MATH 220 — Assignment 3

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Due: January 28, 2015

Question 1

Let $A = \{1, 4, 7, 10, 13, 16, \dots\}$, $B = \{x \in \mathbb{Z} : x \text{ is odd}\}$, $C = \{x \in \mathbb{Z} : x \text{ is prime and } x \neq 2\}$, and $D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$.

(a) $25 \in A$

True: "The number 25 is a member of the set A."

(b) $22 \in A \cup D$

True: "The number 22 is a member of the set given by the union of the sets A and D."

(c) $C \subseteq B$

True: "The set C is a subset of the set B"

(d) $\emptyset \in B \cup D$

False: "The empty set is a member of the set given by the union of the sets B and D."

Although the empty set is a *subset* of every set, the empty set is an *element* of a set only if the set contains the empty set as one of its elements. Here, the union of the sets B and D does not have the empty set as one of its elements. Thus the above statement is false.

Question 2

For the open sentence $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}(\{1, 2, 4\})$, we can determine

(a) all $A \in S$ for which $P(A)$ is true

Let B be the set of all $A \in S$ for which $P(A)$ is true. Since $S = \mathcal{P}(\{1, 2, 4\}) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}$ and $A \subseteq \{1, 2, 3\}$, then there are 4 possible sets $A \in S$ for which $P(A)$ holds true:

$$B = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

(b) all $A \in S$ for which $P(A)$ is false

Let B' be the set of all $A \in S$ for which $P(A)$ is false. Then the sets $A \in S$ for which $P(A)$ is false are the remaining 4 sets in the power set $\mathcal{P}(\{1, 2, 4\})$ that were not used in part (a). Thus B' is given by

$$B' = \{\{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}$$

(c) Let W be the set of all $A \in S$ for which $A \cap \{1, 2, 3\} = \emptyset$. List all the elements of W. Then find the intersection of W with the set of all $A \in S$ for which $P(A)$ is true.

Since A cannot contain the elements 1 or 2, then $W = \{\emptyset, \{4\}\}$. The intersection of W and all the sets of $A \in S$ for which $P(A)$ is true is given by

$$\begin{aligned} W \cap B &= \{\emptyset, \{4\}\} \cap \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \\ &= \{\emptyset\} \end{aligned}$$

Question 3

(a) $P(x)$: At least two of my library books are overdue.

$\sim P(x)$: Less than two of my library books are overdue.

(b) $P(x)$: One of my two friends misplaced his homework assignment.

$\sim P(x)$: One of my two friends did not misplace his homework assignment.

(c) $P(x)$: No one expected that to happen.

$\sim P(x)$: At least one person expected that to happen.

(d) $P(x)$: It's not often that my instructor teaches that course.

$\sim P(x)$: It is often that my instructor teaches that course.

(e) $P(x)$: It's surprising that two students received the same exam score.

$\sim P(x)$: It is not surprising that two students received the same exam score.

Question 4

For the sets $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, then the truth values of the statements

$$P : A \subseteq B. \quad \text{and} \quad Q : |A - B| = 6$$

can be computed. Since $3 \in A$ and $3 \notin B$, then P is **false**. The set $A - B$ can be rewritten in Set-Builder Notation as

$$A - B = \{1, 3, 5, 7, 8, 10\}.$$

Here, we can see that the cardinality $|A - B| = 6$, thus making statement Q **true**. With this knowledge, we can now compute the truth value for the following statements:

(a) $P \vee Q$: **True**

(b) $P \vee \sim Q$: **False**

(c) $P \wedge Q$: **False**

(d) $(\sim P) \wedge Q$: **True**

(e) $(\sim P) \vee (\sim Q)$: **True**

Question 5

Since the truth table for the statement $P \wedge (Q \vee R)$ (Table 1)

