MATH 442 — Assignment 6

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Question 31

Proposition: Let G_1 and G_2 be two homeomorphic graphs. Let G_1 have n_1 vertices and m_1 edges, and let G_2 have n_2 vertices and m_2 edges. Show that $m_1 - n_1 = m_2 = n_2$.

Proof. Let G_1 and G_2 be two homeomorphic graphs. Let G_1 have n_1 vertices and m_1 edges, and let G_2 have n_2 vertices and m_2 edges. For two graphs to be homeomorphic, we can go between the two graphs by performing one or more of the following operations:

1. Inserting vertices in the middle of an edge
2. Erasing vertices of degree 2

Let us examine the first operation. For G_2 of n_2 vertices and m_2 edges, inserting a vertex will give us a new graph G_1 homeomorphic with n_1 vertices and m_1 edges:

$$n_1 = n_2 + 1$$

$$m_1 = m_2 - 1$$

Taking the difference between m_1 and m_2, we have

$$\begin{aligned} m_1 - n_1 &= (m_2 - 1) - (n_2 + 1) \\ &= m_2 - n_2 \end{aligned}$$

For the second operation, let us erase a vertex of degree 2 from G_2. This will give us a new graph G_1 homeomorphic with n_1 vertices and m_1 edges:

$$n_1 = n_2 - 1$$

$$m_1 = m_2 + 1$$

Taking the difference similarly gives us

$$\begin{aligned} m_1 - n_1 &= (m_2 + 1) - (n_2 - 1) \\ &= m_2 - n_2 \end{aligned}$$

■

Question 32

Proposition: No polyhedral graph with exactly 30 edges and 11 faces can exist.

Proof. Assume, to the contrary, that there exists a polyhedral graph with exactly 30 edges and 11 faces. Since a polyhedral graph, by definition, is a simple connected planar graph where each vertex has degree at least 3, we can apply Euler's Theorem to find the number of vertices v of our graph:

$$\begin{aligned} v - e + f &= 2 \\ v &= 2 + e - f \\ &= 2 + (30) - (11) \\ &= 21 \end{aligned}$$

Since each vertex has degree $d_i \geq 3$, we have

$$3v \leq 2e$$

Substituting $v = 21$ and $e = 30$ into our inequality

$$\begin{aligned} 3(21) &\leq 2(30) \\ 63 &\not\leq 60 \end{aligned}$$

we arrive at a contradiction. ■

Question 33

Let us find the dual of each regular polyhedra

1. **Tetrahedron**

   Since the tetrahedron is relatively small (compared to an icosohedron), we can construct the dual graph (Figure 1) following the two stage algorithm as outlined in [1]:

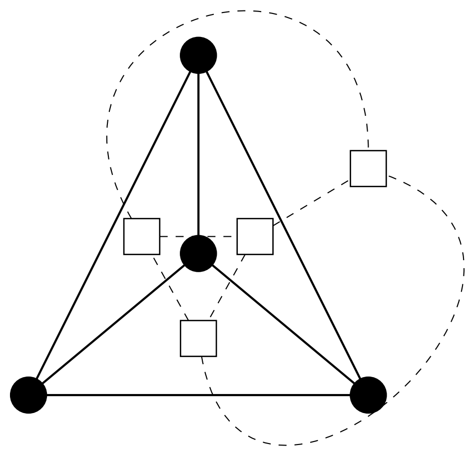
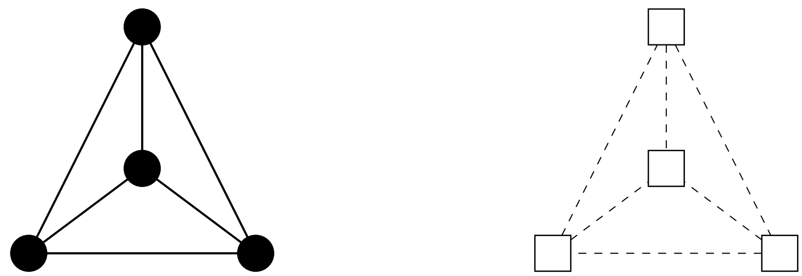


Figure 1: Graph G and dual graph G^*, both of which are isomorphic to a tetrahedron. G has vertices denoted by black circles and edges by solid lines, and G^* denoted by unfilled squares and edges by dashed lines.

   Alternatively, our tetrahedron graph G has number of edges $e = 6$, number of faces $f = 4$, and number of vertices $v = 4$. For our dual graph G^* with number of edges e^*, faces f^*, and vertices v^*, we have by Lemma 4.12 [1] that

$$e^* = e$$
$$f^* = v$$
$$v^* = f$$

Thus the dual of a tetrahedron has $e^* = 6$, $f^* = 4$, and $v^* = 4$, which is a graph isomorphic to a **tetrahedron** (Figure 2).



(a) Graph G of a tetrohedron **(b)** Dual graph G^* of a tetrohedron

Figure 2: Planar graphs G and $\overline{G}$ of 6 vertices.

2. **Cube**

   Our cube graph G has number of edges $e = 12$, number of faces $f = 6$, and number of vertices $v = 8$. For our dual graph G^* with number of edges e^*, faces f^*, and vertices v^*, we have that

$$e^* = e = 12$$
$$f^* = v = 8$$
$$v^* = f = 6$$

   Thus the dual of a cube is a graph isomorphic to an **Octahedron**.

