MATH 220 — Assignment 5

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Question 1

Proposition: $5x - 11$ is even if and only if x is odd.

Proof

$\Rightarrow$ We will express the forward direction in terms of its contrapositive "If x is even, then $5x - 11$ is odd" and prove this logically equivalent statement. Let there exist an integer k such that $x = 2k$. Then

$$\begin{aligned} x &= 2k \\ 5x &= 10k \\ 5x - 11 &= 10k - 11 \\ &= 10k - 12 + 1 \\ &= 2(5k - 6) + 1 \end{aligned}$$

Since $5k-6$ is an integer, then $5x-11$ is odd and we have proved the forward direction.

$\Leftarrow$ Assume that x is odd. Then there exists an integer q such that $x = 2q + 1$ and we have that

$$\begin{aligned} x &= 2q + 1 \\ 5x &= 10q + 5 \\ 5x - 11 &= 10q - 6 \\ &= 2(5q - 3) \end{aligned}$$

Since $5q - 3$ is an integer, then $5x - 11$ is even. ■

Question 2

Proposition: the product of two integers ab is odd if and only if both a and b are odd.

Proof. $\Rightarrow$ We will express the forward direction in terms of its contrapositive "If a and b are both even, then ab is even" and prove this logically equivalent statement. Let there exist integers k_1 and k_2 such that $a = 2k_1$ and $b = 2k_2$. Then the product of these two is given by

$$\begin{aligned} ab &= (2k_1)(2k_2) \\ &= 4k_1k_2 \\ &= 2(2k_1k_2) \end{aligned}$$

Since $2k_1k_2$ is an integer, then ab is even and we have proved the forward direction.

$\Leftarrow$ Assume that a and b are both odd. Then there exists integers q_1 and q_2 such that $a = 2q_1 + 1$ and $b = 2q_2 + 1$. The product of these two integers can be computed as

$$\begin{aligned} ab &= (2q_1 + 1)(2q_2 + 1) \\ &= 4q_1q_2 + 2q_1 + 2q_2 + 1 \\ &= 2(2q_1q_2 + q_1 + q_2) + 1 \end{aligned}$$

Since $2q_1q_2 + q_1 + q_2$ is an integer, then ab is odd. ■

Question 3

Proposition: If n is even, then n^3 is even.

Proof. Let there exist a $k \in \mathbb{Z}$ such that $n = 2k$. Then

$$\begin{aligned} n^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3) \end{aligned}$$

Since $4k^3 \in \mathbb{Z}$, then n^3 is even. ■

Question 4

Proposition: For any sets A and B, $A \Delta B = \emptyset$ iff $A = B$.

Proof. $\Rightarrow$ We will express the forward direction in terms of its contrapositive "If $A \neq B$, then $A \Delta B \neq \emptyset$" and prove this logically equivalent statement. By definition of the symmetric set difference, we have that

$$A \Delta B = (A - B) \cup (B - A)$$

Since $A \neq B$, then both $(A - B)$ and $(B - A)$ do not equal the empty set. By definition, the union of two nonempty sets is not the empty set. Therefore it follows that $A \Delta B = (A - B) \cup (B - A) \neq \emptyset$

$\Leftarrow$ Assume that $A = B$. Then

$$\begin{aligned} A \Delta B &= A \Delta A \\ &= (A - A) \cup (A - A) \\ &= (A - A) \\ &= A \cap \overline{A} \\ &= \emptyset \end{aligned}$$

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Question 5

Conjecture: For any sets A and B, $(A \cup B) - B = A$

Counterexample Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then $(A \cup B) = \{1, 2, 3, 4\}$ and we have that $(A \cup B) - B = \{1\}$. However, this does not equal the set A. Thus the above conjecture is False.

Question 6

Proposition: For any sets A, B, and C, $(A - B) \cup (A - C) = A - (B \cap C)$.

