

# Andreas Ng - MACHINE LEARNING - Stanford

## Linear algebra with one variable - regression ①

Hypothesis:  $h_{\theta}(x) = \theta_0 + \theta_1 x$

Parameters:  $\theta_0, \theta_1$

Cost function:  $J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$

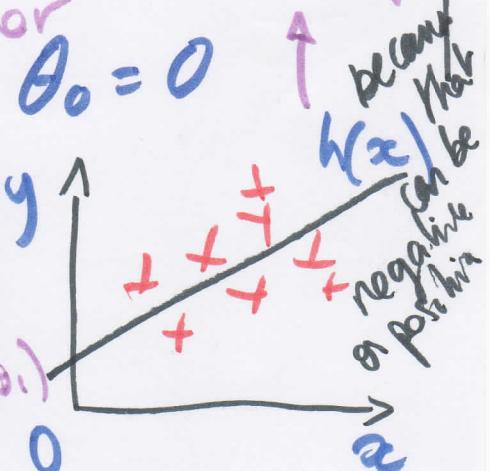
Goal: Minimize  $J(\theta_0, \theta_1)$  over  $\theta_0, \theta_1$  minimize the average error

Simplified:  $h_{\theta}(x) = \theta_1 x$

$J(\theta_1) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$

Minimize  $J(\theta_1)$

find the value  
that minimizes  $J(\theta_1)$



## Gradient Descent

Repeat until convergence for  $j = 0$  and  $j = 1$

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$$

derivative

## Gradient Descent for Linear Regression

Repeat until convergence {

$$\theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) \frac{\partial}{\partial \theta_0} J(\theta)$$

$$\theta_1 := \theta_1 - \alpha \frac{1}{m} \sum_{i=1}^m ((h_{\theta}(x^{(i)}) - y^{(i)}) x^{(i)}) \}$$

Update  $\theta_0$  and  $\theta_1$  simultaneously

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## Gradient Descent for multiple variables

(1)

Hypothesis:  $h_{\theta}(x) = \theta^T x = \theta_0 x_0 + \theta_1 x_1 + \dots + \theta_n x_n$

Parameters:  $\theta_0, \theta_1, \dots, \theta_n$   $[\theta_0, \theta_1, \dots, \theta_n] \leftarrow \theta^T$

Cost function =  $J(\theta_0, \theta_1, \dots, \theta_n) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$

Gradient descent = Repeat {  
 $\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \dots, \theta_n)$  }

(simultaneously update for every  $j = 0, \dots, n$ )  
 $x_0^{(i)} = 1$

• Repeat {

$\theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$  }  
(simultaneously update  $\theta_j$  for  $j = 0, \dots, n$ )

## Feature scaling

• Make sure features are on a similar scale

→ converge faster

ex:  $x_1 = \text{size (0-2000 feet}^2)$   $x_2 = \text{nb of bedrooms (1-5)}$

$x_1 = \frac{\text{size (feet}^2)}{2000}$   $x_2 = \frac{\text{nb of bedrooms}}{5}$

$0 \leq x_1 \leq 1 \quad 0 \leq x_2 \leq 1$

multiple features (ex)  
variables

• Get every feature into approximately a range  
 $-1 \leq x_i \leq 1$

size	Nb of bed	Age	Price
2106	5	!	460
:	:	:	:
:	:	:	:

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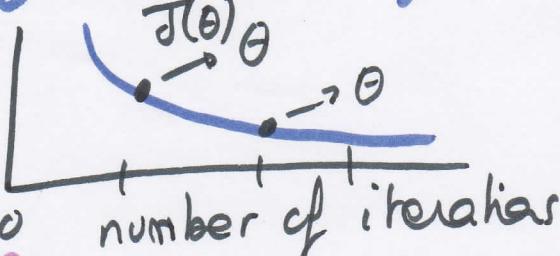
(3)

## Mean normalization

- Replace  $x_i$  with  $\frac{x_i - \bar{x}_i}{s_i}$  to make features have approximately zero mean (Do not apply to  $x_0 = 1$ )
- Ex.  $x_1 = \frac{\text{size} - 1000}{2000}$     $x_2 = \frac{\# \text{bedrooms} - 2}{5}$
- $-0.5 \leq x_1 \leq 0.5$     $-0.5 \leq x_2 \leq 0.5$
- $x_1 \leftarrow \frac{x_1 - \bar{x}_1}{s_1}$  → average value of  $x_1$  in training set → mean  
 $s_1$  → range (max - min) or standard deviation

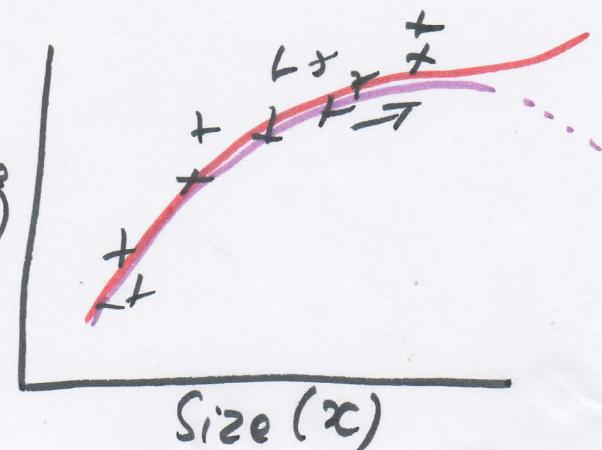
"Debugging": How to make sure gradient descent is working correctly

- For sufficiently small  $\alpha$ ,  $J(\theta)$  should decrease on every iteration
  - But if  $\alpha$  is too small, gradient descent can be slow to converge
  - If  $\alpha$  is too large:  $J(\theta)$  may not decrease on every iteration; may not converge (slow convergence also possible)
- To choose  $\alpha$ , try ..., 0.001, 0.01, 0.1, 1, ...



## Polynomial regression (1/2)

- $\theta_0 + \theta_1 x + \theta_2 x^2$
- $\theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3$



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## Polynomial regression (2/2)

algorithms can help  
to find what polynomial  
regression to use, analyze  
the data

$$h_{\theta}(x) = \theta_0 + \theta_1 x + \theta_2 x^2$$

### Normal equation

- m examples  $(x^{(1)}, y^{(1)}), \dots, (x^{(m)}, y^{(m)})$ ; n features

$$x^{(i)} = \begin{bmatrix} x_0^{(i)} \\ x_1^{(i)} \\ \vdots \\ x_n^{(i)} \end{bmatrix} \in \mathbb{R}^{n+1}$$

$$x_0 = 1$$

$$X = \begin{bmatrix} - & (x^{(1)})^T \\ - & (x^{(2)})^T \\ - & \vdots \\ - & (x^{(m)})^T \end{bmatrix} \quad \begin{array}{l} m \times \\ \text{examples} \\ (n+1) \end{array}$$

e.g. if  $x^{(i)} = \begin{bmatrix} 1 \\ x_1^{(i)} \end{bmatrix}$

$$X = \begin{bmatrix} 1 & x_1^{(1)} \\ 1 & x_1^{(2)} \\ \vdots & \vdots \\ 1 & x_1^{(m)} \end{bmatrix}_{m \times 2} \quad y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

inverse

$$\theta = (X^T X)^{-1} X^T y$$

Octave: ~~pinv(X' \* X) \* X' \* y~~

do not work for  
some complex  
algorithms

- m training examples, n features

## Gradient Descent

- Need to choose  $\alpha$
- Needs many iterations
- Works well even when  $n$  is large

at this point, it might be better  
to use gradient descent

$$n = 10^6$$

## Normal Equation

- No need to choose  $\alpha$
- Don't need to iterate
- Need to compute  $(X^T X)^{-1}$
- Slow if  $n$  is very large

$$n = 100, 1000, 10000 \dots$$

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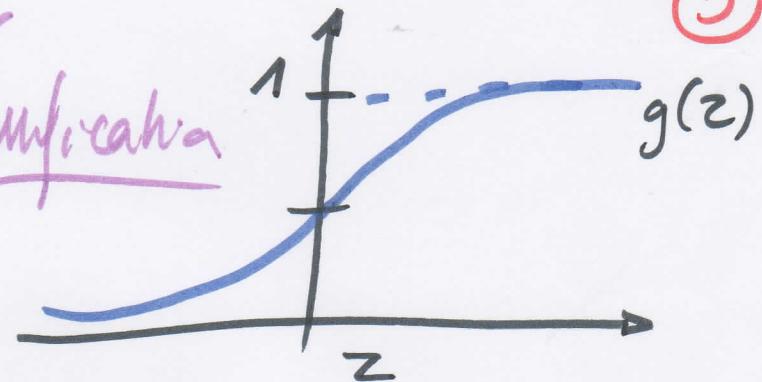
(5)

