QF600 Asset Pricing

Lecture 1
Expected Utility Theory

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Pre-class, in-class, and after-class activities

Pre-class reading materials

Self-paced learning, covering essential contents to be covered in class

Group homework

Takeaway homework to be completed as a group and submitted before the next class

In-class quiz

Reinforce understanding via practice and discussion

Student reflections

Open forum for Q&A and discussions

TA: Cheng Hao
To create telegram group for ease of discussion

Weekly lesson plan

Session	Topics	Assignments/Activities
1	Expected Utility Theory	
2	The Efficient Frontier: Mean-Variance Optimization	
3	The Capital Asset Pricing Model (CAPM)	Group assignment
4	Beyond CAPM: Multi-Factor Models	Group assignment
5	Modern Portfolio Construction	Project presentation
6	The Stochastic Discount Factor and State Prices	Project presentation
7	Consumption-Based Models and Market Puzzles	Project presentation
8	Behavioral Finance: When Rationality Fails	Project presentation
9	Intrinsic Valuation: Discounted Cash Flow Models	Project presentation
10	Relative Valuation and The Big Picture	Project presentation
11	FINAL EXAM (Closed book)	

Course assessment rubics

Class participation

- 20%
- Engagement in discussion, critical thinking

Group assignment

- 40%
- Team work, problem solving, oral communication

Final exam

- 40%
- Problem solving, decision making

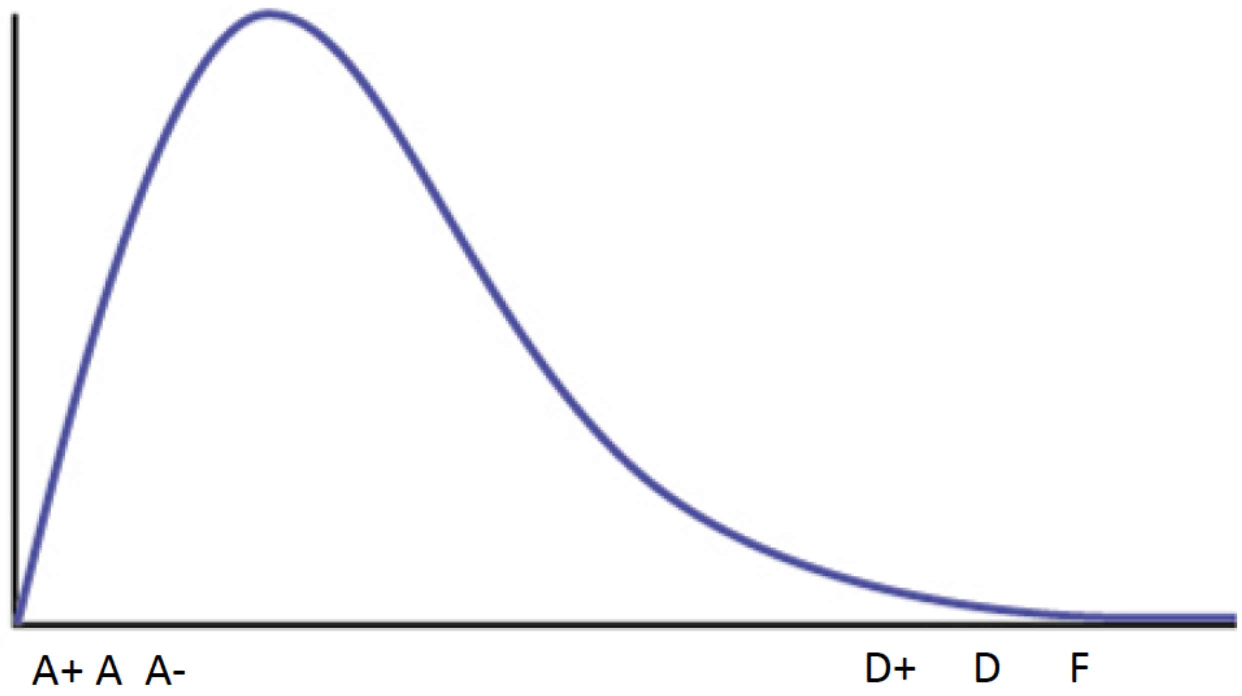
References

- Theory of Asset Pricing (2008), by George Pennacchi
- Financial Decisions and Markets (2018), by John Campbell
- Quantitative Risk Management using Python (2025), by Peng Liu
- Practical Asset Pricing with Python (upcoming), by Peng Liu

Grading curve

Not drawn to scale

Exact distribution is class-specific and confidential



Office consultation hours

10am-12pm, Thursday (tentative)

Please send me an email before you come

Room 5118, LKCSB

Alternatively, can send me an email to book other slots

Quick self-introduction

- Your name and hobby
- Form class groups

Learning outcomes

Course outlook overview

Ways to engage in learning and discussion

Getting to know asset pricing

Understand major applications

In-class quiz

Q1-3

A lottery, L, represents a set of possible outcomes, x_i, each with an associated probability, p_i.

- **Discrete Lottery:** $L = (x_1, p_1), (x_2, p_2), \ldots, (x_n, p_n)$
  - Where x_i is outcome i, and p_i is its probability.
  - Constraints: $p_i \geq 0$ for all i, and $\sum_{i=1}^{n} p_i = 1$.
- **Continuous Lottery:** Defined by a random variable X with a probability density function (PDF) $f(x)$.
  - The probability of an outcome falling within a range $[a, b]$ is $P(a \leq X \leq b) = \int_a^b f(x)dx$.
  - Constraints: $f(x) \geq 0$ for all x, and $\int_{-\infty}^{\infty} f(x)dx = 1$.

speculation, hedging arbitrage and diversification : definition in finance/ meaning

Decision Theory: Choice Under Uncertainty

Describing Lotteries: Expected Value

A basic measure of a lottery's "central tendency" is its Expected Value, $E[L]$ or $E[X]$.

