MATH 442 — Assignment 11

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Question 61

Let $n \geq 3$

(a) *Show the number of pairs (T, e) where T is a spanning tree of K_n and e is an edge of T is $n^{n-2}(n-1)$.*

By our in-class Corollary 6, we know that the number of spanning trees in K_n is $T = n^{n-2}$. By Homework 9, we know that the number of edges e in a tree of n vertices is given by $e = n - 1$. Multiplying these together gives us $n^{n-2}(n-1)$, the number of pairs (T, e).

(b) *Show that the number of spanning trees of K_n containing a specific edge is $2n^{n-2}$.*

To find the number of spanning trees of K_n containing a specific edge, we must take the number of pairs (T, e) and multiply this by the probability of selecting any edge in K_n. Since there are $\frac{n(n-1)}{2}$ edges in K_n, the probability of selecting any edge is $\frac{2}{n(n-1)}$. Multiplying this by the result found in part a) gives us

$$n^{n-2}(n-1) \cdot \frac{2}{n(n-1)} = \frac{2n^{n-2}}{n} = 2n^{n-3}$$

(c) *Show that for some edge e in K_n, we have $\tau(K_n - e) = (n-2)n^{n-3}$*

Deleting any edge e from K_n and calculating the number of spanning trees of the resulting graph $K_n - e$ is equivalent to calculating all possible spanning trees of K_n

and subtracting the number of spanning trees that contain edge e. Thus we have

$$\begin{aligned}\tau(K_n - e) &= \tau(K_n) - 2n^{n-3} \\ &= n^{n-2} - 2n^{n-3} \\ &= (n-2)n^{n-3}\end{aligned}$$

Question 62

For the following graphs Figure 1, we may start at the vertex labelled 1 and conduct a Depth First Search (blue) and a Breadth First Search (red)

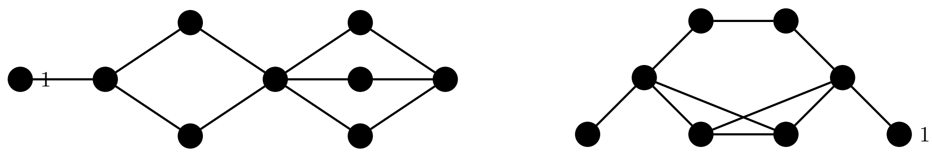


Figure 1: Depth First Search and Breadth First Search of various graphs.

Question 63

For the weighted graph G (Figure 2), we can compute the shortest distance from vertex A to each vertex (Table 1).

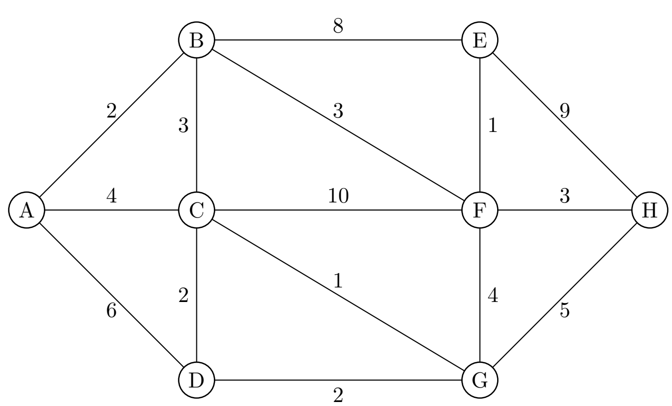


Figure 2: Shortest paths in a weighted graph G.

Start & End	**Distance**	**Path**
$A \to A$	**0**	
$A \to B$	**2**	AB
$A \to C$	**4**	AC
$A \to D$	**6**	AD **or** ACD
$A \to E$	**6**	$ABFE$
$A \to F$	**5**	ABF
$A \to G$	**5**	ACG
$A \to H$	**8**	$ABFH$

Table 1: Table of shortest path distance between vertex A and the remaining vertices in graph G.

Question 64

Assign integer weights to the edges of K_n. Prove that the total weight on every cycle is even if and only if the total weight on every triangle (i.e. cycle of length 3) is even.

Proof. $\Rightarrow$ Assume that the total weight of every cycle is even. Since a triangle is a cycle of length 3, then the total weight is also even.

$\Leftarrow$ Assume that the total weight on every triangle is even. We will do an induction over the size m of a cycle.

Base Case: For a cycle of size $m = 3$, we have a triangle. By our initial assumption, our cycle has an even total weight.

Induction Step: Assume that the statement holds true for cycles of size $3 < m < k$. For a cycle C of size $m = k$, let us choose any path of length 3 from vertices u to v, traversing edges a and b. Deleting edges a and b from our cycle and joining vertices u and v by a new edge c, we have by our induction assumption a cycle D of even weight. Since we also have a triangle of edges a, b, and c we know that this cycle is also of even weight. Thus the parity of the weights $W(a) + W(b)$ and $W(c)$ must be the same. Therefore the cycles D and C also have weights of the same parity, so C is of even weight.

Conclusion: By the principle of induction, the statement is true for all $n \geq 3$.

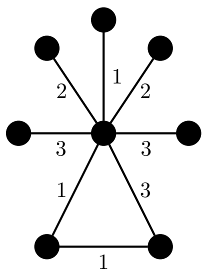
■

Question 65

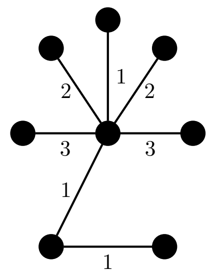
Find a simple connected graph and assign the edge weights (1,1,1,2,2,3,3,3) in two ways: one way so the minimal spanning tree is unique and another way so it is not unique

Minimal Spanning Tree Unique

Here our simple connected graph G is depicted in Figure 3a and its unique minimal spanning tree in Figure 3b.



(a) Graph G



(b) Unique minimal spanning tree of G.

Minimal Spanning Tree Not Unique

Here our simple connected graph H is depicted in Figure 4 and its non-unique minimal spanning trees in Figure 5.

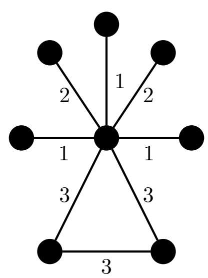


Figure 4: Graph H

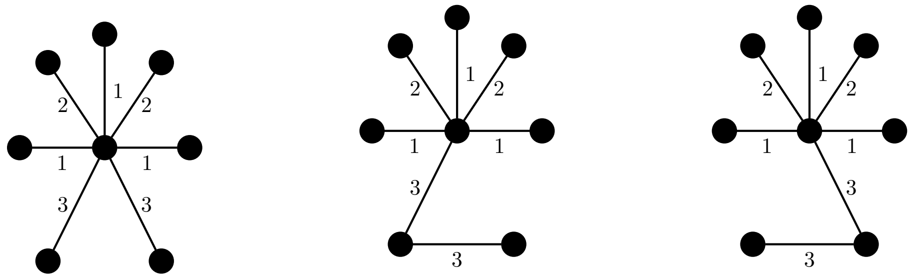


Figure 5: Minimal spanning trees of H.

Question 60

A minimal spanning tree T for the graph is

Here, the weight is

$$W(T) = 1 \cdot (6) + 2 \cdot (2) + 3 \cdot (4) = 22$$