MATH 302 — Assignment 3

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Question 1: Section 3.2 #20

For a six-card hand dealt from an ordinary deck of cards, we have number of cards $n = 52$ and cards dealt $k = 6$. Therefore we can compute the following probabilities:

(a) Probability all 6 cards same suit

$$\begin{aligned} P(\text{all hearts}) &= \frac{\binom{13}{6}}{\binom{52}{6}} \\ &= \frac{\frac{13!}{6!\cdot 7!}}{\frac{52!}{6!\cdot 46!}} \\ &= \frac{33}{391,510} \\ &\approx 8.43 \cdot 10^{-5} \end{aligned}$$

(b) Probability of 3 Aces, 2 Kings, 1 Queen

$$\begin{aligned} P(3\text{A, 2K, 1Q}) &= \frac{\binom{4}{3} \cdot \binom{4}{2} \cdot \binom{4}{1}}{\binom{52}{6}} \\ &= \frac{12}{2,544,815} \\ &\approx 4.72 \cdot 10^{-}6 \end{aligned}$$

(c) Probability of 3 same suit and 3 another suit

$$\begin{aligned} P(3S_1,\ 3S_2) &= \frac{\binom{4}{2}\cdot\binom{13}{3}\cdot\binom{13}{3}}{\binom{52}{6}} \\ &= \frac{4719}{195,755} \\ &\approx 0.0241 \end{aligned}$$

Question 2

For $M \geq 1$ and $n_1, \ldots, n_m$ positive integers, with the notation n for the sum $n_1 + \cdots + n_m$ denoted by $C[n_1, \ldots, n_m]$ the number of ways to place n balls (of labels $1, 2, \ldots, n$) into m urns $U_1, \ldots, U_m$ such that there are n_1 balls falling into $U_1, \ldots, n_m$ balls falling into U_m.

(a) For $m = 2$, we have n_1 balls placed into U_1 and n_2 balls placed into U_2. Therefore the number of combinations for the first urn is given by $C^n_{n_1}$ and the number of combinations for the second urn is given by $C^{n_2}_{n_2}$. Combining these, we have

$$\begin{aligned} C[n_1, n_2] &= C^n_{n_1} \cdot C^{n_2}_{n_2} \\ &= \binom{n}{n_1} \cdot 1 \\ &= \binom{n}{n_1} \end{aligned}$$

(b) Similarly for $m = 3$, we the number of combinations for the first urn is given by $C^n_{n_1}$, the number of combinations for the second urn is given by $C^{n-n_1}_{n_2}$, and the number of combinations for the third urn is given by $C^{n-n_1-n_2}_{n_3} = C^{n_3}_{n_3}$. Combining these, we have

$$\begin{aligned} C[n_1, n_2, n_3] &= C^n_{n_1} \cdot C^{n_2}_{n_2} \cdot C^{n_3}_{n_3} \\ &= \binom{n}{n_1}\binom{n-n_1}{n_2} \cdot 1 \\ &= \binom{n}{n_1}\binom{n-n_1}{n_2} \end{aligned}$$

This simplifies to

$$\begin{aligned} C[n_1, n_2, n_3] &= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \\ &= \frac{n!}{n_1!} \cdot \frac{1}{n_2! \cdot n_3!} \\ &= \frac{n!}{n_1! \cdot n_2! \cdot n_3!} \end{aligned}$$

(c) $C[n_1, n_2, n_3]$ may also be used to find the number of words of length n with letters a appearing n_1 times, b appearing n_2 times, and c appearing n_3 times.

Since $\binom{n}{n_1}$ gives the number of words of length n with a appearing n_1 times, $\binom{n-n_1}{n_2}$ gives the number of words of length n with b appearing n_2 times, and $\binom{n-n_1-n_2}{n_3}$ gives the number of words of length n with c appearing n_3 times, we can combine these to get

$$\begin{aligned} \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3} &= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n_3}{n_3} \\ &= \binom{n}{n_1}\binom{n-n_1}{n_2} \\ &= C[n_1, n_2, n_3] \end{aligned}$$

(d) For the word "assesses", we have $n = 8$ letters with the letter a appearing $n_1 = 1$ times, s appearing $n_2 = 5$ times, and e appearing $n_3 = 2$ times. Thus we have

$$\begin{aligned} C[n_1, n_2, n_3] &= \binom{n}{n_1}\binom{n-n_1}{n_2} \\ &= \binom{8}{1}\binom{8-1}{5} \\ &= 168 \end{aligned}$$

(e) **Proposition:** $C[n_1, \ldots, n_m] = \frac{n!}{n_1! \cdot \cdots \cdot n_m!}$

Proof

$$\begin{aligned}
C[n_1, \ldots, n_m] &= \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_m}{n_m} \\
&= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-\cdots-n_{m-2})!}{n_{m-1}!(n-n_1-\cdots-n_{m-1})!} \\
&= \frac{n!}{n_1!} \cdot \frac{1}{n_2!} \cdots \frac{1}{n_{m-1}! \cdot n_m!} \\
&= \frac{n!}{n_1! \cdots n_m!}
\end{aligned}$$

Question 3: Section 4.1 #3

A fair die is rolled twice, with a sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let E be the event that the sum of the faces is ≥ 7 then we can calculate $P(E|F)$ for different events F:

(a) $F = \{\text{1st die} = 4\}$

Here, the 2nd die needs to be $\{4, 5, 6\}$, thus

$$P(E|F) = 3/6 = 1/2$$

(b) $F = \{\text{1st die} = 4\}$

Here, $P(F) = 3/6 = 1/2$. For $P(E \cap F)$, we have 3 cases for the possible outcomes of the 1st die:

- **Case 1:** 1st die is a 4.