Proof. Using our fundamental properties of set operations, we have

$$\begin{aligned}(A - B) \cup (A - C) &= (A \cap B^c) \cup (A \cap C^c) \quad \textit{(definition of set difference)} \\ &= A \cap (B^c \cup C^c) \quad \textit{(distributive laws)} \\ &= A \cap (B \cap C)^c \quad \textit{(De Morgan's laws)} \\ &= A - (B \cap C) \quad \textit{(definition of set difference)}\end{aligned}$$

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Question 7

Proposition: For sets A, B, C, and D, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. We will first show that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$. Let $(x, y) \in (A \times B) \cap (C \times D)$. Then $x \in A$ and $x \in C$ and $y \in B$ and $y \in D$. Therefore $x \in (A \cap C)$ and $y \in (B \cap D)$. Combining these, we have that

$$(x, y) \in (A \cap C) \times (B \cap D)$$

implying that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.

Let us now show that $(A \times B) \cap (C \times D) \supseteq (A \cap C) \times (B \cap D)$. Let $(u, v) \in (A \cap C) \times (B \cap D)$. Then $u \in A$ and $u \in C$ and $v \in B$ and $v \in D$. Rearranging, we have that $(u, v) \in (A \times B)$ and $(u.v) \in (C \times D)$. Combining these gives us

$$(u, v) \in (A \times B) \cap (C \times D)$$

which completes our proof of $(A \times B) \cap (C \times D) \supseteq (A \cap C) \times (B \cap D)$, thereby proving set equality. ■

Question 8

Proposition: Let $a, b \in \mathbb{Z}, a, b \neq 0$. If $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.

Proof. Assume that $a \mid b$ and $b \mid a$. Then there exists integers k_1 and k_2 such that $a = bk_1$ and $b = ak_2$. Substituting in we have that

$$\begin{aligned}a &= bk_1 \\ &= (ak_2)k_1 \\ k_1k_2 &= 1\end{aligned}$$

Since $k_1, k_2 \in \mathbb{Z}$, then k_1 and k_2 must both equal either 1 or -1. In the first case where $k_1 = k_2 = 1$, we have that $a = b$ and $b = a$. In the second case where $k_1 = k_2 = -1$, $a = -b$ and $b = -a$. Therefore if $a \mid b$ and $b \mid a$, we have that either $a = b$ or $a = -b$. ∎

Question 9

Proposition: Let $x, y \in \mathbb{Z}$. If $3 \nmid x$ and $3 \nmid y$, then $3 \mid (x^2 - y^2)$.

Proof. Assume that $3 \nmid x$ and $3 \nmid y$. Then there exists remainders $r_1, r_2 \in \{1, 2\}$ such that $x \equiv r_1 \bmod 3$ and $y \equiv r_2 \bmod 3$. By the Properties of Congruence Theorem, we have

$$\begin{aligned} x^2 - y^2 &\equiv (r_1 \cdot r_1) - (r_2 \cdot r_2) \bmod 3 \\ &\equiv r_1^2 - r_2^2 \bmod 3 \end{aligned}$$

Since $r_1, r_2 \in \{1, 2\}$, we can break this into 4 cases:

Case 1: $r_1 = r_2 = 1$. Here we have that

$$\begin{aligned} x^2 - y^2 &\equiv (1 - 1) \bmod 3 \\ &\equiv 0 \bmod 3 \end{aligned}$$

Case 2: $r_1 = r_2 = 2$

$$\begin{aligned} x^2 - y^2 &\equiv (4 - 4) \bmod 3 \\ &\equiv 0 \bmod 3 \end{aligned}$$

Case 3: $r_1 = 1$ and $r_2 = 2$

$$\begin{aligned} x^2 - y^2 &\equiv (1 - 4) \bmod 3 \\ &\equiv -3 \bmod 3 \\ &\equiv 0 \bmod 3 \end{aligned}$$

Case 4[1]: $r_1 = 2$ and $r_2 = 1$

$$\begin{aligned} x^2 - y^2 &\equiv (4 - 1) \bmod 3 \\ &\equiv 3 \bmod 3 \\ &\equiv 0 \bmod 3 \end{aligned}$$

Since in each case of values for r_1 and r_2, we have that $x^2 - y^2 \equiv 0 \bmod 3$, we can conclude that $3 \mid (x^2 - y^2)$. ∎

[1] Although this case is equivalent to Case 3 and therefore can be omitted, we have included it for completeness.