P	Q	R	$Q \vee R$	$P \vee (Q \wedge R)$
T	T	T	T	**T**
T	F	T	T	**T**
F	T	T	T	**F**
F	F	T	T	**F**
T	T	F	T	**T**
T	F	F	F	**F**
F	T	F	T	**F**
F	F	F	F	**F**

Table 1: Truth table for the statement $P \wedge (Q \vee R)$.

and the truth table for the statement $(P \wedge Q) \vee (P \wedge R)$ (Table 2)

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	**T**
T	F	T	F	T	**T**
F	T	T	F	F	**F**
F	F	T	F	F	**F**
T	T	F	T	F	**T**
T	F	F	F	F	**F**
F	T	F	F	F	**F**
F	F	F	F	F	**F**

Table 2: Truth table for the statement $(P \wedge Q) \vee (P \wedge R)$.

are equivalent, we can conclude that the distributive law $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ holds true for all statements P, Q, and R. $\blacksquare$

Question 6

For the statements P and Q, we can construct a truth table for the statement $(P \Rightarrow Q) \Rightarrow (\sim P)$ (Table 3):

P	Q	$P \Rightarrow Q$	$\sim P$	$(P \Rightarrow Q) \Rightarrow (\sim P)$
T	T	T	F	**F**
T	F	F	F	**T**
F	T	T	T	**T**
F	F	T	T	**T**

Table 3: Truth table for the statement $(P \Rightarrow Q) \Rightarrow (\sim P)$.

Question 7

Let the sets A and B be non-empty disjoint subsets of a set S. If $x \in S$, then we can find the truth value of the following statements:

(a) It is possible that $x \in A \cap B$.

True: Even though $A \cap B = \emptyset$, the empty set could be an element of the set S. Thus if $x = \emptyset$, then $x \in A \cap B$ (note that although A and B are non-empty sets, they still have an intersection of the empty set).

(b) If x is an element of A, then x can't be an element of B.

True: By definition of disjoint non-empty sets, if an element is in A, then that same element cannot be in B.

(c) If x is not an element of A, then x must be an element of B.

False: Let S be the set of Integers, A the set of even Natural Numbers, B the set of odd Natural Numbers, and let $x = -5$. Although $A \subseteq S$ and $B \subseteq S$, we have that $x \notin A$ and $x \notin B$.

(d) It's possible that $x \notin A$ and $x \notin B$.

True: By similar logic of part (c), even if $x \in S$, we may have $x \notin A, B$.

(e) For each nonempty set C, either $x \in A \cap C$ or $x \in B \cap C$.

False: Let S be the set of Integers, A the set of even Natural Numbers, B the set of odd Natural Numbers, and let $x = -5$. Although $A \subseteq S$ and $B \subseteq S$, we have that $x \notin A$ and $x \notin B$. Then for every non-empty set, $x \notin A \cap C$ and $x \notin B \cap C$

(f) There exists a nonempty set C, such that both $x \in A \cup C$ and $x \in B \cup C$.

True: Since $x \in C$ for every x, then $x \in A \cup C$ and $x \in B \cup C$.

Question 8

Let $P(x)$ be "*Bill takes Sam to the concert.*" and $Q(x)$ be "*Sam will take Bill to dinner.*" Then the statement "*If Bill takes Sam to the concert, then Sam will take Bill to dinner.*", can be expressed as

$$P(x) \Rightarrow Q(x).$$

For this implication statement, it will always be True if $Q(x)$ is True. Similarly it will always be False if $P(x)$ is False. Using this knowledge, the following statements will either imply that $P(x) \Rightarrow Q(x)$ is true or false:

(a) *Sam takes Bill to dinner only if Bill takes Sam to the concert.*

Here, we can rewrite this statement as $P(x) \Rightarrow Q(x)$. Since these are the same statements, this would imply that the statement $P(x) \Rightarrow Q(x)$ is **True**.

