

Functions at Work

Business Calculus

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Dedicated to the champions of Open Educational Resources
at Bridgewater State University and beyond, and to the
developers of PreTeXt who developed and maintain the
software that made this book possible.

How to use this book

Welcome to Business Calculus. You are beginning an exciting adventure into a new kind of mathematics, one with more applications than you have seen in your previous courses, and one which will require new kinds of practice and problem solving.

This book is the result of many years of teaching and refining this course, with a particular focus on highlighting a variety of ways that this course can be of service to you in your future studies.

The large number of examples with full solutions are intended to aid you in your study of the topic.

The Promise of Active Reading. In the online version of this book, the statement of a problem is shown, but the solution starts out hidden. When you see a question like this,

1. Think about the meaning of the question. *What* do the terms used mean? *Why* is this question being asked. Go back to the previous definitions and explanations to refresh yourself.
2. Try to answer the question yourself by hand, on a piece of paper. Do you know *How* to start? Do you get stuck at some point?
3. Once you've been trying for a little bit, then look at the *Solution*. Did you get the same answer? What steps does the solution use? Why does it use those steps?

This might seem like a lot of extra work, since the solutions are already right there in the text, and you already have many other problems and assignments to get to.

One practical resource is [Make It Stick: The Science of Successful Learning](#)² by Brown et.al.

In fact, spending time thinking and struggling is an important part of how humans learn. This has been studied extensively in the theory of learning. The importance of thinking for yourself has also been recognized by great thinkers over thousands of years.

² www.amazon.com/Make-It-Stick-Science-Successful-Learning/dp/0674729013

The Danger of Passive Reading. As Arthur Melzer writes in *Philosophy Between the Lines*,

“A book is a strange and unseemly thing. It delivers into one person’s hands the distilled essence of another’s thinking.”

“The first danger of reading books is that it allows you to skip too many stages, shortcutting the proper intellectual development....

Before you have had a chance to face the questions and live with them a while, you have seen the answers. Books give a false sense of knowledge ... on the belief that you know what you have only read.”

Philosophy Between the Lines, page 211.

Melzer also quotes another author (Schopenhauer) who observes that

“When we read, another person thinks for us: we merely repeat his mental process.... So it comes that if anyone spends almost the whole day reading ... he gradually loses the capacity for thinking; just as the man who always rides, at last forgets how to walk.”

So reading is pointless?

No: we learn quickly with listening and imitation. But we gain the best, most durable learning (like the knowledge you will need during a test) by wrestling with questions and concepts ourselves.

Course Overview. Mathematical knowledge has been described as *synthetic*. This is both because we learn mathematics by doing, and because we must incorporate each new concept into a broader system of knowledge.³

Mastery of *College Algebra*, or an equivalent class, is absolutely essential for success in this course, and *will be assumed throughout*. If you would like a brief summary of selected formulas and ideas from algebra, see Appendix ?? . If your algebraic skills are weaker, talk to your instructor to obtain resources for additional practice.

The goal of this course is to provide a first introduction to two related topics

1. In Part ?? , we will discuss a range of applications of mathematics, including algebra, to economics and business contexts.

Most of the mathematical techniques in this section should be familiar, although the applications will not. The main exception to this will be Chapter ??–??, where we will introduce the mathematics of exponential and logarithmic functions in some detail.

2. In Part ??–??, we will introduce and apply mathematical techniques for studying change, rates, and area. These tools will teach us a *method for calculation* of rates and areas, and are therefore often given the name *calculus*.

³ Much more can be said about this, which gets into the broader foundations of mathematics. You can read a little bit about this [online](#).

Acknowledgements

This book was written during a one semester sabbatical at Bridgewater State University during the Fall of 2022, and expanded during the following few semesters. There may be a range of typographical errors, formatting issues, and so on. If you believe that you have found a mistake, or if you have just found a place where the explanations can be improved or expanded, please contact the author at sflood@bridgew.edu.

The material on graphs of totals and changes is based on previous work in collaboration with my colleague Karen Duseau, also of Bridgewater State University, and funded by a QuAC *Course Development Summer Grant*. I also learned the terminology of AROC and IROC tables from Karen Duseau.

To the best of my knowledge, the other examples and exposition in this book are newly written, and are based on ideas in common use. However, it is always possible that something from a textbook or other resource I have used in the past has been unintentionally incorporated in some form or another. If you believe this may have happened, please contact the author at sflood@bridgew.edu, and I will happily correct the oversight.

Contents

Part I

Algebra For Business

1

Functions of Economics

1.1 Function Concepts

In this section, we will briefly review some key function concepts. However, our main focus will be on beginning to connect these mathematical ideas to the central economic and business concepts, applications, and examples that we will study over the course of this semester.

Exploration 1.1 If you are manufacturing t-shirts, then the *total cost* will be a function of the number of items you make. This is because there will be a rule which, given of the number of items to be made (the input) tells you a unique total cost in \$ (the output).

If x represents the number of items, we often write $C(x)$ to denote the total cost of these x items.

- (a) Write the mathematical expression that says "the cumulative cost of making 97 items costs \$350.42"

Answer. $C(97) = 350.42$

- (b) Translate the statment $C(52) = 253.19$ into natural language.

Solution. This statement says that the cumulative cost of making 52 items is \$253.19.

- (c) Write a mathematical expression that describes the cost of items 53-97 *only*. Use the information from the previous parts to find the cost of these items.

Solution. We have seen already that $C(97)$ tells us the cost of items 0-97, and that $C(52)$ tells us the cost of items 0-52.

To find the cost of *only* items 53-97, we must subtract the cost of items 0-52 from the cost of items 0-97. Mathematically, this is written

$$\text{Cost of items 53-97} = C(97) - C(52)$$

$$\begin{aligned}
&= 350.42 - 253.19 \\
&= 97.23
\end{aligned}$$

The total cost of items 53-97 is \$97.23.

Functions are important because they give us a single language that can describe an enormous number of different situations.

The next example uses the same language of functions to describe the very different situation of a very simple investment.

Example 1.1 Function notation and a simple investment..

You begin with \$1000 in a safe, and add \$365 each year, spread out evenly over the year. Furthermore, you commit to keeping this up for ten years. You want to study the value in the safe as a function of time.

To allow the amount of time to vary, we introduce an independent variable. Often, we use x for the independent variable, which in this case is time. We also often use y for the dependent variable, which in this case is the amount saved. Although x, y are common variables, we could also have chosen other letters, such as v for value and t for time.

Putting this together, we get

$$y = f(x) = 1000 + 365x \text{ for } x \text{ in } [0, 10]$$

Reading from left to right, this says that the output balance y in \$ is equal to a function of the input time x , and that we will use the letter f to refer this function.

It also says that this function is defined by a *rule*, which says to take the number of years x , to multiply that by 365, and then to add \$1000.

Finally, that sentence says that this rule applies only when the input x is in the closed interval $[0, 10]$, which means that x can be any number in the interval $0 \leq x \leq 10$. In other words, this rule specifies its *domain*.

Because our savings are spread out *evenly over the year*, we can find the value of our savings after “four and a half years”, as well as finding the value after three years exactly. In other words, the domain of our savings function contains decimals like 4.5 years, as well as whole numbers like 3 years. This even will us to find the amount in the safe after 40 *days*. Assuming that there are 365 days in a year, day 40 happens at *year* $x = \frac{40}{365} = 0.1096$. \square

Exploration 1.2 Suppose that you begin with \$1000 in a safe, and add \$365 each year for ten years, spread out evenly over the year.

- (a) Write the equation for the value in the safe as a function of time.

Solution. This problem does not specify the variables that we

should use for our function. If y represents value and x represents time in years, then we have the same equation that we found in Example ??.

$$y = f(x) = 1000 + 365x \text{ for } x \text{ in } [0, 10]$$

If we instead used v for value and t for time, the equation would become

$$v = f(t) = 1000 + 365t \text{ for } t \text{ in } [0, 10]$$

We could also use any other variables we decided were most appropriate.

- (b) Graph the value of your savings as a function of time.

