MATH 442 — Assignment 10

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Question 55

If G is a simple connected graph, then for any edge in e in G

$$\tau(G) = \tau(G - e) + \tau(G/e) \tag{1}$$

Proof. no ducking clue ■

Question 51

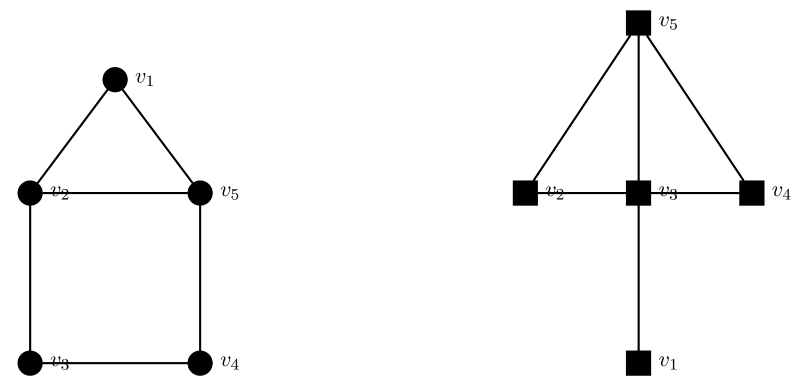
For the graph G of vertices $v_1, v_2, \dots v_5$ in Figure 1a, let us determine the number of non-isomorphic ways to label it. Here, we can label v_1 5 different ways. For the vertex pair v_2 and v_5, we can label them $\binom{4}{2} = 6$ ways, Lastly we can label v_3 2 different ways (due to isomorphism, v_4 simply takes the remaining label). This gives us a total of

$$5 \cdot 6 \cdot 2 = 60$$

non-isomorphic ways to label graph G. For the graph H of vertices $v_1, v_2, \dots v_5$ in Figure 1b, we can label v_1 5 ways, v_3 4 ways, and v_5 3 ways. Since v_2 and v_4 are symmetrical about the y-axis (i.e. same degree), then labelling them the remaining labels does not create any new non-isomorphic graphs. This gives us a total of

$$5 \cdot 4 \cdot 3 = 60$$

non-isomorphic ways to label graph H.



(a) Graph G **(b)** Graph H

Figure 1: Graphs G and H.

Question 57

(a) For the labelled graph (Figure 2), the associate Prüfer sequence is (5,2,2,6,2,2).

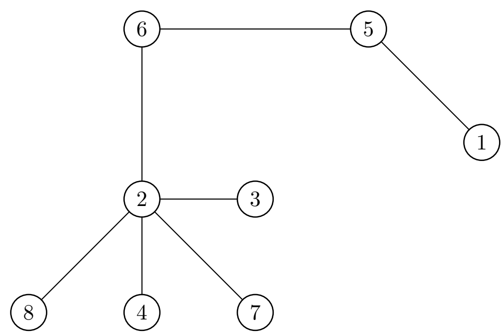


Figure 2: Labelled graph G with associated Prüfer sequence (5,2,2,6,2,2).

(b) For the Prüfer sequence (3,4,8,1,8,8,8), the associated labelled graph H can be seen in Figure 3.

Question 58

A vertex in a labelled graph has degree k if and only if its label appears $k-1$ times in the Prüfer sequence of the graph.

Proof. $\Rightarrow$ Assume that a vertex v in a labeled graph has degree k. Here, we take a leaf

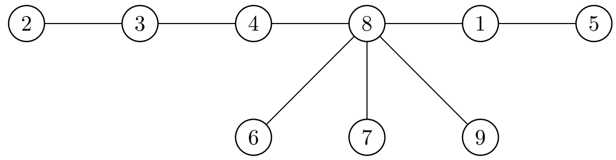


Figure 3: Labelled graph H with associated Prüfer sequence (3,4,8,1,8,8,8).

and put its adjacent vertex in the Prüfer sequence and then delete that leaf. We have two cases for our vertex v of degree k

Case 1: v is a leaf. Then it is never in the Prüfer sequence. Here $k = deg(v) = 1$ and the number of occurrences is $k - 1 = 0$.

Case 2: v is a non-leaf. We add it to the Prüfer sequence and minus its degree by 1. This continues until either it is a leaf (deleted) or it is one of the last two leaves (do not add to Prüfer sequence). Either way, it is only added to the Prüfer sequence $k - 1$ times.

$\Leftarrow$ Assume that a vertex label appears $k - 1$ times in Prüfer sequence of a graph. Then this vertex label was a non-leaf adjacent to a deleted leaf $k - 1$ times until it became a leaf itself. Thus it had original degree $(k - 1) + 1 = k$. ∎

Question 59

There does not exist a tree consisting of 12 vertices, with one vertex of degree 5, two vertices of degree 4, and one vertex of degree 2.

Proof. Assume to the contrary that such a graph exists. Since we cannot form cycles, let us connect these four vertices of degree 5, 5, 5, and 2 together. Without loss of generality, let us attach the vertices of degree 4 and 2 to the vertex of degree 5 (Figure 4).

