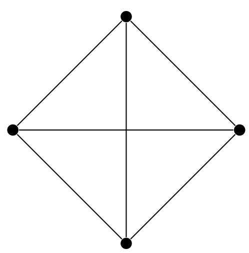
Assignment 6
MORE Graphs

1. Which of the following graphs contain an Euler path? Which contain an Euler circuit? Show your work (including the graph).

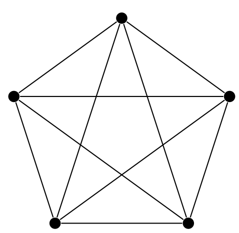
(a) K_4

The degree sequence for K_4 is (3, 3, 3, 3). There is neither an Euler circuit nor a path.



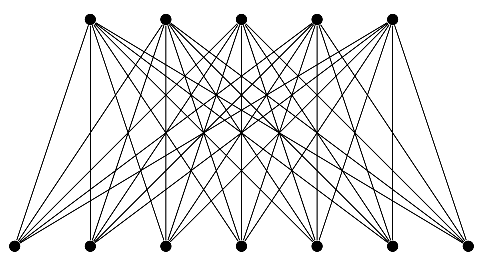
(b) K_5

The degree sequence for K_5 is (4, 4, 4, 4, 4). There is an Euler circuit and path.



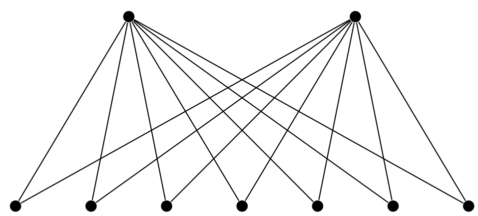
(c) $K_{5,7}$

The degree sequence for $K_{5,7}$ is (7, 7, 7, 7, 7, 5, 5, 5, 5, 5, 5, 5). There is neither an Euler circuit nor path.



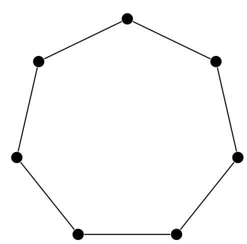
(d) $K_{2,7}$

The degree sequence for $K_{2,7}$ is (7, 7, 2, 2, 2, 2, 2, 2, 2). There is an Euler path, but no Euler circuit.



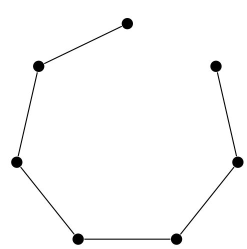
(e) C_7

The degree sequence for C_7 is (2, 2, 2, 2, 2, 2, 2). There is an Euler circuit and path.



(f) P_7

The degree sequence for P_7 is (2, 2, 2, 2, 2, 1, 1). There is an Euler path, but no Euler circuit.



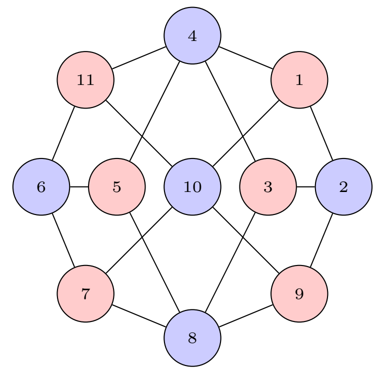
2. Consider the following graph:

(a) Find a Hamiltonian path. Can your path be extended to a Hamiltonian cycle?

A Hamiltonian path is numbered from 1 to 11 in the figure.

(b) Is the graph bipartite? If so, how many vertices are in each "part"?

The graph is bipartite, each "part" is denoted by colors red and blue in the figure. One set contains 6 vertices, the other contains 5.



(c) Use your answer to part (b) to prove that the graph has no Hamilton cycle.

Proof. By contradiction

Assume the graph contains a Hamilton cycle.

A Hamilton cycle visits every vertex exactly once, beginning and ending at the same vertex.

For a bipartite graph, this implies the Hamilton cycle must alternate between vertices in set A and B, as no vertices in the same set are adjacent.

For the given graph $|A| = |B| + 1$.

<u>Case 1:</u> Start at a vertex in set A

The cycle can then be described as:

$$(a_1, b_1, a_2, b_2, ..., a_{n-1}, b_{n-1}, a_n, a_1) \quad \text{s.t. } a \in A \text{ and } b \in B$$

But $a_1, a_n \in A$ and thus cannot be adjacent which is a contradiction.

<u>Case 2:</u> Start at a vertex in set B

The cycle can then be described as:

$$(b_1, a_1, b_2, a_2, ..., b_{n-1}, a_{n-1}, a_n, b_1) \quad \text{s.t. } a \in A \text{ and } b \in B$$

Note vertex b_n does not exist as $|A| = |B| + 1$. $a_{n-1}, a_n \in A$ and thus cannot be adjacent which is a contradiction.

Therefore by contradiction there cannot exist a Hamilton cycle.

□

(d) Suppose you have a bipartite graph G in which one part has at least two more vertices than the other. Prove that G does not have a Hamilton path.

Proof. Let G be a bipartite graph with sets A and B such that $|A| \geq |B| + 2$.

Because G is bipartite, if a Hamilton path exists, the vertices in the path will alternate between vertices in A and vertices in B.

