Operations Research, Spring 2020 (108-2)
Homework 2

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1 Problems

1. (25 points; 5 points each) Consider the following primal LP

$$
\begin{aligned}
\max \quad & 6x_1 + 4x_2 + 5x_3 + 7x_4 \\
\text{s.t.} \quad & 2x_1 + 4x_2 + 3x_3 + x_4 \leq 15 \\
& 3x_1 + 3x_2 + 2x_3 + 2x_4 \geq 18 \\
& 2x_1 + 2x_2 + 3x_3 + 4x_4 \leq 20 \\
& x_j \geq 0 \quad \forall j = 1, ..., 4.
\end{aligned}
$$

   (a) Find its dual LP.

   (b) Recall that "a solution x is optimal to a primal LP if and only if (1) x is feasible to the primal LP, and (2) there is a solution y that is feasible to the dual LP, and (3) x and y satisfies the complementary slackness relationship." Having this in mind, prove or disprove that $\bar{x} = (4, 0, 0, 3)$ is an optimal solution to the primal LP without solving the primal or dual LPs.

   Note. "Solving" an LP means to find its optimal solution from scratch (e.g., using the graphical approach or simplex method). Proving one solution is optimal is not "solving" an LP and is thus allowed.

   (c) Prove or disprove that $\bar{y} = (\frac{5}{3}, 0, \frac{4}{3})$ is an optimal solution to the dual LP without solving the primal or dual LPs.

   (d) Find the shadow prices for the primal constraints without solving the primal or dual LP.

   (e) Suppose that the second constraint of the primal LP becomes

$$3x_1 + 3x_2 + 2x_3 + 2x_4 \leq 18.$$

   Prove or disprove that $\bar{x} = (4, 0, 0, 3)$ is an optimal solution to the new primal LP without solving the primal or dual LP.

2. (25 points) Consider the following integer program, which represents an instance of the "multi-copy knapsack problem:"

$$
\begin{aligned}
\max \quad & 5x_1 + 2x_2 + 4x_3 + 3x_4 + 8x_5 \\
\text{s.t.} \quad & 3x_1 + 3x_2 + 2x_3 + 6x_4 + 3x_5 \leq 14 \\
& x_i \geq 0 \quad \forall i = 1, ..., 5. \\
& x_i \in \mathbb{Z} \quad \forall i = 1, ..., 5.
\end{aligned}
$$

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(a) (5 points) Formulate the linear relaxation of the integer program.
Hint. What should be the upper bound of each x_i?

(b) (5 points) Use the greedy algorithm introduced in class to solve the linear relaxation of this integer program.

(c) (10 points) Use the branch-and-bound algorithm to solve the original integer program. Depict the full branch-and-bound tree. Do not write down the solution process of each node; write down just an optimal solution and its objective value of each node. Choose any branching strategy you like.

(d) (5 points) Prove or disprove the following statement: For any multi-copy knapsack problem, an optimal solution selects at most two items.

3. (30 points; 10 points each) A company needs to send n engineers to m customers for device maintenance in the next day. The customers locate at different locations. Let the company be at location 0 and customer i be at location i, we have the traveling time between locations i and j as d_{ij}. The time it takes to complete the maintenance for customer i is p_i. Each customer has its maintenance *time window*, which is specified by the earliest time for the maintenance to start and the latest time for the maintenance to be completed. We denote customer i's time window by $[s_i, t_i]$. Therefore, to start the maintenance for customer i at time x_i, we have the *time window constraints* $x_i \geq s_i$ and $x_i + p_i \leq t_i$. An engineer may leave the company at most once. In other words, she/he may complete at most one tour.

   (a) Let $n = 1$. Formulate an IP that can find a maintenance plan (the starting time of the maintenance for each customer) to minimize the total traveling time of the engineer while satisfying the time window constraints. Note that to satisfy the time window constraint, you may let the engineer arrive a customer's location earlier than the starting time of the maintenance for that customer. The arrival time is not restricted by the time window constraint.

   (b) Let $n > 1$. Do Part (a) again.

   (c) Let $n > 1$. Now the company may violate the time window constraint. However, if for customer i the maintenance starting time x_i is earlier than s_i, the company must pay $c(s_i - x_i)$ to customer i as a penalty. Similarly, if the maintenance completion time $x_i + p_i$ is later than t_i, the company must pay $d(x_i + p_i - t_i)$ as a penalty. There is no penalty if the time window constraint is satisfied. Formulate an IP that can find a maintenance plan to minimize the total amount of penalty.

4. (20 points; 10 points each) Please answer the following questions.

   (a) Use your own words to explain the meaning and applications of shadow prices.

   (b) May we solve the traveling salesperson problem (TSP) using the branch-and-bound algorithm? Why or why not?

2 Submission rules

The deadline of this assignment is ***1:00 am, May 7***. Works submitted between 1:00 am and 2:00 am will get 10 points deducted as a penalty. Submissions later than 2:00 am will not be accepted. Please submit a PDF file containing your answers for all problems to NTU COOL. Include your student ID and name in the first page of your PDF file.