3, x4), [x1, x2, x3, x4]))
```

```
grad_g1 = matlabFunction(gradient(g1(x1, x2, x3, x4), [x1, x2, x3, x4]))
grad_g2 = matlabFunction(gradient(g2(x1, x2, x3, x4), [x1, x2, x3, x4]))

grad_d(0.5865,-0.0307,2.0192, 0.3408)
grad_g1(0.5865,-0.0307)
grad_g2(2.0192, 0.3408)

g1(0.5865,-0.0307,2.0192, 0.3408)
g2(0.5865,-0.0307,2.0192, 0.3408)
```

1.

$$\begin{pmatrix} -2.8654 \\ -0.7430 \\ 2.8654 \\ 0.7430 \end{pmatrix} = -0.41436 \begin{pmatrix} 6.9152 \\ 1.7934 \\ 0 \\ 0 \end{pmatrix} - 0.55 \begin{pmatrix} 0 \\ 0 \\ -5.2096 \\ -1.3504 \end{pmatrix}$$

2.

$v_1 = -0.41436$ and $v_2 = -0.55$ which both satisfy the condition of $v_i \leq 0$

3.

After evaluating $g1(x^0)$ and $g2(x^0)$, we get $-1.4289e-07 \approx 0$ and $-1.0764e-07 \approx 0$

4.

$-0.41436(0-0) = 0$ and $-0.55(0-0) = 0$

The KKT conditions hold, let's check if x^0 is a minimum or a maximum:

Let's take a point $a = (0, 0, 3, 1)$ and check for feasibility:
$g1(a) = -2 \leq 0$ and $g2(a) = -3 \leq 0$, therefore the point is feasible.
After checking the distance function, it gives $d(a) = 10$, where $10 > 1.4801$.

Therefore x^0 is a minimum.

6

Consider function $f(x) = (x_1 - x_2)^2 + (2 + x_1 + x_2)^2 \leq 10$

By running following script, we find the minimum and maximum x_i values reachable while staying inside the feasible set of x.

```
f1 = @(x) x(1);
f2 = @(x) x(2);
f3 = @(x) -x(1);
f4 = @(x) -x(2);

g = @(x) (x(1) - x(2))^2 + (2 + x(1) + x(2)^2)^2 - 10;

lb = -inf(2,1);
ub = inf(2,1);

nlc = @(x) deal([g(x)], []);

x01 = fmincon(f1, [0,0], [], [], [], [], lb, ub, nlc)
x02 = fmincon(f2, [0,0], [], [], [], [], lb, ub, nlc)
x03 = fmincon(f3, [0,0], [], [], [], [], lb, ub, nlc)
x04 = fmincon(f4, [0,0], [], [], [], [], lb, ub, nlc)
```

The values are $x^01 = (-5.0118, -1.9488)$, $x^02 = (-4.3860, -2.1499)$, $x^03 = (1.0183, 0.1439)$ and $x^04 = (-1.0862, 1.1499)$

Let's show these points are KKT points.
Let's check the KKT conditions:

1. $\nabla f(x) = \sum_{i=1}^{m} v_i \nabla g_i(x)$
2. $v_i \leq 0$
3. $g_i(x) \leq b_i$
4. $v_i(b_i - g_i(x)) = 0$

For all following computations we will be using this matlab code with minor changes:

```
f1 = @(x1,x2) -x1;
g = @(x1, x2) (x1 - x2)^2 + (2 + x1 + x2^2)^2 - 10;

syms x1 x2

grad_f1 = gradient(f1(x1, x2), [x1, x2])
grad_g = matlabFunction(gradient(g(x1, x2), [x1, x2]));
grad_g(1.0183 ,   0.1439)

g(1.0183   ,  0.1439)
```

For x^01:

Compute the gradient of $f1$ and g at point x^01
$(1, 0) = v(-4.5540, -0.0012)$
Assuming Matlab rounding mistakes, let $v = -\frac{1}{4.554}$.
$g(x^01) = -2.0130e-04 \approx 0$
$-\frac{1}{4.554}(0 + 2.0130e-04) = -0.00004 \approx 0$
Point x^01 is a KKT point.

For x^02:

Compute gradient of $f2$ and g at point x^02
$(0, 1) = v(-0.0001, -14.7571)$

Assuming matlab rounding mistakes, let $v = -\frac{1}{14.7571}$.
$g(x^02) = 1.5230e - 04 \approx 0$
$-\frac{1}{14.7571}(0 - 1.5230e - 04) = 0.00001 \approx 0$
Point x^02 is a KKT point.

For x^03:
Compute gradient of $f3$ and g at point x^03
$(-1, 0) = v(7.8268, 0.0005)$
Assuming matlab rounding mistakes, let $v = -\frac{1}{7.8268}$.
$g(x^03) = 1.4018e - 04 \approx 0$
$-\frac{1}{7.8268}(0 - 1.4018e - 04) = 0.00001 \approx 0$
Point x^03 is a KKT point.

For x^04:
Compute gradient of $f4$ and g at point x^04
$(0, -1) = v(-0.0001, 14.7572)$
Assuming matlab rounding mistakes, let $v = -\frac{1}{14.7572}$.
$g(x^04) = 1.5230e - 04 \approx 0$
$-\frac{1}{14.7572}(0 - 1.5230e - 04) = 0.00001 \approx 0$
Point x^04 is a KKT point.

The final rectangle has vertices at $(-5.0118, 1.1499)$, $(1.0183, 1.1499)$, $(1.0183, -2.1499)$ and $(-5.0118, -2.1499)$.
The lengths of sides are 6.0301 x 3.2998.
The area of the rectangle is 19.89812398.

The smallest rectangle containing the set $X = \{x \in R^2 : f(x) \leq 10\}$: