Assignment 5

Trees and More Graphs

1. We often define graph theory concepts using set theory. For example given a graph $G = (V, E)$ and a vertex $v \in V$, we define

$$N(v) = \{u \in V : v, u \in E\}$$

We define $N[v] = N(v) \cup \{v\}$. The goal of this problem is to figure out what all this means.

(a) Let G be the graph with $V = \{a, b, c, d, e, f\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, f\}, \{d, f\}, \{e, f\}\}$. Find $N(a), N[a], N(c), \text{and} N[c]$.

(b) What is the largest and smallest possible values for $|N(v)|$ and $|N[v]|$ for the graph from part (a)? Explain.

(c) Give an example of a graph $G = (V, E)$ (Probably different from the one above) for which $N[v] = V$ for some vertex $v \in V$. Is there a graph for which $N[v] = V$ for *all* $v \in V$? Explain.

(d) Give an example of a graph $G = (V, E)$ for which $N(v) = \emptyset$ for some $v \in V$. Is there an example of such graph for which $N[u] = V$ for some other $u \in V$ as well? Explain.

(e) Describe in words what $N(v)$ and $N[v]$ mean in general.

2. Which of the following graphs are trees

(a) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}$

(b) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}$

(c) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}\}$

(d) $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{d, e\}\}$

3. For each degree sequence below, decide wether it must always, must never, or could possibly be a degree sequence for a tree. Remember, a degree sequence lists out the degrees (number of edges incident to the vertex) of all the vertices in a graph in non-increasing order.

(a) $(4, 1, 1, 1, 1)$

(b) $(3, 3, 2, 1, 1)$

(c) $(2, 2, 2, 1, 1)$

(d) $(4, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1)$

4. Suppose you have a graph with v vertices and e edges that satisfies $v = e + 1$. Must the graph be a tree? Prove your answer.

5. Prove that any graph (not necessarily a tree) with v vertices and e edges that satisfies $v > e + 1$ will NOT be connected.

6. Let T be a rooted tree that contains vertices $v, u, \text{and} w$ (among possibly others). Prove that if w is a descendant of both $u \text{and} v$ then u is a descendant of v or v is a descendant of u.

7. Prove that every connected graph which is not itself a tree must have at last three different spanning trees.