Solution. We are told that this rule only applies between year $x = 0$ and year $x = 10$. That means that we should start our graph at $(0, f(0))$, and end it at $(10, f(10))$. In this case, the value is a linear function, so we can complete the graph by connecting these points.

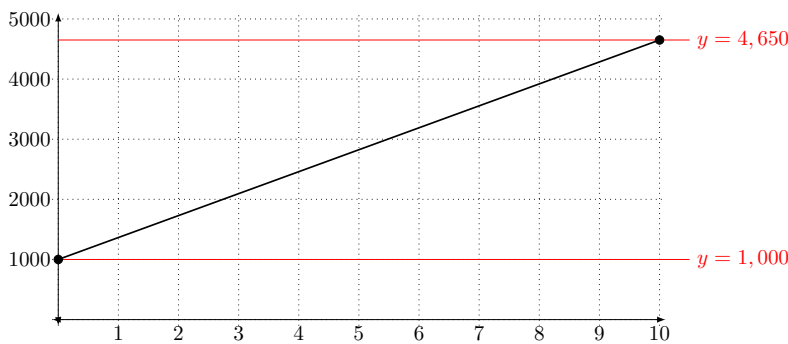


Figure 1.2

- (c) Find the value of savings after four and a half years.

Solution. We know the value after x years is equal to $f(x) = 1000 + 365x$. To find the value after $x = 4.5$ years, we simply compute

$$f(4.5) = 1000 + 365 \cdot 4.5 = 2642.5.$$

- (d) Find the value of savings after 40 *days*.

Solution. We know the value after x years is equal to $f(x) = 1000 + 365x$.

To apply this rule, we must first convert 40 *days* to *years*. In a standard calendar year, there are 365 days, so $1 \text{ day} = 1/365 \text{ year}$. So day 40 is just a little bit over 10% of a year. More precisely, we can do a unit conversion as follows

$$40 \text{ days} = 40 \text{ days} \cdot \frac{1 \text{ year}}{365 \text{ days}}$$

$$\begin{aligned}
&= \frac{40}{365} \frac{\text{days} \cdot \text{year}}{\text{days}} \\
&= \frac{40}{365} \text{year} \\
&= 0.10958 \dots \text{year}
\end{aligned}$$

We can now plug this year into our value function

$$\begin{aligned}
\text{value on day 40} &= \text{value in year } 0.10958 \\
&= f(0.10958) \\
&= 1000 + 365 \cdot 0.10958 \\
&= 1039.9967
\end{aligned}$$

After 40 days, the value will be approximately \$1040.

Definition 1.3 The **domain** of a function is the set of inputs where the rule is defined.

The **range** of a function is the set of all outputs that occur. \diamond

In many applications, a rule only “makes sense” for specific inputs. In this case, we talk about the **applied domain** of the function. For example, you cannot buy a negative number of items, and the price of an item cannot be negative.¹

Exploration 1.3 This example continues Exploration ???. In that example, we started with \$1000 in savings and added \$365 per year for 10 years, spread evenly across the whole year.

¹ You can pay people to take an item, but in this case it is not correct to call that payment a “price”.

- (a) Find the domain of the savings function.

Solution. In Exploration ??, we are explicitly told that our savings function is not defined before year 0, and not defined after year 10. In other words, we are explicitly told that the domain of that function is the interval $[0, 10]$, or equivalently, the set $0 \leq x \leq 10$

- (b) Find the range of the savings function.

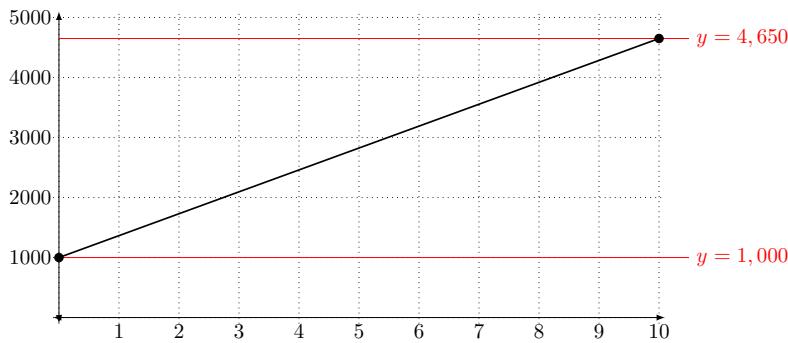
Answer. The range of the savings function is the set of all values that will occur in the account. The smallest value that will occur is \$1000, and the largest value that will occur is \$4,650. That means that the *range* is the set of all numbers between those values. In interval notation, this is $[1000, 4650]$.

Solution. The question of the range in that example is more complex. In words, a number will be in the range of our savings function if there is that amount in the safe at some year x between 0 and 10.

If you think about it, you’ll never have a value of \$900 in your account since it starts at \$1000 and increases. You’ll also never have

\$1,000,000 in your account, since you are only adding about a dollar a day for 10 years. In other words, the numbers 900 and \$1,000,000 are not in the range of our savings function.

On the other hand, the value \$1000 will definitely be in the range because you start on the first day of the first year with $1000 = f(0)$ dollars in your account. And \$4,650 will be in the range because you end the 10th year with $4650 = f(10) = 1000 + 365 \cdot 10$ in your account. Finally, every value between 1000 and 4650 will also be in the range, because you are continuously adding money to the account. But does that mean only the values for whole number years? What about fractional years? To answer that question, it helps to look at the graph of the savings function.



In the graph of our function, we see that the value is added continuously (uniformly), so all y values between 1000 and 4650 can be achieved. Thus, the domain is

$$1000 \leq y \leq 4650$$

In interval notation, this is

$$[1000, 4650]$$

The concept of domain is very important in mathematics. For example, there is an important difference between a function defined using a table, and a function defined using a formula. When you have a function defined using a table, the domain is restricted to the inputs given for all practical purposes. But when you have a function defined using a formula, the domain includes every input where the function is defined.

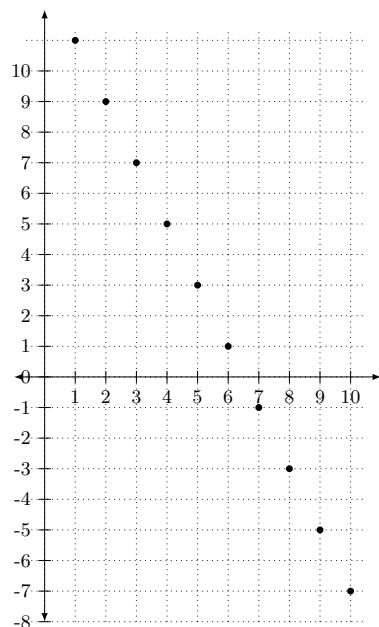
Exploration 1.4 Functions defined by tables and formulas. In this example, we will look at the difference in working with the “same” function $y = -2x + 13$ numerically (when it is given by a few values in a table) and algebraically (using a formula).

- (a) Let $f(x)$ be defined by the following table

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	11	9	7	5	3	1	-1	-3	-5	-7

- (i) Graph this function, and give its domain.

Solution. The only thing the problem gives us is 10 different points on the curve, one for each pair of input x and output y . The best we can do with this information is to plot these points separately.

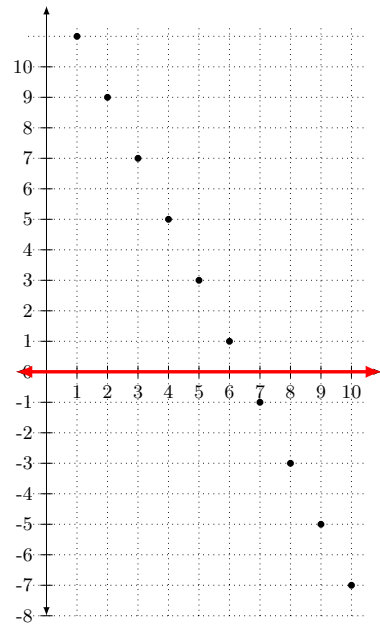


The function is only defined for $x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, so that is the domain of the function.

