Operations Research
Lecture 3: The Simplex Method

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. Convert the following LP to its standard form:

$$
\begin{array}{rl}
\min & 3x_1 + x_2 \\
\text{s.t.} & x_1 \geq 3 \\
& x_1 + x_2 \geq -4 \\
& 2x_1 - x_2 = 3 \\
& x_1 \geq 0, x_2 \text{ urs.}
\end{array}
$$

2. Consider the following LP

$$\begin{aligned} \min \quad & 6x_1 + 4x_2 \\ \text{s.t.} \quad & 9x_1 + 4x_2 \geq 36 \\ & 2x_1 + 8x_2 \geq 24 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) Find the standard form (with no artificial variable).

(b) Find all the bfs.

(c) Draw the feasible region, find all the extreme points, and show how each bfs corresponds to an extreme point.

3. Consider the following LP

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 = 100 \\ & x_1 + x_2 \leq 70 \\ & x_1 \geq 40 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) How many bs and bfs do we have?

(b) Show how each bfs corresponds to an extreme point.

4. Consider the LP

$$
\begin{aligned}
z^* = \max \quad & 2x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\
& 2x_1 + x_2 \leq 8 \\
& x_i \geq 0 \quad \forall i = 1, 2
\end{aligned}
$$

that has been solved in the lecture videos.

(a) Write down the initial tableau.

(b) Instead of entering x_1, enter x_2 to complete one iteration. Write down the tableau after one iteration.

(c) Continue iterating to find an optimal solution.

(d) Depict the route you go through in the above process.

5. When running the simplex method, the *smallest index rule* is a rule to select entering and leaving variables: When multiple variables may enter/leave, choose the one with the smallest index, i.e., choose x_i rather than x_j if $i < j$. Use the simplex method with the smallest index rule to solve the following LP

$$\begin{aligned} z^* = \min \quad & 4x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \leq 8 \\ & -x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0, x_2 \leq 0. \end{aligned}$$

(a) Find an optimal solution $x^* = (x_1^*, x_2^*)$ and the associated objective value z^*. Write down the complete process.

(b) Depict the route you go through in the above process.

6. When you use the simplex method to solve a maximization problem, suppose you get a tableau

$$
\begin{array}{ccccc|c}
c & 2 & 0 & 0 & 0 & 10 \\
\hline
-1 & a_1 & 1 & 0 & 0 & 4 \\
a_2 & -4 & 0 & 1 & 0 & 1 \\
a_3 & 3 & 0 & 0 & 1 & 1
\end{array}
$$

at the end of an iteration. Give conditions on the unknowns c, a_1, a_2, and a_3 to make the following statements true:

(a) The current bfs is optimal.

(b) The current bfs is suboptimal, and we need to do some more iterations to solve this problem.

(c) The problem is unbounded.

7. Suppose that when we run the simplex method for a given linear program with a maximization objective function, a tableau we get is

$$
\begin{array}{cccc|c}
0 & 0 & 0 & 2 & 2 \\
\hline
1 & 0 & -1 & 1 & 6 \\
0 & 1 & -2 & 3 & 3
\end{array}
$$

Answer the following questions with brief explanations.

(a) Is this LP unbounded? Why?

(b) Are there multiple optimal solutions? If no, explain why; if yes, write down two optimal solutions.

8. Consider two LPs

$$(P)\quad \begin{aligned} \min\ & c^{\mathrm{T}}x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned} \qquad \text{and} \qquad (Q)\quad \begin{aligned} \min\ & 1^{\mathrm{T}}y \\ \text{s.t. } & Ax + Iy = b \\ & x, y \geq 0. \end{aligned}$$

Prove or disprove the following statements regarding the two LPs.

(a) If $\bar{x}$ is a feasible bfs to (P), then $(x, y) = (\bar{x}, 0)$ is an optimal bfs to (Q).

(b) If $(x, y) = (\bar{x}, 0)$ is an optimal bfs to (Q), then $\bar{x}$ is a feasible bfs to (P).

(c) If in (P) we are maximizing $c^{\mathrm{T}}x$, what should be an appropriate (Q) that has the above properties?

9. Consider the following LP

$$\begin{aligned} \max \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 = 100 \\ & x_1 \geq 40 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) Find the Phase-I LP and its initial tableau.

(b) Solve the Phase-I LP with the smallest index rule for an initial bfs to the standard form of the original LP.

(c) Find the Phase-II LP and its initial tableau.

(d) Solve the Phase-II LP with the smallest index rule for an optimal solution to the original LP.

(e) Visualize the search path.

10. Use the simplex method with the smallest index rule to solve

$$\begin{aligned} z^* = \min \quad & 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 6 \\ & 2x_1 + 3x_2 = 4 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

Visualize the search path.

11. In general, as we need to choose m out of n variables to be basic, we have at most $\binom{n}{m}$ bases.

(a) Is it possible to have fewer than $\binom{n}{m}$ bases? Why?

(b) Suppose that we have k bases, is it possible to have fewer than k distinct basic solutions? Why?

(c) Suppose that we have k distinct basic solution, is it possible to have fewer than k distinct basic feasible solution? Why?