MATH 302 — Assignment 7

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Question 1: Section 5.2 #2

Let us choose a number U from the unit interval [0,1] with uniform distribution. Then we can find the cumulative distribution and density for the random variables:

(a) $Y = U + 2$

Here, range[Y] $= [1/1, 1/2] = [1/2, 1]$ So the cumulative distribution function in the range[Y] is

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P\Big(\frac{1}{U+1} \leq y\Big) \\
&= P(1 \leq y(U+1)) \\
&= P(U + 1 \geq 1/y) \\
&= P(U \geq 1/y - 1) \\
&= 1 - P(U \leq 1/y - 1) \\
&= 1 - (1/y - 1) \\
&= 2 - 1/y
\end{aligned}$$

Thus the cumulative distribution function is given by

$$F_Y(y) = \begin{cases} 0, & x < 1/2 \\ 2 - 1/y, & x \in [1/2, 1] \\ 0, & x > 1 \end{cases}$$

and the probability density funciton

$$f_Y(y) = \begin{cases} 1/y^2, & x \in [1/2, 1] \\ 0, & x \notin [1/2, 1] \end{cases}$$

(b) $Y = ln(U + 1)$

Here, range[Y] = $[ln(1), ln(2)] = [0, ln(2)]$ So the cumulative distribution function in the range[Y] is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(ln(U+1) \leq y) \\ &= P(e^{ln(U+1)} \leq e^y) \\ &= P(U + 1 \geq e^y) \\ &= P(U \geq e^y - 1) \\ &= e^y - 1 \end{aligned}$$

Thus the cumulative distribution function is given by

$$F_Y(y) = \begin{cases} 0, & x < 0 \\ e^y - 1, & x \in [0, ln(2)] \\ 0, & x > ln(2) \end{cases}$$

and the probability density funciton

$$f_Y(y) = \begin{cases} 1/y^2, & x \in [1/2, 1] \\ 0, & x \notin [1/2, 1] \end{cases}$$

Question 2: Section 5.2 #14

A point P in the unit square has coordinates X and Y chosen at random in the interval [0,1]. Let D be the distance from P to the nearest edge of the square and E the distance to the nearest corner. Then we can compute the following probabilities:

(a) $D < 1/4$

$$\begin{aligned} P(D < 1/4) &= 1 - P(D \geq 1/4) \\ &= 1 - \frac{\text{Area}(E)}{\text{Area(unit square)}} \end{aligned}$$

Here, the E is the event $\{D \geq 1/4\}$. This is the square with edge length

$$1 - (1/4 \cdot 2) = 1/2$$

substituting this in

$$\begin{aligned} P(D < 1/4) &= 1 - \frac{1/2 \cdot 1/2}{1 \cdot 1} \\ &= 3/4 \end{aligned}$$

(b) $E < 1/4$

$$P(E < 1/4) = \frac{\text{Area}(E)}{\text{Area(unit square)}}$$

Here, the F is the event $\{E \geq 1/4\}$. This is 4 quarter circles with radius $r = 1/4$:

$$\text{Area}(E) = 4 \cdot \frac{\pi r^2}{4} = \pi \cdot (1/4)^2 = \frac{\pi}{16}$$

substituting this in

$$\begin{aligned} P(D < 1/4) &= \frac{\frac{\pi}{16}}{1 \cdot 1} \\ &= \frac{\pi}{16} \end{aligned}$$

Question 3: Section 5.2 #16

Let X be a random variable with density function

$$f_X(x) = \begin{cases} cx(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) To find the value of c, let us use the property that

$$\begin{aligned} \int_0^1 f_X(x)dx = \int_0^1 cx(1-x)dx &= 1 \\ c\int_0^1 x - x^2 dx &= 1 \\ 1/6 \cdot c &= 1 \\ c &= 6 \end{aligned}$$