- (ii) Solve the equation $f(x) > 0$.

Solution. Look at the table for all x values where the corresponding output $y = f(x)$ is greater than $y = 0$. This is true for $x = 1, 2, 3, 4, 5, 6$

Alternatively, draw the line $y = 0$ and pick the x values where the dot is above the line.



(iii) Evaluate $f(z)$ where $z = 3$

Solution. $f(3) = 7$

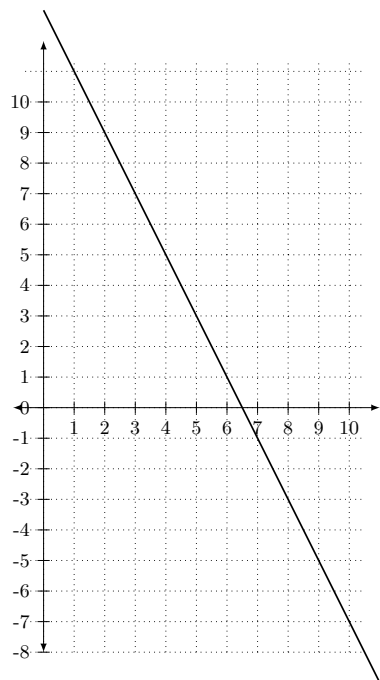
(iv) Solve $f(z) = 3$

Solution. $f(5) = 3$, so the solution is $z = 5$

(b) Let $f(x) = -2x + 13$

(i) Graph this function, and give its domain.

Solution.



The function is defined *at every point* below the curve. That means that the function is *also* defined at $x = 1.2$, and at $x = -5$, and so on. To express this, we can say in words *the domain is the set of all real numbers between $-\infty$ and ∞ , including all decimal numbers in between*. We can also write this in interval notation by writing that the domain is

$$(-\infty, \infty)$$

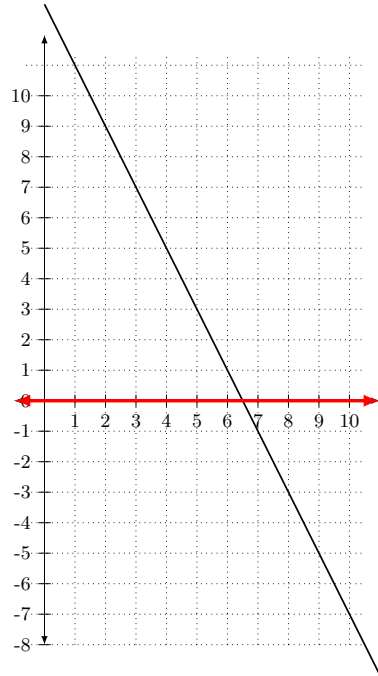
- (ii) Solve the equation $f(x) > 0$.

Solution. We can solve the inequality as follows

$$\begin{aligned} f(x) &> 0 \\ -2x + 13 &> 0 \\ 13 &> 2x && \text{add } 2x \text{ to both sides} \\ 6.5 &> x && \text{divide by positive } 2 \end{aligned}$$

We can also look at the graph of our function, draw the line $y = 0$, and take all x values where the function is above the line. From the graph below, this is the set of x to the left of 6.5. In interval notation, this is

$$(-\infty, 6.5)$$



(iii) Evaluate $f(z)$ where $z = 3$

Solution. $f(3) = -2(3) + 13 = -6 + 13 = 7$

(iv) Solve $f(z) = 3$

Solution. Solving the equation we get

$$\begin{aligned} f(x) &= 3 \\ -2x + 13 &= 3 \\ -2x &= -10 \\ x &= 5 \end{aligned}$$

We can double check this by plugging our answer back into our function and check that the output is indeed 3.

There are also certain *forbidden computations*.

1. **Dividing by Zero:** For any numerator a , the fraction $\frac{a}{0}$ is **undefined**. We often express this by writing $\frac{\#}{0} \text{ DNE}$
2. **Square Roots of Negative Numbers:** The square root of any negative number is undefined. For example, $\sqrt{-1} \text{ DNE}$.

For the purposes of this course, a function is defined on an input as long as neither of the above computations occur.

Exploration 1.5

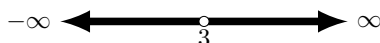
- (a) Write the domain of $f(x) = \frac{2}{3-x}$ in interval notation.

Solution. A fraction $\frac{A}{B}$ is defined as long as the denominator $B \neq 0$.

That means that the fraction $\frac{2}{3-x}$ is defined if and only if the denominator $3-x \neq 0$.

Rewriting this, we see that the the fraction is defined if and only if $3 \neq x$.

To write this in interval notation, draw the points where the function *is* defined on a number line. Use an open circle at $x = 3$ to *exclude* that value, and draw a thick line to include all other numbers



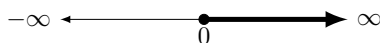
We can now read the domain off of our number line. The function is defined on the interval $(-\infty, 3) \cup (3, \infty)$

- (b) Write the domain of $g(x) = \sqrt{x}$ in interval notation.

Solution. A fraction \sqrt{A} is defined as long as the denominator $A \geq 0$.

That means that the fraction \sqrt{x} is defined if and only if the inside $x \geq 0$.

To write this in interval notation, draw the points where the function *is* defined on a number line. Use a closed circle at $x = 0$ to *include* that value, and draw a thick line to the right to include all positive numbers.



We can now read the domain off of our number line. The function is defined on the interval $[0, \infty)$

- (c) Write the domain of $h(x) = 3x^2 - 2x + 5$ in interval notation.

Solution. At first, this problem can be confusing, because it doesn't have any fractions or roots. But actually, that makes this example very easy.

We are able to multiply, add, and subtract any real number. In other words, the function is defined for *all* inputs. To write this in interval notation, fill in the *entire* real line.

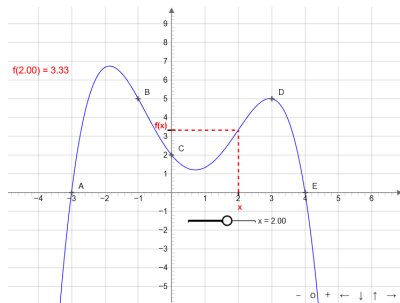


We can now read the domain off of our number line. The function is defined on the interval $(-\infty, \infty)$

1.2 Graphs of Functions

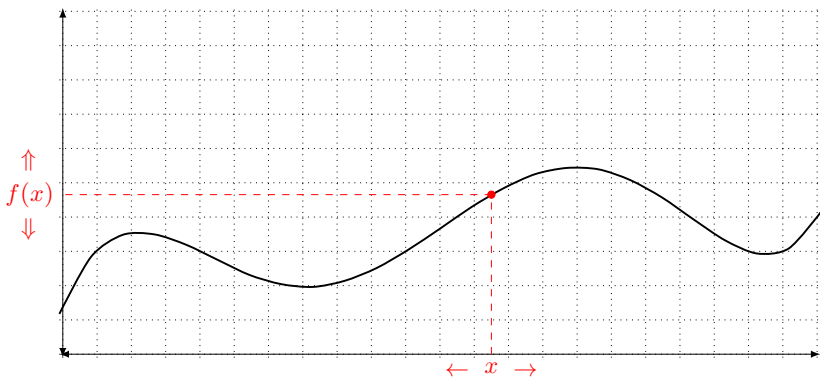
Nothing is truly constant in economics, business, and life. Time and quantities change, and things like cost, revenue and profit change along with them.

In terms of functions, we can say that changing the *input* changes the *output*. On a graph, that means that moving the input x left and right causes the output y to go up and down.



Standalone
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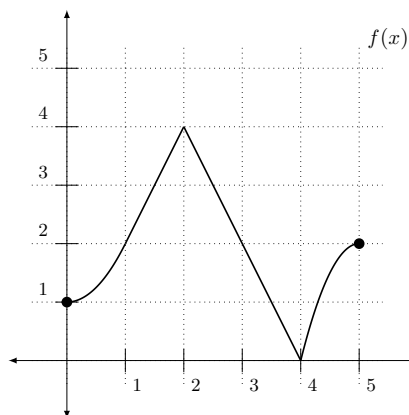
Example 1.4 Here is a static version of the interactive graphic above.



