Operations Research, Spring 2020
Homework 3

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1 Problems

1. (10 points; 5 points each) For each of the following function, find the range over which the given function is convex (or conclude that the function is nowhere convex). You will get full points if and only if the function is convex over the reported region and nonconvex outside the region.

   (a) $f(x_1, x_2, x_3) = x_1^3 + 2x_1x_3 + \frac{x_2}{x_1}$.

   (b) $f(x) = x^4 - 2x^3 + 3x^2 + x - 6$.

2. (30 points) Consider the nonlinear program

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \ & \frac{1}{2}x_1^2 + 3x_2^2 + x_1x_2 \\ \text{s.t. } & x_1 \geq 3 \end{aligned}$$

   (a) (10 points) Show that the objective function is convex.

   (b) (5 points) Ignore the constraint and find the unconstrained optimal solution $\bar{x}$. Show that $\bar{x}$ is infeasible.

   (c) (5 points) Find the feasible point $\tilde{x}$ that is closest to $\bar{x}$. Show that $\tilde{x}$ is infeasible.

   (d) (10 points) Use the KKT condition to find the constrained optimal solution. Do not forget to prove that it is indeed optimal.

3. (30 points) Consider the nonlinear program

$$\min_{x \in \mathbb{R}^2} \ x_1^2 + 3x_2^2 + x_1x_2 - x_1 - 2x_2.$$

   (a) (5 points) Find the gradient and Hessian of the objective function.

   (b) (5 points) Starting at $(1, 1)$, run one iteration of Newton's method to get to the next solution.

   (c) (5 points) Starting at $(1, 1)$, run one iteration of the gradient descent method to get to the next solution. Choose the step size to be $a = \frac{1}{2}$. Compute the improvement in the objective value (which may be positive, zero, or negative) after this iteration.

   (d) (10 points) Starting at $(1, 1)$, run one iteration of the gradient descent method to get to the next solution. Choose the step size to be the one that reaches the lowest point along the improving direction. Compute the improvement in the objective value (which should not be negative) after this iteration.

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(e) (5 points) Find an optimal solution.

Hint. If you have done the previous parts correctly, you should be able to report an optimal solution without doing any more derivations. However, you should clearly explain why your reported solution must be optimal.

4. (20 points; 10 points each) Consider the dynamic program that solves the deterministic inventory problem given in Problem 4 of the lecture problems for Lecture 11 (you may look at the posted solution to obtain the formulation).

   (a) Suppose that $T = 6$, $C = (C_1, C_2, ..., C_6) = (4, 6, 4, 7, 6, 3)$, $H = 1$, $D = (D_1, D_2, ..., D_6) = (2, 1, 2, 3, 3, 2)$, $K = 50$ (so effectively there is no space limitation), and $S = 5$. Please note that the "first" period is period 6 while the "last" period is period 1. Write a computer program to solve the dynamic program. List $V_t(y)$ for all $t = 1, ..., 6$ and $y = 0, 1, ..., 13$ as your solution. Do not include your computer program in your submitted report.

   Note. You may write your computer program by modifying the one provided by the instructor. Please note that the provided computer program only solves the deterministic inventory problem introduced in the lecture videos. Modifications are definitely needed. Please also note that this is just one option, which may or may not save you some time. You may certainly write your own program from scratch.

   (b) Continue from the previous problem. Find $V_6(0)$ and the optimal production/ordering plan (i.e., x_t^* for all $t = 1, ..., 6$) for S ranges from 0, 5, 10, and 15. Depict the impact of S on $V_6(0)$. Comment on the impact of S on the optimal production/ordering plan.

5. (10 points; 5 points each) Consider the stochastic inventory problem introduced in the lecture videos. Ignore the numbers and focus on the abstract model. Suppose that the random demand D_t does not follow a Binomial distribution. Instead, D_t now follows a discrete distribution where $\Pr(D_t = k) = r_k$ for $k = 0, 1, 2$. Obviously, we have $r_0 + r_1 + r_2 = 1$. Formulate a dynamic program that finds $V_T(0)$, the total expected profit in the T periods with no initial inventory, under an optimal production/ordering policy. Please note that a dynamic program consists of a set of boundary conditions and an optimization problem for each period.

   Hint. For period t, first characterize $u(q_t, y)$, the single-period expected profit function, and $\mathbb{E}[V_{t-1}((q_t - D_t)^+)]$, the expected profit from period $t-1$ to period 1. The value-to-go function may then be obtained by combining the two parts.

2 Submission rules

The deadline of this assignment is ***1:00 am, May 28***. Works submitted between 1:00 am and 2:00 am will get 10 points deducted as a penalty. Submissions later than 2:00 am will not be accepted. Please submit a PDF file containing your answers for all problems to NTU COOL. Include your student ID and name in the first page of your PDF file.