Assignment 3
Direct Proofs

1. Prove of Disprove: Every positive integer can be expressed as a sum of three or fewer perfect squares.

Definition 1. An integer n is called a perfect square if, and only if, $n = k^2$ for some integer k.

Proof. The theorem may be restated using the definition of perfect squares1:

$$m = a^2 + b^2 + c^2, \quad \text{for some integers a, b, c, m.}$$

Let $m = 7$:

$$7 = a^2 + b^2 + c^2$$

The square of any integer is positive, and the sum of positive integers is always greater than each individual term of the sum. Therefore, $a, b, c \in [0, 6]$.

Testing all cases:

$$\begin{aligned}
7 &\neq 0^2 + 0^2 + 0^2 = 0, \quad a = 0, b = 0, c = 0 \\
7 &\neq 1^2 + 0^2 + 0^2 = 1, \quad a = 1, b = 0, c = 0 \\
7 &\neq 1^2 + 1^2 + 0^2 = 2, \quad a = 1, b = 1, c = 0 \\
7 &\neq 1^2 + 1^2 + 1^2 = 3, \quad a = 1, b = 1, c = 1 \\
7 &\neq 2^2 + 0^2 + 0^2 = 4, \quad a = 2, b = 0, c = 0 \\
7 &\neq 2^2 + 1^2 + 0^2 = 5, \quad a = 2, b = 1, c = 0 \\
7 &\neq 2^2 + 1^2 + 1^2 = 6, \quad a = 2, b = 1, c = 1 \\
7 &\neq 2^2 + 2^2 + 0^2 = 8, \quad a = 2, b = 2, c = 0
\end{aligned}$$

There are no integers a, b, c such that $7 = a^2 + b^2 + c^2$. Therefore every positive integer cannot be expressed as a sum of three or fewer perfect squares.

□

2. Prove or Disprove: If m is any even integer and n is any odd integer, then $(m+2)^2 - (n-1)^2$ is even.

Property 1. The sum, product, or difference of any two even integers is even.

Property 2. The sum, or difference of any two odd integers is even.

Proof. m is even, so it can be described as:

$$m = 2a, \quad \text{for some integer } a.$$

n is odd, so it can be described as:

$$n = 2b - 1, \quad \text{for some integer } b.$$

Evaluating the first term:

$$(m+2) = (2a+2) = 2(a+1), \quad \text{which is even}$$

Evaluating the second term:

$$(n-1) = ((2b-1)-1) = (2b-2) = 2(b-1), \quad \text{which is even}$$

$(m+2)$ and $(n-1)$ are both even numbers.

$(m+2)(m+2)$ and $(n-1)(n-1)$ are the product of two even numbers, which by property 1 is also even.

Furthermore, by 1, the difference of two even numbers, $(m+2)^2 - (n-1)^2$, is even.

Therefore, if m is any even integer and n is any odd integer, then $(m+2)^2 - (n-1)^2$ is even. $\square$