(b) *Either Bill doesn't take Sam to the concert or Sam takes Bill to dinner.*

Here, the statement can be rewritten as $\sim P(x) \vee Q(x)$. Assuming that this statement is True, then we have 3 cases:

- **Case 1:** $\sim P(x)$ is True and $Q(x)$ is False.

  Since $P(x)$ is False and $Q(x)$ is False, then the implication statement $P(x) \Rightarrow Q(x)$ is **True**.

- **Case 2:** $\sim P(x)$ is False and $Q(x)$ is True.

  Since $P(x)$ is True and $Q(x)$ is True, then the implication statement $P(x) \Rightarrow Q(x)$ is **True**.

- **Case 3:** $\sim P(x)$ is True and $Q(x)$ is True.

  Since $P(x)$ is False and $Q(x)$ is True, then the implication statement $P(x) \Rightarrow Q(x)$ is **True**.

Thus we can conclude that the statement $P(x) \Rightarrow Q(x)$ is always **True** when $\sim P(x) \vee Q(x)$ is True.

(c) *Bill takes Sam to the concert.*

Here, the statement $P(x)$ is True. Thus the implication statement $P(x) \Rightarrow Q(x)$ will be **True** if $Q(x)$ is True and **False** if $Q(x)$ is False.

(d) *Bill takes Sam to the concert and Sam takes Bill to dinner.*

Here, we can express this statement as $P(x) \wedge Q(x)$, which means that both $P(x)$ and $Q(x)$ are True. Therefore $P(x) \Rightarrow Q(x)$ is **True**.

(e) *Bill takes Sam to the concert and Sam doesn't take Bill to dinner.*

Here, we can express the statement as $P(x) \wedge \sim Q(x)$, which means that $P(x)$ is True and $Q(x)$ is False. Thus $P(x) \Rightarrow Q(x)$ is **False**.

(f) *The concert is canceled.*

Since the concert is cancelled, then Bill could not have taken Sam to the concert. Thus, $P(x)$ is False. Regardless of the truth value of $Q(x)$, we can conclude that the implication statement $P(x) \Rightarrow Q(x)$ is **True**.

(g) *Sam doesn't attend the concert.*

Similar to (f), $P(x)$ is False. Thus we can again conclude that $P(x) \Rightarrow Q(x)$ is **True**.

Question 9

For the open sentences $P(n) : 5n + 3$ is prime, and $Q(n) : 7n + 1$ is prime, both over the domain $\mathbb{N}$:

(a) The implication statement $P(n) \Rightarrow Q(n)$ can be expressed in words as "If $5n + 3$ is prime, then $7n + 1$ is prime, for all n in the set of Natural Numbers".

(b) The implication statement $P(2) \Rightarrow Q(2)$ can be expressed in words as "If 13 is prime, then 15 is prime". Since $P(2)$ is True and $Q(2)$ is False, then the statement $P(2) \Rightarrow Q(2)$ is **False**.

(c) The implication statement $P(6) \Rightarrow Q(6)$ can be expressed in words as "If 33 is prime, then 43 is prime". Since $P(6)$ is False and $Q(6)$ is True, then the statement $P(6) \Rightarrow Q(6)$ is **True**.

Question 10

Let the statements P be "*The fish are biting*", Q be "*There are no bugs*", and R be "*It is winter*". Thus we can express the statement "*The fish are biting and there are no bugs, or the fish are not biting and there are bugs, or it is winter*" as

$$(P \wedge \sim Q) \vee (\sim P \wedge Q) \vee R.$$

Thus the negation is given by

$$\begin{aligned} \sim ((P \wedge \sim Q) \vee (\sim P \wedge Q) \vee R)) &= \sim (P \wedge \sim Q) \wedge \sim (\sim P \wedge Q) \wedge \sim R \\ &= (\sim P \vee Q) \wedge (P \vee \sim Q) \wedge \sim R \end{aligned}$$

Converting back to words, we can express this as "*The fish are not biting or there are bugs, and the fish are biting or there are no bugs, and it is not winter.*"