Logistic regression = sigmoid

for

Classification

$$\cdot h_{\theta}(x) = g(\theta^T x)$$



$$\cdot g(z) = \frac{1}{1 + e^{-z}}$$

$$0 \leq h_{\theta}(x) \leq 1$$

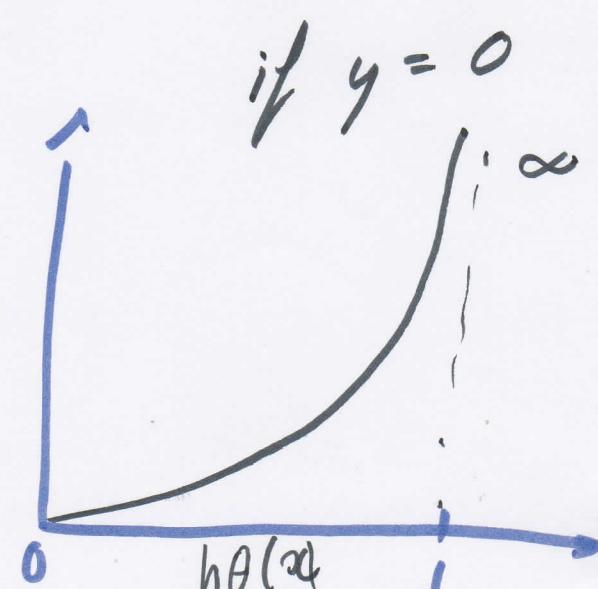
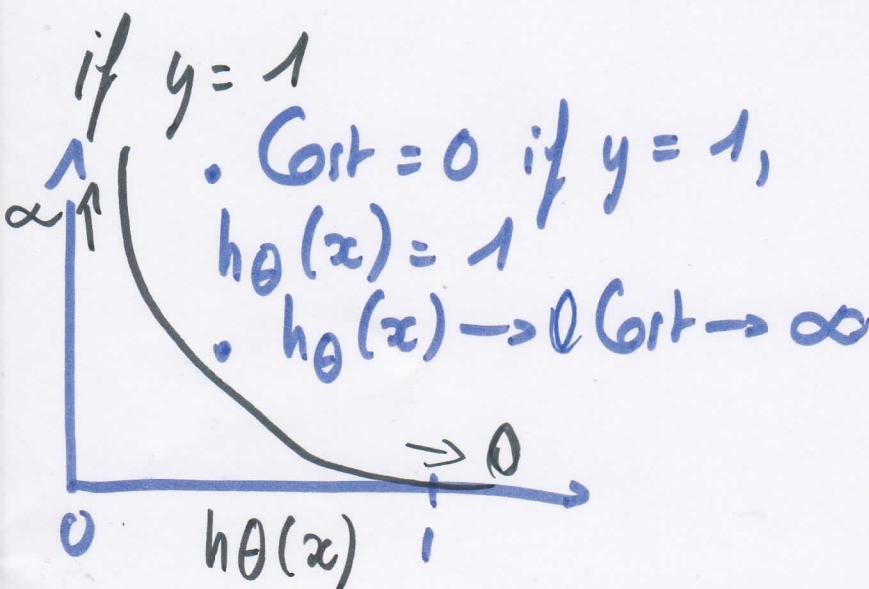
$$\cdot h_{\theta}(x) = \frac{1}{1 + e^{-\theta^T x}}$$

$$\cdot \text{Training set: } \{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})\}$$

$$\cdot m \text{ examples } x \in \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n+1} \quad x_0 = 1, y \in \{0, 1\}$$

Logistic regression cost function (1/2)

$$\cdot \text{Cost}(h_{\theta}(x^{(i)}; y^{(i)})) = \begin{cases} -\log(h_{\theta}(x)) & \text{if } y = 1 \\ -\log(1-h_{\theta}(x)) & \text{if } y = 0 \end{cases}$$



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## Logistic regression cost function (2/2)

⑥

$$\cdot J(\theta) = \frac{1}{m} \sum_{i=1}^m \text{Grt}(h_\theta(x^{(i)}), y^{(i)})$$

Note:  $y = 0$  or  $1$  always

$$\cdot J(\theta) = -\frac{1}{m} \left[ \sum_{i=1}^m y^{(i)} \log h_\theta(x^{(i)}) + (1-y^{(i)}) \log (1-h_\theta(x^{(i)})) \right]$$

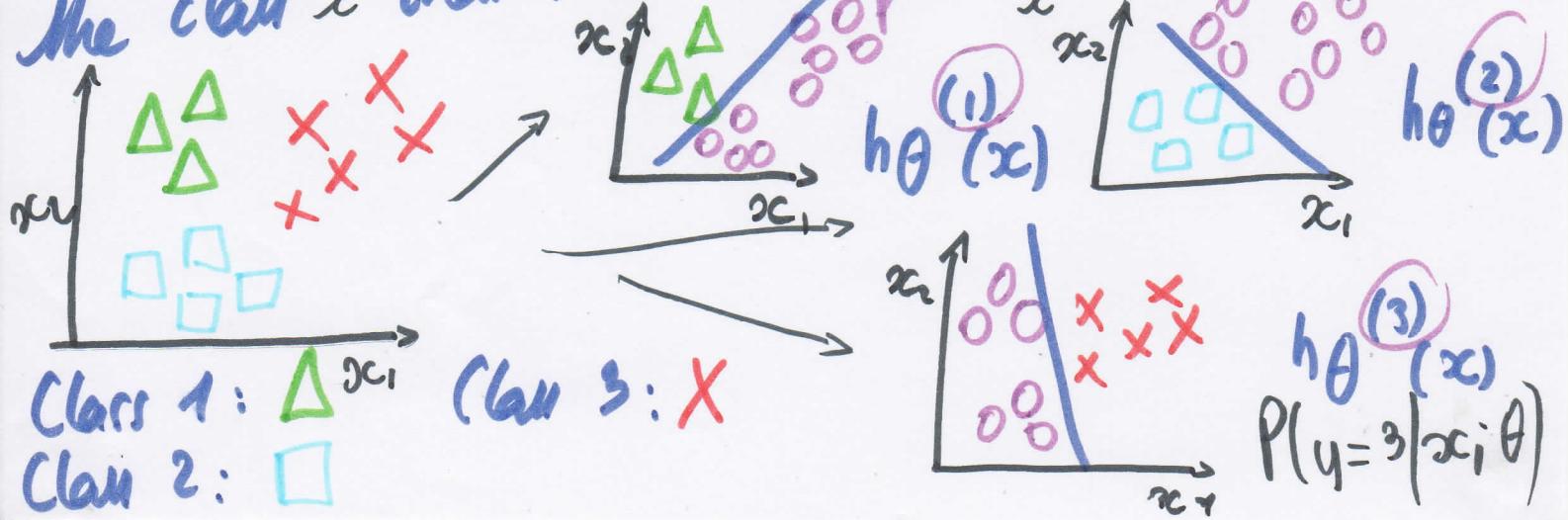
Want  $\min_{\theta} J(\theta)$ :

$$\text{Repeat } \{ \theta_j := \theta_j - \alpha \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)} \}$$

Simultaneously update all  $\theta_j$

## Multiclass classification - one vs all

- Train a logistic regression classifier  $h_\theta^{(i)}(x)$  for each class  $i$  to predict the probability that  $y = i$ .  $h_\theta^{(i)}(x) = P(y=i|x; \theta)$  ( $i = 1, 2, 3$ )
- On a new input  $x$ , to make a prediction, pick the class  $i$  that maximized  $\max_i h_\theta^{(i)}(x)$ .

