# Calculation of coverage intervals: Bayesian inference

A. Chunovkina, A. Stepanov
D.I. Mendeleyev Institute for Metrology, St. Petersburg, Russia

## The aims

— to derive coverage factor $K_{P_0}$ for a confidence level of $P_0 = 90\%, 95\%, 99\%$ using Bayesian approach to uncertainty analysis;
— to check whether coverage factor value $K = 2$ used commonly for $P_0 = 95\%$ is an appropriate choice.

## Prior information; measurement data

Measurement model (two main factors):

$$Y = X + B \quad \Leftrightarrow \quad X = Y - B,$$

where $X$ is measurand, $Y$ is an indication of measuring instrument, and $B$ is bias.

Suppose that $B$ is described by the following class of exponential distributions:

$$p(b) \sim \exp\left\{-\left|\frac{b}{\lambda u_B}\right|^\alpha\right\}, \quad \lambda = \sqrt{\frac{\Gamma(1/\alpha)}{\Gamma(3/\alpha)}}$$

where $\alpha > 0$ is parameter ($\alpha = 2$ corresponds to a normal distribution, $\alpha = +\infty$ corresponds to a uniform distribution). $u_B$ is uncertainty estimated by type B and could be obtained from a calibration certificate.
Let us have $n$ indications $y_1, \dots y_n$ for $Y$ from normal or uniform distribution:

$$y_i \in \mathcal{N}(X + b,\ \sigma) \tag{1}$$

$$y_i \in \mathcal{U}(X + b - \theta, X + b + \theta) \tag{2}$$

Here $\sigma$ or $\theta$ are parameters describing measurement precision.

## Coverage factor: Bayesian inference

Prior pdfs for $\sigma$, $\theta$ are: $p(\sigma) = \sigma^{-1}$, $p(\theta) = \theta^{-1}$, and joint pdf is given by the formula:

$$p\left(x, b, s \,\middle|\, y_1, \dots, y_n\right) \sim L\left(y_1, \dots, y_n \,\middle|\, x, b, s\right)\, p(b)\, s^{-1},$$

where $L$ is a likelihood function and $s$ is a common symbol for $\sigma$ and $\theta$.
So the posterior pdf for $X$ is

$$p(x\,|y_1, \dots, y_n) = C\int_{-\infty}^{+\infty} db \int_{s_0}^{+\infty} L(y_1, \dots, y_n|x, b, s)p(b)s^{-1}ds, \tag{3}$$

where $C$ is norming factor (i.e. $\int_{-\infty}^{+\infty} p(x)dx = C^{-1}$).
Let us find a critical point $\alpha_0$ such that

$$\int_{-\alpha_0}^{\alpha_0} p(x)dx = P_0,$$

then the coverage factor could be obtained by:

$$K = \frac{\alpha_0}{u(x)}, \quad u(x) = \sqrt{\mathrm{Var}(y) + u_B^2}.$$

## Calculation of coverage factors

**Normal distribution for $y_i$:**

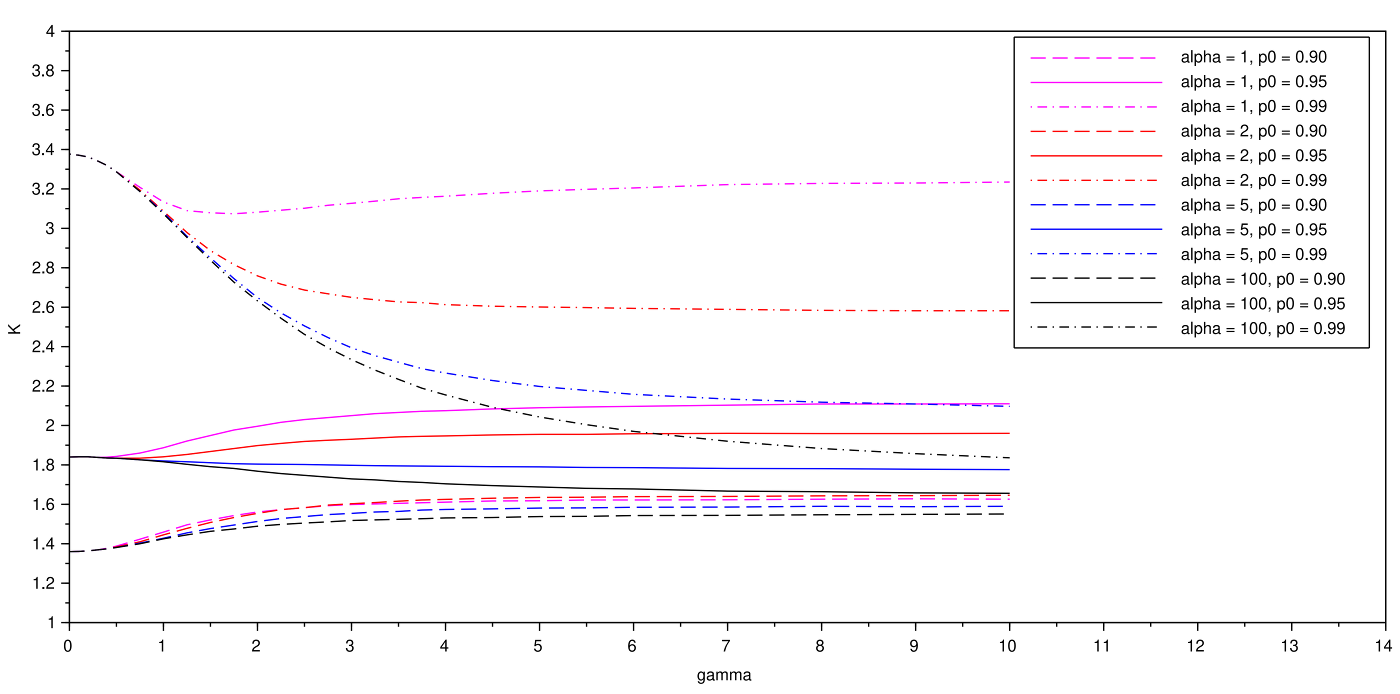
$$(1) \quad \Rightarrow \quad L\left(y_1, \dots, y_n \,\middle|\, x, b, \sigma\right) \sim \frac{1}{\sigma^n}\exp\left\{-\frac{\sum_{i=1}^n (y_i - x - b)^2}{2\sigma^2}\right\}, \quad \sigma > 0.$$

Denote by $\bar{y}$, $S$ sample mean and deviation for $y_i$, $z = \frac{b}{u_B}$; let $\gamma$ be a parameter: $\gamma = \frac{u_B\sqrt{n}}{S}$.
Then for $\tilde{X} = \frac{X - \bar{y}}{S/\sqrt{n}}$ we have:

$$(3) \quad \Rightarrow \quad p(\tilde{x}) \sim \int_{-\infty}^{+\infty}\left(\frac{1}{n-1}(\gamma z + \tilde{x})^2 + 1\right)^{-\frac{n}{2}} \exp\left\{-\left|\frac{z}{\lambda}\right|^\alpha\right\} dz \quad \Rightarrow \quad \alpha_0,$$

$$\text{and} \qquad K = \alpha_0\left(\gamma^2 + \frac{n-1}{n-3}\right)^{-1/2}, \qquad n > 3.$$

Below is an example for $n = 4$.




**Uniform distribution for $y_i$:**

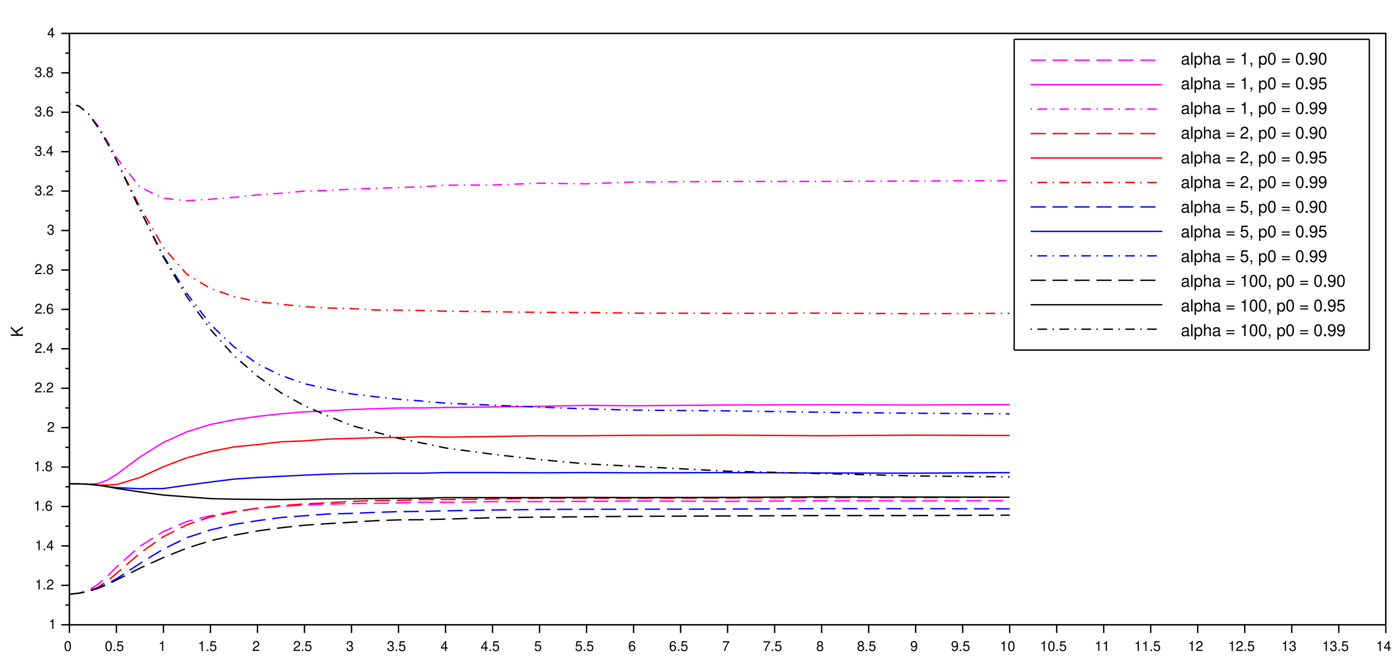
$$(2) \quad \Rightarrow \quad L\left(y_1, \dots, y_n \,\middle|\, x, b, \theta\right) \sim \frac{1}{\theta^n}, \quad \theta \geq \max\left\{x - y_{\min}, y_{\max} - x\right\}.$$