- **Discrete Lottery:** $E[L] = \sum_{i=1}^{n} p_i x_i$
- **Continuous Lottery:** $E[X] = \int_{-\infty}^{\infty} x f(x) dx$
  - *Note:* While intuitive, $E[L]$ alone is often insufficient for decision-making, as it doesn't capture risk.

The core problem

Given a set of available lotteries (choices), $\mathcal{L} = L_1, L_2, \ldots, L_m$, how do we select the "best" or "optimal" lottery? This requires a way to quantify preferences over uncertain outcomes.

Example of Expected Utility – Simple Gamble

- Lottery A: 50% chance of winning $100, 50% chance of winning $0.
- Utility Function: $U(x) = \sqrt{x}$ (Risk-Averse: diminishing marginal utility)
- Expected Utility of Lottery A:

$$E[U(L_A)] \;=\; (0.5 \times 10) + (0.5 \times 0) \;=\; 5 + 0 \;=\; 5 \text{ utils.}$$

- Lottery B: A certain outcome of $49. Expected Utility of Lottery B:

$$E[U(L_B)] \;=\; (1.0 \times U(\$49)) \;=\; 1.0 \times 7 \;=\; 7 \text{ utils.}$$

- Decision: Since

$$E[U(L_B)]\,(7 \text{ utils}) \;>\; E[U(L_A)]\,(5 \text{ utils})\,,$$

a risk-averse individual with $U(x) = \sqrt{x}$ would prefer the certain $49 (Lottery B) over the gamble (Lottery A), even though Lottery A has a higher expected value ($E[L_A] = \$50$).

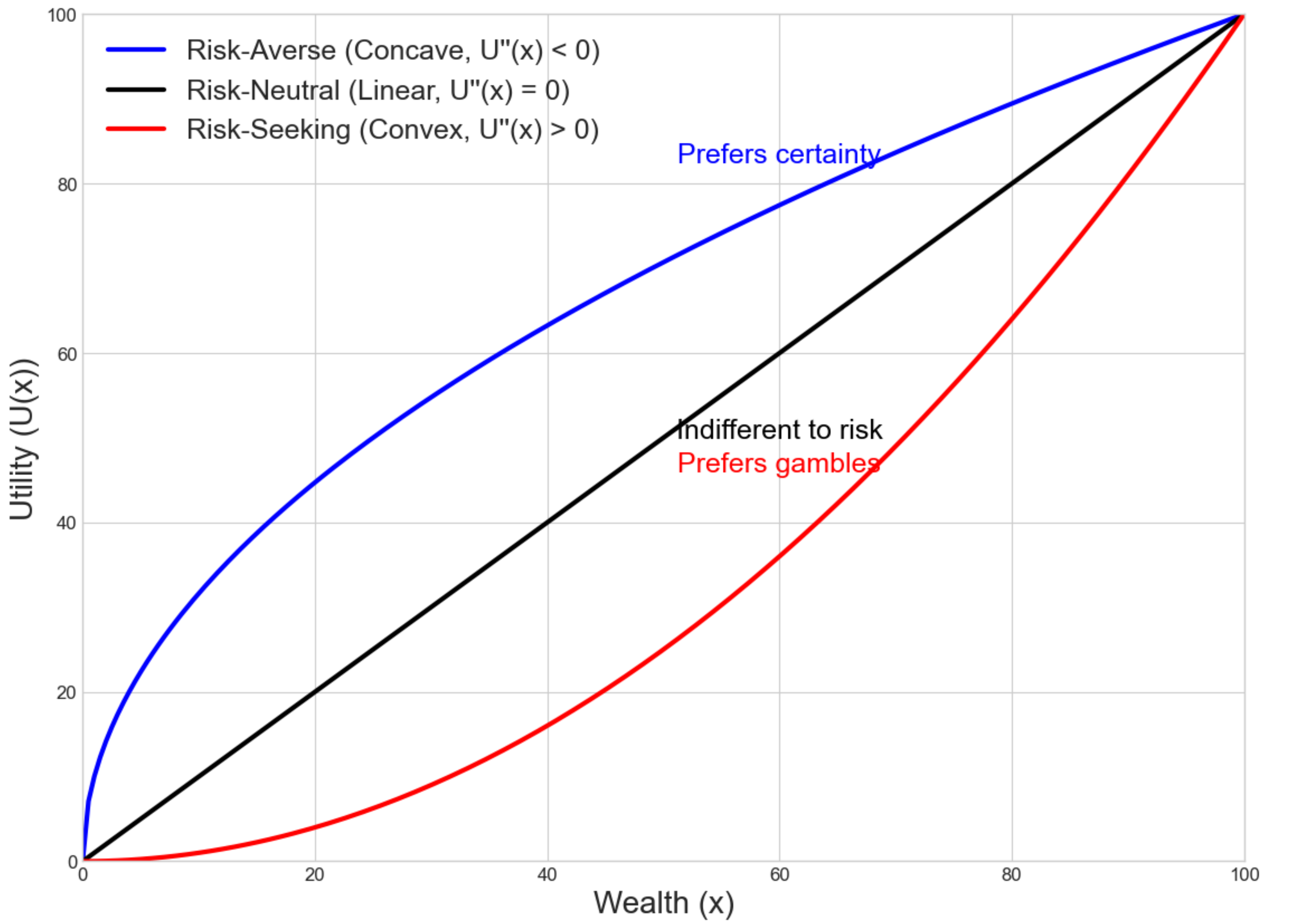
Q: how is risk aversion connected to second derivative?

(hint) $U(\mathbb{E}[X]) \geq \mathbb{E}[U(X)]$

Utility Function: Quantifying Preferences

A utility function, $U(x)$, assigns a numerical value (utility) to each possible outcome x, representing the decision-maker's preference for that outcome.

- $U : \mathbb{R} \to \mathbb{R}$ (typically, outcomes are monetary).
- **Assumption:** Higher utility implies greater preference. If $U(x_1) > U(x_2)$, then outcome x_1 is preferred to x_2.
- **Shape of $U(x)$ reflects risk attitude:**
  - **Risk-Averse:** $U(x)$ is concave ($U''(x) < 0$). Prefers a certain outcome over a lottery with the same expected value.
  - **Risk-Neutral:** $U(x)$ is linear ($U''(x) = 0$). Indifferent between a certain outcome and a lottery with the same expected value.
  - **Risk-Seeking:** $U(x)$ is convex ($U''(x) > 0$). Prefers a lottery over a certain outcome with the same expected value.



The Three Faces of Fortune: Utility Functions

Non-Satiation

1. "More is Better":

- Investors always prefer more wealth; thus, utility strictly increases with wealth.
- For any wealth increase $\theta > 0$,

$$U(W+\theta) > U(W)$$

2. Marginal Utility:

- If $U(W)$ is differentiable, non-satiation implies a positive marginal utility of wealth:

$$U'(W) = \frac{dU(W)}{dW} > 0$$

In Essence: Non-satiation means utility functions are strictly upward-sloping.

In-class quiz

Q4-6

Marginal Utility

1. Marginal Utility ($U'(W)$):

- Additional utility from an infinitesimal wealth increase.

2. Decreasing Marginal Utility:

- $U'(W)$ typically decreases as wealth W increases: $U'(W+\theta) \leq U'(W)$ for $\theta \geq 0$.

3. Implication: Concavity & Risk Aversion

- This implies a **concave** utility function ($U''(W) \leq 0$ if twice-differentiable).
- Concavity reflects risk aversion: $U(\kappa W_1 + (1-\kappa)W_2) \geq \kappa U(W_1) + (1-\kappa)U(W_2)$ for $\kappa \in [0,1]$.

Decreasing marginal utility $\implies$ concave utility $\implies$ risk aversion.

Risk Aversion

1. Fair Lottery ($\tilde{\epsilon}$):

- A lottery $\tilde{\epsilon}$ (e.g., gain ϵ_+ with prob. p, loss ϵ_- with $1-p$) is **fair** if its expected monetary outcome is zero: $E[\tilde{\epsilon}] = p\epsilon_+ + (1-p)\epsilon_- = 0$.
  - Example: 50% chance of $10 win, 50% chance of $10 loss.

2. Jensen's Inequality: For a concave $U(W)$ and fair lottery $\tilde{\epsilon}$ (where $E[W+\tilde{\epsilon}] = W$):

$$U(W) \geq E[U(W+\tilde{\epsilon})]$$

$$U(W) \geq pU(W+\epsilon_+) + (1-p)U(W+\epsilon_-)$$

The utility of certain wealth W is $\geq$ the expected utility of the gamble.

Thus: Concavity in $U(W)$ implies risk aversion, leading to the rejection of fair lotteries because certain wealth is preferred.
Concave Utility $\implies$ Risk Aversion $\implies$ Rejection of Fair Lotteries.

Absolute Risk Premium

- Risk-averse individuals (concave utility $U(W)$) would pay an **absolute risk premium** (π_a) to avoid a fair lottery $\tilde{\epsilon}$ (where $E[\tilde{\epsilon}] = 0$). This premium is the amount of wealth they forgo to achieve the same utility as taking the gamble.
- π_a equates the utility of certain wealth with the expected utility of the lottery:

$$U(W - \pi_a) = E[U(W + \tilde{\epsilon})]$$

- Due to concavity and Jensen's inequality ($E[U(W + \tilde{\epsilon})] < U(W)$ for a fair lottery), $\pi_a > 0$ for the risk-averse. π_a is the monetary cost of the lottery's risk.

Approximation via Taylor Expansion (for small gambles)

Q: what is the sign of pi_a?

- Using Taylor expansions around current wealth W:
  - $U(W-\pi_a) \approx U(W) - \pi_a U'(W)$
  - $E[U(W+\tilde{\epsilon})] \approx U(W) + \frac{1}{2}\sigma_\epsilon^2 U''(W)$ (where $\sigma_\epsilon^2 = \mathrm{Var}(\tilde{\epsilon})$)
- Equating these and solving for π_a (assuming $U'(W) > 0$):

$$\pi_a \approx -\frac{1}{2}\sigma_\epsilon^2 \frac{U''(W)}{U'(W)}$$

- The **Arrow-Pratt coefficient of absolute risk aversion** ($R_a(W)$) measures local risk aversion:

$$R_a(W) = -\frac{U''(W)}{U'(W)}$$

- Substituting this definition into the approximation for π_a:

$$\pi_a \approx \frac{1}{2}\sigma_\epsilon^2 R_a(W)$$

- $R_a(W) > 0$ for risk-averse individuals. Higher $R_a(W)$ means greater aversion to fixed-size gambles.

Relative Risk Aversion

- Investors often assess risk relative to their wealth. For a proportional fair lottery $\tilde{\eta}$ (gambling $\tilde{\eta}W$, with $E[\tilde{\eta}] = 0$), the **relative risk premium** (π_r) is the fraction of wealth an investor forgoes.
- Indifference condition: $U(W(1-\pi_r)) = E[U(W(1+\tilde{\eta}))]$. For small $\tilde{\eta}$ and π_r, Taylor expansion yields:

$$\pi_r \approx \frac{1}{2}\sigma_\eta^2 \left(-W\frac{U''(W)}{U'(W)}\right)$$

where $\sigma_\eta^2 = E[\tilde{\eta}^2]$ is the variance of the proportional lottery.

- The **Arrow-Pratt coefficient of relative risk aversion** is:

$$R_r(W) = -W\frac{U''(W)}{U'(W)}$$

- This can also be expressed in terms of absolute risk aversion: $R_r(W) = W \cdot R_a(W)$.
- Substituting the definition of $R_r(W)$ into the approximation for π_r

$$\pi_r \approx \frac{1}{2}\sigma_\eta^2 R_r(W)$$

In-class quiz

Q7-8

Quadratic Utility

- **Function:** $U(W) = W - \frac{1}{2}bW^2$, for $b > 0$.
- **Marginal Utility:** $U'(W) = 1 - bW$. Utility increases for $W < 1/b$ (satiation point).
- **Absolute Risk Aversion:** $R_a(W) = \frac{b}{1-bW}$.
- **Key Implication:** Quadratic utility exhibits **Increasing Absolute Risk Aversion (IARA):** $\frac{dR_a(W)}{dW} = \frac{b^2}{(1-bW)^2} > 0$.

Exponential Utility

- **Function:** $U(W) = -e^{-bW}$, for $b > 0$.
- **Marginal Utility:** $U'(W) = be^{-bW} > 0$ (strictly increasing).
- **Second Derivative:** $U''(W) = -b^2 e^{-bW} < 0$ (strictly concave, risk-averse).
- **Absolute Risk Aversion ($R_a(W)$):**

$$R_a(W) = b$$

  Exhibits **Constant Absolute Risk Aversion (CARA)**: aversion to fixed-size gambles is constant with wealth.

- **Relative Risk Aversion ($R_r(W)$):**

$$R_r(W) = W \cdot R_a(W) = bW$$

  Exhibits **Increasing Relative Risk Aversion (IRRA)**: aversion to proportional gambles increases with wealth.

Power Utility

- **Function:** $U(W) = \frac{W^{1-\gamma}}{1-\gamma}$, for $\gamma > 0$ and $\gamma \neq 1$.
  - If $\gamma = 1, U(W) = \ln W$ (Logarithmic Utility).
- **Marginal Utility:** $U'(W) = W^{-\gamma} > 0$ (strictly increasing).
- **Second Derivative:** $U''(W) = -\gamma W^{-(\gamma+1)} < 0$ (strictly concave, risk-averse).
- **Absolute Risk Aversion ($R_a(W)$):**

$$R_a(W) = -\frac{-\gamma W^{-(\gamma+1)}}{W^{-\gamma}} = \frac{\gamma}{W}$$

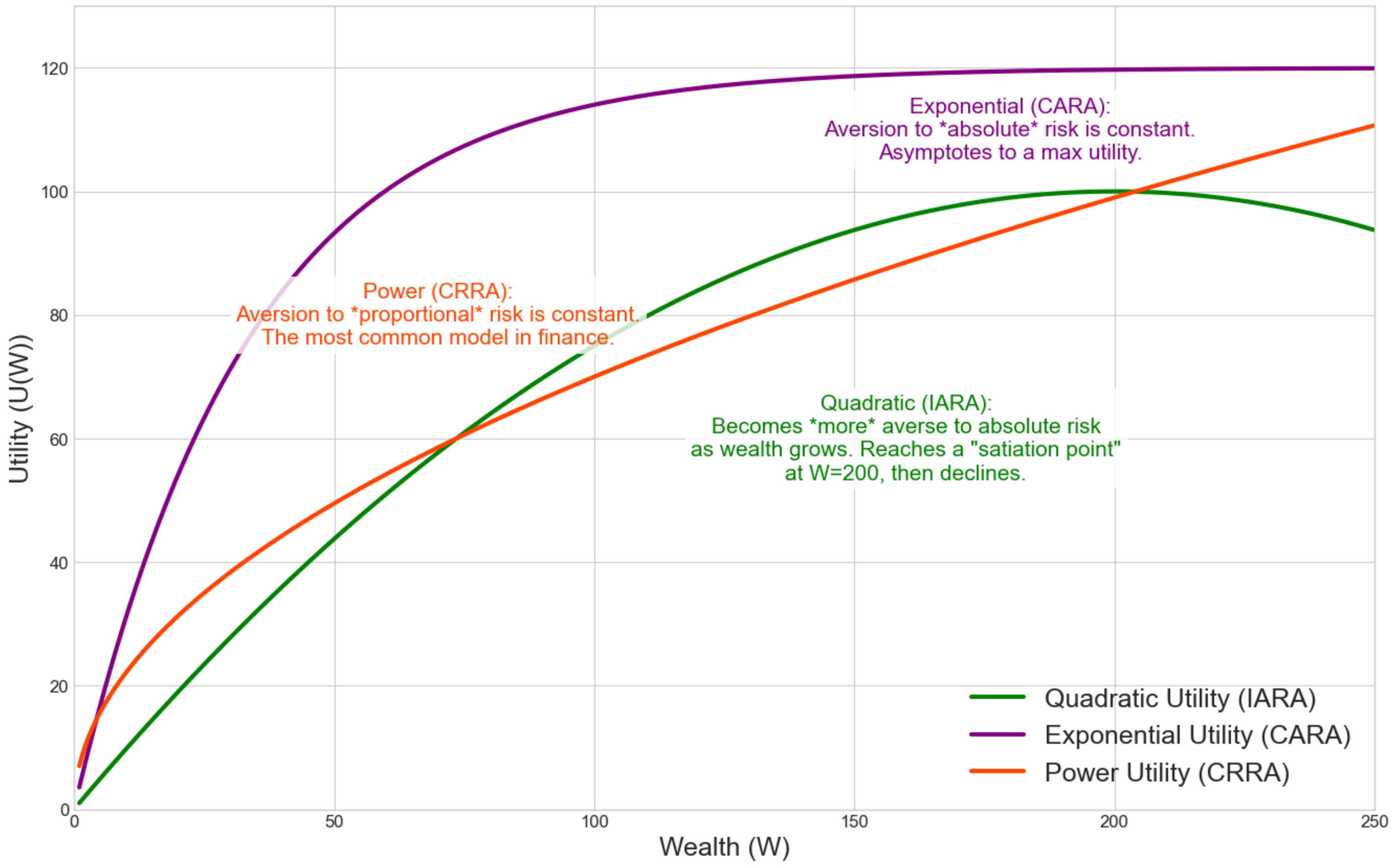
  Exhibits **Decreasing Absolute Risk Aversion (DARA)**: aversion to fixed-size gambles decreases as wealth increases.

- **Relative Risk Aversion ($R_r(W)$):**

$$R_r(W) = W \cdot R_a(W) = W \cdot \frac{\gamma}{W} = \gamma$$

  Exhibits **Constant Relative Risk Aversion (CRRA)**: aversion to proportional gambles is constant regardless of wealth. This is a widely used property in financial modeling.

Three Common Families of utility function



Normal Returns & Utility - Setup

- **Initial Wealth & Return:** Investor has initial wealth W_0. Let $\tilde{R}$ be the (one plus) random return on the portfolio. Final wealth $\tilde{W} = W_0\tilde{R}$.
- **Simplified Utility:** If $W_0 = 1$ (or for a single investor context), utility can be seen as a function of portfolio return: $U(\tilde{W}) = U(\tilde{R})$.
- **Taylor Expansion of Utility:** Let μ be mean portfolio return and σ^2 be its variance.

$$U(\tilde{R}) \approx U(\mu) + U'(\mu)(\tilde{R} - \mu) + \frac{1}{2}U''(\mu)(\tilde{R} - \mu)^2 + \dots$$

- **Expected Utility Approximation:**

$$E[U(\tilde{R})] \approx U(\mu) + \frac{1}{2}\sigma^2 U''(\mu) + \dots$$

This shows expected utility depends on mean and variance (and higher moments if the expansion continues or utility is not quadratic).

Normal Returns & Utility - Distribution Choice

- **Quadratic Utility:** If utility is quadratic, $E[U(\tilde{R})]$ depends *only* on μ and σ^2.
- **Non-Quadratic Utility:** Expected utility also depends on higher moments (skewness, kurtosis), which is less convenient.
- **Distribution Assumption:** To simplify, we can assume a portfolio return distribution that is fully described by its mean and variance.
  - **Normal Distribution ($\tilde{R} \sim N(\mu, \sigma^2)$):**
    - Stable under addition (portfolio of normal returns is normal).
    - Unbounded below, implying potential for unlimited liability (returns can be less than -100%).
  - **Lognormal Distribution:**
    - Bounded below (returns cannot be less than -100%).
    - Not stable under addition.
- **Assumption Made:** We often assume portfolio returns are normally distributed: $\tilde{R} \sim N(\mu, \sigma^2)$.

Expected Utility with Normal Returns

- **Standard Normal Variable:** Let $\tilde{z} = \frac{\tilde{R}-\mu}{\sigma}$ be a standard normal variable ($\tilde{z} \sim N(0,1)$).
- **Expected Utility Integral:**

$$E[U(\tilde{R})] = \int_{-\infty}^{\infty} U(\mu + z\sigma)\phi(z)dz$$

where $\phi(z)$ is the standard normal probability density function (PDF).

- **Impact of Expected Return (μ):** If the investor is never satiated ($U'(W) > 0$), expected utility increases with expected return:

$$\frac{\partial}{\partial \mu} E[U(\tilde{R})] = \int_{-\infty}^{\infty} U'(\mu + z\sigma)\phi(z)dz > 0$$

Since $U' > 0$ and $\phi(z) > 0$, the integral is positive.

Impact of Risk (Standard Deviation) on Expected Utility (optional)

- **Risk Aversion Implication:** If an investor is risk-averse, higher standard deviation (risk) σ should lead to lower expected utility.

$$\frac{\partial}{\partial \sigma} E[U(\tilde{R})] = \int_{-\infty}^{\infty} U'(\mu + z\sigma) z\phi(z) dz < 0$$

- **Splitting the Integral:** To show this, the integral is split:

$$\int_{-\infty}^{\infty} U'(\mu + z\sigma) z\phi(z) dz = \int_{0}^{\infty} U'(\mu + z\sigma) z\phi(z) dz + \int_{-\infty}^{0} U'(\mu + y\sigma) y\phi(y) dy$$

Impact of Risk - Proof Continuation (optional)

- **Substitution:** In the second integral, let $y = -z$. Then $dy = -dz$. The limits change from $(-\infty, 0)$ for y to $(\infty, 0)$ for z. Also, $\phi(y) = \phi(-z) = \phi(z)$ due to symmetry of the normal PDF.

$$\int_{-\infty}^{0} U'(\mu + y\sigma)y\phi(y)dy = \int_{\infty}^{0} U'(\mu - z\sigma)(-z)\phi(z)(-dz)$$

$$= \int_{\infty}^{0} U'(\mu - z\sigma)z\phi(z)dz = -\int_{0}^{\infty} U'(\mu - z\sigma)z\phi(z)dz$$

$$\frac{\partial}{\partial\sigma}E[U(\tilde{R})] = \int_{0}^{\infty} U'(\mu + z\sigma)z\phi(z)dz - \int_{0}^{\infty} U'(\mu - z\sigma)z\phi(z)dz$$

$$= \int_{0}^{\infty} U'(\mu + z\sigma) - U'(\mu - z\sigma)z\phi(z)dz$$

- **Risk Aversion Condition:** For a risk-averse investor, $U'' < 0$. Thus, for $z > 0$: $U'(\mu + z\sigma) < U'(\mu - z\sigma)$
- Therefore, $U'(\mu + z\sigma) - U'(\mu - z\sigma)$ is negative. Since $z\phi(z)$ is positive for $z > 0$:

$$\frac{\partial}{\partial\sigma}E[U(\tilde{R})] < 0$$

This confirms that for a risk-averse investor, expected utility decreases as portfolio risk (σ) increases, assuming normal returns.

In-class quiz

Q9-10

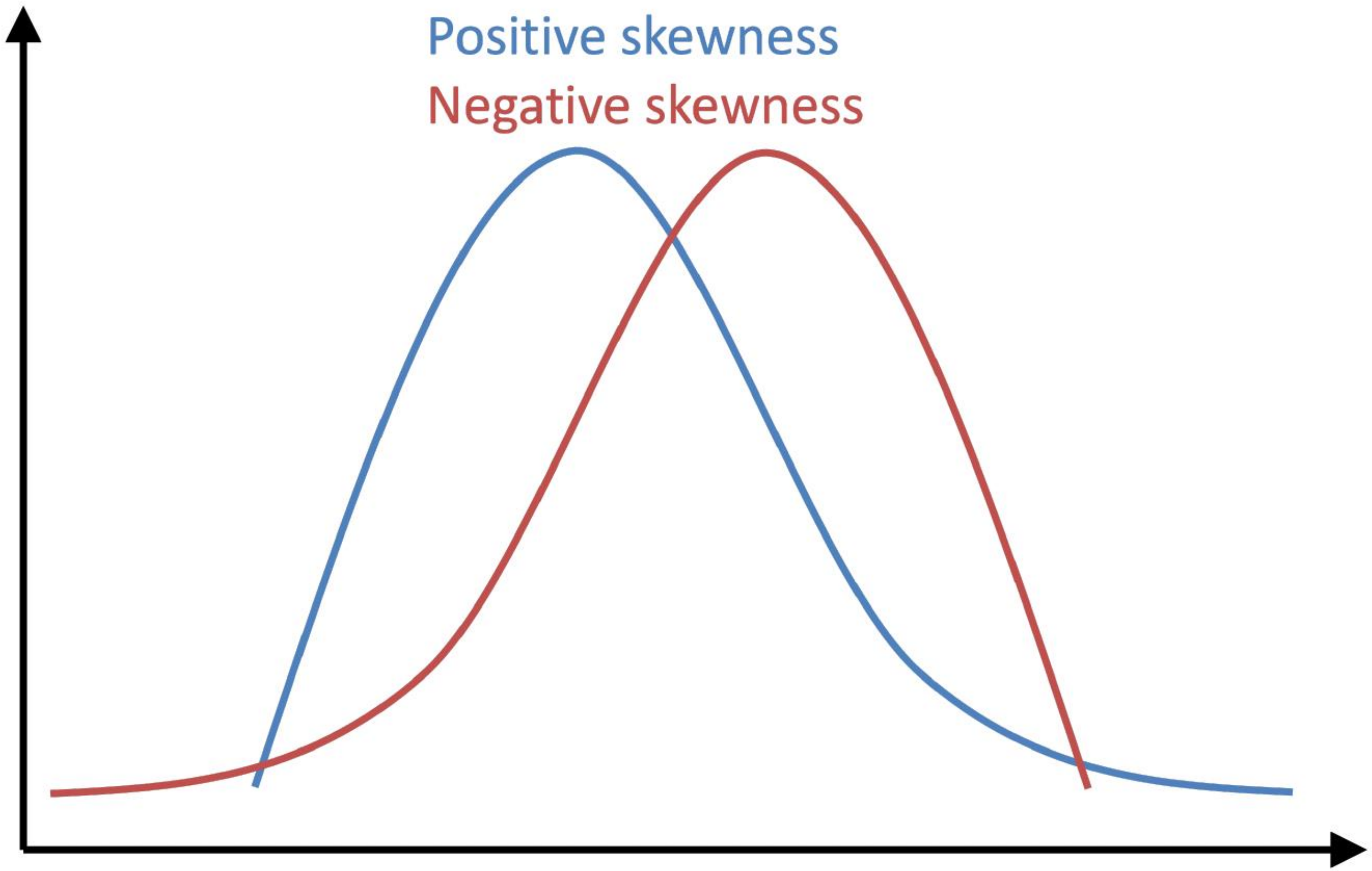
Skewness

- Skewness measures the asymmetry of a return distribution ($\tilde{R}$) around its mean (μ). It indicates the likelihood of extreme outcomes.
- **Formula:** $\mathrm{Skew}(\tilde{R}) = E\left[\left(\frac{\tilde{R}-\mu}{\sigma}\right)^3\right]$
- **Positive Skew (Right-Skewed):** Long tail to the right; potential for large positive outliers (e.g., lottery win). *Mean > Median*.
- **Negative Skew (Left-Skewed):** Long tail to the left; potential for large negative outliers (e.g., market crash). *Mean < Median*.
- Zero skew indicates a symmetric distribution (e.g., normal distribution).