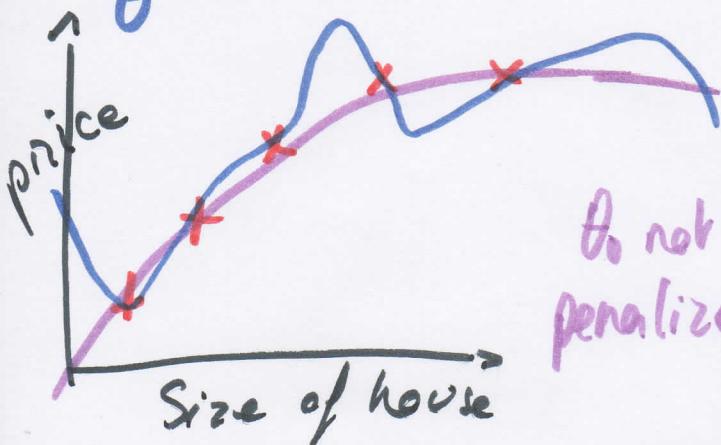


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## Regularization - Regularized linear regression (7)

$$\cdot J(\theta) = \frac{1}{2m} \left[ \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)})^2 + \lambda \sum_{j=1}^n \theta_j^2 \right]$$

$$\cdot \min_{\theta} J(\theta)$$



it is important to choose well  
the regularization parameter  $\lambda$

$\theta_0$  not penalized  $\rightarrow$  gradient descent

$$\cdot \theta_j := \theta_j \left(1 - \alpha \frac{\lambda}{m}\right) - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

$$\cdot \text{Normal equation} \quad \theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)} \rightarrow \min_{\theta} J(\theta)$$

$$X = \begin{bmatrix} (x^{(0)})^T \\ (x^{(1)})^T \\ \vdots \\ (x^{(m)})^T \end{bmatrix} \quad y = \begin{bmatrix} y^{(0)} \\ \vdots \\ y^{(m)} \end{bmatrix} \in \mathbb{R}^m$$

$$\theta = (X^T X + \lambda \begin{bmatrix} 0 & \dots & 0 \end{bmatrix})^{-1} X^T y$$

## Regularized logistic regression

$$J(\theta) = \left[ -\frac{1}{m} \sum_{i=1}^m y^{(i)} \log(h_\theta(x^{(i)})) + (1-y^{(i)}) \log(1 - h_\theta(x^{(i)})) \right] + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$$

$\rightarrow$  penalize  $\theta_1, \theta_2, \dots$   
for being too large

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(8)

data

## Supervised learning

Gives the "right answer" for each example in the

## Regression problem

Predict real valued output

## Classification

Discrete-valued output

## Supervised learning - training set

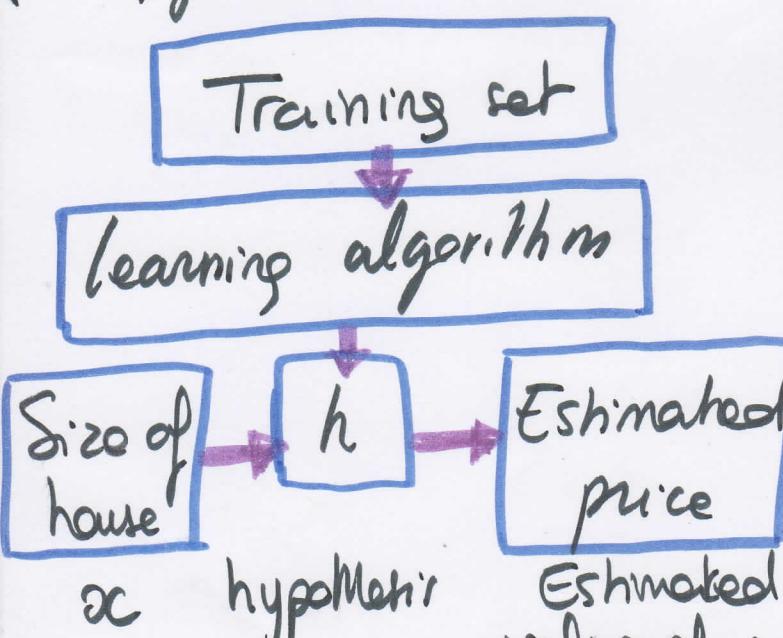
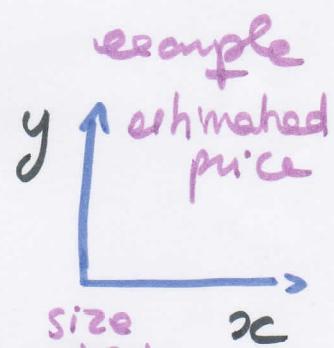
$m$  = number of training examples

$x$ 's = "input" variable / features

$y$ 's = "output" variable / "target" variable

$(x, y)$  = one training example

$(x^{(i)}, y^{(i)})$  =  $i^{\text{th}}$  training example

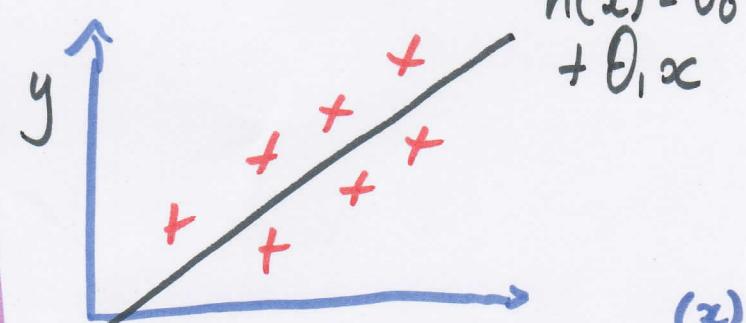


$h$  maps from  $x$ 's to  $y$ 's

How do we represent  $h$ ?

$$h_{\theta}(x) = \theta_0 + \theta_1 x$$

Shorthand:  $h(x)$



Linear regression w/ one variable  
Univariate linear regression  
to one variable

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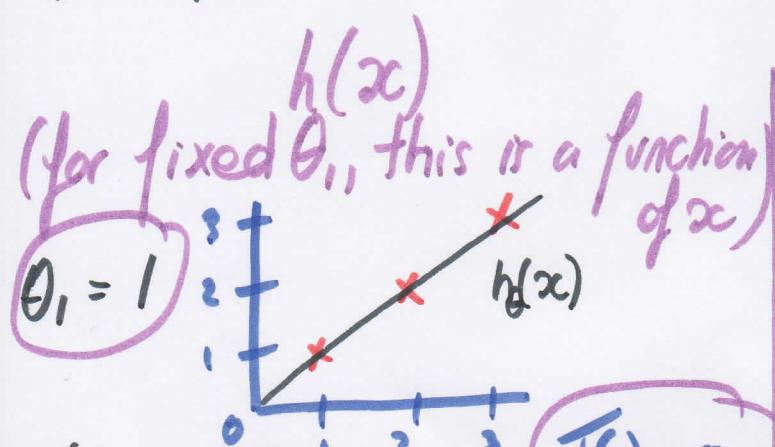
(9)

## Cost function

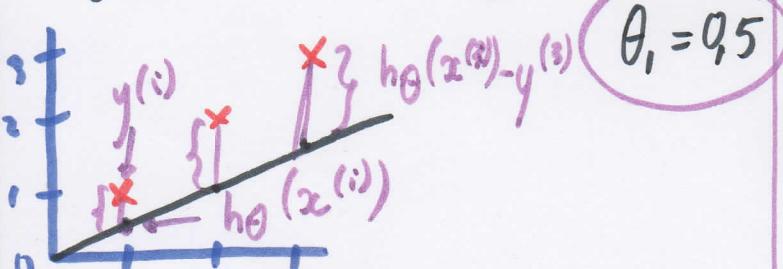
Hypothesis:  $h\theta(x) = \theta_0 + \theta_1 x$  - parameters:  $\theta_0, \theta_1$

Idea: Choose  $\theta_0, \theta_1$  so that  $h\theta(x)$  is close to  $y$  for our training examples  $(x, y)$

