MATH 302 — Assignment 6

Stephanie Knill
54882113
Due: March 2, 2016

Question 1: Section 5.1 #21

In a class of 80 students, a professor calls on 1 student chosen at random for a citation in each class period. There are 32 class periods in a term.

(a) Let the random variable X be the number of times a student is called upon in a term. Then the exact probability that a given student is called upon j times during the term is given by

$$\begin{aligned} P(X = j) &= Bin(n, p) \\ &= Bin(32, 1/80) \\ &= \binom{n}{j} p^j \cdot (1-p)^{n-j} \\ &= \binom{32}{j} (1/80)^j \cdot (79/80)^{32-j} \end{aligned}$$

(b) Using the poisson approximation to model X, we have $\lambda = np = 32 \cdot 1/80 = 2/5$. Thus for j times

$$\begin{aligned} P(X = j) \approx Pois(\lambda) &= Pois(2/5) \\ &= e^{-\lambda} \cdot \frac{\lambda^j}{j!} \\ &= e^{-\frac{2}{5}} \cdot \frac{(2/5)^j}{j!} \end{aligned}$$

Similarly for the probability a given student is called upon more than twice

$$\begin{aligned}
P(X > 2) &\approx 1 - P(X = 0) - P(X = 1) \\
&\approx 1 - e^{-\frac{2}{5}} \cdot \frac{(2/5)^0}{0!} - e^{-2/5} \cdot \frac{(2/5)^1}{1!} \\
&\approx 1 - e^{-\frac{2}{5}}(1 - \frac{2}{5}) \\
&\approx 1 - e^{-\frac{2}{5}} \cdot \frac{3}{5} \\
&\approx 0.598
\end{aligned}$$

Question 2: Section 5.1 #23

For a certain experiment, the Poisson distribution with parameter $\lambda = m$ has been assigned. To find the most probable outcome for this experiment, let us first compute the ratio of successive probabilities:

$$\begin{aligned}
\frac{P(X = k + 1)}{P(X = k)} &= \frac{e^{-\lambda} \cdot \lambda^{k+1}}{(k + 1)!} \cdot \frac{k!}{e^{-\lambda} \cdot \lambda^k} \\
&= \frac{\lambda}{k + 1}
\end{aligned}$$

For $k = \lambda - 1$, we have that

$$\frac{P(X = (\lambda - 1) + 1)}{P(X = \lambda - 1)} = \frac{P(X = \lambda)}{P(X = \lambda - 1)} = \frac{\lambda}{(\lambda - 1) + 1} = 1$$

Thus the successive probabilities $k = \lambda - 1$ to $k = \lambda$ are equal, which means the most probable (maximal) outcome of the experiment $k \in \mathbb{Z}$ lies within the range $\lambda - 1 \leq k \leq \lambda$. Since k is an integer, the most probable outcome is $k = \lfloor \lambda \rfloor$

Two most probable values for k can only occur if λ is also an integer. In this case, we have two integer values for k that lie within the most probable range. In addition to the maximal outcome $k = \lfloor \lambda \rfloor$, we also have the other integer value $k = \lfloor \lambda - 1 \rfloor$

Question 3: Section 5.1 #26

Let the random variable X be the number of hits on a square. Let the total number of hits be $n = 537$ and the probability of being hit $p = \frac{1}{576}$. Assuming the hits are purely random, we can use the Poisson approximation with $\lambda = np = 537 \cdot \frac{1}{576} = \frac{537}{576}$ to find the probability that a particular square would have exactly k hits:

$$\begin{aligned} P(X=k) &\approx Pois(\frac{537}{576}) \\ &\approx e^{-\frac{537}{576}} \cdot \frac{\frac{537}{576}^k}{k!} \end{aligned}$$

Let us now compute the expected number of squares that would have 0, 1, 2, 3, 4, and 5 or more hits. The probability a square has 0 hits is

$$\begin{aligned} P(X=0) &\approx e^{-\frac{537}{576}} \cdot \frac{\frac{537}{576}^0}{0!} \\ &\approx e^{-\frac{537}{576}} \\ &\approx 0.39 \end{aligned}$$

Multiplying this by the number of square n gives us the expected number of squares with 0 hits

$$\begin{aligned} \text{Expected \# Squares} &= P(X=0) \cdot n \\ &\approx 211.39 \end{aligned}$$

Thus the expected number of squares to be hit 0 times is approximately 211.

Although midterm season is no longer upon us, I honestly just don't give a duck. The remaining calculations were done in Python (Figure 1).

We have 211 squares with 0 hits, 197 with 1 hit, 92 with 0 hits, 29 with 3 hits, 7 with 4 hits, and 1 with 5 or more hits. Although only an approximation, they are still comparable to the observed results (229 squares with 0 hits, 211 with 1 hit, 93 with 0 hits, 35 with 3 hits, 7 with 4 hits, and 1 with 5 or more hits).