Denote by $\hat{y} = \frac{1}{2}(y_{\min} + y_{\max})$, $r = y_{\max} - y_{\min} > 0$; let $\mu$ be a parameter: $\mu = \frac{u_B\sqrt{n}}{r}$.
Then for $\tilde{X} = \frac{\tilde{X} - \hat{y}}{r/\sqrt{n}}$ we have:

$$(3) \quad \Rightarrow \quad p(\tilde{x}) \sim \int_{-\infty}^{+\infty}\left(\frac{2}{\sqrt{n}}|\mu z + \tilde{x}| + 1\right)^{-n} \exp\left\{-\left|\frac{z}{\lambda}\right|^\alpha\right\} dz \quad \Rightarrow \quad \alpha_0,$$

$$\text{and} \qquad K = \alpha_0\left(\mu^2 + \frac{n}{2(n-2)(n-3)}\right)^{-1/2}, \qquad n > 3.$$

Below is an example for $n = 4$.




Note that the $K$ values vary in a wide range (especially for higher $P_0$), so we cannot provide a common recommended value here.

## Asymptotics for coverage factors

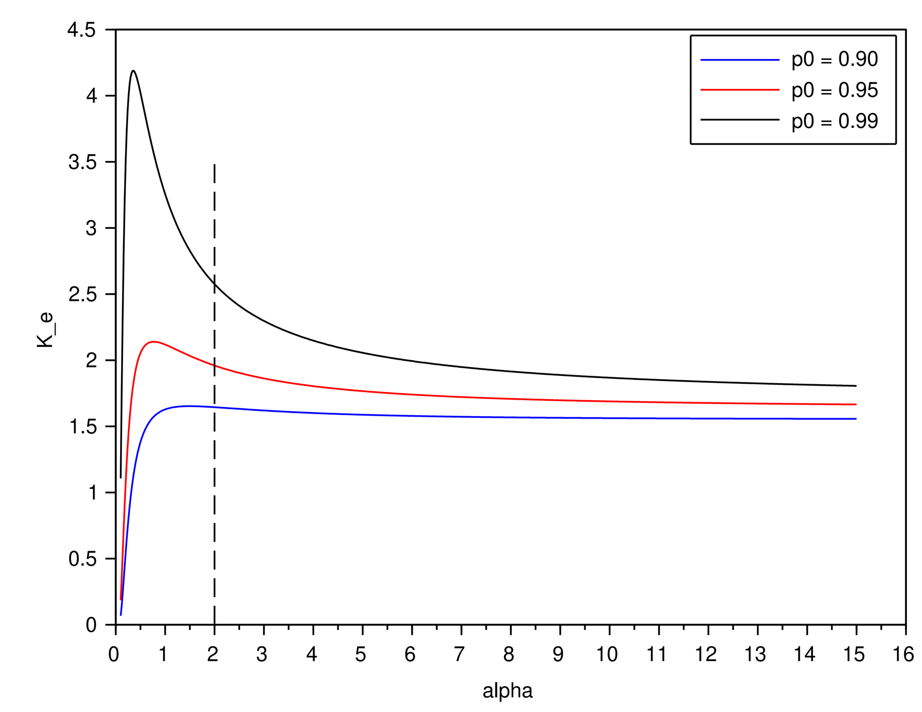
Asymptotics:

$$(1) \quad \Rightarrow \quad K \to \sqrt{\frac{n-3}{n-1}}\, t_{P_0}(n-1), \quad \gamma \to 0$$

$$(2) \quad \Rightarrow \quad K \to \sqrt{\frac{(n-2)(n-3)}{2}}\left(\frac{1}{\sqrt[n-1]{1 - P_0} - 1}\right), \quad \mu \to 0$$

$$K \to K_e(\alpha)\big|_{P_0}, \quad \mu, \gamma \to +\infty,$$

here $K_e(\alpha)$ is a coverage factor for the particular distribution from the exponential family.
The plots for $K_e(\alpha)$ are presented below.

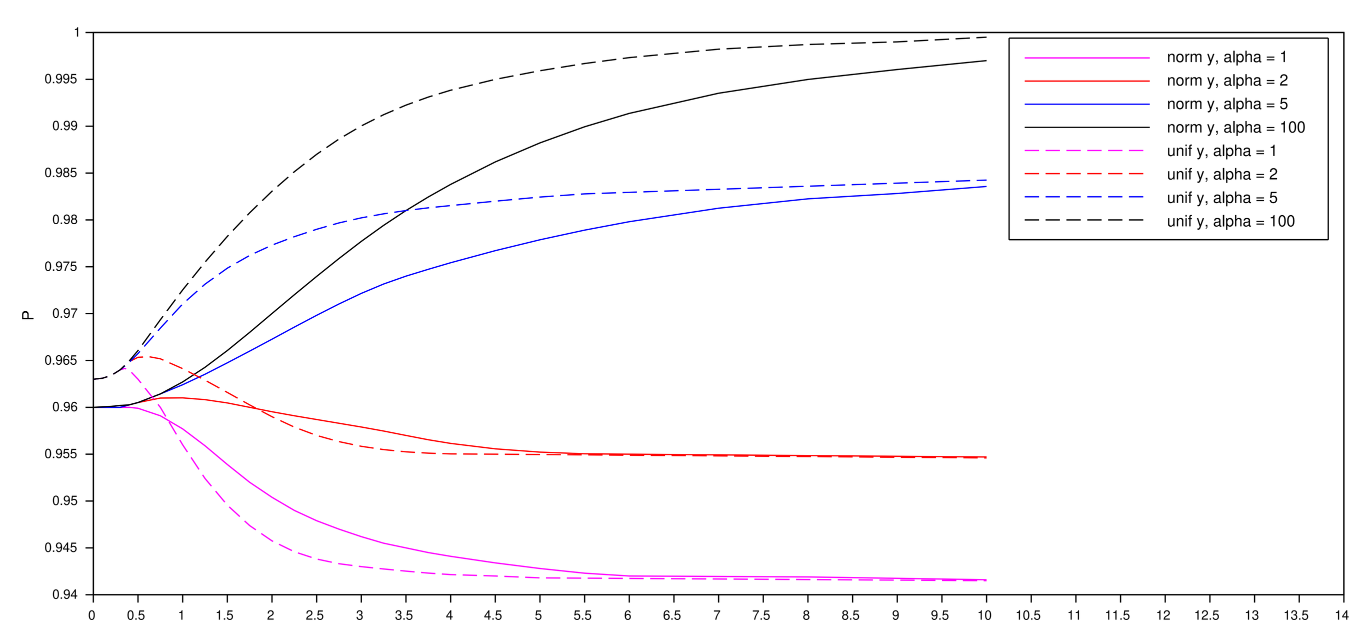



## Coverage probabilities for $K = 2$

Conside a coverage probability

$$P\big|_{K=2} = \int_{-2u(x)}^{2u(x)} p(x)dx.$$

Example for $n = 4$ is given below (the probabilities are given as functions of parameters $\gamma$, $\mu$).




$K = 2$ seems to be an appropriate choice for $P_0 = 0.95$ (as well as for other $n$).

**References:**
[1] A. Chunovkina, A. Stepanov. Calculation of coverage intervals for repeated measurements (Bayesian inference). *Journal of Physics: Conference Series*. Vol 1065, 2018
[2] A. Stepanov, A. Chunovkina, N. Burmistrova. Calculation of coverage intervals: some study cases. *Series on Advanced Mathematics for Applied Sciences*. Vol 86, 2015