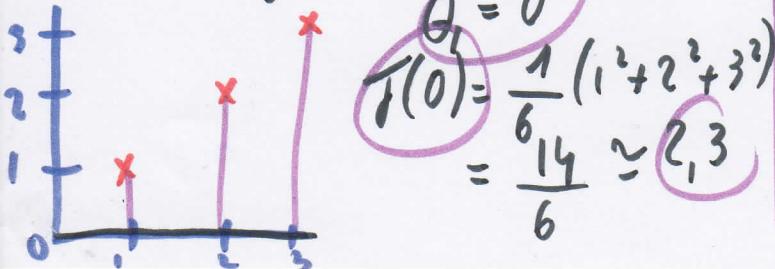
Cost function = squared error function



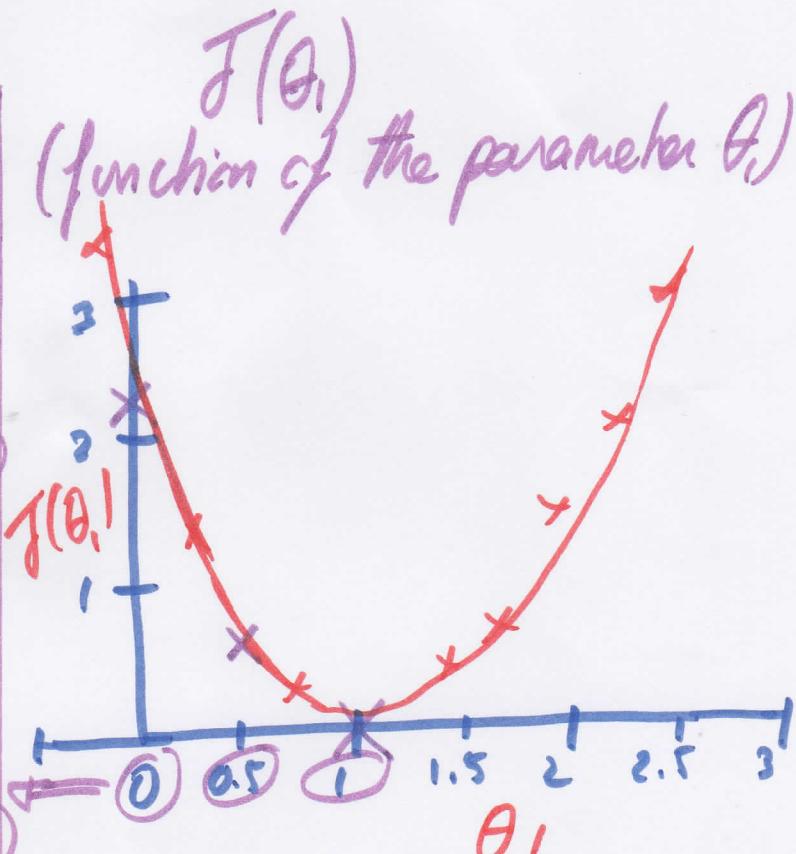
$$\begin{aligned} h\theta(x^{(i)}) &= y^{(i)} \\ J(\theta_1) &= \frac{1}{2M} \sum_{i=1}^m (h\theta(x^{(i)}) - y^{(i)})^2 \\ &= \frac{1}{2 \times 3} (0^2 + 0^2 + 0^2) = 0^2 = 0 \end{aligned}$$



$$\begin{aligned} J(0.5) &= \frac{1}{2 \times 3} [(0.5 \cdot 1 - 1)^2 + (0.5 \cdot 2 - 2)^2 + (0.5 \cdot 3 - 3)^2] \\ &= \frac{3.5}{6} \approx 0.58 \end{aligned}$$



$$\begin{aligned} J(0) &= \frac{1}{2 \times 3} (1^2 + 2^2 + 3^2) \\ &= \frac{14}{6} \approx 2.3 \end{aligned}$$



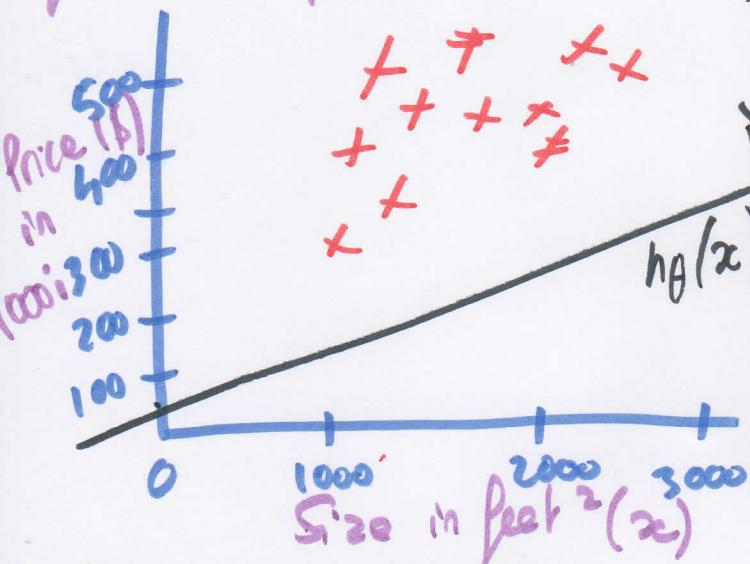
Each value of  $\theta_1$  corresponds to a hypothesis  $h\theta_1(x)$ , which means finding a straight line that fits the data well.

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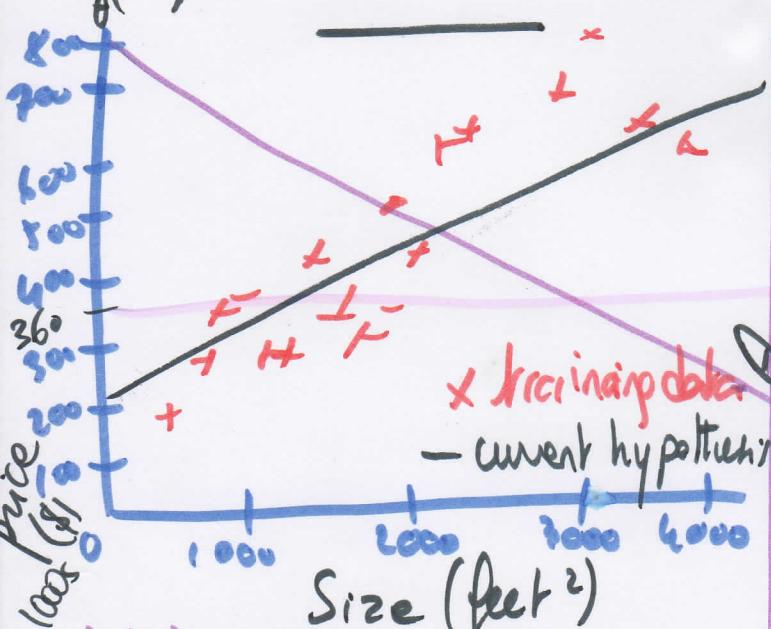
(10)

$$h_{\theta}(x)$$

(for fixed  $\theta_0, \theta_1$ , this is a function of  $x$ )



$$h_{\theta}(x) = 50 + 0.06x$$



$$h(x) = 800 - 0.15x$$

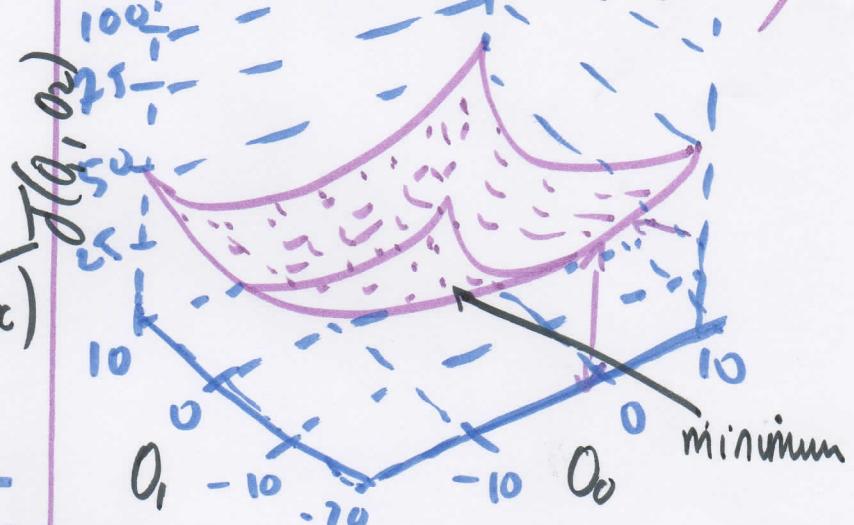
$$h(x) = 360 + 0x$$

$$h(x) = 220 + 0.15x$$

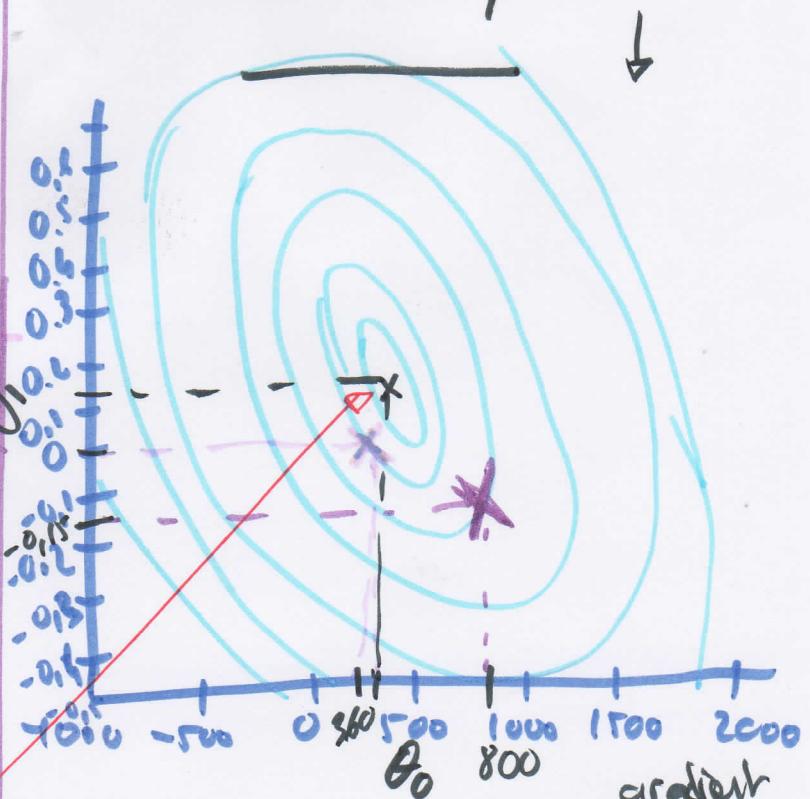
fit better  $\uparrow$  the training set

$$J(\theta_0, \theta_1)$$

(function of the parameters  $\theta_0, \theta_1$ )



contour plot



Outline

- Start with some  $\theta_0, \theta_1$
- Keep changing  $\theta_0, \theta_1$  to reduce  $J(\theta_0, \theta_1)$  until we hopefully end up at a minimum

gradient descent

## Gradient descent

- Minimize the cost function  $J(\theta_0, \theta_1)$  and other functions
- Have some function  $J(\theta_0, \theta_1)$  Want  $\min_{\theta_0, \theta_1} J(\theta_0, \theta_1)$  as well with  $J(\theta_0, \theta_1, \theta_2, \dots, \theta_n)$
- Outline:
  - ⇒ Start with some  $\theta_0, \theta_1$
  - ⇒ Keep changing  $\theta_0, \theta_1$  to reduce  $J(\theta_0, \theta_1)$  until we hopefully end up at a minimum
  - Repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1) \quad (\text{for } j=0 \text{ and } j=1)$$

Simultaneous update

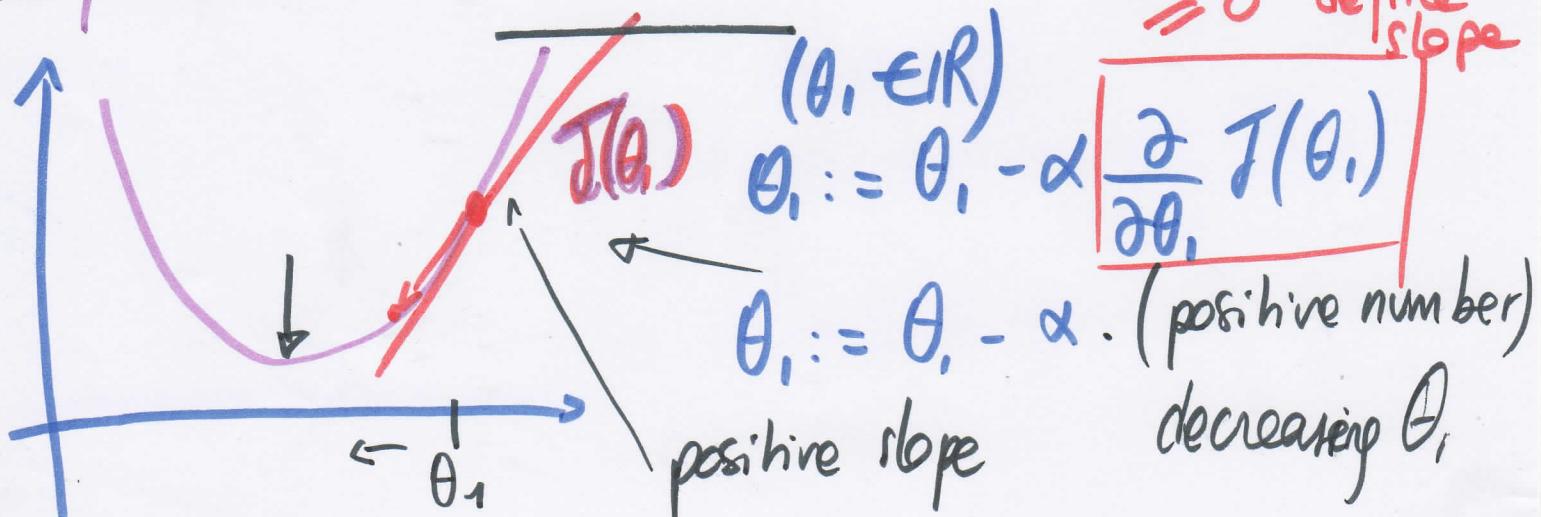
alpha = learning rate

it controls how big the step is when going down with gradient descent

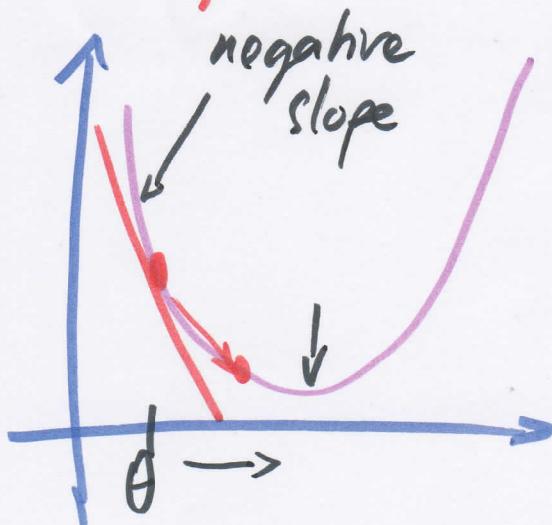
if small = baby step  
if large = huge step

$$\begin{aligned}\theta_0 &:= \text{temp}_0 \\ \theta_1 &:= \text{temp}_1\end{aligned}$$

assignment



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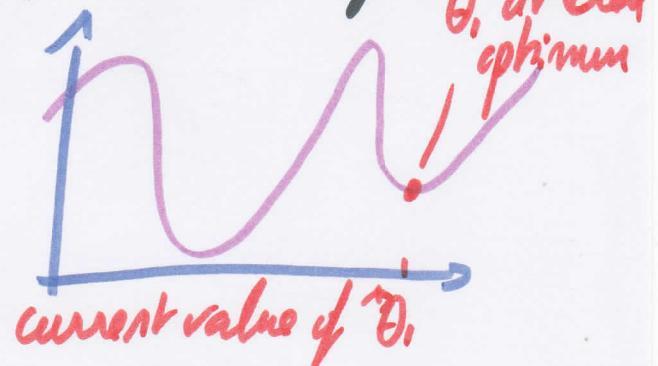
$$\theta_0 := \theta_0 - \alpha \cdot \frac{d \frac{\partial}{\partial \theta} J(\theta_0)}{d \theta} \quad \text{(2)}$$

$\theta_0 := \theta_0 - \alpha \cdot \frac{d \frac{\partial}{\partial \theta} J(\theta_0)}{d \theta}$  (negative number)  
increasing  $\theta_0$

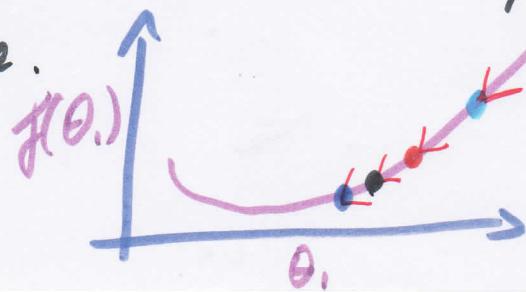
$\alpha$  too small or too large

$$\theta_0 := \theta_0 - \alpha \cdot \frac{d}{d \theta} J(\theta_0)$$

- If  $\alpha$  is too small, gradient descent can be slow.
- if  $\alpha$  is too large, gradient descent can overshoot the minimum. It may fail to converge, or even diverge.
- If  $\theta_0$  is at a local optimum of  $J(\theta_0)$ , one step of gradient descent will leave  $\theta_0$  unchanged



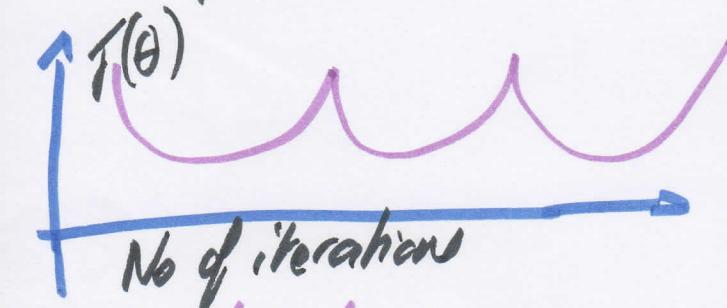
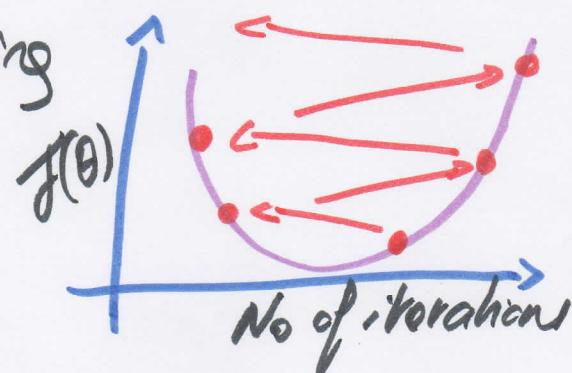
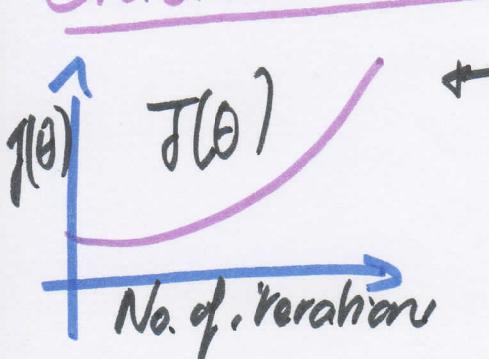
- Gradient descent can converge to a local minimum, even with the learning rate  $\alpha$  fixed.  
As we approach a local minimum, gradient descent will automatically take smaller steps. So no need to decrease  $\alpha$  over time.



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(3)

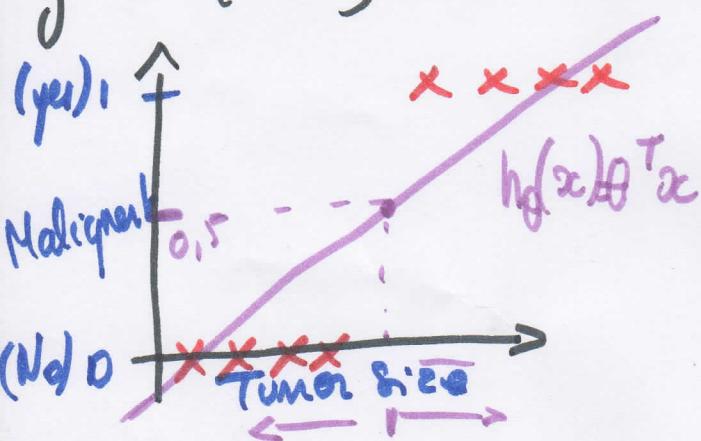
Gradient descent not working correctly



## Classification

$y \in \{0, 1\}$

0: "Negative Class" (e.g. No)  
1: "Positive Class" (e.g. Yes)



Threshold classifier output  
 $h_\theta(x)$  at 0.5:

- If  $h_\theta(x) \geq 0.5$ , predict " $y=1$ "
- If  $h_\theta(x) < 0.5$ , predict " $y=0$ "

Classification:  $y = 0$  or  $1$ , however  $h_\theta(x)$  can be  $\geq 1$   
so: no linear regression but logistic regression

$$\text{where: } 0 \leq h_\theta(x) \leq 1$$

Interpretation of Hypothesis Output with logistic regression

$h_\theta(x)$  = estimated probability that  $y=1$  on input  $x$

Example: If  $x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \text{Tumor size} \end{bmatrix}$   $h_\theta(x) = 0.7$

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Tell patient that 70% chance of tumor being malignant

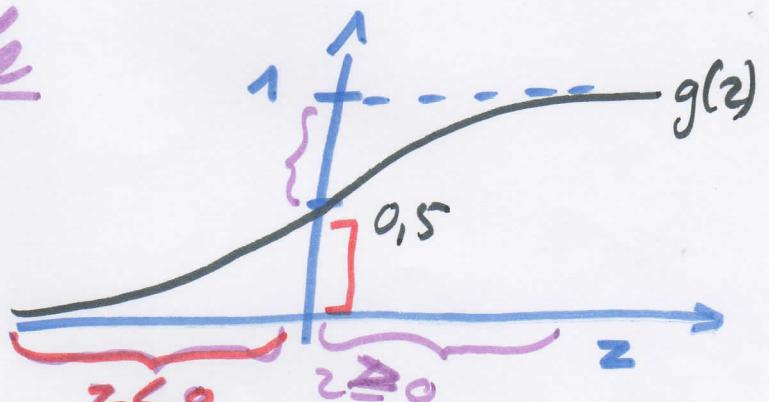
$h_{\theta}(x) = P(y=1|x; \theta)$  probability that  $y=1$ , given  $x$ , parameterized by  $\theta$

- $P(y=0|x; \theta) + P(y=1|x; \theta) = 1$
- $P(y=0|x; \theta) = 1 - P(y=1|x; \theta)$

## logistic regression example

$$h_{\theta}(x) = g(\theta^T x);$$

$$g(z) = \frac{1}{1+e^{-z}}$$

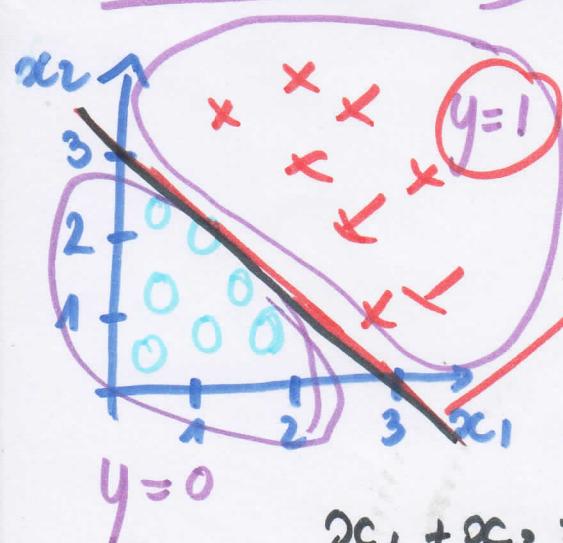


Suppose predict "y = 1" if  $h_{\theta}(x) \geq 0.5$

$g(z) \geq 0.5$  when  $z \geq 0$   $h_{\theta}(x) = g(\theta^T x) \geq 0.5$   
whenever  $\theta^T x \geq 0$

predict "y = 0" if  $h_{\theta}(x) < 0.5$   $h_{\theta}(x) = g(\theta^T x)$   $g(z) < 0.5$   
 $\theta^T x < 0$

## Decision boundary



$$h_{\theta}(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2)$$

**Decision boundary**  $\theta = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$   
property of the hypothesis  
but not the data set (in this case)

