MATH 220 — Assignment 8

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Question 1

Find the domain and range of the function $f(x) = \frac{\sqrt{x-1}}{x}$ (assume x is real).

Domain

For the numerator we have that $x - 1 \geq 0$, so $x \geq 1$. In the denominator $x \neq 0$. Thus overall we have

$$dom(f) = \{x \in \mathbb{R} \mid x \geq 1\}$$

Proof. Assume to the contrary that $x \notin dom(f)$. Let $x = 1 - \delta$, where δ non-negative constant. Substituting this into $f(x)$ we have

$$f(x) = \frac{\sqrt{(x-\delta)-1}}{x}$$

which is undefined. Thus we have arrived at a contradiction and $x \in dom(f) = \{x \in \mathbb{R} \mid x \geq 1\}$. ■

Range

Since $x \geq 1$, we have the minimal value of $f(x)$ to be at $f(1) = 0$ and the maximal value of $f(x)$ to be at $f(1) = 1/2$. Thus we have that

$$range(f) = \{f(x) \in \mathbb{R} \mid 0 \leq y \leq 1/2\}$$

Proof. Assume to the contrary that $f(x) \notin range(f)$. The we have that $f(x) = 0 - \delta$ or $f(x) = 1/2 + \alpha$, where δ, α non-negative constant. Substituting in we have

$$0 - \delta = \frac{\sqrt{x-1}}{x}$$

$$\delta = -\frac{\sqrt{x-1}}{x}$$

which is a contradiction since $\delta > 0$. Similarly in the case of $f(x) = 1/2 + \alpha$ we also derive a contradiction. Thus we have that $range(f) = \{f(x) \in \mathbb{R} \mid 0 \leq y \leq 1/2\}$. ■

Question 2

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 3x + 4$.

(a) f is not injective.

Proof. Let $x_1 = 0$ and $x_2 = -3$. Then $f(x_1) = 4 = f(x_2)$, thus f is not injective. ■

(b) Find all pairs a, b of real numbers so that $f(a) = f(b)$.

Let $f(a) = f(b)$, where $a, b \in \mathbb{R}$. Then

$$\begin{aligned} a^2 + 3a + 4 &= b^2 + 3b + 4 \\ a^2 + 3a - b^2 - 3b &= 0 \\ (a^2 - b^2) + 3(a - b) &= 0 \\ (a - b)(a + b + 3) &= 0 \end{aligned}$$

Thus for $f(a) = f(b)$, $a = b$ or $a = -(b + 3)$. In other words, we have that f is injective when

$$\{(a, b) \in \mathbb{R} \mid a = b \text{ or } a = -(b + 3)\}$$

Question 3

Let $h : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $h(n) = 3n - 8$.

(a) Prove: f is injective.

Proof. Assume $h(a) = h(b)$, where $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} 3a - 8 &= 3b - 8 \\ 3a &= 3b \\ a &= b \end{aligned}$$

Thus f is injective. ■

(b) Disprove: f is surjective.

Let $h(n) = 0 \in \mathbb{Z}$. Then

$$3n - 8 = 0$$
$$n = \frac{8}{3}$$

Since $n = \frac{8}{3} \notin \mathbb{Z}$, f is not surjective.

Question 4

Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is

(a) one-to-one and onto

$$f(x) = x$$

(b) one-to-one but not onto

$$f(x) = x + 1$$

. Here, range$(f) = \mathbb{N} - \{1\}$.

(c) onto but not one-to-one

$$f(x) = \begin{cases} x - 1 & \text{if } x \geq 2 \\ x & \text{if } x = 1 \end{cases}$$

(d) neither one-to-one or onto

$$f(x) = \begin{cases} 1 & \text{if } x \text{ odd} \\ 2 & \text{if } x \text{ even} \end{cases}$$

Question 5

Prove or disprove: for every set A there is an injective function $f : A \to \mathcal{P}(A)$

Proof. [1] For a function $f : A \to \mathcal{P}(A)$ to be injective, then

$$|A| \geq |\mathcal{P}(A)|$$

However we know that $\mathcal{P}(A) = 2^{|A|}$. Thus $|A| \geq 2^{|A|}$ and the function f is not injective. ∎

[1] A counterexample may have also sufficed. Let $A = \{1, 2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Thus $|A| = 2 < 4 = \mathcal{P}(A)$ and f is not injective. ∎

