Operations Research
Lecture 7: Integer Programming

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1. Consider the integer program

$$\begin{array}{rrcrcr} \max & 3x_1 & + & 2x_2 & & \\ \text{s.t.} & 2x_1 & + & 5x_2 & \leq & 20 \\ & 2x_1 & - & x_2 & \leq & 4 \\ & & & x_i \in \mathbb{Z}_+ & & \forall i = 1, 2. \end{array}$$

   (a) Graphically solve its linear relaxation.

   (b) Graphically solve the IP. Show that the linear relaxation provides an upper bound.

   (c) As the LR-optimal solution is fractional, it is not IP-feasible. Among the four ways to round up or down the two variables, how many of them are IP-feasible?

2. Let's use the branch-and-bound algorithm to solve this IP:

$$\begin{array}{rrcrcr} \max & 3x_1 & + & 2x_2 & & \\ \text{s.t.} & 2x_1 & + & 5x_2 & \le & 20 \\ & 2x_1 & - & x_2 & \le & 4 \\ & & & \multicolumn{3}{l}{x_i \in \mathbb{Z}_+ \quad \forall i = 1, 2.} \end{array}$$

(a) After you solve the linear relaxation, branch on x_1. Solve the two subproblems. Write down the candidate solution for the IP, if there is one, and its objective value. Otherwise, write down "None."

(b) Keep branching for one more level. Write down the candidate solution for the IP, if there is one, and its objective value. Otherwise, write down "None."

(c) Conclude that you have solved the IP in Part (b) or continue until you find an IP-optimal solution. What is your complete branching tree?

(d) After you solve the linear relaxation, branch on x_2. Continue and solve the IP.

3. For the IP discussed in the lecture video

$$\begin{array}{rrcrcr}
\max & 8x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 6 \\
& 9x_1 & + & 5x_2 & \leq & 45 \\
& & & \multicolumn{3}{l}{x_i \in \mathbb{Z}_+ \quad \forall i = 1, 2,}
\end{array}$$

use the branch-and-bound algorithm to solve it by branching on x_2 after you solve the linear relaxation.

4. Consider the following knapsack instance

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 + 4x_3 + x_4 + 3x_5 \\ \text{s.t.} \quad & 4x_1 + 5x_2 + 3x_3 + x_4 + 4x_5 \leq 11 \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 5. \end{aligned}$$

(a) Consider the following algorithm: For item i, calculate its value-weight ratio

$$r_i = \frac{v_i}{w_i},$$

where v_i and w_i are the value and weight of item i, respectively. Then select those items with the largest r_i until no more item can be selected. Apply this algorithm to this instance and show that it does *not* find an optimal solution.

(b) Consider the linear relaxation of the knapsack problem in general. Does the above algorithm always find an optimal solution?

5. The previous observation on the knapsack problem makes it easier to apply the branch-and-bound algorithm for knapsack instances. Consider the same instance:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 + 4x_3 + x_4 + 3x_5 \\ \text{s.t.} \quad & 4x_1 + 5x_2 + 3x_3 + x_4 + 4x_5 \leq 11 \\ & x_i \in \{0,1\} \quad \forall i = 1, ..., 5. \end{aligned}$$

(a) Solve the linear relaxation by the proposed algorithm. Then branch on the only fractional variable. What are the two additional constraints to add?

(b) Solve the two subproblems generated in Part (a).

(c) Keep branching until an IP-optimal solution is found. Depict the full branching tree.

6. When there are multiple nodes to branch, how to select one?

   (a) One common approach is to branch the node with the highest objective value (for a maximization problem). Why?

   (b) Another popular approach is "once a node is branched, all its descendants are branched before any nondescendant." Why?

7. A manufacturer can sell product 1 at a price of \$5 per unit and product 2 at a price of \$7 per unit. Nine units of raw material are needed to manufacture one unit of product 1, and seven units of raw material are needed to manufacture one unit of product 2. A total of 120 units of raw material are available. The setup costs for producing products 1 and 2 are \$30 and \$40, respectively.

   (a) Formulate an IP that maximizes the profit.

   (b) If both products are produced for positive amounts, there is a saving of \$20 in the setup cost (i.e., in total \$50 are paid). Formulate an IP that maximizes the profit.

8. You are deciding how to utilize the coming weekend. In total you have 16 hours for studying the following four subjects: Algorithms, Statistics, Management Information Systems (MIS), and Operating Systems (OS). You must study for Algorithms and OS in total for at least 6 hours. Moreover, you must either study Statistics for at least 4 hours or study MIS for at least 3 hours. If you fail to study OS for at least 5 hours, you must study Statistics and MIS in total for at least 8 hours.

   (a) Suppose that you want to minimize your total study time. Formulate an integer program that can solve your problem.

   (b) Suppose that you want to maximize the number of subjects that you study for at least 4 hours. Formulate an integer program that can solve your problem.

9. In AMPL, we may use the keywords **binary** and **integer** to declare binary and integer variables, respectively. The syntax is below:

```
var x1 integer;
var x2 binary;
```

(a) Write an AMPL program to solve the IP

$$\begin{array}{rrcrcr} \max & 8x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \le & 6 \\ & 9x_1 & + & 5x_2 & \le & 45 \\ & & & \multicolumn{3}{l}{x_i \in \mathbb{Z}_+ \quad \forall i = 1, 2.} \end{array}$$

Do not forget to use the solver CPLEX.

(b) Use the default solver MINOS (by typing `option solver minos;`) to solve your IP model. What do you observe?