MATH 442 — Assignment 7

Stephanie Knill
54882113
Due: March 3, 2016

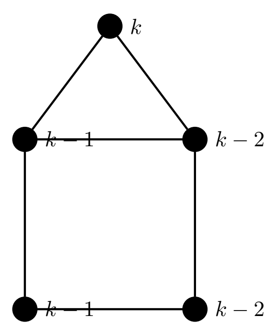
Question 37

Multiplying the colour assignments of the vertices in Figure 1a gives us the chromatic polynomial for our graph G:

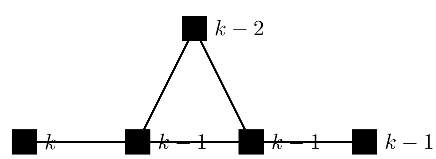
$$P_G(k) = k \cdot (k-1)^2 \cdot (k-2)^2$$

By similar procedure, we have that the chromatic number of Graph G' (Figure 1b) is

$$P_{G'}(k) = k \cdot (k-1)^3 \cdot (k-2)$$



(a) Graph G



(b) Graph G'

Figure 1: Graphs G and G' where each vertex is labelled by the number of ways it can be coloured k colours.

Question 38

For the complete graph K_n, where $n \geq 2$, we have that each vertex is adjacent to every other vertex, thereby giving a chromatic number of

$$\chi(K_n) = n$$

If we remove an edge e joining vertices v_1 and v_2 from our complete graph, then these vertices are no longer adjacent in our new graph $K_n - e$. Since we can now colour v_1 and v_2 the same colour, the chromatic number of $K_n - e$ is equivalent to the chromatic number of a complete graph of $n-1$ vertices:

$$\begin{aligned}\chi(K_n - e) &= \chi(K_{n-1}) \\ &= n - 1\end{aligned}$$

Question 39

Conjecture: The chromatic polynomial of any *tree* (a connected graph that contains no cycles) with n vertices is $k(k-1)^{n-1}$.

Proof. We will proceed by strong induction on the number of vertices n.

Base Case: $n = 1$

Here we have a null graph of 1 vertex, which has a chromatic polynomial of $k^n = k^1 = k$. Using our formula for the chromatic polynomial of a tree with $n = 1$ vertices likewise gives us

$$\begin{aligned}P_G(k) &= k(k-1)^{n-1} \\ &= k(k-1)^0 \\ &= k\end{aligned}$$

Induction Step: assume that all trees with up to $n-1$ vertices has a chromatic polynomial of $k(k-1)^{n-1}$.

For our tree of n vertices, we know there must be at least one vertex v of degree 1, otherwise there would be a cycle. Let us remove this vertex v. Then we are left with a graph G' of $n-1$ vertices and we have

$$P_{G'}(k) = k(k-1)^{n-2}$$

by our induction assumption. Adding our vertex v back in, we can assign it $k-1$ possible colours. Multiplying this by the chromatic polynomial of G' gives us the chromatic polynomial of the graph G of n vertices

$$\begin{aligned} P_G(k) &= P_{G'}(k) \cdot (k-1) \\ &= k(k-1)^{n-2} \cdot (k-1) \\ &= k(k-1)^{n-1} \end{aligned}$$

■

Question 40

The windmill graph $Wd(n, N)$ on $N(n-1)+1$ vertices is the connected simple graph formed by taking N copies of K_n and joining them at a common vertex.

Proposition: The chromatic polynomial of the windmill graph $Wd(n, N)$ is

$$P_{Wd(n,N)} = k \prod_{i=1}^{n-1} (k-i)^N$$

Proof. For our windmill graph, let us colour our common vertex k colours.

Now let us look at each K_n copy separately. Since we have already coloured one of the vertices in it, we will only examine the remaining $n-1$ vertices. Here the next vertex can be coloured $k-1$ ways, the next $k-2$ ways, ... , and the $(n-1)$ vertex $k-(n-1)$ ways. Thus the chromatic polynomial of one K_n copy is

$$(k-1)(k-2) \cdot \ldots \cdot (k-(n-1)) = \prod_{i=1}^{n-1} (k-i)$$

However, we have N copies of each of these components, so we must multiply the chromatic number of each copy by the chromatic number of all other copies.

$$\prod_{i=1}^{n-1} (k-i)^N$$

Multiplying this by our common vertex coloured k ways gives us the chromatic polynomial of our windmill graph

$$P_{Wd(n,N)} = k \prod_{i=1}^{n-1} (k-i)^N$$

■

Question 41

Proposition: Let G be a simple graph with n vertices. Prove that the coefficient in $P_G(k)$ of k^n is 1 and of k^{n-1} is $-|E(G)|$.

Proof. We will proceed by induction on the number of edges m.

Base Case: $m = 0$

Here we have a null graph of n vertices. Thus the chromatic polynomial of the null graph G of n vertices is

$$P_G(k) = k^n$$

Here the coefficient of k^n is 1 and the coefficient of k^{n-1} is $0 = -|E(G)|$.

Induction Step: assume the proposition holds true for less than m edges, where $m > 0$. Let G be a graph with m edges. Using the Deletion-Contraction Theorem, we will choose any edge e in our graph G to delete and contract, which gives us

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

Coefficient of k^n

Since $P_{G-e}(k)$ has less than m edges, then by the induction assumption we know that the coefficient of k^n in $P_{G-e}(k)$ is 1. Since we contracted an edge in $P_{G/e}(k)$, we only have $n-1$ vertices. Thus the highest degree is k^{n-1}, which means the coefficient of k^n is 0. Taking the difference of these gives us 1, which is the coefficient of k^n in $P_G(k)$

Coefficient of k^{n-1}

Again, $P_{G-e}(k)$ has less than m edges, so by the induction assumption the coefficient of k^{n-1} in $P_{G-e}(k)$ is $-|E(G-e)|$. Similarly, since $P_{G/e}(k)$ has $n-1$ vertices the highest degree is k^{n-1}, which has a coefficient of 1. Taking the difference between the two gives us the coefficient of k^{n-1} in $P_G(k)$:

$$\begin{aligned} -|E(G-e)| - 1 &= -(|E(G)| - 1) - 1 \\ &= -|E(G)| \end{aligned}$$

■

Question 42

Let G be a simple graph. Then the chromatic polynomial $P_G(k)$ is the product of the chromatic polynomials of its components.

Proof. Let G be a simple graph.

Case 1: G is connected

Since G consists of 1 component, then the chromatic polynomial of this one component is the chromatic polynomial of the graph G.

Case 2: G is not connected

Since G is a union of disjoint components $C_1, C_2, \ldots, C_n$, then—similar to the Null Graph—no two *components* are adjacent. Thus we can colour C_1 the chromatic polynomial of that component, C_2 the chromatic polynomial of that polynomial, ... , and C_n the chromatic polynomial of the component. So the total number of colourings is

$$P_{C_1}(k) \cdot P_{C_2}(k) \cdot \ldots \cdot P_{C_n}(k) = P_G(k)$$

which is the chromatic polynomial of the graph G.

■