Question 6

Give an example of

(a) A bijection from (0,1) to $(1, \infty)$

$$f(x) = tan\left(\frac{\pi x}{2}\right) + 1$$

Proof. Let us first prove that f is injective. Assume that $f(a) = f(b)$. Then

$$tan\left(\frac{\pi a}{2}\right) + 1 = tan\left(\frac{\pi b}{2}\right) + 1$$
$$tan\left(\frac{\pi a}{2}\right) = tan\left(\frac{\pi b}{2}\right)$$

Since $a, b \in (0, 1)$ and the tangent has a period of π, then for $tan\left(\frac{\pi a}{2}\right) = tan\left(\frac{\pi b}{2}\right)$, we must have that $a = b$. Thus f is injective. Let us now prove that f is also surjective. Let $r \in (1, \infty)$ and $x = \frac{2}{\pi}tan^{-1}(r-1)$. Then

$$f(x) = tan\left(\frac{\pi}{2} \cdot \frac{2}{\pi}tan^{-1}(r-1)\right) + 1 = (r-1) + 1 = r$$

thereby making f also surjective. Since f is both injective and surjective, we can conclude that f is bijective. ■

(b) A surjection from $\mathbb{R}$ to $\mathbb{Q}$

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Proof. Let $r \in \mathbb{Q}$. We will us examine the case where $x \in \mathbb{Q}$. Here

$$f(x) = f(r) = r$$

Which means we hit every point in the range, thus making it surjective. [2] ■

(c) A bijection from $\mathbb{N}$ to $\mathbb{Z}$

$$f(x) = \begin{cases} -\frac{x}{2} & \text{if } x \text{ even} \\ \frac{x-1}{2} & \text{if } x \text{ odd} \end{cases}$$

[2] Although we do not need to examine the case where $x \notin \mathbb{Q}$, we have $f(x) = f(0) = 0 \in \mathbb{Q}$ which makes this function only surjective, rather than bijective.

Proof. Let us first prove that f is injective. Assume that $f(a) = f(b)$. In the case where x is even

$$-\frac{a}{2} = -\frac{b}{2}$$
$$a = b$$

Similarly in the case where x is odd

$$\frac{a-1}{2} = \frac{b-1}{2}$$
$$a = b$$

Thus f is injective. Let us now prove that f is also surjective. Let $r \in \mathbb{Z}$. In the first case where x is even, let $x = -2r$. Then

$$f(-2r) = -\frac{-2r}{2} = r$$

Similarly in the case where x is odd, let $x = 2r + 1$. Then

$$f(2r+1) = \frac{(2r+1)-1}{2} = r$$

thereby making f also surjective. Since f is both injective and surjective, we can conclude that f is bijective. ■

Question 7

Let $f : A \to B$ be a function and if $D \subset B$, recall that the inverse image of D under f is by definition the set $f^{-1}(D) = \{x \in A : f(x) \in D\}$. Note that $f^{-1}(D)$ is a *set*, and is defined for any function f, even if f does not have an inverse.

(a) If $D_1, D_2 \subseteq B$, prove that

$$f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$$

Proof. $\subseteq$ Let $x \in f^{-1}(D_1 \cap D_2)$. So $f(x) \in D_1 \cap D_2$. Since $f(x) \in D_1$ and $f(x) \in D_2$, we have that $x \in f^{-1}(D_1)$ and $x \in f^{-1}(D_2)$. Thus $x \in f^{-1}(D_1) \cap f^{-1}(D_2)$ and we have that $f^{-1}(D_1 \cap D_2) \subseteq f^{-1}(D_1) \cap f^{-1}(D_2)$

$\supseteq$ Let $x \in f^{-1}(D_1) \cap f^{-1}(D_2)$. Then $x \in f^{-1}(D_1)$ and $x \in f^{-1}(D_2)$, so we have that $f(x) \in D_1$ and $f(x) \in D_2$. Since $f(x) \in D_1 \cap D_2$, then $x \in f^{-1}(D_1 \cap D_2)$. Thus we can conclude that $f^{-1}(D_1 \cap D_2) \supseteq f^{-1}(D_1) \cap f^{-1}(D_2)$

■

(b) If $D_1, D_2 \subseteq B$, prove that

$$f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$$

Proof. $\subseteq$ Let $x \in f^{-1}(D_1 \cup D_2)$. So $f(x) \in D_1 \cup D_2$. Since $f(x) \in D_1$ or $f(x) \in D_2$, we have that $x \in f^{-1}(D_1)$ or $x \in f^{-1}(D_2)$. Thus $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$ and we have that $f^{-1}(D_1 \cup D_2) \subseteq f^{-1}(D_1) \cup f^{-1}(D_2)$

