MATH 442 — Assignment 9

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Question 49

From our lecture timetable, we can construct a graph G (Figure 1) where each vertex represents a lecture period and two lecture periods are adjacent if they must not coincide.

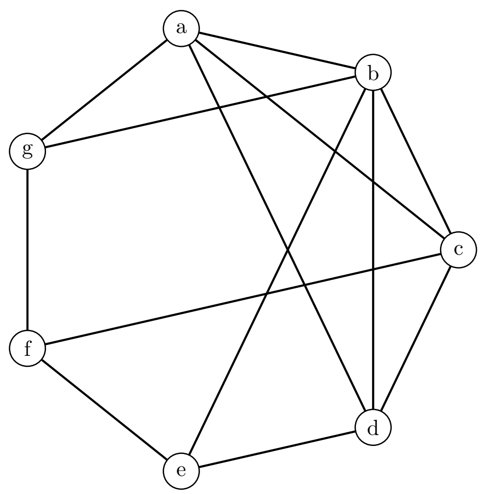

Figure 1: Graph G where each vertex represents a lecture period and two lecture periods are adjacent if they must not coincide.

To determine the minimum number of lecture periods needed to timetable all 7 lectures, we need to calculate the chromatic index $\chi(G)$—the minimum number of colours needed to colour the vertices of G. Using Brooke's Theorem, the largest degree $\Delta = 5$, so G is 5-colourable. Now that we have estabilished an upper bound on $\chi(G)$, let us see if we can do better. Using the 4-colour Theorem, we know that if G is planar, then G is 4-colourable. Using Glasgow's Algorithm (Figure 2)

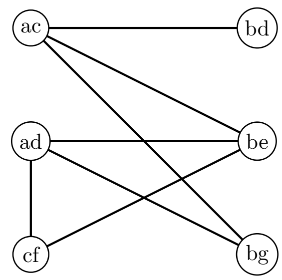

Figure 2: Graph H generated from G using Glasgow's Algorithm.

We see there exists a cycle of length 3 $ad \to be \to cf \to ad$, thus H is not bipartite and G is not planar. Since we cannot do better, G is 5-chromatic and we need 5 lecture periods.

Question 50

There are five teams playing in a tournament, each playing the other four teams once and with two pitches available. Let us express our tournament as the connected graph K_5 of number of vertices $n = 5$. To determine the number of periods needed to schedule all the matches, we need to find the minimum number of colours required to colour the edges, or $\chi'(K_5)$. Since n is odd, we have that $\chi'(K_5) = n = 5$. So we need 5 periods if we have an unlimited number of pitches to play on. However, having an unlimited or 2 pitches is the same as a team cannot play 2 matches simultaneously. This means we can have at most 2 games played in a single period, thus we need 5 periods.

Question 51

A simple connected graph T is a tree if and only if $v = e + 1$.

Proof. $\Rightarrow$ Assume the simply connected graph T is a tree. Since it is planar, we can apply Euler's Theorem to it

$$v - e + f = 2$$

Here, no cycles exist, so we only have the infinite face and $f = 1$. Substituting this in gives us

$$\begin{aligned} v - e + 1 &= 2 \\ v &= e + 1 \end{aligned}$$

$\Leftarrow$ Assume to the contrary that the graph T has $v = e + 1$ and it is not a tree. Thus we must have a cycle in T. To connect our v vertices into a cycle, we need at least v edges. However, we only have $e = v - 1$ edges, thereby giving us the necessary contradiction. ■

Question 52

Prove that a simple connected graph T is a tree if and only if adding an edge between two existing vertices of T creates exactly one cycle.

Proof. $\Rightarrow$ Assume that T is a tree. Then there exists a path between any 2 vertices v_i and v_j. If we join vertex v_i and v_j we will form a cycle of the form $v_i \to \cdots \to v_j \to v_i$. By Lemma, this is the only cycle formed.

Lemma Assume that adding an edge e in a tree creates more than 1 cycle. Adding e between v_i and v_j forms at least 2 cycles (Figure 3)

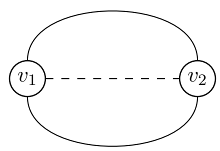

Figure 3: Cycles formed from adding edge e dashed between vertices v_i and v_j.

However, this implies that a cycle already existed in T of the form $v_i \to \cdots \to v_j \to \cdots \to v_i$, which contradicts the definition of a tree.

$\Leftarrow$ Assume that adding an edge e between any two existing vertices of T creates exactly one cycle. Then we have 3 cases. In the first case, we added e to the same vertex, forming a loop. This cycle would result in any graph. Similarly in the case where we add e to any two adjacent vertices, we form multiple edges, thus again resulting a cycle. Again, this can occur in any graph.

Let us examine the final case where we add an edge e between any two non-adjacent vertices v_i and v_j. To form a cycle, then there must exist exactly 1 path between v_i and v_j. Otherwise, if there were none, we would not form a cycle. If there were more than 1 path, a cycle would have already existed involving v_i and v_j and we would form more than 1 cycle. By definition then T must be a tree. ■

Question 53

Prove that a forest with n vertices and $k < n$ has $n - k$ connected components.

Proof. Since a forest is a union of trees $T_1, T_2, \ldots, T_m$, where each tree T_i has number of vertices $n_i = k_i + 1$ by Lemma. So we have that $n_i - k_i = 1$. Let $n = n_1 + \ldots + n_m$ and $k = k_1 + \ldots k_m$, where m is the number of components. Then

$$\begin{aligned} m &= 1 + 1 + \ldots + 1 \\ &= (n_1 - k_1) + (n_2 - k_2) + \ldots + (n_m - k_m) \\ &= n - k \end{aligned}$$

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Lemma: Any tree with n vertices contains precisely $n - 1$ edges.

We will proceed by induction over the number of vertices n.

Base Case: When $n = 1$, we have the null graph N_1 of 0 edges.

Induction Step: Let $k \in \mathbb{N}$ be given and suppose the statement is true for $n = k$ vertices. Let G denote the tree of k vertices that has $k - 1$ edges by the induction assumption. Let us add a vertex v to G. Since the resulting graph must be connected, we have two cases:

Case 1: Join v to G with 1 edge. Here we have $k + 1$ vertices and k edges, so we are done.

Case 2: Join v to G with at least 2 edges. Since there exists a path between any two vertices v_i and v_j in G, if we join vertex v to both v_i and v_j we will form a cycle of the form

$$v \to v_i \to \cdots \to v_j \to v$$

which is not allowed.

Thus, the statement holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction the statement is true for all n.

Question 54

Let T be a tree with no vertices of degree 2. Prove that T has more leaves than non-leaf vertices.

Proof. We will proceed by induction over the number of vertices n.

Base Case: When $n = 2$, we have 2 leaves and 0 non-leaf vertices.

Induction Step: Assume true for $2 < n \leq k$ vertices. Let G be such a graph of k vertices. Let us add a vertex v to our graph to form G' of $n = k + 1$ vertices. Since no vertex can be of degree 2, then we cannot add v to a leaf. Adding v to a non-leaf means we have the same number of non leafs and an additional leaf. Since be the induction assumption we have more leaves than non-leaves in G, then G' also has more leaves.

Conclusion: By the principle of induction the statement is true for all n. ■