

Assignment 3

Direct Proofs

1. Prove or Disprove: Every positive integer can be expressed as a sum of three or fewer perfect squares.

Definition 1. An integer n is called a perfect square if, and only if, $n = k^2$ for some integer k .

Proof. The theorem may be restated using the definition of perfect squares:

$$m = a^2 + b^2 + c^2, \quad \text{for some integers } a, b, c, m.$$

Let $m = 7$:

$$7 = a^2 + b^2 + c^2$$

The square of any integer is positive, and the sum of positive integers is always greater than each individual term of the sum. Therefore, $a, b, c \in [0, 6]$.

Testing all cases:

$$7 \neq 0^2 + 0^2 + 0^2 = 0, \quad a = 0, b = 0, c = 0$$

$$7 \neq 1^2 + 0^2 + 0^2 = 1, \quad a = 1, b = 0, c = 0$$

$$7 \neq 1^2 + 1^2 + 0^2 = 2, \quad a = 1, b = 1, c = 0$$

$$7 \neq 1^2 + 1^2 + 1^2 = 3, \quad a = 1, b = 1, c = 1$$

$$7 \neq 2^2 + 0^2 + 0^2 = 4, \quad a = 2, b = 0, c = 0$$

$$7 \neq 2^2 + 1^2 + 0^2 = 5, \quad a = 2, b = 1, c = 0$$

$$7 \neq 2^2 + 1^2 + 1^2 = 6, \quad a = 2, b = 1, c = 1$$

$$7 \neq 2^2 + 2^2 + 0^2 = 8, \quad a = 2, b = 2, c = 0$$

There are no integers a, b, c such that $7 = a^2 + b^2 + c^2$. Therefore every positive integer cannot be expressed as a sum of three or fewer perfect squares.

□

2. Prove or Disprove: If m is any even integer and n is any odd integer, then $(m + 2)^2 - (n - 1)^2$ is even.

Property 1. The sum, product, or difference of any two even integers is even.

Property 2. The sum, or difference of any two odd integers is even.

Proof. m is even, so it can be described as:

$$m = 2a, \quad \text{for some integer } a.$$

n is odd, so it can be described as:

$$n = 2b + 1, \quad \text{for some integer } b.$$

Evaluating the first term:

$$(m + 2) = (2a + 2) = 2(a + 1), \quad \text{which is even}$$

Evaluating the second term:

$$(n - 1) = ((2b + 1) - 1) = (2b) = 2(b), \quad \text{which is even}$$

$(m + 2)$ and $(n - 1)$ are both even numbers.

$(m + 2)(m + 2)$ and $(n - 1)(n - 1)$ are the product of two even numbers, which by property 1 is also even.

Furthermore, by 1, the difference of two even numbers, $(m + 2)^2 - (n - 1)^2$, is even.

Therefore, if m is any even integer and n is any odd integer, then $(m + 2)^2 - (n - 1)^2$ is even. \square