$\supseteq$ Let $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$. Then $x \in f^{-1}(D_1)$ or $x \in f^{-1}(D_2)$, so we have that $f(x) \in D_1$ or $f(x) \in D_2$. Since $f(x) \in D_1 \cup D_2$, then $x \in f^{-1}(D_1 \cup D_2)$. Thus we can conclude that $f^{-1}(D_1 \cup D_2) \supseteq f^{-1}(D_1) \cup f^{-1}(D_2)$

■

(c) If $D \subset B$, prove that

$$f^{-1}(B - D) = A - f^{-1}(D)$$

Proof. $\subseteq$ Let $x \in f^{-1}(B - D)$. Then

$$\begin{aligned}
&x \in f^{-1}(B \cap D^c) \\
&x \in f^{-1}(B) \cap f^{-1}(D^c) \qquad \text{(by proof 7b)} \\
&x \in A \cap f^{-1}(D^c) \\
&x \in A - f^{-1}(D)
\end{aligned}$$

$\supseteq$ Let $x \in A - f^{-1}(D)$. Then

$$\begin{aligned}
&x \in A \cap f^{-1}(D^c) \\
&x \in f^{-1}(B) \cap f^{-1}(D^c) \\
&x \in f^{-1}(B \cap D^c) \\
&x \in f^{-1}(B - D)
\end{aligned}$$

■

Question 8

Let A be a well-ordered set. Let $f : \mathcal{P}(A) \to A$ be the function that assigns to every $B \subseteq A$ the smallest element of B.

(a) Is f a function?

Proof. Let $x \in B$. Since $B \subseteq A$ and A is a well-ordered set, then B contains a minimal element. Thus for every input B we have a minimal element (we need not worry about the case where B is the empty set), thus a single output. By definition then f is a function. ■

(b) Is f injective?

Proof. Let $A = \{1, 2\}$. Then for $B = \{1\}$ and $B = \{1, 2\}$, both have a minimal element of 1, thereby making f not injectvie. ■

(c) Is f surjective?

Proof. Since A is a well-ordered set, then every subset of A, in other words the power set of A, contains a minimal element. Thus each subset within the power set $\mathcal{P}(A)$ will map onto an element in A. Since a subset of cardinality 1 will always have a minimal element of itself, we know that $\mathcal{P}(A)$ has the additional property of mapping onto all elements of A, thereby making f surjective. ■

Question 9

(a) Prove that if $f : A \to B$ is an injective function, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ for all $C_1, C_2 \subseteq A$.

Proof. $\subseteq$ Let $x \in f(C_1 \cap C_2)$. Then

$$f^{-1}(x) \in C_1 \cap C_2$$

$$f^{-1}(x) \in C_1 \text{ and } f^{-1}(x) \in C_2$$

$$x \in f(C_1) \text{ and } x \in f(C_2)$$

$$x \in f(C_1) \cap f(C_2)$$

Thus $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.

$\supseteq$ Let $x \in f(C_1) \cap f(C_2)$ Then $x \in f(C_1)$ and $x \in f(C_2)$. Since $x \in f(C_1)$, $\exists y_1 \in A$ such that $f(y_1) = x$. Similarly, since $x \in f(C_2)$, $\exists y_2 \in A$ such that $f(y_2) = x$. Since $f(y_1) = x = f(y_2$ and f is an injection, then $y_1 = y_2$. Therefore $y_1 \in C_2$ and $y_1 \in C_1 \cap C_2$. Thus we have that $x = f(y_1) \in f(C_1 \cap C_2)$ and we have shown that $f(C_1 \cap C_2) \supseteq f(C_1) \cap f(C_2)$.

■

(b) Prove that if $f : A \to B$ is a surjective function, then $f(f^{-1}(D)) = D$ for every subset $D \subseteq B$.

Proof. $\subseteq$ Let $x \in f(f^{-1}(D))$. Then $\exists a \in f^{-1}(D)$ such that $f(a) = x$. By the definition of pre-image, $x \in D$ and we are done.

$\supseteq$ Let $x \in D$. Since f is a surjection, there exists $a \in A$ such that $f(a) = x$. Thus $a \in f^{-1}(D)$ and we have that $f(a) \in f(f^{-1}(D))$. Here $x = f(a)$, so $x \in f(f^{-1}(D))$.

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