In the minimum case:

$$A = \{a_1, a_2, ..., a_n\} \text{ and } B = \{b_1, b_2, ..., b_{n-2}\} \quad \text{if } |A| = |B| + 2$$

We pair up vertices from sets A and B to create a path.
<u>Case 1:</u> Start at a vertex in set A
The path can be described as:

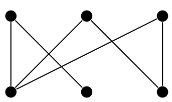
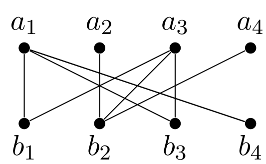
$$(a_1, b_1, a_2, b_2, \dots, a_{n-2}, b_{n-2}, a_{n-1}, a_n) \quad \text{s.t. } a \in A \text{ and } b \in B$$

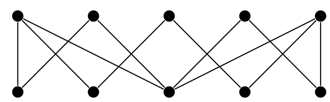
But $a_{n-1}, a_n \in A$ and thus cannot be adjacent which is a contradiction.
<u>Case 2:</u> Start at a vertex in set B
The path can be described as:

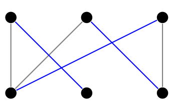
$$(b_1, a_1, b_2, a_2, ..., b_{n-2}, a_{n-2}, a_{n-1}, a_n) \quad \text{s.t. } a \in A \text{ and } b \in B$$

But $a_{n-2}, a_{n-1}, a_n \in A$ and thus cannot be adjacent which is a contradiction.
In any case where $|A| > |B| + 2$ A would have more vertices and more would be unpaired.
Therefore by contradiction there cannot exist a Hamilton path. $\square$

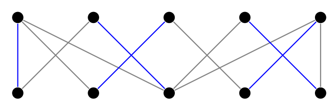
3. Find a matching of the bipartite graphs below or explain why no matching exists.



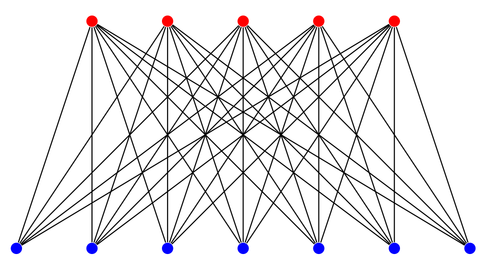


No matching exists because a_2 and a_4 are only adjacent to b_2.

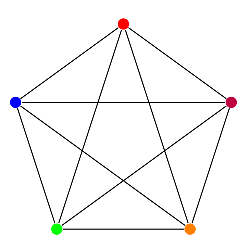


4. What is the smallest number of colors you need to properly color the vertices of $K_{5,7}$?

Two as it is a bipartite graph.



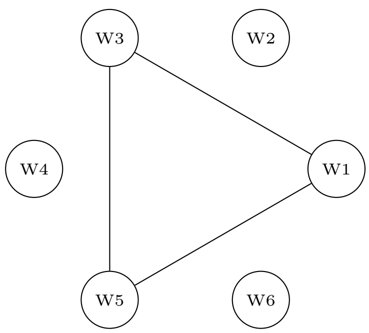
5. Draw a graph with chromatic number 5. Could your graph be planar? Explain. No because by the Four Color Theorem which states that "If G is a planar graph, then the chromatic number of G is less than or equal to 4."



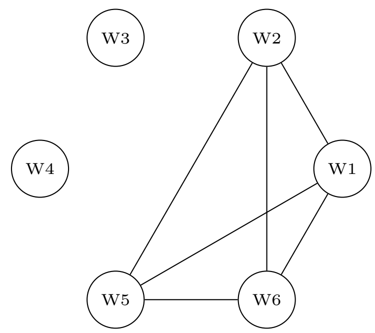
6. The local Entomology Club has six work groups, each meeting once a month. How many different meeting times must be used to ensure that no member is scheduled to attend two meetings at the same time if the work groups consist of the following members?

W1 = {John, Voormann, Ringo}
W2 = {Voormann, George, Paul}
W3 = {John, Paul, Ringo}
W4 = {George, Paul, Ringo}
W5 = {John, Voormann}
W6 = {Voormann, Paul, Ringo}

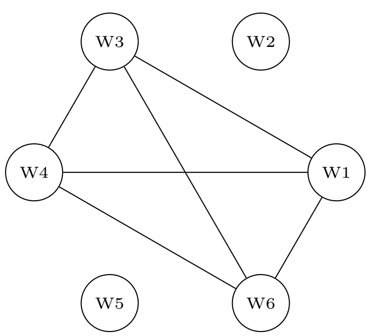
John



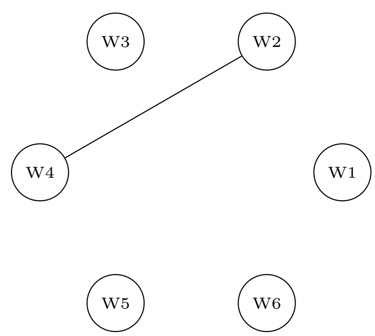
Voormann



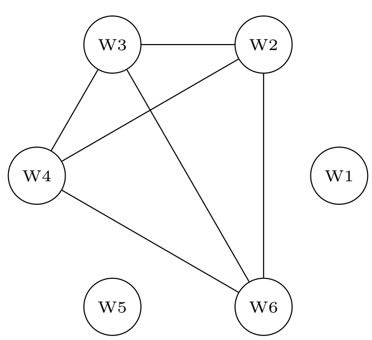
Ringo



George



Paul



Eight meeting times are required.