MATH 220 — Assignment 9

Stephanie Knill
54882113
Due: April 5, 2016

Question 1

Proposition 1. *There does not exist a continuous bijective function* $f : \mathbb{R} \mapsto \mathbb{R} - \{1\}$.

Proof. Assume, to the contrary, that there exists a continuous bijective function $f : \mathbb{R} \mapsto \mathbb{R} - \{1\}$. Since f is bijective, then f is also surjective. So there exists $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = 0$ and $f(x_2) = 2$. Since f is continuous and $0 < 1 < 2$, then by the Intermediate Value Theorem, there exists an $x_3 \in \mathbb{R}$ such that $f(x_3) = 1$. However, $1 \notin \mathbb{R} - \{1\}$, thereby giving us the necessary contradiction. ■

Question 2

Let A_1, A_2, B_1, B_2 be non-empty sets such that $|A_i| = |B_i|$ for $i = 1, 2$. Then

(a) $|A_1 \times A_2| = |B_1 \times B_2|$

Proof. Case 1: A_1, A_2, B_1, B_2 finite. Let $|A_1| = |A_2| = |B_1| = |B_2| = m$, for some $m \in \mathbb{N} \cup \{0\}$. Then $|A_1 \times A_2| = m^2 = |B_1 \times B_2|$.

Case 2: A_1, A_2, B_1, B_2 infinite. Thus $|A_1| = |A_1 \times A_2|$ and $|B_1| = |B_1 \times B_2|$. Since $|A_1| = |B_1|$, then $|A_1 \times A_2| = |B_1 \times B_2|$. ■

(b) If $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$, Then $|A_1 \cup A_2| = |B_1 \cup B_2|$.

Proof. Case 1: A_1, A_2, B_1, B_2 finite. Let $|A_1| = |A_2| = |B_1| = |B_2| = m$, for some $m \in \mathbb{N} \cup \{0\}$. Since the intersection is empty, then $|A_1 \cup A_2| = 2m = |B_1 \cup B_2|$.

Case 2: A_1, A_2, B_1, B_2 infinite. Thus $|A_1| = |A_1 \cup A_2|$ and $|B_1| = |B_1 \cup B_2|$. Since $|A_1| = |B_1|$, then $|A_1 \cup A_2| = |B_1 \cup B_2|$.

■

Question 3

Proposition 2. *Let A be an non-empty set. Prove that $|A| \leq |A \times A|$*

Proof. *Case 1:* A finite. Since $A \neq \emptyset$, then $|A| < |A \times A|$.

Case 2: A infinite. By our in class Theorem, $|A| = |A \times A|$. Thus for any set A, we have that $|A| \leq |A \times A|$. ■

Question 4

Proposition 3. *If A is a denumerable set and there exists a surjective function from A to B (and B is infinite), then B is denumerable.*

Proof. If B is an infinite set, then it is either denumerable or uncountable. Let us assume to the contrary that B is denumerable. Since there exists a surjective function $f : A \mapsto B$, then for all $b \in B$ there exists an $a \in A$ such that $f(a) = b$. Since B is uncountable, we know that A is also uncountable. However, we know that A is denumerable, thus giving us the necessary contradiciton. ■

Question 5

Proposition 4. *If A and B are denumerable sets, and C is a finite set, then $A \cup B \cup C$ is denumerable.*

Proof. Since A and B are denumerable, then $A \cup B$ is denumerable. Thus the union of a denumerable set $(A \cup B)$ with a finite set C is also denumerable. That is, $A \cup B \cup C$ is denumerable. ■

Question 6

Proposition 5. *If a set A contains an uncountable subset, then A is uncountable*

Proof. Let $B \subseteq A$, where B is uncountable. Assume to the contrary that A is countable. Then by our in class Theorem, we know that any subset of a countable set is countable. However, $B \subseteq A$ and B is uncountable, thereby giving us the necessary contradiction. ■

Question 7

Proposition 6. *Let A be any uncountable set, and let $B \subset A$ be a countable subset of A. Prove that $|A| = |A - B|$.*

Proof. Assume that A is any uncountable set and $B \subset A$ be a countable subset of A. So $A - B$ is infinite. Let us define a new denumerable subset $C \subseteq A - B$. From our in class proof, we know that $B \cup C$ is also denumerable. Thus there exists a bijective function $f : B \cup C \mapsto C$. Now, let us define a function $g : A \mapsto A - B$ such that

$$f(x) = \begin{cases} f(x) & \text{if } x \in B \cup C \\ x & \text{if } x \in (A - B) - C \end{cases}$$

which is bijective. ■

Question 8

(a) If $\mathcal{P}_{fin}(\mathbb{N})$ denotes the set of finite subsets of N, show that $\mathcal{P}_{fin}(\mathbb{N})$ is denumerable.

Proof. Since $\mathcal{P}_{fin}(\mathbb{N}) \subset \mathbb{N}$ and $\mathbb{N}$ is denumerable, we have that

$$|\mathcal{P}_{fin}(\mathbb{N})| = |\mathbb{N}|$$

Thus $\mathcal{P}_{fin}(\mathbb{N})$ is also denumerable. ■

(b) If $\mathcal{P}_{inf}(\mathbb{N})$ denotes the set of finite subsets of N, show that $\mathcal{P}_{inf}(\mathbb{N})$ is uncountable.

Proof. Since $\mathcal{P}_{fin}(\mathbb{N}) \subset \mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})$ is uncountable, we have that

$$|\mathcal{P}_{inf}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$$

Thus $\mathcal{P}_{inf}(\mathbb{N})$ is also uncountable. ■

Question 9

Proposition 7. *Let A, B be sets. If $|A - B| = |B - A|$ then $|A| = |B|$.*

Proof. *Case 1:* $A \cap B = \emptyset$.

Then $A - B = A$ and $B - A = B$, so we have that $|A - B| = |A|$ and $|B - A| = |B|$. Therefore $|A - B| = |B - A| = |A| = |B|$.

Case 2: $A \cap B \neq \emptyset$

Since $A = (A - B) \cup (A \cap B)$ and $B = (B - A) \cup (A \cap B)$, then we have that

$$|A| = |(A - B) \cup (A \cap B)|, \text{ and}$$

$$|B| = |(B - A) \cup (A \cap B)|$$

However, since $|A - B| = |B - A|$, then $|A| = |(A - B) \cup (A \cap B)| = |(B - A) \cup (A \cap B)| = |B|$ ■

Question 10

Proposition 8. *Let $\{0, 1\}^{\mathbb{N}}$ be the set of all possible sequences of 0s and 1s. We have proved in class that this set is uncountable. Corollary 10.22 in the text states that in fact, the cardinality of this set is continuum: $|\mathbb{R}| = |\{0, 1\}^{\mathbb{N}}|$. Using this fact, prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.*

Proof. ■