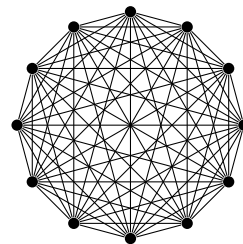


## Assignment 4

### Graphs

1. If 12 people each shake hands with each other, how many handshakes took place?

Let each vertex represent a person, and the edges between them a handshake. This problem may be represented by a graph of 12 nodes, each of degree 11—the complete graph  $K_{12}$ . Each vertex is degree 11, and there are 12 vertices. There are  $n(n-1)/2$  edges in graph  $K_n$ . Therefore, there are  $12(12-1)/2 = 66$  edges in graph  $K_{12}$ . 66 handshakes took place.

Figure 1:  $K_{12}$ 

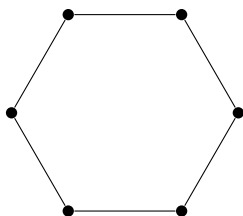
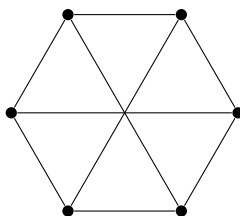
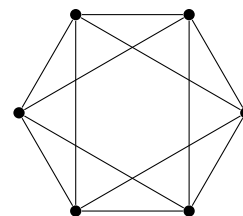
2. In a group of six people, is it possible for everyone to be friends with exactly two other people in the group? How about three other people? Four?

Let each person be represented by a vertex in a graph, with friendships illustrated as edges between vertices. Note that in any graph, the number of vertices with an odd degree must be even.

$G_1$  Graph of 6 vertices, each of degree 2. The sum of degrees in  $G_1$  is  $6 \times 2 = 12$ . The graph would have 6 edges which is possible. Also see Figure 2.

$G_2$  Graph of 6 vertices, each of degree 3. The sum of degrees in  $G_2$  is  $6 \times 3 = 18$ . The graph would have 9 edges which is possible. Also see Figure 3.

$G_3$  Graph of 6 vertices, each of degree 4. The sum of degrees in  $G_3$  is  $6 \times 4 = 24$ . The graph would have 12 edges which is possible. Also see Figure 4.

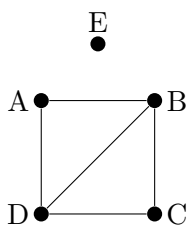
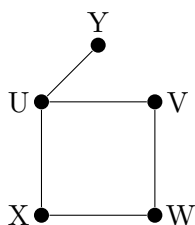
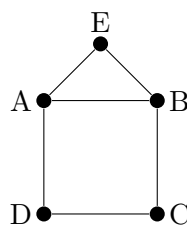
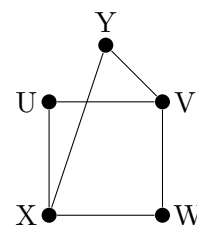
Figure 2:  $G_1$ Figure 3:  $G_2$ Figure 4:  $G_3$ 

3. Is it possible for two graphs with the same number of nodes and edges to not be isomorphic? What if the degrees of the nodes in the two graphs are the same? Give example graphs or explain why.

Two graphs are isomorphic if there is a one-to-one relationship (bijection) between their vertices that preserves adjacency—that is, the graphs have the same structure and differ only in the labeling of their vertices.

Graphs  $G_1$  (5) and  $G_2$  (6) have  $|E| = 5$  and  $|V| = 5$ . E is not reachable in  $G_1$ , however every node is reachable in  $G_2$ , therefore they cannot be isomorphic.

Graphs  $G_2$  (7) and  $G_4$  (8) have  $|E| = 6$ ,  $|V| = 5$ , and degree sequence  $(3, 3, 2, 2, 2)$ . For  $G_3$  and  $G_4$  to be isomorphic, there must exist a bijective mapping between the vertices which preserves adjacency. This implies that vertices with the same degree must be mapped to each other. In  $G_3$ , vertices A and B are degree three and are adjacent. In  $G_4$  vertices X and V are degree three but are not adjacent. Therefore, the graphs cannot be isomorphic.

Figure 5:  $G_1$ Figure 6:  $G_2$ Figure 7:  $G_3$ Figure 8:  $G_4$ 

4. What is the largest number of edges possible in a graph with 10 vertices? What is the largest number of edges possible in a bipartite graph with 10 vertices? What is the largest number of edges possible in a tree with 10 vertices?

