MATH 220 — Assignment 4

Stephanie Knill
54882113
Due: February 2, 2016

Question 1

(a) For ever rational number r, the number $1/r$ is rational.

$P:\ \forall r \in \mathbb{Q},\ 1/r$ is rational.

$\sim P:\ \exists r \in \mathbb{Q}\ :\ 1/r$ is irrational.

(b) There exists a rational number r such that $r^2 = 2$.

$P:\ \exists r \in \mathbb{Q}\ :\ r^2 = 2.$

$\sim P:\ \forall r \in \mathbb{Q},\ r^2 \neq 2.$

Question 2

Let P be the statement "$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ y^2 = x$".

(a) In words: "For all real numbers x, there exists a real number y such that $y^2 = x$"

(b) This statement P is **True**.

(c) $\sim P:\ \exists x \in \mathbb{R}\ \forall y \in \mathbb{R},\ y^2 \neq x$

Question 3

(a) $\exists x \in \mathbb{R}, x^2 - x = 0$

True: $x = 0, 1$

(b) $\forall x \in \mathbb{R}, \sqrt{x^2} = x$

True

(c) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y + 3 = 8$

True: $x = 0$ and $y = 5$.

(d) $\forall x, y \in \mathbb{R}, x + y + 3 = 8$

False: $x = y = 0$.

(e) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y + 3 = 8$

True: $y = 5 - x$

(f) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y + 3 = 8$

True: $x = 5 - y$

(g) $\exists m, n \in \mathbb{N}, n^2 + m^2 = 25$

True: $m = 3, n = 4$

(h) $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}, n^2 + m^2 = 25$

False: $m = 100$

Question 4

Let $P(x)$ and $Q(x)$ be open sentences where the domain of the variable x is a set S. Then which of the following implies $\sim P(x) \Rightarrow Q(x)$ is false for some $x \in S$?

If $P(x)$ is False and $Q(x)$ is False, then $\sim P(x) \Rightarrow Q(x)$ is False. Thus, for the following:

(a) $P(x) \wedge Q(x)$ is false for all $x \in S$: **does imply False**.

Here, we have 3 cases for the truth values of $P(x)$ and $Q(x)$:

1. Both $P(x)$ and $Q(x)$ are False
2. $P(x)$ is True and $Q(x)$ is False
3. $P(x)$ is False and $Q(x)$ is True

Since the first case would make $\sim P(x) \Rightarrow Q(x)$ False, then "$P(x) \wedge Q(x)$ is false for all $x \in S$" implies that $\sim P(x) \Rightarrow Q(x)$ is False for some $x \in S$.

(b) $P(x)$ is true for all $x \in S$: **does not imply False.**

Since $P(x)$ always True, then $\sim P(x) \Rightarrow Q(x)$ can never be False.

(c) $Q(x)$ is true for all $x \in S$: **does not imply False.**

Since $Q(x)$ always True, then $\sim P(x) \Rightarrow Q(x)$ can never be False.

(d) $P(x) \vee Q(x)$ is false for all $x \in S$: **does imply False.**

Here both $P(x)$ and $Q(x)$ are False, therefore implying that $\sim P(x) \Rightarrow Q(x)$ is False.

Question 5

Let $S = [1, 2]$ and $T = (3, \infty)$.

(a) $\exists x \in S$ s.t. $\exists y \in T$ s.t. $|x - y| > 3$: "There exists an x in the set S such that there exists a y in the set T such that $|x - y| > 3$."

Proof: Let $x = 1$ and $y = 100$. Since the inequality

$$3 < |1 - 100| = |-99| = 99$$

holds true, then statement is also true. ■

(b) $\exists x \in S$ s.t. $\forall y \in T, |x - y| > 3$: "There exists an x in the set S such that for all y in the set T, $|x - y| > 3$."

Proof: Let $x = 1$. Then the inequality can be expressed as

$$3 < |1 - y| = 1 + |-y|$$

and we have that $|y| > 2$. Since $y \in T$, then $y > 3$. Therefore the statement holds true. ■

(c) $\forall x \in S, \exists y \in T$, s.t. $|x - y| > 3$: "For all x in the set S, there exists a y in the set T such that $|x - y| > 3$."