Question 4: Section 2.2 #1

Let us choose a *random* (i.e. uniform/equiprobable density) real number X from the interval [2,10].

(a) The density function $f(x)$ can be computed as

$$f(x) = \frac{1}{\text{length of interval}} = \frac{1}{10-2} = \frac{1}{8}$$

and the probability of an event E that is a subinterval $[a, b]$ of [2.10] as

$$P(E) = \frac{\text{length of } E}{\text{length of interval}} = \frac{b-a}{8}$$

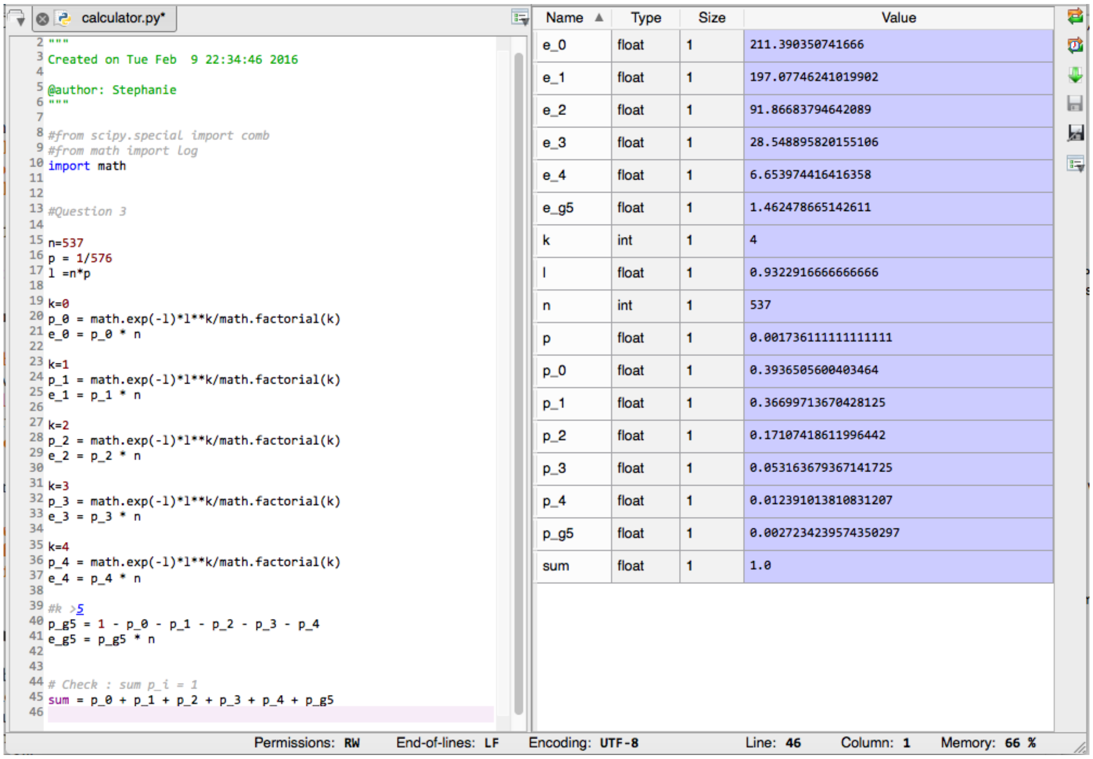

Figure 1: Computation for the expected number of squares that would have 0, 1, 2, 3, 4, and 5 or more hits in Question 3.

(b) For the probability that $X > 5$, we have

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - \frac{5-2}{8} \\ &= \frac{5}{8} \end{aligned}$$

Similarly for $5 < X < 7$

$$\begin{aligned} P(5 < X < 7) &= 1 - P(X \leq 5) - P(X \geq 7) \\ &= 1 - \frac{5-2}{8} - \frac{10-7}{8} \\ &= \frac{1}{4} \end{aligned}$$

For $X^2 - 12X + 35 > 0$, we can simplify this to

$$X^2 - 12X + 35 = (X-7)(X-5) > 0$$

Thus $X > 7$ or $X < 5$. Computing the probability

$$\begin{aligned}
P(5 < X < 7) &= P(X > 7 \cup X < 5) \\
&= P(X > 7) + P(X < 5) - P(X > 7 \cap X < 5) \\
&= P(X > 7) + P(X < 5) \\
&= 1 - P(X \leq 7) + 1 - P(X \geq 5) \\
&= 2 - \frac{7-2}{8} - \frac{10-5}{8} \\
&= \frac{3}{4}
\end{aligned}$$

Question 5: Section 2.2 #2

Let us choose a real number X from the interval [2,10] with a density function of the form

$$f(x) = Cx,$$

where C is a constant.