  Then the 2nd die must be $\{4, 5, 6\}$. Therefore $1/6 \cdot 3/6 = 1/12$.

- **Case 2:** 1st die is a 5.

  Then the 2nd die must be $\{3, 4, 5, 6\}$. Therefore $1/6 \cdot 4/6 = 1/9$.

- **Case 3:** 1st die is a 6.

  Then the 2nd die must be $\{2, 3, 4, 5, 6\}$. Therefore $1/6 \cdot 5/6 = 5/36$.

Adding these up, we have

$$P(E \cap F) = 1/12 + 1/9 + 5/36 = 1/3$$

Therefore we can now calculate

$$\begin{aligned} P(E|F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{1/3}{1/2} \\ &= 2/3 \end{aligned}$$

(c) $F = \{1\text{st die} = 1\}$

Here, there are no possible rolls of the 2nd die that can make the event E true. Thus

$$P(E|F) = 0$$

(d) $F = \{1\text{st die} < 5\} = \{1, 2, 3, 4\}$

Here, $P(F) = 4/6 = 2/3$. For $P(E \cap F)$, the 2nd die needs to be $\{4, 5, 6\}$. Again we will have three cases:

$$P(E \cap F) = 1/6 \cdot 1/6 + 1/6 \cdot 2/6 + 1/6 \cdot 3/6 = 1/6$$

Therefore we can now calculate

$$P(E|F) = \frac{1/6}{2/3} = 1/4$$

Question 4: Section 4.1 #9

For a family of 2 children (with equiprobability of it being a girl or boy) with sample space $\Omega = \{BB, BG, GB, GG\}$, we can compute $P(E|F)$ given various events E and F:

(a) $P(E|F)$, where E is the event that the family has 2 boys and F is the event of having at least one boy.

Since $F = \{BB, BG, GB\}$ we have that

$$P(F) = \frac{|F|}{|\Omega|} = \frac{3}{4}$$

likewise we know that $E = \{BB\}$, so we also have

$$P(E \cap F) = P(E) = \frac{|E|}{|\Omega|} = \frac{1}{4}$$

Therefore we can compute the probability of a family having 2 boys given that it has at least 1 boy as

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{3/4} = 1/3$$

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(b) $P(E|F)$, where E is the event that the family has 2 boys and F is event that the first child is a boy.

Since $F = \{BB, BG\}$ we have that

$$P(F) = \frac{|F|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$$

likewise we know that $E = \{BB\}$, so we also have

$$P(E \cap F) = P(E) = \frac{|E|}{|\Omega|} = \frac{1}{4}$$

Therefore we can compute the probability of a family having 2 boys given that it has at least 1 boy as

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{1/2} = 1/2$$

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Question 5: Section 4.1 #15

In a game of bridge with number of cards $n = 52$ and $k = 13$ cards dealt to each player, we can compute $P(E|F)$ given various events E and F:

(a) $P(E|F)$, where E is the event that your bridge partner has exactly 2 aces and F is the event he/she has at least one ace.

To compute the probability of having at least one ace, let us use the Law of Total Probability:

$$\begin{aligned} P(F) &= 1 - P(\text{no ace}) \\ &= 1 - \frac{\binom{48}{13}}{\binom{52}{13}} \\ &= \frac{14,498}{20,825} \end{aligned}$$

Next the intersection of the two events is given by

$$P(E \cap F) = P(E) = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} = \frac{4,446}{20,825}$$

Therefore we can compute the probability of your bridge partner has exactly 2 aces given he/she has at least one ace as

$$\begin{aligned} P(E|F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{\frac{4,446}{20,825}}{\frac{14,498}{20,825}} \\ &= \frac{2,223}{7,249} \\ &\approx 0.307 \end{aligned}$$

(b) $P(E|F)$, where E is the event that your bridge partner has exactly 2 aces and F is the event he/she has ace of spades.

To compute the probability of having the ace of spades, we have

$$P(F) = \frac{\binom{1}{1}\binom{51}{12}}{\binom{52}{13}}$$

Next the intersection of the two events is given by

$$P(E \cap F) = \frac{\binom{1}{1}\binom{3}{1}\binom{48}{11}}{\binom{52}{13}} = \frac{3 \cdot \binom{48}{11}}{\binom{52}{13}}$$

Therefore we can compute the probability of your bridge partner has exactly 2 aces given he/she has at least one ace as

$$\begin{aligned} P(E|F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{\frac{3\cdot\binom{48}{11}}{\binom{52}{13}}}{\frac{\binom{1}{1}\binom{51}{12}}{\binom{52}{13}}} \\ &= \frac{8,892}{20,825} \\ &\approx 0.427 \end{aligned}$$

Question 6: Section 4.1 #18

Let E be the event that test positive, d_i be event that the patient has disease d_i, and $\overline{d_i}$ be event that the patient *does not* have disease d_i, for $i \in \{1,2,3\}$. Additionally, we have the following known probabilities: $P(E \mid d_1) = .8, P(E \mid d_2) = .6, P(E \mid d_2) = .4$, and $P(d_1) = P(d_2) = P(d_3) = 1/3$.

