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Continuous Optimization
Hemtentamen
Submit by 2022-10-03, 17:00
Hjälpmedel: literature, Matlab

The problems are to be solved individually. You may use Matlab and refer to the course book(s) or the hand-outs on Canvas. Solutions must be resonably complete, clearly presented in English and easy to follow, with references where needed. Giving just a Matlab computation is *not* sufficient. Upload solutions to Canvas as *one (1) pdf-file*, including relevant Matlab-code.

1. Define $f(x) = (x_1 - x_2)^2 + (2 + x_1 + x_2^2)^2$.

a) Prove that f is convex for $x_1 \geq -2$. (2p)

b) Is the set $X = \{x \in \mathbb{R}^2 : f(x) \leq 10\}$ convex? Prove your answer. (2p)

2. The function f from **1** has a stationary point x^0. Find it (just prove that it *is* a stationary point) and give the second order approximation $p(x)$ around x^0. Let $d(x) = p(x) - f(x)$, find $\min d(x)$ and $\max d(x)$ over the set $X = \{x \in \mathbb{R}^2 : \|x - x^0\| = 1\}$. Prove your answer. (4p)

3. Define the functions:

$$\begin{aligned}
f(x) &= x_1^2 + 3x_2^2 + 5x_3^2 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3 + x_1 - 2x_2 + 3x_3 \\
g_1(x) &= x_1 + x_2 + x_3 - 1 \\
g_2(x) &= x_1^2 - x_2 + 1
\end{aligned}$$

and consider the two problems

a) $P_1 : \min_{x \in X_1} f(x), \quad X_1 = \{x \in \mathbb{R}^3 : g_1(x) \leq 0\}$

b) $P_2 : \min_{x \in X_2} f(x), \quad X_2 = \{x \in \mathbb{R}^3 : g_1(x) \leq 0, g_2(x) \leq 0\}$

Formulate and solve the corresponding dual problems. Give the dual functions $h(v)$ as explicit formulas (also give $x(v)$), find v^* and show that strong duality holds. Plot both dual functions in separate figures over relevant regions. (4p)

4. Let D be the disk with centre $(0, 2)$ and radius 1. Given a point $p(t) = (t, t^2)$ on a parabola, what is the distance $d(t)$ between $p(t)$ and D? Solve the dual problem and give the dual function $h(v)$, $x(v)$, v^* and $h(v^*)$ explicitly (where v^* and $h(v^*)$ will only depend on t). Show strong duality. (4p)

5. Define the sets

$$\begin{aligned}
X_1 &= \{x \in \mathbb{R}^2 : 6x_1^2 + 9x_2^2 + 4x_1x_2 - 2 \leq 0\} \\
X_2 &= \{x \in \mathbb{R}^2 : 4x_1^2 + 4x_2^2 - 4x_1x_2 - 20x_1 + 4x_2 + 25 \leq 0\}
\end{aligned}$$

What is the distance $\operatorname{dist}(X_1, X_2)$ between the sets? Prove optimality. (4p)

More $\Longrightarrow\Longrightarrow\Longrightarrow$

6. Let X be the set in problem **1b** above. Find the smallest rectangle, with sides parallel to the coordinate axes, that fits around X. Divide the problem into subproblems: $\min\limits_{x \in X} \pm x_i$ and show that the minima are KKT-points. Give the vertex points, side lengths and area of the rectangle. Make a figure. (4p)