(a) To compute C, we will use the fact that

$$\int_a^b f(x)\ \mathrm{d}x = 1$$

Here, $[a, b] = [2, 10]$ so we have

$$\begin{aligned}
\int_2^{10} Cx\ \mathrm{d}x &= 1 \\
C \cdot \frac{x^2}{2}\bigg|_{x=2}^{10} &= 1 \\
C &= \frac{1}{48}
\end{aligned}$$

(b) The probability of an event E that is a subinterval $[a, b]$ of [2.10] is given by

$$\begin{aligned}
P(E) &= \int_a^b f(x)\ \mathrm{d}x \\
&= \int_a^b \frac{1}{48}x\ \mathrm{d}x \\
&= \frac{1}{96}x^2\ \bigg|_{x=a}^{b} \\
&= \frac{b^2 - a^2}{96}
\end{aligned}$$

(c) For the probability that $X > 5$, we have

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - \frac{5^2 - 2^2}{96} \\ &= \frac{25}{32} \end{aligned}$$

Similarly for $X < 7$

$$\begin{aligned} P(5 < X < 7) &= 1 - P(X \geq 7) \\ &= 1 - \frac{10^2 - 7^2}{96} \\ &= \frac{15}{32} \end{aligned}$$

For $X^2 - 12X + 35 > 0$, we have

$$\begin{aligned} P(5 < X < 7) &= P(X > 7) + P(X < 5) \\ &= 1 - P(X \leq 7) + 1 - P(X \geq 5) \\ &= 2 - \frac{7^2 - 2^2}{96} - \frac{10^2 - 5^2}{96} \\ &= \frac{3}{4} \end{aligned}$$

Question 6

Let there be players A and B whom initially start with $i = \$10$ and $j = \$5$ respectively, giving us a total of $M = i + j = \$15$. Rolling a 6-sided fair die, player A is paid \$1 from player B if it is a 1 or 2, otherwise player A pays \$1 to player B. So we have the probability of success for player A to be $p = 2/6 = 1/3$. Let P_i be the probability player A wins if he or she has initially $i\$$ and $l_i = P_i - P_{i-1}$. To find an equation for P_i, we first compute l_i to be

$$
\begin{aligned}
P_i &= p \cdot P_{i+1} + (1-p) \cdot P_{i-1} \\
p \cdot P_i + (1-p) \cdot P_i &= p \cdot P_{i+1} + (1-p) \cdot P_{i-1} \\
(1-p) \cdot [P_i - P_{i-1}] &= p \cdot [P_{i+1} - P_i] \\
(1-p) \cdot l_i &= p \cdot l_{i+1} \\
l_{i+1} &= \frac{1-p}{p} \cdot l_i \\
l_i &= \frac{1-p}{p} \cdot l_{i-1} \\
l_i &= \alpha \cdot l_{i-1}
\end{aligned}
$$

where $\alpha = \frac{1-p}{p}$. Since P_i can also be defined as the summation of all intervals up to l_i, we have

$$
\begin{aligned}
P_i &= l_1 + l_2 + \ldots + l_i \\
&= l_1 + \alpha \cdot l_1 + \ldots + \alpha^{i-1} \cdot l_1 \\
&= l_1 \cdot (1 + \alpha + \ldots + \alpha^{i-1}) \\
&= l_1 \cdot \frac{\alpha^i - 1}{\alpha - 1} \\
&= l_1 \cdot \frac{1 - \alpha^i}{1 - \alpha}
\end{aligned}
$$

Since α is known, let us compute the last remaining unknown, l_1. Using our boundary condition $P_M = 1$ gives us

$$
\begin{aligned}
P_M &= l_1 \cdot \frac{1-\alpha^M}{1-\alpha} \\
1 &= l_1 \cdot \frac{1-\alpha^M}{1-\alpha} \\
l_1 &= \frac{1-\alpha}{1-\alpha^M}
\end{aligned}
$$

Substituting back into P_i gives us the desired equation

$$
\begin{aligned}
P_i &= \frac{1-\alpha}{1-\alpha^M} \cdot \frac{1-\alpha^i}{1-\alpha} \\
&= \frac{\alpha^i - 1}{\alpha^M - 1}
\end{aligned}
$$

Here, we have $i = 10$, $M = 15$, and $\alpha = \frac{1-p}{p} = \frac{2/3}{1/3} = 2$. Plugging this in

$$P_i = \frac{2^{10} - 1}{2^{15} - 1} \approx 0.0312$$

gives us the probability that player A wins all his or her adversary's money to be approximately 3%.