Using these, we can calculate $P(d_i \mid E)$, for $i \in \{1,2,3\}$:

For $P(d_1 \mid E)$, we can express it as

$$P(d_1 \mid E) = \frac{P(d_1 \cap E)}{P(E)}$$

Here,

$$P(d_1 \cap E) = P(d_1) \cdot P(E \mid d_1) = \frac{1}{3} \cdot \frac{8}{10} = \frac{4}{15}$$

To find the probability of testing positive, we have

$$\begin{aligned} P(E) &= P(d_1 \mid E) \cdot P(d_1) + P(d_2 \mid E) \cdot P(d_2) + P(d_3 \mid E) \cdot P(d_3) \\ &= (8/10 \cdot 1/3) + (6/10 \cdot 1/3) + 4/10 \cdot 1/3) \\ &= \frac{3}{5} \end{aligned}$$

Plugging these back into $P(d_1 \mid E)$ we now have

$$P(d_1 \mid E) = \frac{4/15}{3/5} = \frac{4}{9}$$

Similarly for $P(d_2 \mid E)$, we have

$$P(d_2 \mid E) = \frac{P(d_2 \cap E)}{P(E)}$$

Here,

$$P(d_2 \cap E) = P(d_2) \cdot P(E \mid d_2) = \frac{1}{3} \cdot \frac{6}{10} = \frac{1}{5}$$

Plugging this back into $P(d_2 \mid E)$ we now have

$$P(d_2 \mid E) = \frac{1/5}{3/5} = \frac{1}{3}$$

Finally for $P(d_3 \mid E)$, we have

$$P(d_3 \mid E) = \frac{P(d_3 \cap E)}{P(E)}$$

Here,

$$P(d_3 \cap E) = P(d_3) \cdot P(E \mid d_3) = \frac{1}{3} \cdot \frac{4}{10} = \frac{2}{15}$$

Plugging this back into $P(d_3 \mid E)$ we now have

$$P(d_3 \mid E) = \frac{2/15}{3/5} = \frac{2}{9}$$

Question 7: Section 4.1 #22

For a collection of $n = 65$ coins, one coin has two heads while the rest are fair. A coin is then selected at random and tossed 6 times. Let E be the event that an unfair coin is chosen and F the event that a heads turns up 6 times in a row.

Here, $P(F)$ will be given by

$$\begin{aligned} P(F) &= P(\text{6H and fair}) + P(\text{6H and unfair}) \\ &= \left(\frac{1}{2}\right)^6 \cdot \frac{64}{65} + 1 \cdot \frac{1}{65} \\ &= \frac{2}{65} \end{aligned}$$

For the intersection of the two events, we have that

$$P(E \cap F) = P(E) \cdot P(F \mid E) = 1/65 \cdot 1 = 1/65$$

Plugging these in gives us

$$\begin{aligned} P(E \mid F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{1/65}{2/65} \\ &= \frac{1}{2} \end{aligned}$$

Question 8: Section 4.1 #24

Proposition: For a fair coin tossed n times, the conditional probability of a head on any specified trial, given a total of k heads over the n trials, is k/n for $k > 0$.

Proof

Let E be the event that you get a head on the i-th trial and F be the event that you toss k heads in n trials. Then we can express $P(F)$ as a Binomial Distribution, where the random variable X is the number of times have heads when tossing a coin n times:

$$P(F) = P(X = k) = \binom{n}{k} p^k \cdot q^{n-k}$$

Since $p = q = 1/2$, we can rewrite it as

$$P(F) = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

Next we know the intersection of the two events involves the probability of having a head on the i-th trial multiplied by the probability of getting $k-1$ heads on the remaining $n-1$ tosses. Thus we have

$$\begin{aligned} P(E \cap F) &= \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \\ &= \left(\frac{1}{2}\right)^n \cdot \binom{n-1}{k-1} \end{aligned}$$

Combining these, we have

$$\begin{aligned} P(E \mid F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{\left(\frac{1}{2}\right)^n \cdot \binom{n-1}{k-1}}{\left(\frac{1}{2}\right)^n \cdot \binom{n}{k}} \\ &= \frac{(n-1)!}{(k-1)!(n-1-(k+1))!} \cdot \frac{k!(n-k)!}{n!} \\ &= \frac{(n-1)! \cdot k! \cdot (n-k)!}{(k-1)! \cdot (n-k)! \cdot n!} \\ &= \frac{(n-1)! \cdot k!}{(k-1)! \cdot n!} \\ &= \frac{k}{n} \end{aligned}$$

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