Proof: Let $y = 100$. Then the inequality can be expressed as

$$3 < |x - 100| = |x| + 100$$

and we have that $|x| > -97$. Since $x \in T = [1, 2]$, then the inequality holds true for all x. ■

(d) $\forall x \in S, \forall y \in T, |x-y| > 3$: "For all x in the set S and for all y in the set T, $|x-y| > 3$."

Proof: We can express the inequality as

$$\begin{aligned} |x-y| &> 3 \\ |x| + |-y| &> 3 \\ |x| + |y| &> 3 \end{aligned}$$

Since $x \in S = [1,2]$ and $y \in T = (3, \infty)$, then $|x| \geq 1$ and $|y| > 3$. Thus

$$|x| + |y| > 1 + 3 > 3$$

and the inequality holds true for all x and y. ■

Question 6

Let $I = \{n^2 | n \in \mathbb{Z}\}$. Let P be the statement

$$\bigcup_{k \in I} [k, 2k] = \mathbb{R}.$$

(a) Expressing $P(x)$ using quantifiers, we have

$$\begin{aligned} \bigcup_{k \in I} [k, 2k] &= \{x : \exists k \in I \text{ s.t. } x \in [k, 2k]\} = \mathbb{R} \\ &= \{x : \forall x \in \mathbb{R},\ \exists k \in I \text{ s.t. } x \in [k, 2k]\} \\ &= \{x : \forall x \in \mathbb{R}, \exists n \in \mathbb{Z} \text{ s.t. } x \in [n^2, 2n^2]\} \end{aligned}$$

(b) $P(x)$: **False**

Proof: Using Roster Notation, let us re-express the set I as

$$I = \{0, 1, 4, 9, 16, 26, \ldots\}.$$

Then the statement P is given by

$$\bigcup_{k \in I} [k, 2k] = [0,0] \cup [1,2] \cup [4,8] \cup [9,18] \cup \cdots$$

which is not equivalent to the set of real numbers $\mathbb{R}$. ■

Question 7

Let $A = \{x \mid \forall n \geq 3,\ 1/n < x < 1 - 1/n\}$.

(a) Let $I = \{n \in \mathbb{N} : n \geq 3\}$ and $A_n = (1/n,\ 1 - 1/n)$. Then the set A can be represented as an intersection of an indexed collection of sets:

$$\begin{aligned} A &= \bigcap_{n=3}^{\infty} \left(\frac{1}{n},\ 1 - \frac{1}{n}\right) \\ &= \bigcap_{n \in I} A_n \end{aligned}$$

(b) We can also express the set A as an interval in $\mathbb{R}$: $A = (0, 1)$ or equivalently, $A =]0, 1[$.

Question 8

For the two statements

- Among the inhabitants of QE220 who can watch TV, not all have antennae on their head.
- The inhabitants of QE220 that are green and do not have antennae, cannot watch TV.

we will express them using quantifiers. For the universal set Ω of inhabitants of the planet QE220, let T, A, and G be subsets of Ω such that T is the set that can watch TV, A the set that have an antennae on their head, and G the set that are green. Then we can express the two statements as

- Statement 1: If $x \in T$, then $\exists x \notin A$.
- Statement 2: If $y \in (G \cap \overline{A})$, then $y \notin T$.

and the statement "not all the inhabitants of QE220 that can watch TV are green" that we want to determine the truth value of as

- Statement 3: $\exists z \in T$, such that $z \in G$.

Taking the contrapositive of Statement 2, we have that

$$\text{If } y \in (G \cap \overline{A}),\ \text{then } y \notin T \equiv \text{If } y \in T, \text{then } y \notin (G \cap \overline{A})$$

Thus for both Statement 1 and Statement 2, if an element is in T then there is no information as to whether it is also in the set G. Therefore the statement "not all the inhabitants of QE220 that can watch TV are green" does not follow from the given two statements. ■

Question 9

Proposition: If x is an odd integer, then $9x + 5$ is even.

Proof: Since x is an odd integer, then there exists $k \in \mathbb{Z}$ such that

$$\begin{aligned} x &= 2k + 1 \\ 9x &= 18k + 9 \\ 9x + 5 &= 18k + 9 + 5 \\ &= 18k + 14 \\ &= 2(9k + 7) \end{aligned}$$

Since $9k + 7 \in \mathbb{Z}$, then $9x + 5$ is even. ■