MATH 220 — Assignment 7

Stephanie Knill
54882113
Due: March 3, 2016

Question 1

Proposition: Any non-empty subset of a well-ordered set of real numbers is well-ordered.

Proof. Let S be a well-ordered set of real numbers and S' a non-empty subset of S. Then we can express S as

$$S = \{x_1, x_2, \ldots, x_n\},$$

where $x_i > x_j$, for all $i > j$. Since S' is a finite set of real numbers that always contain a minimal element, then every subset of S' will also contain a minimal element. Thus S' is also well-ordered. ■

Question 2

Proof. We will prove by induction, that for all $n \in \mathbb{N}$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \tag{1}$$

Base Case: $n = 1$

For the left hand side we have

$$\frac{n}{1(n+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

which equates to the right hand side

$$\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$$

Induction Step: assume the statement holds true for $1 < n < k$. So

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{k}{k+1}$$

For $n = k+1$, we have that

$$\begin{aligned}
\sum_{n=1}^{k+1} &= \sum_{n=1}^{k} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{(by inductive assumption)} \\
&= \frac{m(m+1)+1}{(m+1)(m+2)} \\
&= \frac{m+1}{m+2}
\end{aligned}$$

Thus, (1) holds true for $n = k+1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, (1) is true for all $n \in \mathbb{N}$. ■

Question 3

Proof. We will prove by induction, that for $a, b, m \in \mathbb{Z}$, if $a \equiv b \bmod m$, then $a^n \equiv b^n \bmod m$ for all $n \in \mathbb{N}$.

Base Case: $n = 1$

$$a \equiv b \bmod m$$

Induction Step: assume the statement holds true for $n = k$. So

$$a^k \equiv b^k \bmod m$$

For $n = k+1$, we can multiply our inductive assumption congruence with $a \equiv b \bmod m$

$$\begin{aligned}
a^k \cdot a &\equiv b^k \cdot b \bmod m \\
a^{k+1} &\equiv b^{k+1} \bmod m
\end{aligned}$$

Thus, the statement holds true for $n = k+1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, the statement is true for all $n \in \mathbb{N}$. ■

Question 4

Proof. We will prove by induction on the finite set A of cardinality n, that for the power set $|\mathcal{P}(A)|$

$$|\mathcal{P}(A)| = 2^{|A|} \tag{2}$$

Base Case: $n = 0$. Here, A is the empty set and $|A| = 0$. Thus we have

$$|\mathcal{P}(A)| = |\mathcal{P}(\emptyset)| = 1 = 2^0 = 2^{|A|}$$

Induction Step: assume the statement holds true for $n = k$. Then $|A| = k$ and $|P(A)| = 2^k$.

For $n = k + 1$, let us consider the set $A' = A \setminus \{x\}$, where $x \in A$. Here $|A'| = k$ and $|\{x\}| = 2$. In the case where x is not in a subset of A, we have 2^k possible subsets by the induction hypothesis. In the case where x is a a subset of A, let us add the element x to each of the 2^k subsets. Then the total number of subsets didn't change and we again have 2^k possible subsets. Thus the total number of subsets is

$$\begin{aligned} 2^k + 2^k &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

Thus, (2) holds true for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, (2) is true for all $n \in \mathbb{N}$. ■

Question 5

Proof. We will prove DeMorgan's laws by induction on the number of sets n, that for sets $A_1, \ldots A_n$

$$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \tag{3}$$

Base Case: When $n = 1$, both the left hand side and right hand side of (3) is $\overline{A_1}$.

Induction Step: Let $k \in \mathbb{N}$ be given and suppose (3) is true for $n = k$. Then by the induction assumption

$$\begin{aligned} \overline{A_1 \cup \cdots \cup A_k} &= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \\ \overline{\bigcup_{i=1}^{k}} A_i &= \bigcap_{i=1}^{k} \overline{A_i} \end{aligned}$$

For $n = k + 1$, we have

$$
\begin{aligned}
\overline{\bigcup_{i=1}^{k+1} A_i} &= \overline{\bigcup_{i=1}^{k} A_i \cup A_{k+1}} \\
&= \overline{\bigcup_{i=1}^{k} A_i} \cap \overline{A_{k+1}} \\
&= \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}} \qquad \text{(by inductive assumption)} \\
&= \bigcap_{i=1}^{k+1} \overline{A_i}
\end{aligned}
$$

Thus, (3) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, (3) is true for all n. ■

Question 6

Proof. We will prove by induction over the natural numbers $n \geq 10$ that

$$2^n > n^3 \tag{4}$$

Base Case: When $n = 10$, we have

$$2^{10} = 1024 > 1000 = 10^3 = n^3$$

Induction Step: Let $k \in \mathbb{N}$ be given and suppose (4) is true for $n = k$. Then by the induction assumption

$$2^k > k^3$$

For $n = k + 1$, we have

$$
\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&> 2k^3 \qquad \text{(by inductive assumption)} \\
&= k^3 + k^3
\end{aligned}
$$

Since $k > 10$, then $k^3 > 7k^2$ and we have

$$\begin{aligned} 2^{k+1} &> k^3 + 7k^2 \\ &= k^3 + 3k^2 + 3k^2 + k^2 \\ &> k^3 + 3k^2 + 3k + 1 \\ &> (k+1)^3 \end{aligned}$$

Thus, (4) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, (4) is true for all n. ■

Question 7

Let $F_1, F_2, \dots, F_N$ be a sequence of Fibonacci numbers. Then

$$F_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big] \tag{5}$$

Proof. We will proceed by induction over the natural numbers n.

Base Case: When $n = 1$, we have

$$F_1 = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^1 - \Big(\frac{1-\sqrt{5}}{2}\Big)^1\Big] = \frac{1}{\sqrt{5}}\Big(\frac{2\sqrt{5}}{2}\Big) = 1$$

Induction Step: Let $k \in \mathbb{N}$ be given and suppose (5) is true for $n = k$. For notation purposes, let $x = \frac{1+\sqrt{5}}{2}$ and $y = \frac{1-\sqrt{5}}{2}$. For $n = k + 1$, we have

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &= \frac{1}{\sqrt{5}}(x^n - y^n) + \frac{1}{\sqrt{5}}(x^{n-1} - y^{n-1}) \\ &= \frac{1}{\sqrt{5}}[x^{n-1}(x+1) - y^{n-1}(y+1)] \end{aligned}$$

Here, $x + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = x^2$ and $y + 1 = \frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2} = y^2$. Substituting this back in

$$\begin{aligned} F_{n+1} &= \frac{1}{\sqrt{5}}(x^{n-1}x^2 - y^{n-1}y^2) \\ &= \frac{1}{\sqrt{5}}(x^{n+1} - y^{n+1}) \end{aligned}$$

Thus, (5) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, (5) is true for all n. ■

Question 8

The complete graph K_n of n vertices contains precisely $\frac{n(n-1)}{2}$ edges.

Proof. By definition, the complete graph has n vertices, each of which are connected to $n-1$ other vertices. So the degree of every vertex is $n-1$. Since we have n vertices, we have $n(n-1)$ edges emerging from the vertices of K_n. However, each edge has 2 ends, so we have counted each edge twice. Hence, the total number of edges of K_n is

$$\frac{n(n-1)}{2}$$

■

Question 9

(a) Any tree must contain at least one vertex of degree 1.

Proof. Assume to the contrary that a tree of vertices $v_1, v_2, \ldots, v_n$ contains no vertex of degree 1. Let us take a walk along the longest path from vertex v_i to vertex v_j. Since v_j is not of degree 1, it is adjacent to another vertex v. Since there exists a path between vertex v and v_i, we have another path from v_j to v_i thereby creating a cycle in our tree which is a contradiction. ■

(b) Any tree with n vertices contains precisely $n-1$ edges.

Proof. We will proceed by induction over the number of vertices n.

Base Case: When $n = 1$, we have the null graph N_1 of 0 edges.

Induction Step: Let $k \in \mathbb{N}$ be given and suppose the statement is true for $n = k$ vertices. Let G denote the tree of k vertices that has $k-1$ edges by the induction assumption. Let us add a vertex v to G. Since the resulting graph must be connected, we have two cases:

Case 1: Join v to G with 1 edge. Here we have $k+1$ vertices and k edges, so we are done.

Case 2: Join v to G with at least 2 edges. Since there exists a path between any two vertices v_i and v_j in G, if we join vertex v to both v_i and v_j we will form a cycle of the form

$$v \to v_i \to \cdots \to v_j \to v$$

which is not allowed.

Thus, the statement holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction the statement is true for all n.

$\blacksquare$