Operations Research
Lecture 8: Single-variate Nonlinear Programming

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1. (a) What is the maximum area inside a rectangle whose lengths of the four edges sum to 1? Formulate an NLP whose optimal solution answers this question.

   (b) What is the minimum total length of a rectangle whose area is 1? Formulate an NLP whose optimal solution answers this question.

2. Graphically or intuitively determine which of the following sets are convex:

   (a) An interval $[a, b] \subseteq \mathbb{R}$ for some $a, b \in \mathbb{R}$.
   (b) $[0, 10] \cup [20, 30]$.
   (c) $\{x \in \mathbb{R}^2 | x_1 + x_2 \leq 2, x_1^2 + x_2^2 \leq 9\}$.
   (d) $\{x \in \mathbb{R}^2 | x_1 + x_2 \leq 2, x_1^2 + x_2^2 \geq 9\}$.
   (e) $\{x \in \mathbb{R}^3 | x_1 + 2x_2 + 4x_3 \leq 8, x_2 \geq 0, x_3 \geq 0\}$.
   (f) $\{x \in \mathbb{Z}^2 | x_1 \geq 0, x_2 \geq 0, x_1 + 2x_2 \leq 6\}$.

3. Graphically or intuitively determine which of the following functions are convex, concave, or neither over the given domain S:

   (a) $f(x) = x^3$, $S = \mathbb{R}$.
   (b) $f(x) = x^3$, $S = [0, \infty)$.
   (c) $f(x) = \frac{1}{x}$, $S = (0, \infty)$.
   (d) $f(x) = x^a$ for some $a \in (0, 1)$, $S = [0, \infty)$.
   (e) $f(x) = x^a$ for some $a \in (1, 2)$, $S = [0, \infty)$.
   (f) $f(x) = 2^x$, $S = \mathbb{R}$.

4. For each of the following functions over given domains, graphically find all local and global minima.

   (a) $f(x) = x^3 + 2x^2 - 2$ over $[-2, 2]$.

   (b) $f(x) = -x^2$ over $(-1, 0] \cup [1, 2]$.

   (c) $f(x) = e^x$ over $\mathbb{R}$.

5. For each of the following single-variate twice-differentiable functions, analytically determine whether it is convex, concave, or neither in the given domain:

   (a) $f(x) = x(x-1)(x+2)$ over $(-\infty, -1]$.
   (b) $f(x) = e^x + x^3$ over $[0, \infty)$.
   (c) $f(x) = \frac{1}{x} + x$ over $(0, \infty)$.

   Is it true that the sum of two twice-differentiable convex functions is still a convex function? Why or why not?

6. The problem for finding the maximum area inside a rectangle whose lengths of the four edges sum to 1 can be formulated as follows. Let x and y be the height and width of the rectangle, the formulation is

$$\begin{aligned} \max_{x\geq 0, y\geq 0} \quad & xy \\ \text{s.t.} \quad & x + y = \frac{1}{2}. \end{aligned}$$

(a) Is the feasible region convex?

(b) Explain why the NLP is equivalent to the one below:

$$\begin{aligned} \max \quad & x\left(\frac{1}{2} - x\right) \\ \text{s.t.} \quad & 0 \leq x \leq \frac{1}{2}. \end{aligned}$$

(c) Solve the above single-variate NLP and find an optimal solution.

7. A retailer is importing a product from an overseas supplier. If there are q units on the market, the unit price of the product will be $a - bq$ dollars (this is called the *market clearing price*). The unit procurement cost of the product is c. The retailer wants to determine a procurement quantity that maximizes its profit.

   (a) Formulate an NLP that maximizes the retailer's profit.

   (b) Solve the NLP.

   (c) Determine how a, b, and c affect the optimal procurement quantity and the associated profit.

   (d) Provide economic interpretations for your answers above.

8. Each month, a gas station sells 4,000 gallons of gasoline. Each time the parent company refills the station's tanks, it charges the station a fixed cost \$50 plus a variable cost \$0.7 per gallon. The annual cost of holding a gallon of gasoline is \$0.3. Suppose the demand rate is constant and no shortage is allowed. We want to minimize the annual total cost.

   (a) How large should the station's one order be?

   (b) How many orders per year will be placed in average?

   (c) How long will it be between orders (how long is the cycle time)?

   (d) Suppose that there is an *ordering lead time* $L > 0$, which is the amount of time it takes to get the ordered gasoline after an order is placed. Suppose that $L \leq T^*$, where T^* is the optimal cycle time. Explain why the optimal *reorder point* is to order when the on-hand inventory level is LD, as D is the annual demand.

(e) If the lead time is a half month, what is the reorder point?

(f) If the lead time is 2.2 months, what is the reorder point?

9. Consider an EOQ problem with annual demand D, unit holding cost h per year, and unit ordering cost K. Let $TC(q)$ be the total holding and ordering cost under an order quantity q and q^* be the optimal order quantity. For some $r > 0$, let

$$f(r) = \frac{TC(rq^*)}{TC(q^*)}.$$

   (a) Express $f(r)$ as a function of r.
   (b) Depict $f(r)$ for $r \in [\frac{1}{2}, 2]$. What are the values of $f(\frac{1}{2})$ and $f(2)$?
   (c) Suppose the estimation of K was wrong: It is K, but we thought it is $2K$. Let q^* be the "true" EOQ, what is our "wrong" EOQ as a function of q^*?
   (d) How much do we lose by using the wrong EOQ? Please compare the annual total costs by using q^* and the wrong EOQ.