OR-Inclass9

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1. Problem 1

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Problem 2

   (a) $\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & 0 \\ 0 & 6x_2 \end{bmatrix}$
   Therefore $f(x_1, x_2)$ is convex when $x_1 \in R$ and $x_2 \geq 0$

   (b) $\nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$
   This function is never convex

   (c) $\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 - 2x_2 & -2x_1 \\ -2x_1 & 2 \end{bmatrix}$
   $f(x_1, x_2)$ is convex when $3x_1 \geq x_2 + x_1^2$

   (d) $\nabla^2 f(x_1, x_2, x_3) \begin{bmatrix} 2 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{bmatrix}$
   This function is convex when $x_2 \in R$ and $x_1, x_3 = 0$.

   (e) $\nabla^2 f(x_1, x_2, x_3) \begin{bmatrix} \frac{1}{4}x_1^{-\frac{3}{2}} & 0 & 0 \\ 0 & 2x_2^{-}3 & 0 \\ 0 & 0 & e^{-x_3} \end{bmatrix}$
   The function is convex when $x_3 \in R$, $x_1, x_2 \geq 0$

3. Problem 3

   (a) Yes, since there is only one constraint and the model will be unbounded if there is no constraint, we can conclude that if the optimal exists, the constraint will always binding at the optimal solution.

   (b) $\mathcal{L}(x|\lambda) = x_1 + x_2 + \lambda(12 - x_2^2 - 2x_2^2)$

   (c) Lagrangian relaxation is $z^L(\lambda) = max(\mathcal{L}(x|\lambda)$

   (d) $\nabla\mathcal{L} = 0$ requires $1 - 2\lambda x_1 = 0$ and$1 - 4\lambda x_2 = 0$, which then imply $x_1 = 2x_2$.

(e) Based on complementary slackness we have $\lambda(12 - x_2^2 - 2x_2^2) = 0$. AS we know the constraint $12 - x_2^2 - 2x_2^2 \leq 12$ must be binding at the optimal solution,we can obtain the solution based on $x_1 = 2x_2$ derived in (d). The solution is $(2\sqrt{2}, \sqrt{2})$ with objective value $3\sqrt{2}$.

4. Problem 5

   (a) There is no multiplier with a specif sign that can reward feasibility or penalize infeasibility for $Ax = b$.

   (b) If $(c^T)_i \geq (\lambda^T A)_i$ for any i , we will get infinity.

   (c) Fro the minimization problem to be meaningful , λ must satisfy $c^T \leq \lambda^T A$, otherwise that λ will lead to positive infinity as a objective value , which is definitely not optimal. Is λ satisfies $c^T \leq \lambda^T A$, we know $max_{x \geq 0}(c^T - \lambda^T A) = 0$ and the Lagrangian dual becomes

$$\begin{array}{ll} \min & z^*(\lambda) = \lambda^T b \\ \text{s.t.} & \lambda^T A \geq c^T \end{array}$$

   Which is exactly th dual LP.

5. Problem 6

   (a) Suppose $b \geq 1$, it means other product's price will have the equal or even more impact on your own product's price , which is not reasonable. For $b < 0$ means the higher other product's price the lower your quantities,which may be another model(complementary product).

   (b) $max_{p_1,p_2} P_1(a - P_1 + bP_2) + P_2(a + bP_1 - P_2)$.

   (c) Let $f(P) = -[P_1(a-P_1+bP_2)+P_2(a+bP_1-P_2)]$, we have $\nabla^2 f(P) = \begin{bmatrix} 2 & -2b \\ -2b & 2 \end{bmatrix}$,which is PSD because $b \in [0, 1)$.Therefore, $f(P)$ is convex and $-f(P)$,our objective function is concave.

   (d) $\nabla f(P) = 0$ requires $-a + 2P_1 - 2bP_2 = 0$ and $-a + 2P_2 - 2bP_1 = 0$, which lead to $P_1 = P_2 = \frac{a}{2(1-b)}$.

   (e) Yes, if a increase means demand increase, so price increase is reasonable.If b increase means the product is easier to substitute the other product , so price increase is reasonable.

6. Problem 8

   (a) First consider $i = 1$ and collect terms with $x_1 or y_1$. We obtain $f_1(\alpha\beta) = \alpha^2 + y_1^2 + \beta^2 x_1^2 - 2\alpha y_1 - 2\beta x_1 y_1 + 2\alpha\beta x_1$. We than have$\nabla^2 f_1(\alpha\beta) = \begin{bmatrix} 2 & 2x_1 \\ 2x_1 & 2x_1^2 \end{bmatrix}$ which means f_1 is a convex function.As the summation of convex functions is also convex,the proof is complete.

(b) $\nabla f(\alpha\beta) = 0$ requires $\sum_{i=1}^{n}[y_1 - (\alpha + \beta x_1)] = 0 \to (\sum_{i=1}^{n} y_i) - n\alpha - \beta \sum_{i=1}^{n} x_1 = 0$ and $\sum_{i=1}^{n} -2x_i[y_i - (\alpha + \beta x_i)] = 0 \to (\sum_{i=1}^{n} x_i y_i) - (\sum_{i=1}^{n} x_i) - \beta \sum_{i=1}^{n} x_i^2 = 0$.

By above we can acquire $\beta = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$

$\alpha = \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$