(b) The cumulative density function is given by

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{0} 6t(1-t)dt \\ &= 6\Big[\frac{x^2}{2} - \frac{x^3}{3}\Big] \\ &= 3x^2 - 2x^3 \\ &= x^2(3-2x) \end{aligned}$$

(c) The probability $\mathrm{P}(X < 1/4)$ can be computed as

$$\begin{aligned} P(X \le x) &= P(X \in [0, 1/4]) \\ &= F(1/4) \\ &= (1/4)^2(3 - 2(1/4)) \\ &= \frac{5}{32} \end{aligned}$$

Question 4: Section 5.2 #17

Let X be a random variable with cumulative distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ sin^2(\pi x/2), & 0 \le x \le 1 \\ 1, & 1 < x \end{cases}$$

(a) The density function is given by

$$\begin{aligned} f_X(x) &= F_X'(x) \\ &= \begin{cases} \pi \cdot sin(\pi x/2) \cdot cos(\pi x/2), & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases} \end{aligned}$$

(b) The probability $\mathrm{P}(X < 1/4)$ can be computed as

$$\begin{aligned} P(X \leq x) &= P(X \in [0, 1/4]) \\ &= F(1/4) \\ &= sin^2(\pi/2 \cdot 1/4) \\ &= \frac{1}{4}(2 - \sqrt{2}) \\ &\approx 0.146 \end{aligned}$$

Question 5: Section 5.2 #18

Let X be a random variable with cumulative distribution function $F_X(x)$, and let $Y = X+b$, $Z = aX$, $W = aX + b$.

In the case where $a > 0$, we have that

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X + b \leq y) \\ &= P(X \leq y - b) \\ &= F_X(y - b) \end{aligned}$$

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(aX \leq z) \\ &= P(X \leq \frac{z}{a}) \\ &= F_X\left(\frac{z}{a}\right) \end{aligned}$$

$$\begin{aligned} F_W(w) &= P(W \leq w) \\ &= P(aX + b \leq w) \\ &= P(X \leq \frac{w - b}{a}) \\ &= F_X\left(\frac{w - b}{a}\right) \end{aligned}$$

Question 6: Section 5.2 #19

Let X be a random variable with cumulative distribution function $F_X(x)$, and let $Y = X+b$, $Z = aX$, $W = aX + b$.

In the case where $a > 0$, we have that

$$\begin{aligned} f_Y(y) &= \int F_Y(y)\,dy \\ &= \int F_X(y-b)\,dx \\ &= f_X(y-b) \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \int F_Y(z)\,dz \\ &= \int F_X\Big(\frac{z}{a}\Big)\,\frac{1}{a}\cdot dx \\ &= \frac{1}{a}\cdot f_X\Big(\frac{z}{a}\Big) \end{aligned}$$

$$\begin{aligned} f_W(w) &= \int F_W(w)\,dw \\ &= \int F_X\Big(\frac{w-b}{a}\Big)\,\frac{1}{a}\cdot dx \\ &= \frac{1}{a}\cdot f_X\Big(\frac{w-b}{a}\Big) \end{aligned}$$

Question 7: Section 5.2 #19

Let X be a random variable uniformly distributed over $[c,d]$and $Y = aX + b$. Let us find a, b such that Y is uniformly distributed over $[0.1]$.

Since range$[X] = [c,d]$, then range$[Y] = [ac+b, ad+b] = [0,1]$. Thus we have a system of equatinos

$$\begin{aligned} ac + b &= 0 \qquad (1) \\ ad + b &= 1 \qquad (2) \end{aligned}$$

Subtracting equation (2) from equation (1)

$$\begin{aligned} ac + b - (ad+b) &= 0 - 1 \\ a(c-d) &= -1 \\ a &= \frac{1}{d-c} \end{aligned}$$

Solving now for b

$$\begin{aligned} b &= -ac \\ &= \frac{c}{d-c} \\ &= \frac{c}{d} - 1 \end{aligned}$$

Question 8: Section 5.2 #21

I have no ducking clue. Pls don't mark.