Investor Preferences

- **Preference for Positive Skewness:** Risk-averse investors may still find positive skew attractive due to the small chance of a very large gain (the "lottery ticket" effect). They might be willing to pay for this (e.g., accept a lower mean or higher variance).
    - *Example:* Investing in venture capital or buying lottery tickets.
- **Aversion to Negative Skewness:** Risk-averse investors strongly dislike negative skewness because it implies a small chance of a catastrophic loss. They are willing to pay to avoid this (e.g., accept lower returns or pay for insurance).
    - *Example:* Buying portfolio insurance or hedging against disaster risk.

Skewed Returns



Indifference Curves in Mean-Variance Space

- **Definition:** An indifference curve represents a set of portfolios that provide an investor with the same level of expected utility ($\bar{U}$).
- These curves are typically plotted in a graph where the x-axis is the standard deviation of return (σ, representing risk) and the y-axis is the expected return (μ). This is often referred to as the (σ, μ)-space or mean-variance space.
- **Constant Utility:** If two portfolios, P_1 (with risk σ_1 and expected return μ_1) and P_2 (with risk σ_2 and expected return μ_2), lie on the same indifference curve, it means:

$$E[U(\tilde{R}_1)] = E[U(\tilde{R}_2)] = \bar{U}$$

  The investor is equally satisfied with either portfolio.
- **Non-Intersecting Curves:** Different indifference curves represent different levels of utility. A fundamental property is that indifference curves for a given investor cannot intersect.

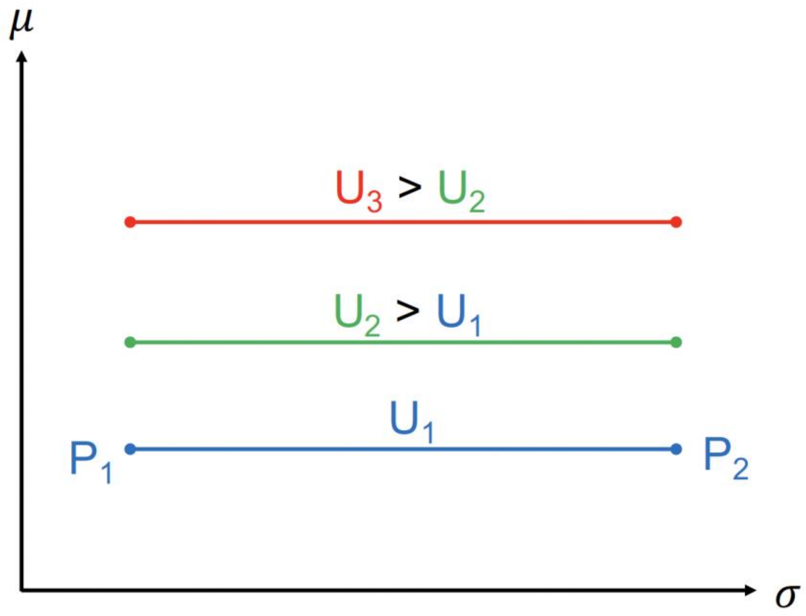
Indifference Curves for a Risk-Neutral Investor

- **Risk-Neutrality Defined:** A risk-neutral investor is solely concerned with the expected return of a portfolio and is indifferent to the level of risk (variance or standard deviation).
- Their utility function is linear in wealth (or return), for example, $U(W) = aW + b$ with $a > 0$, which means $U'(W) = a > 0$ and $U''(W) = 0$.
- Therefore, for a risk-neutral investor, expected utility is simply the utility of the expected return (if utility is linear in return) or directly proportional to expected return:

$$E[U(\tilde{R})] = U(E[\tilde{R}]) = U(\mu)$$

Or, more generally, if $U(W)$ is linear, $E[U(W_0\tilde{R})] = U(W_0E[\tilde{R}]) = U(W_0\mu)$.

- The key is that only μ matters for expected utility, not σ.



Convex Indifference Curves for a Risk-Averse Investor

- Ploting risk (σ) vs. expected return (μ) combinations yield the same expected utility $E[U(W)]$. They slope upwards because higher risk requires higher expected return for constant utility.
- **Increasing Compensation for Risk:** Investors demand increasingly more extra expected return for each additional unit of risk as their total risk exposure grows.
- Adding risk when already facing high risk is more "painful" (due to concave utility – potential larger losses impact utility more severely). This demands greater compensation ($\Delta\mu$) compared to adding the same risk at low existing risk levels.
- **Analogy:** Carrying weights (risk) for reward (return). Adding 10kg when already carrying 50kg requires a much larger reward than adding the first 10kg, as discomfort (disutility) rises disproportionately.