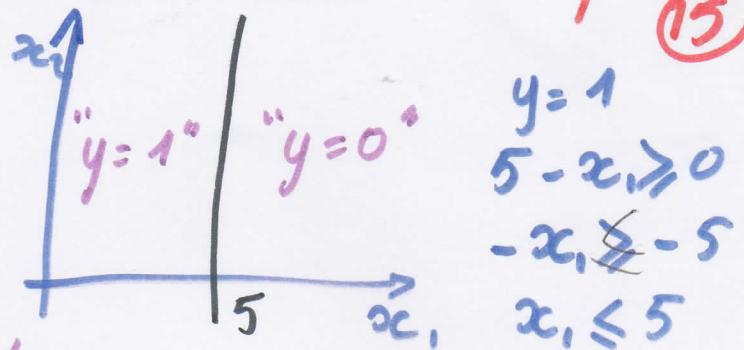
Predict "y = 1" if  $\frac{-3 + x_1 + x_2 \geq 0}{\theta^T x}$

$$x_1 + x_2 = 3 \quad h_{\theta}(x) = 0.5 \Rightarrow x_1 + x_2 \geq 3$$

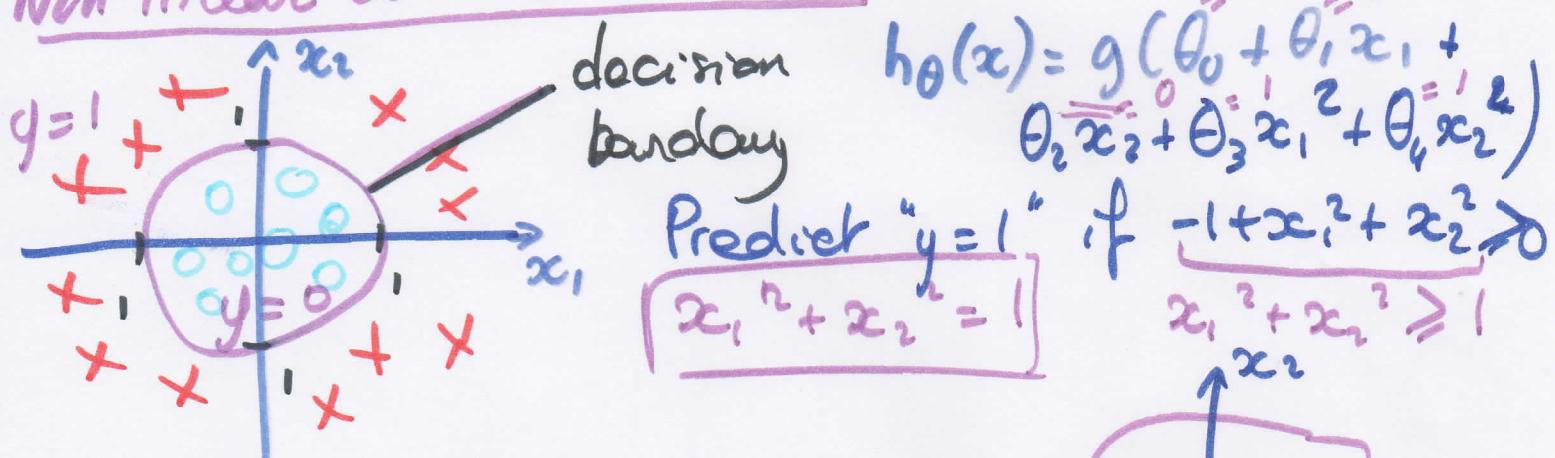
$$x_1 + x_2 < 3 \quad y = 0$$

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Decision boundary  
 of  $h_{\theta}(x) = g(\sum \theta_j x_j)$   
 with  $\theta_0 = 5, \theta_1 = -1, \theta_2 = 0$

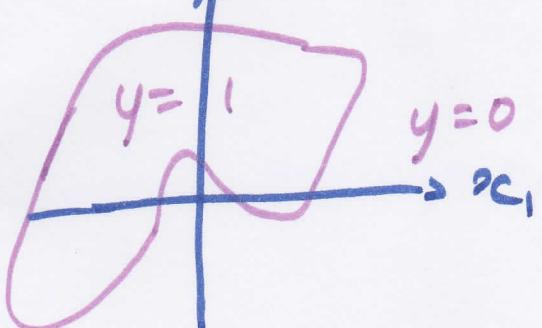


## Non linear decision boundaries



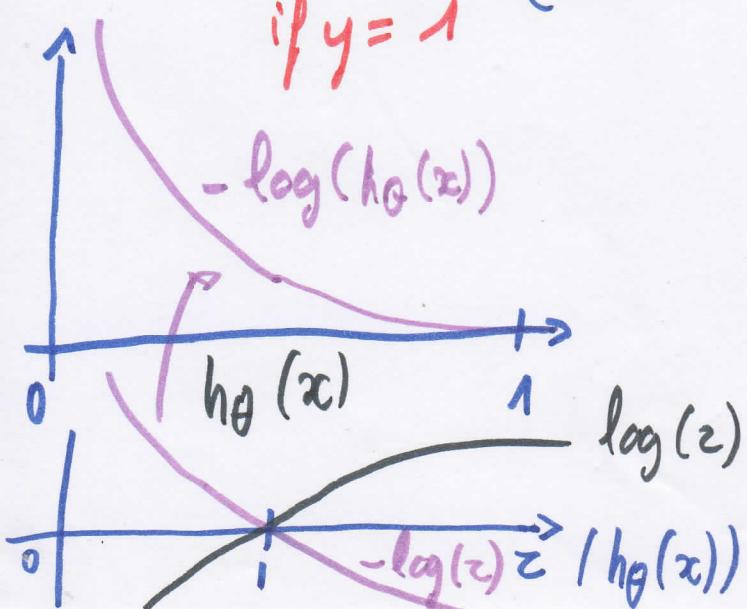
$$h_{\theta}(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_1 x_2 + \theta_5 x_2^2 \dots)$$

→ can be very complex



## Logistic regression cost function

$$\text{Cost}(h_{\theta}(x), y) = \begin{cases} -\log(h_{\theta}(x)) & \text{if } y = 1 \\ -\log(1 - h_{\theta}(x)) & \text{if } y = 0 \end{cases}$$



Cost = 0 if  $y = 1, h_{\theta}(x) = 1$

But as  $h_{\theta}(x) \rightarrow 0$

Cost  $\rightarrow \infty$

Gives intuition that  $h_{\theta}(x) = 0$  (predict  $P(y=1 | x; \theta) = 0$ ), but  $y=1$  will penalize learning algorithm by a very large cost

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## Logistic regression cost function

- if  $h_{\theta}(x) = y$ , then  $\text{cost}(h_{\theta}(x), y) = 0$  (for  $y=0$  and  $y=1$ )
- $\text{Cost}(h_{\theta}(x^{(i)}), y^{(i)}) = \frac{1}{2} (h_{\theta}(x^{(i)}) - y^{(i)})^2$  but not convex so
- if  $y=0$ , then  $\text{cost}(h_{\theta}(x), y) \rightarrow \infty$  as  $h(x) \rightarrow 1$  (not working)
- Regardless of whether  $y=0$  or  $y=1$  if  $h_{\theta}(x)=0.5$ , then  $\text{cost}(h_{\theta}(x), y) > 0$

## Optimization algorithm

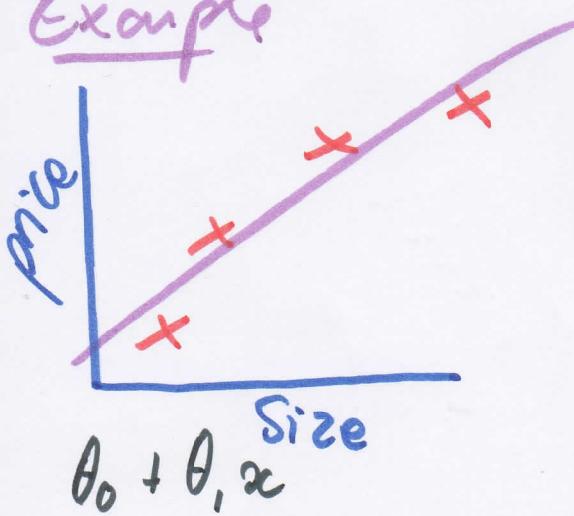
- Gradient descent
- Conjugate gradient
- BFGS
- L-BFGS

