MATH 442 — Assignment 10

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Due: March 24, 2016

Question 55

If G is a simple connected graph, then for any edge in e in G

$$\tau(G) = \tau(G - e) + \tau(G/e) \tag{1}$$

Proof. no ducking clue ■

Question 51

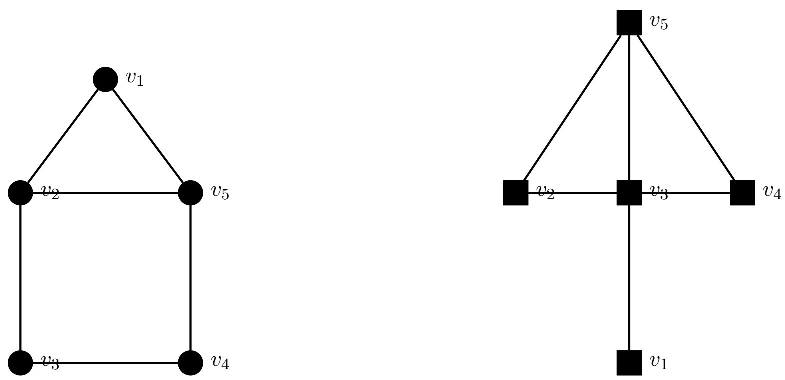
For the graph G of vertices $v_1, v_2, \dots v_5$ in Figure 1a, let us determine the number of non-isomorphic ways to label it. Here, we can label v_1 5 different ways. For the vertex pair v_2 and v_5, we can label them $\binom{4}{2} = 6$ ways, Lastly we can label v_3 2 different ways (due to isomorphism, v_4 simply takes the remaining label). This gives us a total of

$$5 \cdot 6 \cdot 2 = 60$$

non-isomorphic ways to label graph G. For the graph H of vertices $v_1, v_2, \dots v_5$ in Figure 1b, we can label v_1 5 ways, v_3 4 ways, and v_5 3 ways. Since v_2 and v_4 are symmetrical about the y-axis (i.e. same degree), then labelling them the remaining labels does not create any new non-isomorphic graphs. This gives us a total of

$$5 \cdot 4 \cdot 3 = 60$$

non-isomorphic ways to label graph H.



(a) Graph G **(b)** Graph H

Figure 1: Graphs G and H.

Question 57

(a) For the labelled graph (Figure 2), the associate Prüfer sequence is (5,2,2,6,2,2).

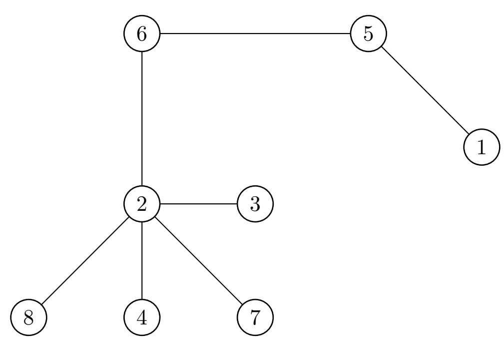


Figure 2: Labelled graph G with associated Prüfer sequence (5,2,2,6,2,2).

(b) For the Prüfer sequence (3,4,8,1,8,8,8), the associated labelled graph H can be seen in Figure 3.

Question 58

A vertex in a labelled graph has degree k if and only if its label appears $k-1$ times in the Prüfer sequence of the graph.

Proof. $\Rightarrow$ Assume that a vertex v in a labeled graph has degree k. Here, we take a leaf

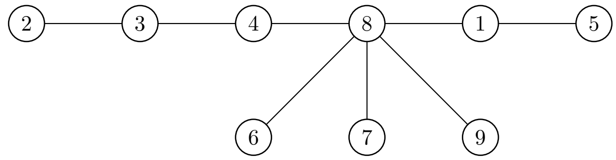


Figure 3: Labelled graph H with associated Prüfer sequence (3,4,8,1,8,8,8).

and put its adjacent vertex in the Prüfer sequence and then delete that leaf. We have two cases for our vertex v of degree k

Case 1: v is a leaf. Then it is never in the Prüfer sequence. Here $k = deg(v) = 1$ and the number of occurrences is $k - 1 = 0$.

Case 2: v is a non-leaf. We add it to the Prüfer sequence and minus its degree by 1. This continues until either it is a leaf (deleted) or it is one of the last two leaves (do not add to Prüfer sequence). Either way, it is only added to the Prüfer sequence $k - 1$ times.

$\Leftarrow$ Assume that a vertex label appears $k - 1$ times in Prüfer sequence of a graph. Then this vertex label was a non-leaf adjacent to a deleted leaf $k-1$ times until it became a leaf itself. Thus it had original degree $(k - 1) + 1 = k$. ■

Question 59

There does not exist a tree consisting of 12 vertices, with one vertex of degree 5, two vertices of degree 4, and one vertex of degree 2.

Proof. Assume to the contrary that such a graph exists. Since we cannot form cycles, let us connect these four vertices of degree 5, 5, 5, and 2 together. Without loss of generality, let us attach the vertices of degree 4 and 2 to the vertex of degree 5 (Figure 4).

Since this is a tree, none of the edges from these four vertices can connect back to v_1, v_2, v_3 or v_4. Thus we must attach a new vertex on the ends of each of them. However we have 9 edges which will result in a total of 13 vertices in T, thereby giving us the necessary contradiction. ■

Question 60

How many labelled trees exist with 6 vertices such that the degree of ever vertex is 1 or 3?

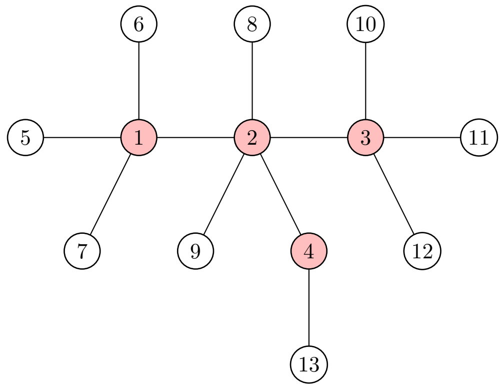


Figure 4: Graph of tree T with vertices v_1 and v_3 of degree 4, v_2 of degree 5, and v_4 of degree 2 (denoted by pink).

Since we can only have vertices of degree 1 or 3, this gives us one possible configuration: two vertices of degree 3 attached with vertices of degree 1 on the remaining edges (Figure 5).

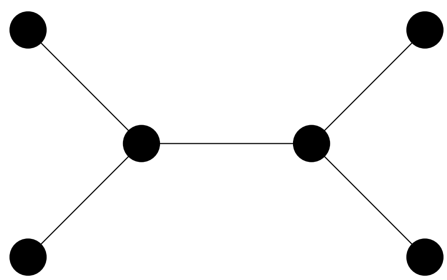


Figure 5: Graph of 6 vertices with vertices of degree 1 or 3.

ask stephanie: isomorphic or non-isomorphic???