# Assignment 4
Graphs

1. If 12 people each shake hands with each other, how many handshakes took place?

Let each vertex represent a person, and the
edges between them a handshake. This prob-
lem may be represented by a graph of 12 nodes,
each of degree 11—the complete graph K_{12}.
Each vertex is degree 11, and there are 12 ver-
tices. There are $n(n-1)/2$ edges in graph K_n.
Therefore, there are $12(12-1)/2 = 66$ edges in
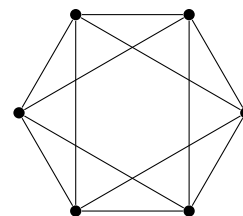graph K_{12}. 66 handshakes took place.
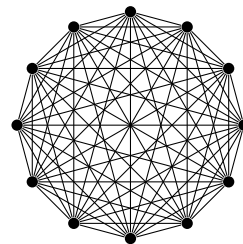
Figure 1: K_{12}

2. In a group of six people, is it possible for everyone to be friends with exactly two other people
in the group? How about three other people? Four?

Let each person be represented by a vertex in a graph, with friendships illustrated as edges
between vertices. Note that in any graph, the number of vertices with an odd degree must be
even.

G_1 Graph of 6 vertices, each of degree 2. The sum of degrees in G_1 is $6 \times 2 = 12$. The graph
would have 6 edges which is possible. Also see Figure 2.

G_2 Graph of 6 vertices, each of degree 3. The sum of degrees in G_2 is $6 \times 3 = 18$. The graph
would have 9 edges which is possible. Also see Figure 3.

G_3 Graph of 6 vertices, each of degree 4. The sum of degrees in G_3 is $6 \times 4 = 24$. The graph
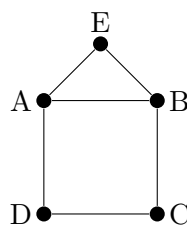would have 12 edges which is possible. Also see Figure 4.
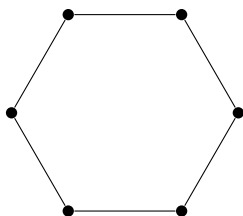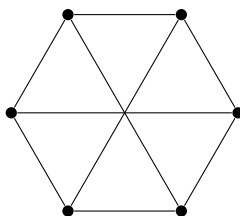
Figure 2: G_1 Figure 3: G_2 Figure 4: G_3

3. Is it possible for two graphs with the same number of nodes and edges to not be isomorphic?
What if the degrees of the nodes in the two graphs are the same? Give example graphs or
explain why.

Two graphs are isomorphic if there is a one-to-one relationship (bijection) between their vertices
that preserves adjacency—that is, the graphs have the same structure and differ only in the
labeling of their vertices.

Graphs G_1 (5) and G_2 (6) have $|E| = 5$ and $|V| = 5$. E is not reachable in G_1, however every node is reachable in G_2, therefore they cannot be isomorphic.

Graphs G_2 (7) and G_4 (8) have $|E| = 6$, $|V| = 5$, and degree sequence $(3, 3, 2, 2, 2)$. For G_3 and G_4 to be isomorphic, there must exist a bijective mapping between the vertices which preserves adjacency. This implies that vertices with the same degree must be mapped to each other. In G_3, vertices A and B are degree three and are adjacent. In G_4 vertices X and V are degree three but are not adjacent. Therefore, the graphs cannot be isomorphic.
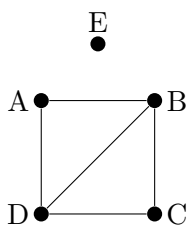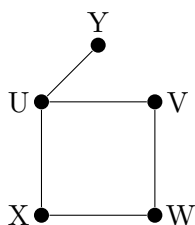
Figure 5: G_1 Figure 6: G_2 Figure 7: G_3 Figure 8: G_4
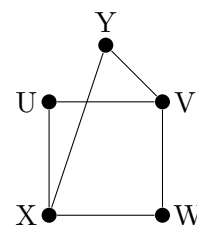
4. What is the largest number of edges possible in a graph with 10 vertices? What is the largest number of edges possible in a bipartite graph with 10 vertices? What is the largest number of edges possible in a tree with 10 vertices?

(a) The largest number of edges in a graph with 10 vertices is the complete graph K_{10} with $|E| = 10(10 - 1)/2 = 45$.

(b) The largest number of edges in a bipartite graph with 10 vertices will be a complete bipartite graph $K_{m,n}$ such that $m + n = 10$. The number of edges in a complete bipartite graph is $m \times n$ by the Handshake Lemma. Therefore the constraint $m + n = 10$, we wish to maximize $m \times n$.

