MATH 442 — Assignment 8

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Question 43

Let G be a simple graph with at least one edge. Then the sum of the coefficients of the chromatic polynomial $P_G(k)$ is 0.

Proof. We will proceed by induction on the number of vertices v.

Base Case: For a simple graph G of $v = 2$ vertices with at least one edge (Figure 1) we have

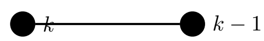

Figure 1: Base case of $v = 2$ vertices.

the chromatic polynomial $P_G(k) = k \cdot (k-1) = k^2 - k$. Here the coefficients are 1 and -1, which sum to 0.

Induction Step: Let us assume the statement holds true for $v = k$ vertices. Then for a graph G of k vertices, we have its chromatic polynomial $P_G(k)$ coefficients sum to 0.

For a graph G' of $v = k + 1$ vertices, the chromatic polynomial is

$$P_{G'}(k) = P_G(k) \cdot (k - i)$$

where i represents the number of edges between our new vertex and our graph G. By the induction assumption, the coefficients of $P_G(k)$ sums to 0. Multiplying this by $(k - i)$ will similarly yield the sum of coefficients to be 0. Thus, the statement holds true for $v = k+1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, the statement is true for all $n \in \mathbb{N}$. ■

Question 44

Try to prove the four colour theorem by adapting the proof of the five colour theorem from class

Proof. We will proceed by induction on the number of vertices V in a graph G.

Base Case: For $V = 1$ we have a single vertex, which is 4-colourable.

Induction Step: Assume all simple connected planar graphs with up to $n - 1$ vertices are 4-colourable.

Suppose G is a simple graph connected planar graph with n vertices. Then we know that G has a vertex v of degree less than or equal to 5.

Let us examine the case where $\deg(v) < 4$. Here, the proof follows exactly with the in-class proof and we find that G is 4-colourable

For the case of $\deg(v) = 4$, we cannot say that all vertices surrounding v are *not* adjacent, since the non-planar subgraph K_5 does not arise. Since our graph will remain planar even if all vertices are adjacent (we would have the planar K_4 subgraph) our five colour theorem proof breaks down here when we try to extend it to the four colour theorem. ∎

Question 45

A graph G is k-critical if $\chi(G) = k$ and the deletion of any vertex yields a graph with a smaller chromatic number.

Proposition: If G is k-critical, then every vertex has degree at least $k - 1$.

Proof. Assume to the contrary that G is k-critical and there exists a vertex v of at most $k - 2$ degree. Since G is k-critical, the graph $G - v$ can be coloured with $k - 1$ colours. Inserting vertex v back in, we see that at least one of the $k-1$ colours is not adjacent to v. Colouring v this gives us a $k-1$ colouring of the graph G, thereby giving us the necessary contradiction.

∎

Question 46

An example graph that is both 3-colourable(f) and 3-colourable is the null graph of one vertex N_1 (Figure 2).

Figure 2: Null graph N_1 which is both 3-colourable(f) and 3-colourable.

Question 47

For the edge colouring of the hypercube Q_k of k colours, the chromatic index is

$$\chi'(Q_k) = k$$

Proof. We will proceed by induction on k.

Base Case: For $k = 1$, we have Q_1 which consists of 2 vertices joined by a single edge.

Figure 3: Hypercube Q_1.

Here, we can colour our edge with 1 colour, namely pink[1].

Induction Step: assume the proposition holds true for a n-dimensional hypercube Q_n, where $1 < k \leq n$. For notation purposes, let a sequence of 0's of length m be denoted by 0^m and a sequence of 1's of length m be denoted by 1^m (for example, 0^m where $m = 3$ would be the vertex 000).

For the hypercube Q_{n+1}, we can imagine it as two separate cycles $0v$ and $1v$, where v is the set of all possible permutations of 0's and 1's of length n, with a single edge joining the corresponding vertices in $0v$ and $1v$. Since we can colour Q_n with n colours, then we can colour $0v$ and $1v$ with n colours. Using our $(n+1)$-th colour, we can colour the edges joining $0v$ to $1v$, thereby allowing us to colour Q_{n+1} with $n+1$ colours. This gives us a chromatic number $\chi'(Q_{k+1}) = k + 1$ and the proof of the induction step is complete.

Conclusion: By the principle of induction, the statement is true for all $k \in \mathbb{N}$.

■

[1] Or in Stephanie's words, ”PINKKKKK!”

Question 48

Let G be a simple graph with an odd number of vertices. If G is regular of degree $d \geq 2$, then $\chi'(G) = d + 1$.

Proof. Assume that G is a simple graph with an odd number of vertices, of which are regular of degree $d \geq 2$. By Vizing's Theorem, the chromatic index $\chi'(G)$ is d or $d+1$. Let us show that $X'(G) \neq d$.

Assume to the contrary that $X'(G) = d$. Since G has an odd number of vertices and the sum of all vertex degrees is even, then d must be even. For a G of even regular degree, the largest number of edges of the same colour is $d/2$, otherwise the edges would emerge from over half the number of vertices and meet up. Then G has at most

$$\frac{d}{2} \cdot \chi'(G) = \frac{n \cdot d}{2}$$

edges. Since $\chi'(G) = d$, we have that

$$\begin{aligned} \chi'(G) &= n \\ d &= n \end{aligned}$$

However, d is even but we have know that the number of vertices n is odd. Thus we have a contradiction and the chromatic index cannot be d. Therefore the chromatic index of G is $\chi'(G) = d + 1$. ■