3. **Octahedron**

   By Theorem 4.13 in [1], we have that the dual of a dual graph (G^{**}) is isomorphic to the original graph (G^*). From above, we know that a cube's dual is an octahedron. Thus the dual of an octahedron is a **Cube**.

4. **Dodecahedron**

   Our dodecahedron graph G has number of edges $e = 30$, number of faces $f = 20$, and number of vertices $v = 12$. For our dual graph G^* with number of edges e^*, faces f^*, and vertices v^*, we have that

$$e^* = e = 30$$
$$f^* = v = 12$$
$$v^* = f = 20$$

   Thus the dual of a dodecahedron is a graph isomorphic to an **Icosohedron**.

5. **Icosohedron**

   Again by Theorem 4.13 in [1], we have that the dual of a dual graph (G^{**}) is isomorphic to the original graph (G^*). From above, we know that a dodecahedron's dual is an icosohedron. Thus the dual of an icosohedron is a **Dodecahedron**.

Question 34

Show the line graph of a tetrahedron graph is the octahedron graph

Superimposing the line graph $L(G)$ on top of our tetrahedron graph G (Figure 3)

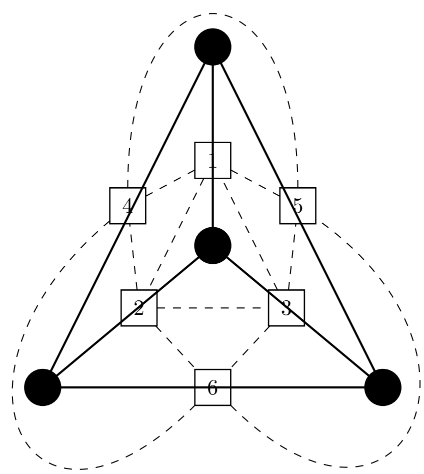


Figure 3: Graph G of a tetrahedron and its dual graph G^*. G has vertices denoted by black circles and edges by solid lines, and G^* denoted by unfilled squares and edges by dashed lines.

Next, let us separate $L(G)$ from our original graph. Rearranging the orientation of the vertices gives us a graph isomorphic to the octahedron graph (Figure 4).

Prove that if a simple graph G is regular of degree k, then $L(G)$ is regular of degree $2k-2$.

Proof. Assume that G is a graph of regular degree k. Let us choose any edge e of G which will correspond to a vertex v^l in the line graph $L(G)$. Since each edge connects 2 vertices,

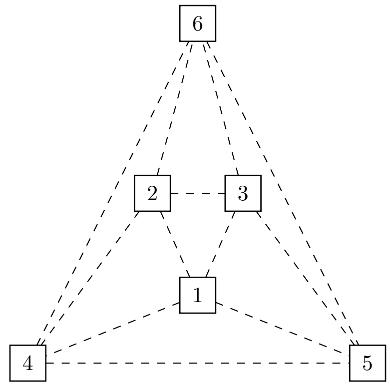


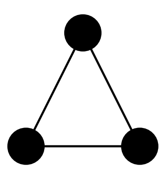
Figure 4: Line graph $L(G)$ of a tetrahedron is an octahedron.

we have a total of $2k$ edges meeting at the vertex ends of edge e. This means that the vertex v^l is adjacent to $2k$ other vertices in $L(G)$, which gives us a vertex degree of $k^l = 2k$. However we have double counted the edge e twice: once for the first vertex end of degree k and a second time for the other vertex of degree k. Thus our line graph $L(G)$ has vertices regular of degree $k^l = 2k - 2$. ■

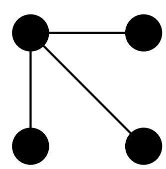
Question 35

Find two simple graphs G and H such that G and H are not isomorphic but $L(G)$ and $L(H)$ are isomorphic.

Since $L(G)$ is isomorphic to $L(H)$, we have that $e = e^l$. Utilizing proof by inspector gadget, we see that a cycle graph C_3 (Figure 5a) and a simple connected graph of 4 vertices with one vertex degree 3 and the rest degree 1 (Figure 5b) are not isomorphic to each other



(a) Graph G



(b) Graph H

Figure 5: Non-isomorphic graphs G and H with isomorphic line graphs.

but whose line graphs are isomorphic (Figure 6).

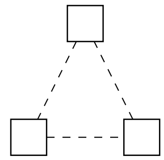


Figure 6: Line graphs $L(G)$ and $L(H)$.

Question 36

Proof. Let G be a simple graph with vertices $v_1, \ldots, v_n$ of corresponding vertex degree d_i for $1 \leq i \leq n$. To find the total number of edges in $L(G)$, let us first find the number of edges contributed to $L(G)$ for each vertex v_i. For a vertex v_i of degree d_i, we have d_i edges emerging from it. This means that each of these d_i edges (corresponding to a vertex in the line graph) is adjacent to $d_i - 1$ other edges going into v_i. Since there are d_i of these edges, this gives us a total of $d_i(d_i - 1)$ edges contributed to $L(G)$ from each vertex v_i. However, each edge has been double counted. Thus we only have

$$\frac{d_i(d_i - 1)}{2}$$

edges contributed per vertex v_i. Summating over all vertices $v_1, \ldots, v_n$ gives us the total number of edges in $L(G)$ to be

$$\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2}$$

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Bibliography

[1] Wilson, Robin J. (2010). *Introduction to Graph Theory* (5th ed.). Harlow, England: Pearson Education Limited.