## Advantages

- No need to manually pick  $\alpha$
- Often faster than gradient descent
- Disadvantages
- More complex

## Underfitting or overfitting for linear regression

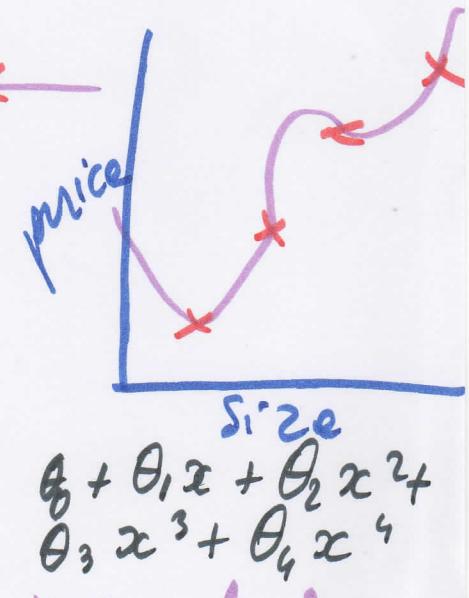
### Example



"Underfit" "High bias"  
(not fitting)



"Just right"

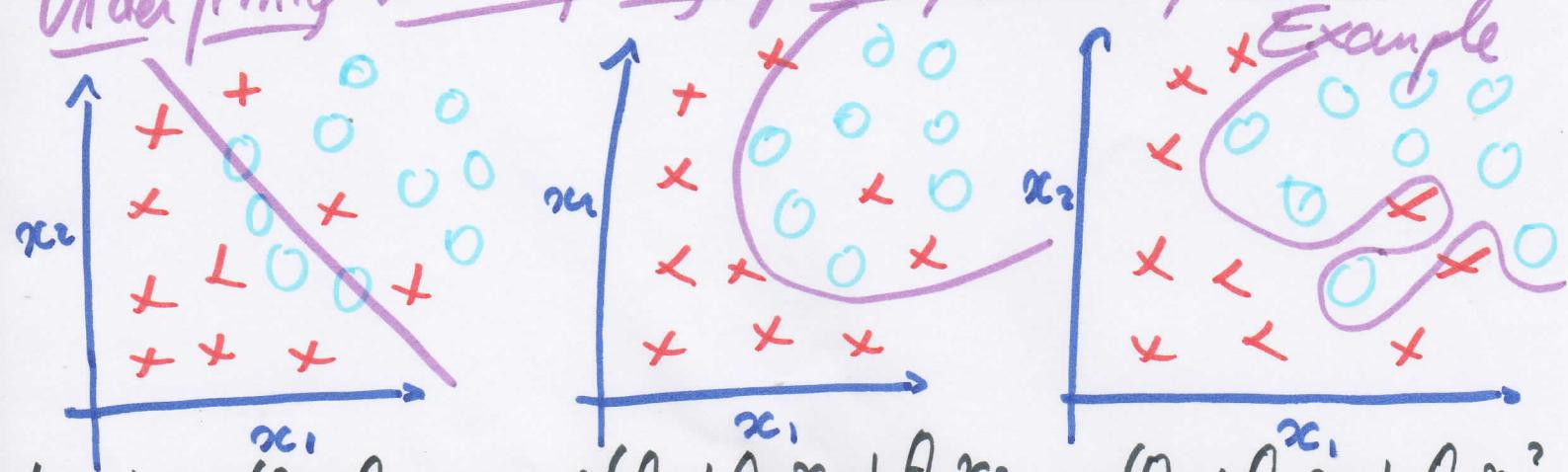


"Overfit"  
"High Variance"

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Overfitting = if we have too many features, the learned hypothesis may fit the training set very well ( $J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)})^2 \approx 0$ ), but fail to generalize to new examples (predict prices on new examples).

## Underfitting & overfitting for logistic regression



$$h_\theta(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2)$$

( $g$  = sigmoid function)

"Underfit"

$$g(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_2^2 + \theta_5 x_1 x_2)$$

"Just right"

$$g(\theta_0 + \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3 x_2 + \theta_4 x_1^4 x_2^2 + \theta_5 x_1^5 x_2 + \dots)$$

"Overfit"

- When we have many features & not a lot of training examples, overfitting can become a problem.
- ex:  $x_1$  = size of house,  $x_2$  = nb. of bedrooms,  $x_3$  = nb. of floors...

## Addressing overfitting: Options

### 1. Reduce no. of features

- Manually select features to keep
- Model selection algorithm

### 2. Regularization

- Keep all the features, but reduce magnitude values of  $\theta_j$
- Works well when we have a lot of features, contribute predicting  $y$

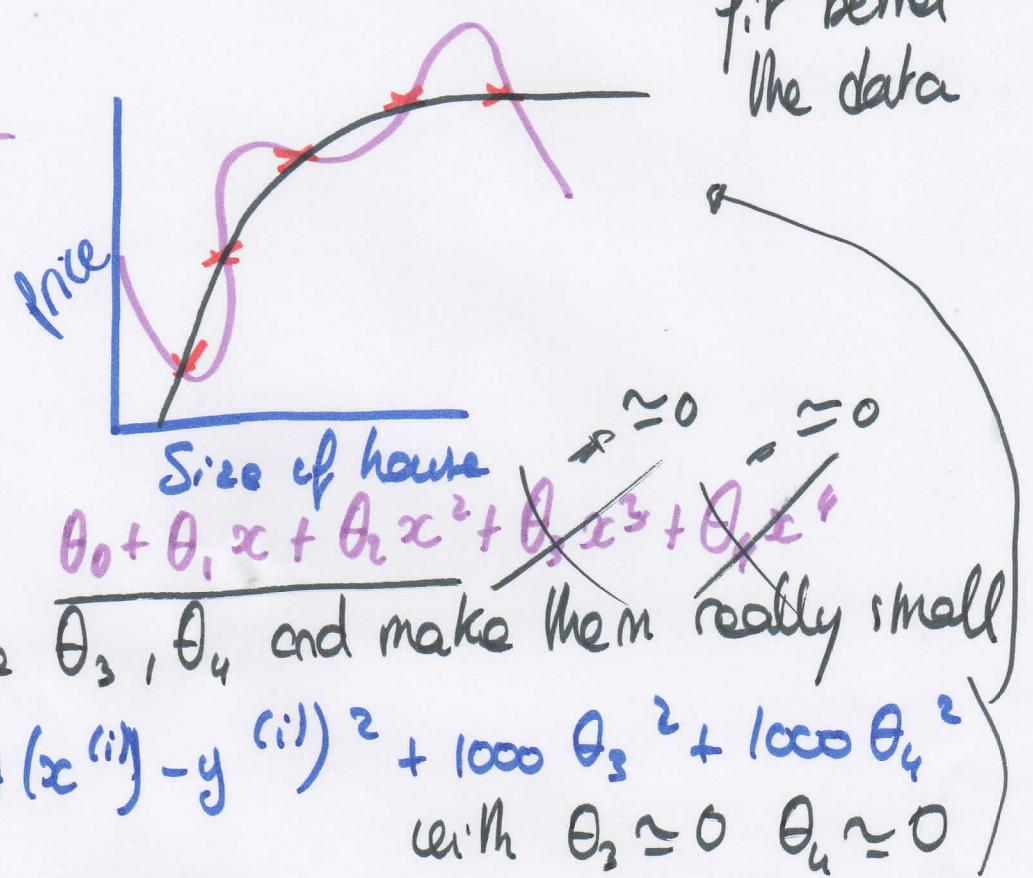
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## Regularization

- Small values for parameters  $\theta_0, \theta_1, \dots, \theta_n$

- "Simpler" hypothesis  
less prone to overfitting

- Intuition



## Regularized gradient descent

$$\text{Repeat } \left\{ \theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)} \right.$$

$$\left. \theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)} + \frac{\lambda}{m} \theta_j \right\}$$

$j = 1, 2, 3, \dots, n$

$$\theta_j := \theta_j \left(1 - \alpha \frac{\lambda}{m}\right) - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

$\theta_1, \theta_2, \dots, \theta_n$  are penalized. Not  $\theta_0$ , that is why  $\neq$  treatment