MATH 442 — Assignment 2

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Question 7

In order to have an Euler cycle—a cycle utilizing each edge of a graph exactly once—in a graph, all vertices must be of even degree. In the case of the town Köonigsburg, we can express it as a graph where each land mass is represented by a vertex and each bridge by an edge (Figure 1).

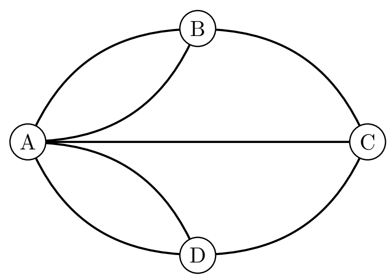

Figure 1: Graph of Köonigsburg. Landmasses are represented by vertices and bridges by edges.

Since all vertices are of odd degree ($deg(A) = 5, deg(B) = deg(C) = deg(D) = 3$), we need to remove at least **two** edges between two different pairs of odd degree vertices. Without loss of generality, let us remove the edges between vertices A and B and vertices C and D. This leaves us with vertices of $deg(A) = 4, deg(B) = deg(C) = deg(D) = 2$. Since all vertices are now even, an Euler cycle is feasible. An example Euler tour of the form $A \to B \to C \to A \to D \to A$ is shown in Figure 2.

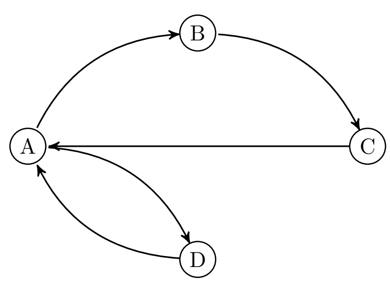

Figure 2: Directed graph of Köonigsburg with two bridges removed.

Question 8

In order to construct an Euler path in the town of Köonigsburg, we must have only 2 odd degree vertices. At an even degree vertex, you go both in *and* out of the vertex. Whereas at an odd vertex, you either go in *or* out of the vertex. Therefore with 2 odd degree vertices, one will be the starting vertex and the other will be the terminal vertex. Since we have 4 vertices of odd degree, we can without loss of generality demolish any one of the bridges. For example, let us demolish the bridge connecting vertex A to vertex C. This leaves us with vertices of $deg(A) = 4, deg(B) = deg(C)3, deg(D) = 2$. An example Euler tour of the form $B \to C \to D \to A \to B \to A \to D$ is shown in Figure 3. Note how we start and end at the two odd degree vertices B and D.

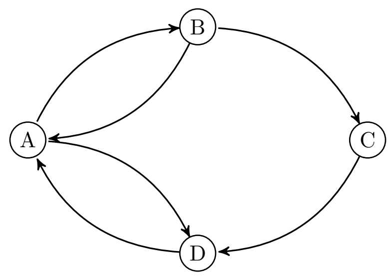

Figure 3: Directed graph of Köonigsburg with one bridge removed.

Question 9

On a 3 x 4 chessboard, it is possible as a knight to visit each of the squares exactly once; however, you will not be able to start and finish at the same place. An example Euler walk is depicted in Table 1.

10	7	2	5
1	4	9	12
8	11	6	3

Table 1: Euler walk for a knight on a 3 x 4 chessboard.

Here, each number $i = 1, 2, \ldots, 12$ represents the order in which a square is visited. This means we start at $i = 1$ and end at $i = 12$.

According to the nature governing how a knight moves, if a knight starts on a *white* square it can only move to a *black* square; conversely, if a knight starts on a *black* square it can only move to a *white* square. In a board of odd number of squares, there is an unequal number of black and white squares. Therefore if a knight were to start on a white square, it will end on a black square; likewise, if a knight were to start on a black square, it will end on a white square. Thus, a knight's tour on an odd number of squares board cannot be conducted such that the knight starts and ends on the same square.

Question 10

The given parse tree can be alternatively expressed as

$$((4t - 5w)(x + y)) * ((y + w + z) + (2x + 1 + y)/(3 + 5w^2))$$

Question 11

Conjecture: *The game of Sprouts with n vertices must terminate after at most $3n - 1$ moves.*

Proof by Construction

Since no vertex can have more than 3 edges coming out of it, our initial isolated graph of n vertices has $3n$ degrees available to "play" with. If we are to draw an edge, we will have one less degree available, or $3n - 1$ degrees. Thus with each successive edge drawn, we will have $3n - 1, 3n - 2, 3n - 3, \ldots$ degrees available.

Since the loser is the first player who cannot make a move, we have 2 possible termination cases: 1) one vertex has degree less than three; or 2) two or more vertices have degree less than three but joining them would cross an edge. The first case has the maximum number of moves in a given game, so we will only consider this case. In this scenario, we only have 1 degree available after the last move. Hence the set of degrees available after each move is given by $S = 3n - 1, 3n - 2, \ldots, 1$. To find the total number of moves in this maximal moves case, we need only take the cardinality of the set. Because $|S| = 3n - 1$, it follows that the game of Sprouts with n vertices must terminate after at most $3n - 1$ moves. ■

Thanks to a collaborative effort[1] between Ian Yihang Zhu, Thomas Roehrl and myself, a game of sprouts with 3 vertices that terminated after 6 moves was finally produced:

Although the graph could have been visually simplified, it was deemed that the game in all of its original, swirly glory must be preserved.

[1] Or to put it more accurately, a "sprouts-off".

Question 12

Claim: *In a party of 6 people it is true there exists 4 people who all do know each other or there exists 4 people who all do not know each other.*

Let a solid edge between two vertices represent if the people know each other and a dotted edge between two vertices represent if the people do not know each other. For this claim to hold true, there must always exist a subgraph that is a complete graph K_4 compose entirely of solid or dotted edges. In the following counterexample (Figure 4)

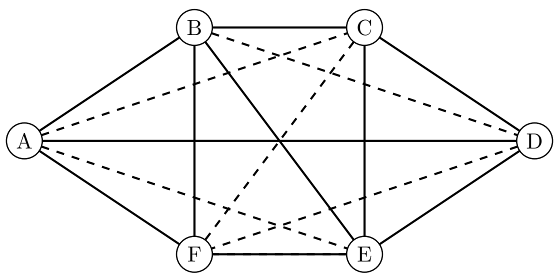

Figure 4: Graph of party of 6 people, where each person is represented by a node. Relationship between two people is indicated by an edge: solid line if they know each other or dashed line if they do not.

we can see that no such subgraph K_4 exists, thereby disproving the claim.