□

Reading graphs dynamically will play an essential role in the course, beginning in Chapter ???. But this skill also plays an important role in answering much more basic questions about graphs.

Exploration 1.6 The graph of a function f is given below.



- (a) Compute $f(0)$, $f(2)$, $f(4)$, and $f(5)$.

Answer. $f(0) = 1$, $f(2) = 4$, $f(4) = 0$, and $f(5) = 2$

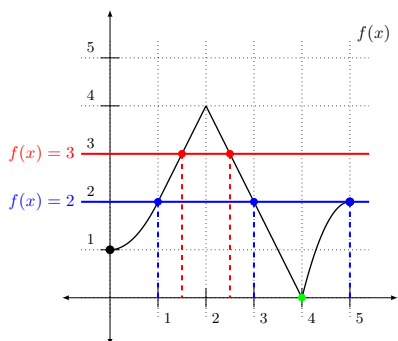
- (b) For what x is $f(x) = 2$? For what x is $f(x) = 3$? For what x is $f(x) = 0$?

Answer. $f(x) = 2$ when $x = 1$ or when $x = 3$ or when $x = 5$.

$f(x) = 3$ when $x = 1.5$ or when $x = 2.5$.

$f(x) = 0$ when $x = 4$.

Solution.



Remember that the equation $f(x) = 2$ means that you are looking for the *inputs* x which correspond to an *output* of 2. Looking at the graph, we see that $f(1) = 2$ and that $f(3) = 2$ and that $f(5) = 2$. That means that the solution set is $x = 1$ or $x = 3$ or $x = 5$.

Note: Endpoints require special care. In this case, there was a *filled in dot* at the point $(5, 2)$, which means that $f(5)$ is defined and equal to 2. If there was an *open circle* at the point, then the function would be defined *near* $x = 5$, but not *at* $x = 5$.

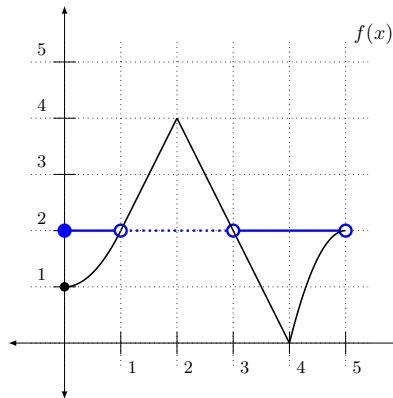
Similarly, solving the equation $f(x) = 3$ means finding the inputs x that correspond to an output (height) y of 3. Looking at the graph, we see that $f(1.5) = 3$ and $f(2.5) = 3$. That means that the solutions set is $x = 1.5$ or $x = 2.5$

Finally, solving $f(x) = 0$ means finding the inputs x that have a height y of zero. Looking at the graph, this happens only for $x = 4$.

(c) For what x is $f(x) < 2$?

Answer. The output of the function is *strictly less than* 2 on the set $[0, 1) \cup (3, 5)$, or equivalently when $0 \leq x < 1$ or $3 < x < 5$.

Solution.



Remember that the equation $f(x) < 2$ means that you are looking for the *inputs* x which correspond to an output of *strictly less than* 2. To understand this inequality, follow the graph of the function, moving through x -values from left to right.

Above $x = 0$, the height is 1 which is less than 2. Moving right, the height of the function increases, but stays below 2 until we reach $x = 1$. That means that the function is *less than or equal to* 2 on the interval $[0, 1]$. But because we want the height to be *strictly less than* 2, we must exclude 1, giving us the interval $[0, 1)$.

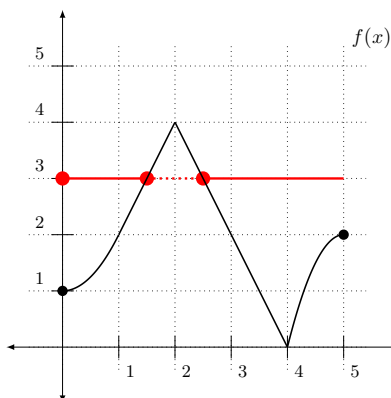
Similarly, the height is also less than 2 between $x = 3$ and $x = 5$, and equal to 2 at those endpoints, so the function is *strictly less than* 2 on the interval $(3, 5)$.

Putting it all together, we have that $f(x) < 2$ on the set $[0, 1) \cup (3, 5)$, or equivalently when $0 \leq x < 1$ or $3 < x < 5$.

(d) For what x is $f(x) \leq 3$?

Answer. Remember that the equation $f(x) \leq 3$ means that you are looking for the *inputs* x which correspond to an output of *less than or equal to* 3. This happens for values of x satisfying $0 \leq x \leq 1.5$ or $2.5 \leq x$. In interval notation, this corresponds to the set $[0, 1.5] \cup [2.5, 5]$.

Solution.



(e) For what x is $f(x) > 0$?

Answer. The inequality $f(x) > 0$ is true for all inputs that correspond to an output that is strictly greater than 0. This is true for all values of x *except* $x = 4$. So this is true for values of x in the interval $[0, 4) \cup (4, 5]$.

(f) What is the domain and range of f ?

Answer. The domain of $f(x)$ is the set of x -values where the function is defined. In this example, the function is defined starting at $x = 0$ and stopping at $x = 5$. Therefore the domain is $[0, 5]$.

The range of $f(x)$ is the set of y -values that are output by the function. Looking at the function, note that you can achieve outputs as small as $f(x) = 0$ (at $x = 4$) and as large as $f(x) = 4$ (at $x = 2$), as well as every output between those y -values. Furthermore, no larger or smaller y can be achieved. As a result, the range of the function is the interval $[0, 4]$.

2

Working with Functions

2.1 Combining Functions

The “big idea” behind all of algebra is the use of a symbols to represent a different object. The simplest thing you can do in algebra is to replace the symbol with the value it represents, often called “plugging in” the value.

Exploration 2.1 Let $f(x) = x^2 - 3x$.

- (a) Evaluate $f(5)$, and simplify completely.

Solution. The expression $f(5)$ is really an instruction to take the definition of $f(x) = x^2 - 3x$, and to replace every occurrence of x with the value 5. Performing this substitution, and the simplifying, gives

$$\begin{aligned} f(5) &= 5^2 - 3 \cdot 5 \\ &= 25 - 15 \\ &= 10 \end{aligned}$$

- (b) Evaluate $f(5 + h)$, and simplify completely.

Solution. Again, the expression $f(5 + h)$ is really an instruction to take the definition of $f(x) = x^2 - 3x$, and to replace every occurrence of x with the value $(5 + h)$. Performing this substitution, gives the expression

$$f(5 + h) = (5 + h)^2 - 3 \cdot (5 + h)$$

It is **very important** to include parentheses when you are substituting x with the term $(5 + h)$.

To simplify this expression, you should separately simplify the first term and the second term. Expanding out (FOIL-ing) the first term gives

$$(5 + h)^2 = (5 + h) \cdot (5 + h) = 5 \cdot 5 + 5 \cdot h + h \cdot 5 + h \cdot h = 25 + 10h + h^2.$$

Distributing out the second term gives

$$3 \cdot (5 + h) = 3 \cdot 5 + 3 \cdot h = 15 + 3h$$

Putting it all together we get

$$\begin{aligned} f(5 + h) &= (5 + h)^2 - 3 \cdot (5 + h) \\ &= (25 + 10h + h^2) - (15 + 3h) \\ &= 25 + 10h + h^2 - 15 - 3h \\ &= 10 + 7h + h^2 \end{aligned}$$

Note: Whenever substituting an expression into another expression, such as when we substituted the simplifications for $(5 + h)^2$ and $3 \cdot (5 + h)$ above, you must always include parentheses around each term. Above, that helped us remember that we needed to distribute the subtraction to *both* terms of $15 + 3h$.

(c) Evaluate $f(x^5)$, and simplify completely.

Solution. Again, the expression $f(x^5)$ is really an instruction to take the definition of $f(x) = x^2 - 3x$, and to replace every occurrence of x with the value x^5 . Performing this substitution, gives the expression

$$f(x^5) = (x^5)^2 - 3 \cdot (x^5)$$

We could separately simplify the first and second term as before, or we can do it all together in a single sequence of steps. You can take whichever approach you are most comfortable with.

$$\begin{aligned} f(x^5) &= (x^5)^2 - 3 \cdot (x^5) \\ &= x^5 \cdot x^5 - 3x^5 \\ &= x^{5+5} - 3x^5 \\ &= x^{10} - 3x^5 \end{aligned}$$

Note that in our last example, we plugged a expression for x into another expression for x . In the language of Chapter ??, that means we plugged one function into another. In mathematics, we call this process of plugging one function into another "composing" the functions.

Exploration 2.2 Let $f(x) = \frac{1}{x}$ and $g(x) = x + 5$. Compute $f(g(x))$.

Solution. The problem asks us to evaluate the result of plugging the function $g(x) = x + 5$ into the every occurrence of x in the equation $f(x) = \frac{1}{x}$

First, we can look at $f(g(x))$, and notice that we have already been told that the expression $g(x)$ refers to $(x + 5)$. As a result, we can

replace the argument to f as follows

$$f(g(x)) = f(x + 5)$$

Next, we just replace every occurrence of x in the definition $f(x) = \frac{1}{x}$ with the expression $(x + 5)$. This gives

$$f(x + 5) = \frac{1}{(x + 5)}$$

Putting it all together, we get

$$\begin{aligned} f(g(x)) &= f(x + 5) \\ &= \frac{1}{(x + 5)} \end{aligned}$$

We'll need to do a lot of composition later in the semester. For now, we will use these concepts to practice our algebra skills.

Exploration 2.3 Let $f(x) = 3x^2 - x + 1$. Simplify the following expressions completely

(a) $f(2)$

Solution. $f(2) = 3 \cdot (2)^2 - (2) + 1 = 12 - 2 + 1 = 11$

(b) $f(2 + h)$

Solution.

$$\begin{aligned} f(2 + h) &= 3 \cdot (2 + h)^2 - (2 + h) + 1 \\ &= 3 \cdot (2 + h) \cdot (2 + h) - 2 - h + 1 \\ &= 3 \cdot (4 + 4h + h^2) - 2 - h + 1 \\ &= 12 + 12h + 3h^2 - 2 - h + 1 \\ &= 3h^2 + 11h + 11 \end{aligned}$$

(c) $\frac{f(2 + h) - f(2)}{h}$

Solution. This expression is slightly more complex. The key idea is to realize that the numerator contains two *separate* expressions $f(2 + h)$ and $f(2)$, and that we have already found that they equal $(3h^2 + 11h + 11)$ and (11) respectively. Making the appropriate replacements gives

$$\frac{f(2 + h) - f(2)}{h} = \frac{(3h^2 + 11h + 11) - (11)}{h}$$

Now we can use basic algebra to simplify the expression as before.

$$= \frac{3h^2 + 11h}{h}$$

$$\begin{aligned}
&= \frac{h(3h+11)}{h} \\
&= \frac{h}{h} \cdot \frac{3h+11}{1} \\
&= 3h+11
\end{aligned}$$

The previous example is complicated because it combines a number of different kinds of steps. To see what's going on, it can help to think of these different steps separately.

Exploration 2.4 Let $f(x) = 2x^2 - 4x$ and $g(x) = 2x - 4$. Simplify the following expressions completely.

(a) $f(x) + g(x)$

Solution. To evaluate the expression $f(x) + g(x)$, first replace $f(x)$ with the expression $(2x^2 - 4x)$, and replace $g(x)$ with the expression $(2x - 4)$. Then, simplify the remaining expression completely using algebra.

$$\begin{aligned}
f(x) + g(x) &= (2x^2 - 4x) + (2x - 4) \\
&= 2x^2 - 4x + 2x - 4 \\
&= 2x^2 - 2x - 4
\end{aligned}$$

(b) $f(x) - g(x)$

Solution. To evaluate the expression $f(x) - g(x)$, first replace $f(x)$ with the expression $(2x^2 - 4x)$, and replace $g(x)$ with the expression $(2x - 4)$. Then, simplify the remaining expression completely using algebra.

$$\begin{aligned}
f(x) - g(x) &= (2x^2 - 4x) - (2x - 4) \\
&= 2x^2 - 4x - 2x + 4 \\
&= 2x^2 - 6x + 4
\end{aligned}$$

Note that subtracting both terms of $2x - 4$ was the same thing as adding $-2x$ and adding a positive $4x$

(c) $f(x) \cdot g(x)$

Solution. To evaluate the expression $f(x) \cdot g(x)$, first replace $f(x)$ with the expression $(2x^2 - 4x)$, and replace $g(x)$ with the expression $(2x - 4)$. Then, simplify the remaining expression completely using algebra.

$$\begin{aligned}
f(x) \cdot g(x) &= (2x^2 - 4x) \cdot (2x - 4) \\
&= 2x^2 \cdot 2x + 2x^2 \cdot (-4) + (-4x) \cdot 2x + (-4x) \cdot (-4) \\
&= 4x^3 - 8x^2 - 8x^2 + 16x \\
&= 4x^3 - 16x^2 + 16x
\end{aligned}$$

(d) $\frac{f(x)}{g(x)}$

Solution. To evaluate the expression $\frac{f(x)}{g(x)}$, first replace $f(x)$ with the expression $(2x^2 - 4x)$, and replace $g(x)$ with the expression $(2x - 4)$. Then, simplify the remaining expression completely using algebra.

$$\frac{f(x)}{g(x)} = \frac{(2x^2 - 4x)}{(2x - 4)}$$

The only way to simplify a fraction is to factor the top and bottom separately, and see if any of the terms cancel. The top factors as $2x^2 - 4x = 2 \cdot (x^2 - 2x) = 2 \cdot x \cdot (x - 2)$. The bottom factors as $2x - 4 = 2 \cdot (x - 2)$. Using this, we get

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{(2x^2 - 4x)}{(2x - 4)} \\ &= \frac{2 \cdot x \cdot (x - 2)}{2 \cdot (x - 2)} \\ &= \frac{x}{1} \\ &= x \end{aligned}$$

Surprisingly, the expression $\frac{f(x)}{g(x)}$ simplifies down to the identity function $\frac{f(x)}{g(x)} = x$

(e) $f(g(x))$

Solution. To evaluate the expression $f(g(x))$, first replace $g(x)$ with the expression $(2x - 4)$. This gives

$$f(g(x)) = f(2x - 4)$$

But the expression $f(2x - 4)$ is just the instruction to take the equation $f(x) = 2x^2 - 4x$, and to replace each copy of x with the expression $(2x - 4)$. This gives

$$f(2x - 4) = 2(2x - 4)^2 - 4(2x - 4).$$

Simplifying this gives

$$\begin{aligned} f(g(x)) &= f(2x - 4) \\ &= 2(2x - 4)^2 - 4(2x - 4) \end{aligned}$$

$$\begin{aligned}
&= 2(2x - 4)(2x - 4) - 4(2x - 4) \\
&= 2(4x^2 - 16x + 16) - 8x + 16 \\
&= 8x^2 - 32x + 32 - 8x + 16 \\
&= 8x^2 - 40x + 48
\end{aligned}$$

In the examples above, we have seen how to simplify various combinations of functions algebraically.

For a few ways of combining of functions, it is also possible to explain very precisely how the combination changes the graph of the function. Let $y = f(x)$ be the graph of some function, and let a, b, c, d be constant numbers.

- The graph of $y = a \cdot f(x)$ is obtained by scaling the graph of $y = f(x)$ by a .
- The graph of $y = f(x + b)$ is obtained by scaling the graph of $y = f(x)$ left b units when b is positive (and right when b is negative).
- The graph of $y = f(x) + c$ is obtained by shifting the graph of $y = f(x)$ up by c units when c is positive (and down when c is negative).

2.2 *Graphs of Totals and Changes

Many things change over time, such as the cost of gasoline. There are different ways of thinking about describing this change. For example, suppose the price of gas over the course of a single week is given by the following table:

day (x)	1	2	3	4	5	6	7
cost of gas (y)	4	3.75	3.25	3.75	2	2.5	3

This gives us the **total** cost of gas per day. There are also another way to think about this: how much does the price change each day? Over the course of the week the price decreases 25 cents, then decreases another 50 cents, then increases 50 cents, then increases \$1.25, then decreases \$2.50, and finally increases another 50 cents.

We will write Δy to emphasize that this is the **change** in price each day in a table.

day (x)	1	2	3	4	5	6	7
change in gas (Δy)	N/A	-0.25	-0.50	0.50	-1.75	0.50	0.50

Example 2.1 A Strategy for Computing Δy . In the example above, we described the change in words, and summarized this in a table. But we can also find the change Δy using mathematical language.

Day 1 is a special case. Because it is our first data point, there's nothing to compare it to.

On day 2 our cost y was 3.75, and on day 1 our cost y was 4. Then the **change in y on day 2** is $\Delta y = (3.75) - (4) = -0.25$. We can also translate this into function language.

- The cost on day 1 is \$4, so we write $f(1) = 4$
- The cost on day 2 is \$3.75, so we write $f(2) = 3.75$
- To compute Δy on day 2, we compute $\Delta y = f(2) - f(1) = 3.75 - 4.00 = -0.25$

Repeating this process, we can find Δy for days 3, 4, 5, 6, and 7.

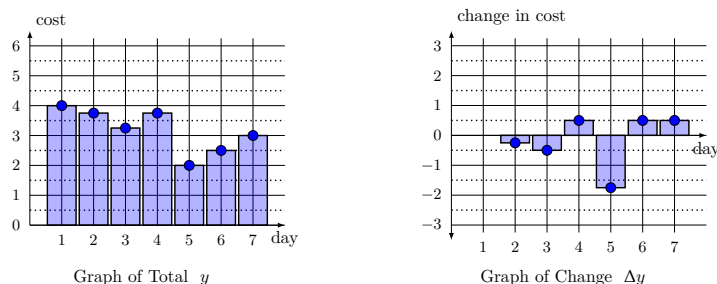
We can also summarize this technique using even more general mathematical language by defining the **change in y** due to **changing from x_1 to x_2** to be the expression

$$\Delta y = f(x_2) - f(x_1)$$

□

Now we have two tables, which *both give functions* that describe a different perspective on how the cost of gas changes over time: the total value (y) of gas graphed with respect to time, and the change in value (Δy) graphed with respect to time.

To see how they're related, we can graph both functions side by side:



In the example above, the biggest drop in the total y (left) occurs on day 5. This matches the graph on the right, where day 5 has the largest negative movement (change) Δy . On the other hand, there is almost no change on day 2 in the total graph on the left (only a small drop in y), which matches the small value of Δy in the change graph on the right.

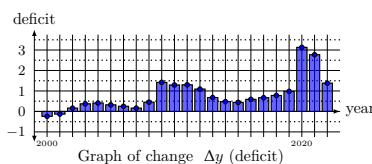
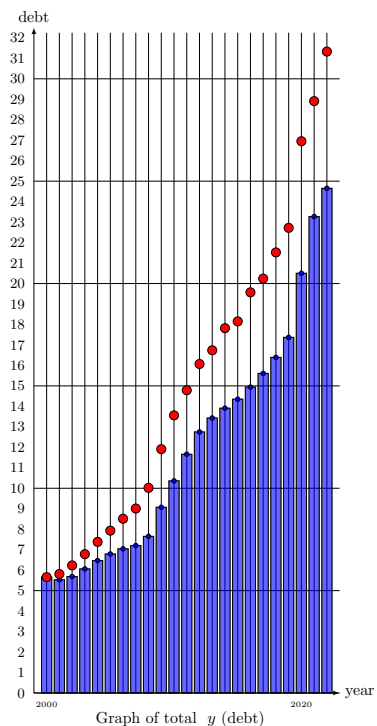
Graphs of totals and changes occur in a wide variety of economic, business, and other contexts. But one of the most common and important applications is in the important distinction between *debts* and *deficits*.

Debts and Deficits. The **US national debt** in a given year is the total amount of money that the government owed in that year. The

net revenue from a given year is the difference between money taken in (through taxes) and money spent by the government. The government is running an **annual deficit** if it spends more than it takes in. In other words, having negative net revenue in a given year is the same thing as a positive deficit in that year.

The page <https://datalab.usaspending.gov/americas-finance-guide/deficit/trends/>¹ lists US budget deficits in *trillions of dollars* since the year 2000, rounded to two decimal places. The same page indicates that the US debt in 2000 was 5.65 trillions.

As before, we can compute the debt y for each year since 2000 by starting with 5.65 and repeatedly adding the Δy for each year.² This gives us our familiar graphs of the total y and change Δy .

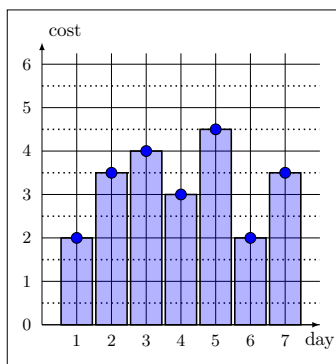
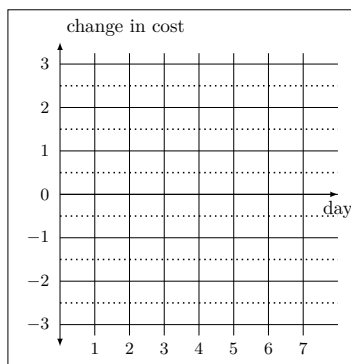


¹ datalab.usaspending.gov/americas-finance-guide/deficit/trends/

² **Note:** Actually, the debt is always more than the sum of the deficits, since the government must also pay **interest** on the money it borrows to cover a deficit. The true debt for each year is shown using a red dot. The actual value of the debt (not adjusted for inflation) can be found at <https://fiscaldata.treasury.gov/datasets/debt-to-the-penny/debt-to-the-penny>.

In most examples, we will only be given one of these types of graphs, and we will need to create the other type of graph. In the next example you are only given the graph of the total value (y), and asked to find the graph of the change (Δy).

Exploration 2.5 Using Total Value to Compute Change. The graph of gas price as a function of the day x is given below on the left.

Graph of Total y Graph of Change Δy

- (a) What is the cost of gas (y) on days 5 and 6? What is the change of cost (Δy) between days 5 and 6?

Solution. From our graph, we see that $f(5) = 4.5$ and that $f(6) = 2$. That means that the change is

$$\Delta y = f(6) - f(5) = 2 - 4.5 = -2.5$$

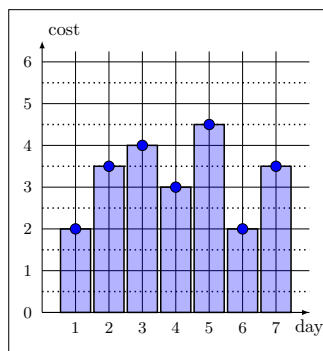
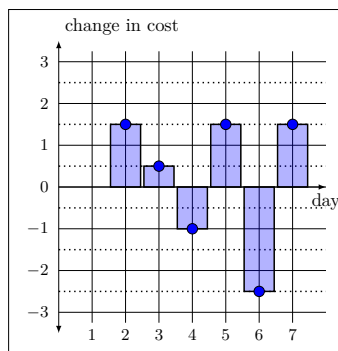
- (b) Write down a table giving the actual price y and change in price Δy on all seven days.

Solution.

day (x)	1	2	3	4	5	6	7
price (y)	2	3.5	4	3	4.5	2	3.5
change in gas (Δy)	N/A	1.50	0.50	-1.00	1.50	-2.50	1.50

- (c) Graph the “Change in Cost” function. (Use the graph provided on the right).