Proof of Convexity 1

- **Concave Utility's Impact:** $U''(W) < 0$ means wealth variance (risk) reduces expected utility.
- **Expected Utility Approximation:** For risk-averse investors, expected utility can be approximated via Taylor expansion as:

  $$E[U(\tilde{R})] \approx \mu - \frac{1}{2}A\sigma^2$$

  Where $A > 0$ is a measure of risk aversion.
- **Indifference Curve Equation:** For constant expected utility $\bar{U}$:

  $$\bar{U} \approx \mu - \frac{1}{2}A\sigma^2 \implies \mu \approx \bar{U} + \frac{1}{2}A\sigma^2$$

  This describes the indifference curve as a quadratic relationship between μ and σ.

Proof of Convexity 2

- **Recall Indifference Curve:** $\mu(\sigma) = \bar{U} + \frac{1}{2}A\sigma^2$ (where $A > 0$).
- **Slope (Marginal Rate of Substitution, $MRS_{\sigma,\mu}$):**

$$\frac{d\mu}{d\sigma} = A\sigma$$

  The slope is positive ($A\sigma > 0$ for $\sigma > 0$), so curves are upward sloping. The slope increases with σ, meaning curves steepen as risk rises.

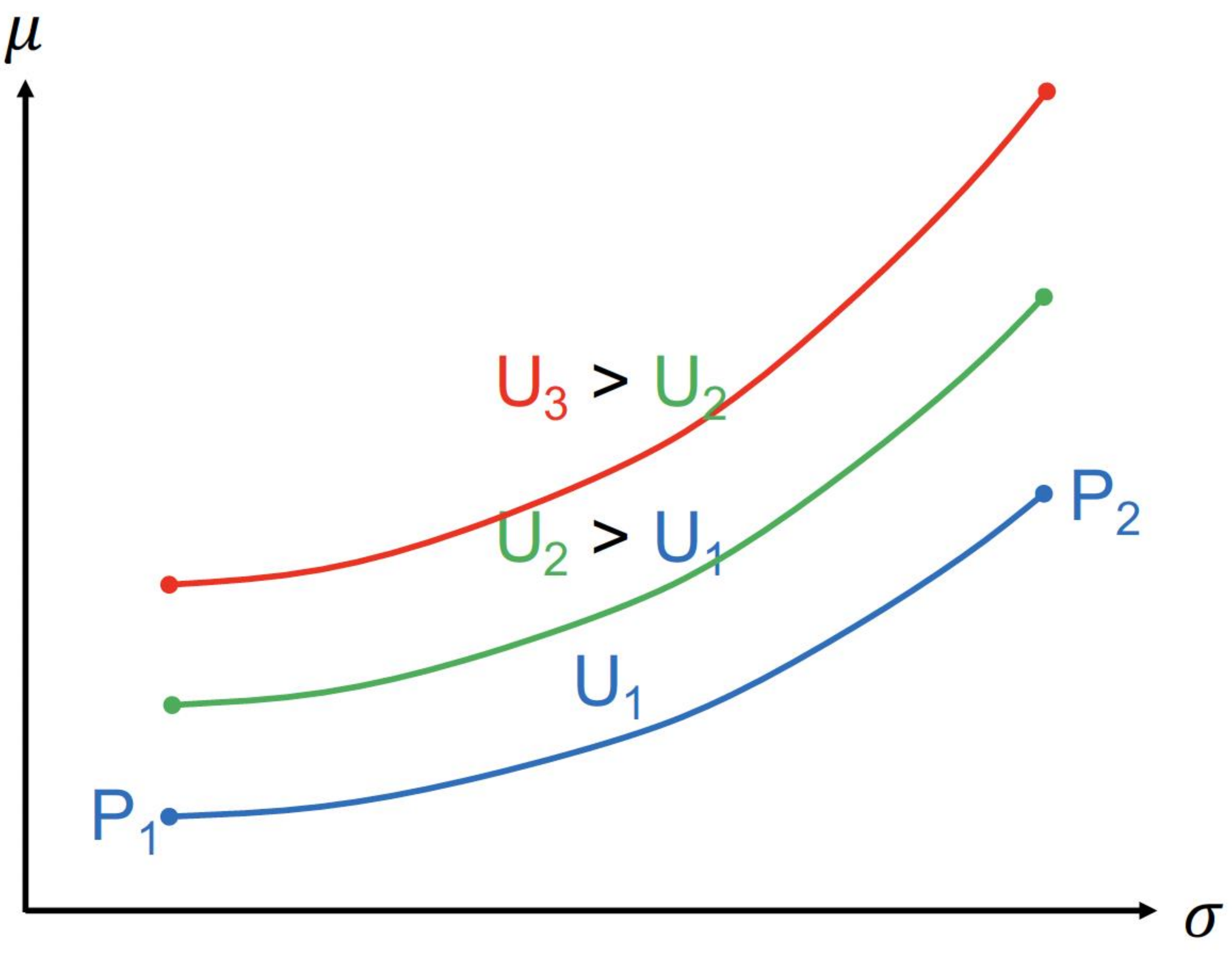
- **Convexity (Curvature):**

$$\frac{d^2\mu}{d\sigma^2} = A$$

  Since $A > 0$ (from $U''(W) < 0$), the positive second derivative confirms the indifference curve is **convex**.

- **Conclusion:** The utility function's concavity ($U''(W) < 0$) directly causes convex indifference curves, as investors require progressively larger return compensation for incremental risk increases, especially at higher overall risk levels.

Indifference Curves for Risk-Averse Investor



In-class quiz

Q11-12

Coding session

Homework

- Watch/review video tutorials and class recording for week 1 lecture (if you have not done so)
- Post learning reflections and questions if any
- Get to know your teammates and discuss ideas for final project