Let

$$f(m, n) = mn \quad \text{and} \quad g(m, n) = m + n - 10.$$

Using the method of Lagrange multipliers:

$$\nabla f(m, n) = \langle n, m \rangle, \quad \nabla g(m, n) = \langle 1, 1 \rangle.$$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} n = \lambda, \\ m = \lambda, \end{cases} \Rightarrow m = n.$$

Substitute into the constraint:

$$m + n = 10 \Rightarrow 2m = 10 \Rightarrow m = 5.$$

Because $m = n$ then:

$$m = 5 \Rightarrow n = 5.$$

The bipartite graph with 10 vertices and the most possible edges is $K_{5,5}$, which has 25 edges.

(c) In any tree, the number of edges is given by $|E| = |V| - 1$. In a tree with 10 vertices, the number of edges is therefore $|E| = 10 - 1 = 9$.

5. Consider graphs with n vertices. Remember, graphs do not need to be *connected*.

(a) How many edges must the graph have to guarantee at least one vertex has degree two or more. Prove your answer.

Proof. By direct proof.

Let G be a graph with $|V| = n$.

The maximum number of edges G can have while no vertex has degree 2 is achieved when every vertex has degree exactly 1.

By the Handshake Lemma, the total number of edges in a graph is given by:

$$\sum_{v \in V} d(v) = 2e \quad \Rightarrow \quad e = \frac{1}{2} \sum_{v \in V} d(v)$$

Since every vertex has degree 1, the sum of the degrees is:

$$\sum_{i=1}^{n} 1 = n \quad \Rightarrow \quad e = \left\lfloor \frac{n}{2} \right\rfloor$$

Now, suppose we add one more edge to this graph. This edge must connect to at least one existing vertex, increasing its degree to 2.

Therefore, G must have $\lfloor \frac{n}{2} \rfloor + 1$ edges to guarantee at least one vertex has degree two or greater. □

(b) How many edges must the graph have to guarantee all vertices have degree two or more? Prove your answer.

Proof. By direct proof.

Let G be a graph with $|V| = n$.

The maximum number of edges G can have while one vertex v is disconnected occurs when when the remaining $n - 1$ vertices form a complete subgraph K_{n-1}. This subgraph has:

$$e = \frac{(n-1)(n-2)}{2}$$

Every vertex in K_{n-1} has degree $n - 2$, which is at least 2 for all $n \geq 4$.

Now, add one edge connecting v to a vertex $u \in K_{n-1}$, so that v has degree 1. The total number of edges is now:

$$e = \frac{(n-1)(n-2)}{2} + 1.$$

Now, add one more edge connecting v to another vertex $w \in K_{n-1}$, so that v is now adjacent to two vertices.

Therefore $\deg(v) = 2$ and all other vertices still have degree at least $n - 2$ because edges were never removed.

The minimum total number of edges is then:

$$e = \frac{(n-1)(n-2)}{2} + 2.$$

G must have $\frac{(n-1)(n-2)}{2} + 2$ to guarantee all vertices have degree two or more.

□

6. Which of the graphs below are bipartite? Justify your answers.

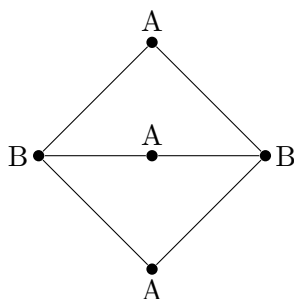G_1 (9), G_2 (10), and G_4 (12) are bipartite. G_3 (11) is not bipartite.
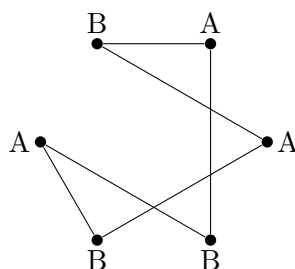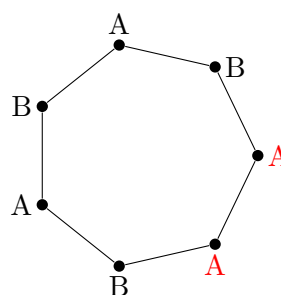

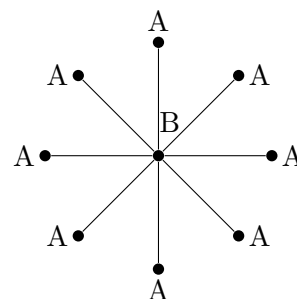Figure 9: G_1 Figure 10: G_2 Figure 11: G_3 Figure 12: G_4