Solution.

Graph of Total y Graph of Change Δy

Often, we are working with *formulas* that define (or model) the quantity of interest.

Exploration 2.6 Changes and Formulas. Suppose that on day x , the value of a certain stock is given by $f(x) = x - 0.1x^2$.

- (a) Use the formula you are given to fill in the following table. Note that the difference between consecutive days x is given by $\Delta x = 1$.

day (x)	1	2	3	4	5	6	7
price (y)							
change in price (Δy)							

Hint: You can use the “Table” mode of a graphing calculator to speed this up significantly.

Solution.

day (x)	1	2	3	4	5	
price (y)	$f(1) = 0.9$	$f(2) = 1.6$	$f(3) = 2.1$	$f(4) = 2.4$	$f(5) = 2.5$	
change in price (Δy)	N/A	$f(2) - f(1) = 0.7$	$f(3) - f(2) = 0.5$	$f(4) - f(3) = 0.3$	$f(5) - f(4) = 0.1$	$f(6) - f(5) = -0.1$

- (b) Note that the change $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$ implicitly depends on the distance $\Delta x = x_2 - x_1$. Here, all our x values are consecutive, but you could imagine computing Δy (the change in y) in different increments, such as for $x = 1, 3, 5, 7$.

Using the same function $f(x) = x - 0.1x^2$ as above, complete the following table. Note that the difference between consecutive days x is now given by $\Delta x = 2$.

day (x)	1	3	5	7
price (y)				
change in price (Δy)				

Solution.

day (x)	1	3	5	7
price (y)	$f(1) = 0.9$	$f(3) = 2.1$	$f(5) = 2.5$	$f(7) = 2.1$
change in price (Δy)	N/A	$f(3) - f(1) = 1.2$	$f(5) - f(3) = 0.4$	$f(7) - f(5) = -0.4$

2.3 *Connecting Δy and Δx

This section is inspired by Thompson’s *Calculus Made Easy*

We can now introduce one of the central ideas in calculus, which we will come back in more detail in Part III of the class. For now, our main goal is to better see the connection between change graphs (which have clear economic and business applications) and functions and their combinations (which does not).

There are three key ideas.

1. The independent variable x and dependent variable y both change and grow.
2. The change in input x is written (denoted) Δx , and the change in output y is written Δy
3. Because x and y are related by some function $y = f(x)$, then Δy and Δx are also related.

Our goal is to understand how to find Δy from x and Δx and, eventually, to understand the relationship between Δy and Δx .

We can make these ideas precise in a definition

Definition 2.2 Given a function $y = f(x)$ and two inputs x_1, x_2 , and corresponding outputs $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we say that the *change in y between x_1 and x_2* is written Δy . There are three different, but equivalent, ways that we will think about Δy

$$\Delta y = y_2 - y_1 \quad (2.1)$$

$$\Delta y = f(x_2) - f(x_1) \quad (2.2)$$

$$\Delta y = f(x_1 + \Delta x) - f(x_1) \quad (2.3)$$

These first two are saying the same thing because we always write $y_1 = f(x_1)$ and $y_2 = f(x_2)$. We need the third equation when we only know only x_1 and $\Delta x = x_2 - x_1$, but not x_2 . In that case, we solve for $x_2 = x_1 + \Delta x$, and compute $y_2 = f(x_2) = f(x_1 + \Delta x)$. \diamond

Exploration 2.7 Suppose that $y = f(x) = x^2$, that x_1 and x_2 are two unknown inputs, and that $\Delta x = x_2 - x_1$ is the change in input.

- (a) Find the value y_1 of the function at input x_1

Solution. To find the value of $y = f(x)$ at $x = x_1$, just plug this value into our function $f(x) = x^2$

$$y_1 = f(x_1) = (x_1)^2$$

- (b) Find the value y_2 of the function at input x_2 .

Solution. As before, we could just plug $x = x_2$ into our function $f(x) = x^2$ to get

$$y_2 = f(x_2) = (x_2)^2$$

But we can get a more informative answer by using the fact that because $\Delta x = x_2 - x_1$, then $x_2 = x_1 + \Delta x$. Plugging this into our equation gives

$$y_2 = f(x_1 + \Delta x)$$

$$\begin{aligned}
&= (x_1 + \Delta x)^2 \\
&= (x_1 + \Delta x) \cdot (x_1 + \Delta x) \\
y_2 &= (x_1)^2 + 2(x_1)(\Delta x) + (\Delta x)^2
\end{aligned}$$

(c) Find Δy in terms of Δx and x_1

Solution. Recall that Δy is just the change in y that occurs as a result of changing the input from x_1 to x_2 . Mathematically, this gives us

$$\Delta y = (y_2) - (y_1)$$

Using our work above, we know $y_1 = (x_1)^2$ and that $y_2 = (x_1)^2 + 2(x_1)(\Delta x) + (\Delta x)^2$. This gives us

$$\begin{aligned}
\Delta y &= (y_2) - (y_1) \\
&= ((x_1)^2 + 2(x_1)(\Delta x) + (\Delta x)^2) - (x_1)^2 \\
\Delta y &= 2(x_1)(\Delta x) + (\Delta x)^2
\end{aligned}$$

We now have a formula for how the change in output Δy depends on the change in input Δx

(d) Find an equation for $\frac{\Delta y}{\Delta x}$.

Solution. We have already see that

$$\Delta y = 2(x_1)(\Delta x) + (\Delta x)^2$$

We can factor a term of Δx out of the right hand side.

$$\Delta y = (\Delta x) \cdot (2(x_1) + (\Delta x))$$

Now, we can divide both sides by Δx to get a $\frac{\Delta y}{\Delta x}$ on the left hand side.

$$\frac{\Delta y}{\Delta x} = 2(x_1) + (\Delta x)$$

We will see what this term really means later in the course, when we discuss rates of change and slopes.

In the next chapter, we will see a special case of $\frac{\Delta y}{\Delta x}$ when we review linear functions. We will also introduce a number of functions that play an important role in economics and business. We close with a quick example using the total cost function $C(x)$

Exploration 2.8 Suppose that $C(x) = 0.01x^2 - x + 50$ gives the total cost of purchasing x barrels of oil.

In the past, you have purchased $x = 300$ barrels of oil each month. But now you are thinking of buying a bit more oil. You want to know how much your costs will change as a result of this increase in purchase volume.

- (a) Find the cost of 300 barrels of oil.

Solution. $C(300) = 0.01(300)^2 - 300 + 50 = 650$

- (b) Find the change in cost ΔC between $x_1 = 300$ barrels and $x_2 = 400$ barrels.

Solution. The formula for change in cost is

$$\Delta C = C_2 - C_1$$

Here, the first cost is the cost of 300 barrels

$$C_1 = C(300) = 0.01(300)^2 - 300 + 50 = 650$$

The second cost is the cost of 400 barrels

$$C_2 = C(400) = 0.01(400)^2 - 400 + 50 = 1250$$

The change in the cost is therefore

$$\Delta C = C(400) - C(300) = 1250 - 650 = 600$$

The $\Delta x = 100$ additional barrels will increase the total cost by \$600 (an average of \$6 per barrel).

- (c) Suppose you want to increase your purchase volume from $x_1 = 300$ by $\Delta x = 10$ barrels. What is the change in cost going to be?

Solution. We want to compare the cost of

$$x_1 = 300$$

and

$$x_2 = 300 + \Delta x = 300 + 10 = 310$$

barrels of oil. We can compute the costs separately, and then subtract them.

$$C_1 = C(300) = 650$$

$$C_2 = C(300 + 10) = C(310) = 0.01(310)^2 - 310 + 50 = 701$$

Subtracting these total costs gives us the change in cost

$$\Delta C = C_2 - C_1 = C(310) - C(300) = 51$$

The cost will increase $\Delta C = 51$ dollars

- (d) You are not sure how much you want to increase the purchase by. In other words, $x_1 = 300$, but Δx is unknown. Find an equation for the change in cost ΔC .

