MATH 442 — Assignment 5

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Question 25

Proposition: If a graph has no closed paths of odd length, then it is bipartite.

Proof by Contradiction

Let there be a graph G that has no closed paths of odd length and is not bipartite. We will denote a closed path of even length by $v_1 \to v_2 \to \ldots \to v_m \to v_1$.

For a bipartite graph, we can colour all vertices 2 colours, say black and white, such that each edge joins a black and a white vertex. Without loss of generality, we will colour a vertex v_i black if $i = 2k+1$ and white if $i = 2k$, for some $k \in \mathbb{Z}$. Then we have that v_1 is black, v_2 is white, v_3 is black, ... , and v_m is white. However, v_m and v_1 are different colours, so they can be joined by an edge thereby making it bipartite and contradicting our initial assumption. $\blacksquare$

Question 26

If a simple connected planar graph consists of 10 vertices of degree 3, then by Euler's Theorem the number of faces is given by

$$\begin{aligned} v - e + f &= 2 \\ f &= 2 + e - v \\ &= 2 + (15) - (10) \\ &= 7 \end{aligned}$$

An example graph of this can be seen in Figure 1.

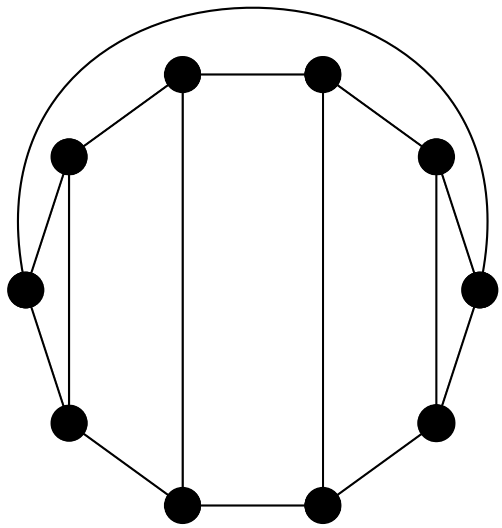

Figure 1: Simple connected planar graph of 10 vertices of degree 3.

Question 27

Proposition For $k < 4$, the graph Q_k is planar.

Lemma: In Q_k, no cycles of length 3 exist.

Assume that Q_k contains a cycle of length 3, such as $v_1 \to v_2 \to v_3 \to v_1$. For the starting point v_1, it differs by 1 digit from v_2 and by 1 digit from v_3. This means that vertices v_2 and v_3 differ by 2 digits, thus they cannot be joined by an edge. This contradits our initial assumption that a cycle of length 3 exists and we are done. ■

Proof

Since no cycles of length 3 exist by Lemma and the number of vertices v is greater than 2, then by Corollary 3 we have that the inequality

$$e \leq 2v - 4$$

holds true if Q_k is planar. From our previous assignment, we know that the number of vertices in Q_k is given by 2^k and the number of edges by $2^{k-1} \cdot k$. Thus we can rewrite our inequality as

$$\begin{aligned} 2^{k-1} \cdot k &\leq 2(2^k) - 4 \\ k &\leq \frac{2^{k+1}}{2^{k-1}} - \frac{2^2}{2^{k-1}} \\ k &\leq 2^2 - 2^{3-k} \\ k - 4 &\leq -2^{3-k} \end{aligned}$$

Since the right hand side is always negative, this inequality holds true for all $k \geq 4$. Thus we can conclude that Q_k is not planar for all $k \geq 4$ and is planar for $k < 4$. ■

Question 28

A graph G with 6 vertices such that G and its complement are both planar is depicted in Figure 2.

Question 29

Proposition: Every simple planar graph has at least 1 vertex of degree ≤ 5.

Proof

Case 1: number of vertices $v \leq 6$.

Since a graph of v vertices can have vertex degree of $v - 1$, then all vertices must be of degree ≤ 5.

Case 2: number of vertices $v > 6$.

Assume that every planar graph has no vertices of degree ≤ 5. Since the minimum degree of each vertex is 6, then the lower bound on the number of edges is given by

$$\# \text{ edges} \geq \frac{6 \cdot v}{2} = 3v$$

For a graph of vertices $v > 2$, the upper bound on the number of edges can be calculated by Corollary 2:

$$e \leq 3v - 6$$

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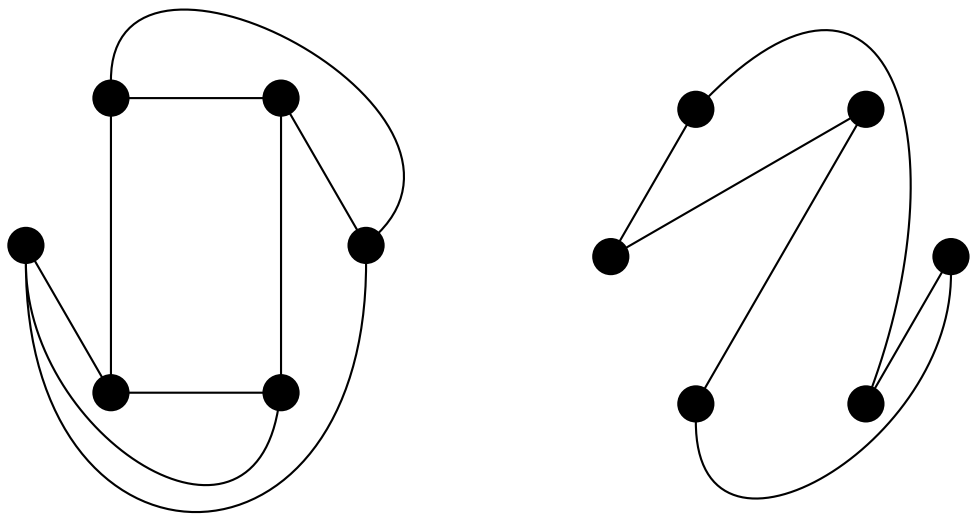

(a) Graph G **(b)** Graph $\overline{G}$

Figure 2: Planar graphs G and $\overline{G}$ of 6 vertices.

Combining these we have

$$3v \leq e \leq 3v - 6$$

which cannot hold true. Thus our initial assumption is false and we can conclude that every simple planar graph has at least 1 vertex of degree ≤ 5. $\blacksquare$

Question 30

Proposition: If G is a connected planar simple graph with $v > 2$ such that $e \geq g > 2$ and no cycle exists of length $< g$ then

$$e \leq \frac{g}{g-2}(v-2)$$

Proof

Since G is simple with more than 2 vertices and no cycles of length $< g$, then each face has $\geq g$ edges. So

$$g \cdot f \leq 2e$$

Since G is a connected planar graph, we have

$$\begin{aligned} v - e + f &= 2 \\ f &= 2 + e - v \end{aligned}$$

by Euler's Theorem. Substituting in we get

$$\begin{aligned} gf &\leq 2e \\ g \cdot (2 + e - v) &\leq 2e \\ g \cdot (2 - v) + ge &\leq 2e \\ g \cdot (2 - v) &\leq e \cdot (2 - g) \\ e \cdot (g - 2) &\leq g(v - 2) \\ e &\leq \frac{g}{g-2} \cdot (v - 2) \end{aligned}$$

■