- (a) The largest number of edges in a graph with 10 vertices is the complete graph  $K_{10}$  with  $|E| = 10(10 - 1)/2 = 45$ .
- (b) The largest number of edges in a bipartite graph with 10 vertices will be a complete bipartite graph  $K_{m,n}$  such that  $m + n = 10$ . The number of edges in a complete bipartite graph is  $m \times n$  by the Handshake Lemma. Therefore the constraint  $m + n = 10$ , we wish to maximize  $m \times n$ .

Let

$$f(m, n) = mn \quad \text{and} \quad g(m, n) = m + n - 10.$$

Using the method of Lagrange multipliers:

$$\nabla f(m, n) = \langle n, m \rangle, \quad \nabla g(m, n) = \langle 1, 1 \rangle.$$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} n = \lambda, \\ m = \lambda, \end{cases} \Rightarrow m = n.$$

Substitute into the constraint:

$$m + n = 10 \Rightarrow 2m = 10 \Rightarrow m = 5.$$

Because  $m = n$  then:

$$m = 5 \Rightarrow n = 5.$$

The bipartite graph with 10 vertices and the most possible edges is  $K_{5,5}$ , which has 25 edges.

- (c) In any tree, the number of edges is given by  $|E| = |V| - 1$ . In a tree with 10 vertices, the number of edges is therefore  $|E| = 10 - 1 = 9$ .

5. Consider graphs with  $n$  vertices. Remember, graphs do not need to be *connected*.

- (a) How many edges must the graph have to guarantee at least one vertex has degree two or more. Prove your answer.

*Proof.* By direct proof.

Let  $G$  be a graph with  $|V| = n$ .

The maximum number of edges  $G$  can have while no vertex has degree 2 is achieved when every vertex has degree exactly 1.

By the Handshake Lemma, the total number of edges in a graph is given by:

$$\sum_{v \in V} d(v) = 2e \quad \Rightarrow \quad e = \frac{1}{2} \sum_{v \in V} d(v)$$

Since every vertex has degree 1, the sum of the degrees is:

$$\sum_{i=1}^n 1 = n \quad \Rightarrow \quad e = \left\lfloor \frac{n}{2} \right\rfloor$$

Now, suppose we add one more edge to this graph. This edge must connect to at least one existing vertex, increasing its degree to 2.

Therefore,  $G$  must have  $\left\lfloor \frac{n}{2} \right\rfloor + 1$  edges to guarantee at least one vertex has degree two or greater.  $\square$

- (b) How many edges must the graph have to guarantee all vertices have degree two or more? Prove your answer.

*Proof.* By direct proof.

Let  $G$  be a graph in which  $|V| = n$  and every vertex is degree 2.

By the Handshake Lemma, the total number of edges in a graph is given by:

$$\sum_{v \in V} d(v) = 2e \quad \Rightarrow \quad e = \frac{1}{2} \sum_{v \in V} d(v)$$

Since every vertex has degree 2, the sum of the degrees is:

$$\sum_{i=1}^n 2 = 2n \quad \Rightarrow \quad e = \frac{2n}{2} = n$$

Therefore, the  $G$  must have at least  $n$  edges to ensure every vertex is at least degree 2.  $\square$

6. Which of the graphs below are bipartite? Justify your answers.

$G_1$  (9),  $G_2$  (10), and  $G_4$  (12) are bipartite.  $G_3$  (11) is not bipartite.

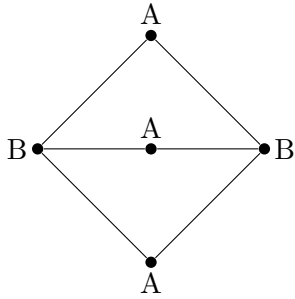


Figure 9:  $G_1$

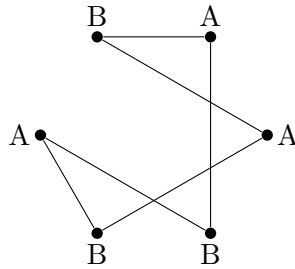


Figure 10:  $G_2$

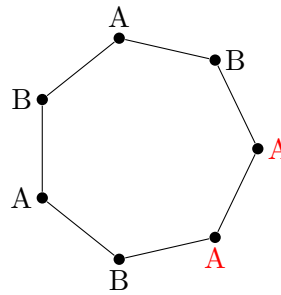


Figure 11:  $G_3$

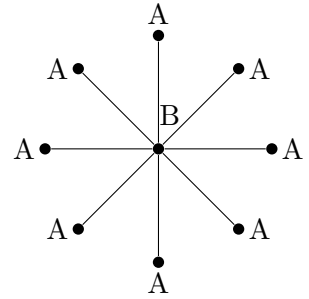


Figure 12:  $G_4$