Since this is a tree, none of the edges from these four vertices can connect back to v_1, v_2, v_3 or v_4. Thus we must attach a new vertex on the ends of each of them. However we have 9 edges which will result in a total of 13 vertices in T, thereby giving us the necessary contradiction. ∎

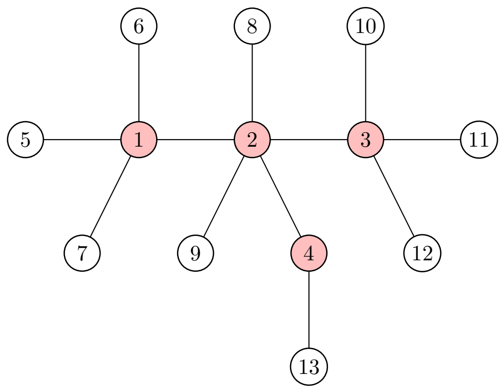


Figure 4: Graph of tree T with vertices v_1 and v_3 of degree 4, v_2 of degree 5, and v_4 of degree 2 (denoted by pink).

Question 60

How many labelled trees exist with 6 vertices such that the degree of ever vertex is 1 or 3?

Since we can only have vertices of degree 1 or 3, this gives us one possible configuration: two vertices of degree 3 attached with vertices of degree 1 on the remaining edges (Figure 6). If we were to have less vertices of degree 3—in this case, zero or one—we will have less than 6 vertices total. If we were to have greater than two vertices of degree 3, we will have more than 6 vertices. Thus we can only have a configuration with exactly two vertices of degree 3.

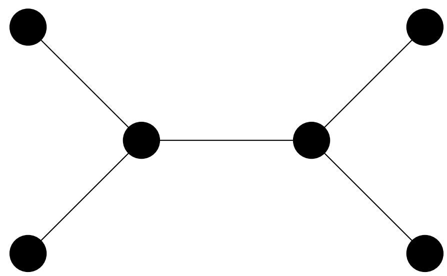
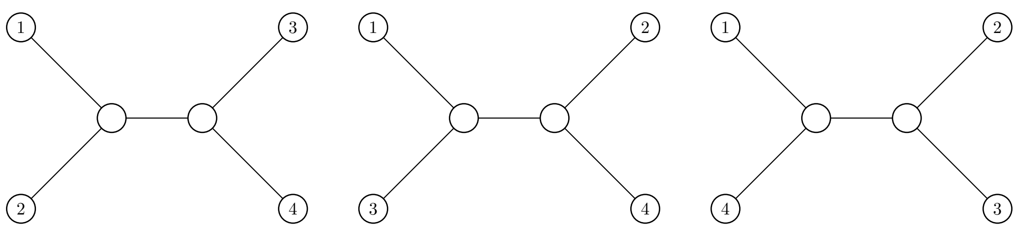


Figure 5: Graph of 6 vertices with vertices of degree 1 or 3.

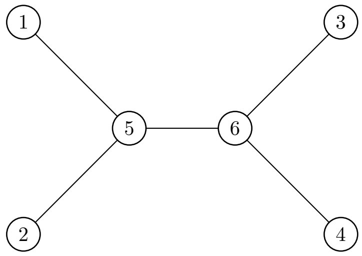
Let us first fix the labels of the vertices of degree 1. Since we can rotate about the vertices of degree 3 and flip the graph about its line of symmetry in the y-axis, two graphs are isomorphic if they have the same labels attached to a vertex of degree 3. Let the possible labels be 1, 2, 3, and 4. Then for labels i, j, where $i \neq j$ and $i, j \in \{1, 2, 3, 4\}$, we must assign each i and j such that they are adjacent to the same vertex of degree 3 and adjacent to different vertices of degree 3. This gives us 3 possible configurations (Figure 6) for the fixing of the labels for vertices of degree 1.



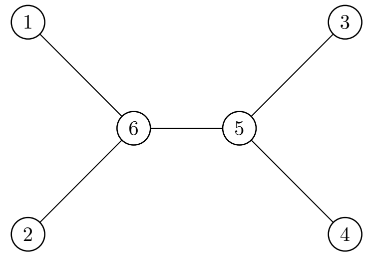
(a) Configuration 1 **(b)** Configuration 2 **(c)** Configuration 3

Figure 6: Possible configurations for the fixing of the labels for vertices of degree 1.

Now, let us fix the labels for the two vertices of degree 3. Although we have a line of symmetry about the y-axis along the edge adjacent to our two vertices of degree 3, switching the assigned labels for these vertices will yield non-isomorphic graphs (Figure 7). Thus for each of the remaining labels, we have 2 possible arrangements.



(a) Configuration 1



(b) Configuration 2

Figure 7: Swapping labels for the vertices of degree 3 yields non-isomorphic graphs.

However, for the remaining vertices we have 6 different labels for which we must choose 2. This gives us $\binom{6}{2} = 15$ possibilities. Multiplying all of these togethers gives us a total of

$$3 \cdot 2 \cdot 15 = 90$$

non-isomorprhic labelled trees with 6 vertices of degree 1 or 3.