Assignment 5

Trees and More Graphs

1. We often define graph theory concepts using set theory. For example given a graph $G = (V, E)$ and a vertex $v \in V$, we define

$$N(v) = \{u \in V : \{v, u\} \in E\}$$

We define $N[v] = N(v) \cup \{v\}$. The goal of this problem is to figure out what all this means.

(a) Let G be the graph with $V = \{a, b, c, d, e, f\}$ and
$E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, f\}, \{d, f\}, \{e, f\}\}$. Find $N(a), N[a], N(c),$ and$N[c]$.

$$N(a) = \{b, e\} \quad N[a] = \{a, b, e\} \quad N(c) = \{b, d, f\} \quad N[c] = \{b, c, d, f\}$$

(b) What is the largest and smallest possible values for $|N(v)|$ and $|N[v]|$ for the graph from part (a)? Explain.
$N(v)$ is the set of vertices incident to v, therefore:

$$|N(v)| = \deg(v)$$

$N[v]$ is the set of vertices incident to v union v itself, therefore:

$$|N(v)| = \deg(v) + 1$$

Given the degree sequence of the graph G is:

$$(2,\ 2,\ 3,\ 3,\ 3,\ 3)$$

We compute the minimum and maximum values as follows:

$$\min(|N(v)|) = 2 \quad \max(|N(v)|) = 3 \quad \min(|N[v]|) = 3 \quad \max(|N[v]|) = 4$$

(c) Give an example of a graph $G = (V, E)$ (Probably different from the one above) for which $N[v] = V$ for some vertex $v \in V$. Is there a graph for which $N[v] = V$ for *all* $v \in V$? Explain.
An example of a graph for which $N[v] = V$ for some $v \in V$ is any graph for which v is adjacent to all other vertices in the graph. Example: 1
An example of a graph for which $N[v] = V$ for *all* $v \in V$ is a graph in which for all $v \in V$, v is adjacent to all other vertices. This is the case for any complete graph. Example: 2.

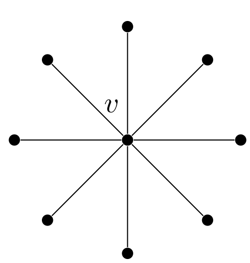

Figure 1: $N[v] = V$ for $v \in V$

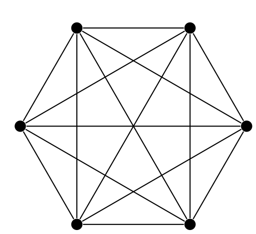

Figure 2: K_6, $N[v] = V$ for all $v \in V$

(d) Give an example of a graph $G = (V, E)$ for which $N(v) = \emptyset$ for some $v \in V$. Is there an example of such graph for which $N[u] = V$ for some other $u \in V$ as well? Explain.

An example of a graph for or which $N(v) = \emptyset$ for some $v \in V$ is any graph with an unconnected vertex v. Consider the example:

$$G \text{ is a graph such that } V = \{a, b, c, d\} \text{ and } E = \{\{a, b\}, \{b, c\}\} \text{ then } N(d) = \emptyset$$

An example of such graph for which $N[u] = V$ for some other $u \in V$ as well does not exist. As discussed previously, for $N[u] = V$ for some $u \in V$, u must be adjacent all other vertices in the graph. However, from the first part we know v must be unconnected, and therefore u cannot be adjacent to v. Thus, there exists no such graph.

(e) Describe in words what $N(v)$ and $N[v]$ mean in general.

$N(v)$ is the set of vertices incident to v.
$N[v]$ is the set of vertices incident to v union v itself.

2. Which of the following graphs are trees

(a) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}$
Not a tree–there exists a cycle.

(b) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}$
Tree.

(c) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}\}$
Tree (rooted tree).

(d) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{d, e\}\}$
Not a tree–graph is disconnected. (However, there is a forest of two trees)

3. For each degree sequence below, decide wether it must always, must never, or could possibly be a degree sequence for a tree. Remember, a degree sequence lists out the degrees (number of edges incident to the vertex) of all the vertices in a graph in non-increasing order.

We know given a degree sequence, the number of edges is given by the Handshake Lemma:

$$\sum_{v \in V} d(v) = 2e \quad \Rightarrow \quad e = \frac{1}{2} \sum_{v \in V} d(v)$$

And tree always satisfies:

$$e + 1 = v$$

(a) $(4, 1, 1, 1, 1)$
5 vertices, 4 edges. $v = e + 1$. Always a tree.

(b) $(3, 3, 2, 1, 1)$
5 vertices, 5 edges. $v \neq e + 1$ Not a tree.

(c) $(2, 2, 2, 1, 1)$
5 vertices, 4 edges. $v = e + 1$. Possibly a tree.

(d) $(4, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1)$
14 vertices, 14 edges. $v \neq e + 1$. Not a tree.

4. Suppose you have a graph with v vertices and e edges that satisfies $v = e + 1$. Must the graph be a tree? Prove your answer.

Proof. By counterexample

Consider the graph $G = (V, E)$ with

$$V = \{a, b, c, d\} \quad \text{and} \quad E = \{\{a, b\}, \{b, c\}, \{c, a\}\}$$

In this case, $v = 4$ and $e = 3$, so the condition $v = e + 1$ holds:

$$4 = 3 + 1$$

However, G is not a tree. It contains a cycle between the vertices a, b, and c, which violates the definition of a tree. Furthermore, G is disconnected, as vertex d is isolated.

While G satisfies $v = e + 1$, it is not a tree.

Therefore, if there exists a graph with v vertices and e edges that satisfies $v = e + 1$, it is not always a tree. □

5. Prove that any graph (not necessarily a tree) with v vertices and e edges that satisfies $v > e + 1$ will NOT be connected.

Proof. By contradiction

Assume there exists a connected graph G with v vertices and e edges that satisfies $v > e + 1$.

We know every connected graph has a spanning tree. A spanning tree will have $v' = v$ vertices and $e' \leq e$ edges.

Thus the spanning tree will also have $v' > e' + 1$.

But a tree must have $v = e + 1$, which is a contradiction, thus G is not connected.

Therefore any graph with v vertices and e edges that satisfies $v > e + 1$ will NOT be connected. □

6. Let T be a rooted tree that contains vertices v, u, and w (among possibly others). Prove that if w is a descendant of both u and v then u is a descendant of v or v is a descendant of u.

Proof. By contradiction

Assume w is a descendant of both u and v in a rooted tree.

This implies that w is not the root as it is a descendant of other vertices. Additionally, u and v are on a path from w to the root by the definition of a descendant.

By the definition of a rooted tree, there is exactly one simple path from any node to the root. Let the path from w to the root be denoted by P.

P must either pass through both u and v or there exists another path to the root which would contradict the definition of a rooted tree.

Therefore, either u is a descendant of v or v is a descendant of u. □

7. Prove that every connected graph which is not itself a tree must have at least three different spanning trees.

Proof. Direct proof

Let G be a connected graph which is not a tree.

G must therefore contain at least one cycle C_n connecting some n vertices.

To obtain a spanning tree we must remove an edge from the cycle C_n. Since the cycle contains n edges, there are at least n distinct ways to remove an edge and each may produce a unique spanning tree.

Consider the minimum example in which G contains only one cycle C_3. C_3 has three vertices and three edges. In this minimal case, there are three edges which can be removed, each producing a distinct spanning tree.

Therefore, every connected graph that is not itself a tree must have at least three distinct spanning trees. □