Solution. We want to find

$$\Delta C = C_2 - C_1 = C(x_2) - C(x_1).$$

We know that $x_1 = 300$, but we do not know x_2 . Therefore, we must write

$$x_2 = x_1 + \Delta x = 300 + \Delta x$$

We can now compute C_1 and C_2

$$C_1 = C(300) = 0.01(300)^2 - 300 + 50 = 650$$

and

$$\begin{aligned} C_2 &= C(300 + \Delta x) \\ &= 0.01(300 + \Delta x)^2 - (300 + \Delta x) + 50 \\ &= 0.01(300 \cdot 300 + 2 \cdot 300 \cdot \Delta x + (\Delta x)^2) - (300 + \Delta x) + 50 \\ &= 900 + 6 \cdot \Delta x + 0.01(\Delta x)^2 - 300 - \Delta x + 50 \\ &= 650 + 5 \cdot \Delta x + 0.01(\Delta x)^2 \end{aligned}$$

We have found that

$$\begin{aligned} C_1 &= 650 \\ C_2 &= 650 + 5 \cdot \Delta x + 0.01(\Delta x)^2 \end{aligned}$$

We can now compute

$$\begin{aligned} \Delta C &= C_2 - C_1 \\ &= (650 + 5\Delta x + 0.01(\Delta x)^2) - (650) \\ \Delta C &= 5\Delta x + 0.01(\Delta x)^2 \end{aligned}$$

- (e) You are not sure how much you want to increase the purchase by. In other words, $x_1 = 300$, but Δx is unknown. Find an equation for $\frac{\Delta C}{\Delta x}$.

Solution. We found above that for $C(x) = 0.01x^2 - x + 50$, $x_1 = 300$, and unknown (arbitrary) Δx that

$$\Delta C = 5\Delta x + 0.01(\Delta x)^2$$

To find $\frac{\Delta C}{\Delta x}$, we first need to factor Δx out of every term on the right side.

$$\Delta C = \Delta x(5 + 0.01\Delta x)$$

Now we can divide both sides by Δx

$$\frac{\Delta C}{\Delta x} = 5 + 0.01\Delta x$$

- (f) You want to look at how changing quantity impacts the change in cost in general. In other words, find a formula for ΔC and $\frac{\Delta C}{\Delta x}$ terms of an *unknown* starting quantity x_1 and *unknown* change Δx .

Solution. We want to find

$$\Delta C = C_2 - C_1 = C(x_2) - C(x_1).$$

We do not know x_1, x_2 , so we must write

$$x_2 = x_1 + \Delta x$$

We can now compute C_1 and C_2

$$C_1 = C(x_1) = 0.01(x_1)^2 - x_1 + 50$$

and

$$\begin{aligned} C_2 &= C(x_1 + \Delta x) \\ &= 0.01(x_1 + \Delta x)^2 - (x_1 + \Delta x) + 50 \\ &= 0.01(x_1^2 + 2 \cdot x_1 \cdot \Delta x + (\Delta x)^2) - (x_1 + \Delta x) + 50 \\ &= 0.01x_1^2 + 0.02x_1 \cdot \Delta x + 0.01(\Delta x)^2 - x_1 - \Delta x + 50 \end{aligned}$$

We have found that

$$\begin{aligned} C_1 &= 0.01(x_1)^2 - x_1 + 50 \\ C_2 &= 0.01x_1^2 + 0.02x_1 \cdot \Delta x + 0.01(\Delta x)^2 - x_1 - \Delta x + 50 \end{aligned}$$

We can now compute

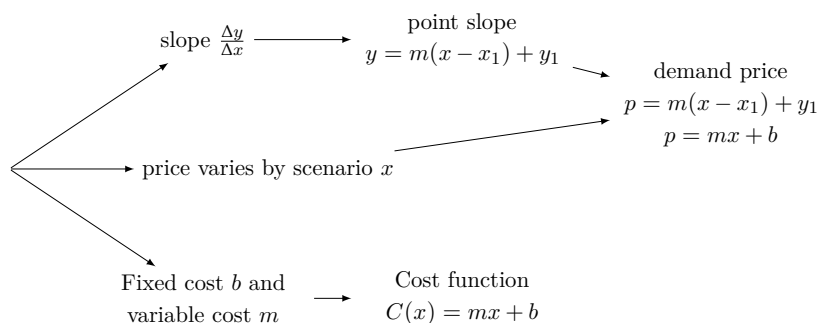
$$\begin{aligned} \Delta C &= C_2 - C_1 \\ &= (0.01x_1^2 + 0.02x_1 \cdot \Delta x + 0.01(\Delta x)^2 - x_1 - \Delta x + 50) - (0.01(x_1)^2 - x_1 + 50) \\ \Delta C &= 0.02x_1 \cdot \Delta x + 0.01(\Delta x)^2 - \Delta x \end{aligned}$$

To find $\frac{\Delta C}{\Delta x}$, we need to factor a Δx out of every term on the right, and then divide by it.

$$\begin{aligned} \Delta C &= (\Delta x)(0.02x_1 + 0.01\Delta x - 1) \\ \frac{\Delta C}{\Delta x} &= 0.02x_1 + 0.01\Delta x - 1 \end{aligned}$$

Lines and Linear Functions

Linear functions show up everywhere, and have very nice mathematical properties. In this section we will begin by briefly reviewing the basic properties of lines. But our main focus will be on introducing and understanding a number of important applications of linear functions to topics in economic and business contexts.



3.1 Finding the Equation of a Line

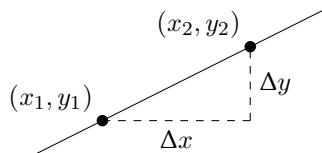
Objectives

- Review the equation for the point-slope form of a line
- Find the equation of a line under a variety of circumstances.

Definition 3.1 Any two points (x_1, y_1) and (x_2, y_2) define a line with slope

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Graphically, we get the following picture.



The **point slope form** of a line through the point (x_1, y_1) with slope m is the equation

$$y = m(x - x_1) + y_1$$

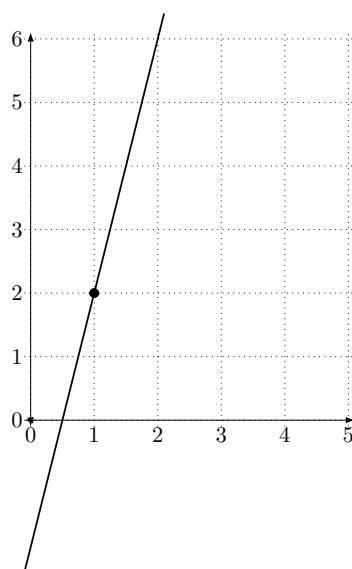
The **slope intercept form** of a line has the form $y = mx + b$, and is what you get if you use algebra to simplify the point slope form of the line. \diamond

Exploration 3.1 Suppose $y = f(x)$ is a line with slope $m = 4$ through the point $(1, 2)$. Find the equation of this line, and draw the graph of the function.

Solution. The point slope form for the line is $y = m(x - x_1) + y_1$.

We are also given values for the three constants m , x_1 , and y_1 . Putting this together, we get

$$y = 4(x - 1) + 2.$$



Important: there is only *one* independent variable, x in the right hand side of the point slope form. The letters m , x_1 , and y_1 all refer to *specific numbers*.

$$y = m(x - x_1) + y_1$$

Whenever you are asked for the *equation for a line*, you just need to find *numerical values* for the slope m and a single point on the line (x_1, y_1) .

Exploration 3.2 Find the equation for a line through the points $(1, 3)$ and $(3, 2)$, and draw a graph of this line.

Solution. You are asked to find the *equation* of the line described. That means to find a rule defining the function as $y = f(x)$

First, write down the point-slope definition of the line.

$$y = m(x - x_1) + y_1$$

You must find numbers for m , for x_1 , and for y_1 . In the problem, the first point $(1, 3)$ corresponds to $x_1 = 1$ and $y_1 = 3$. You could also use the second point $(3, 2)$, which would correspond to the point $x_2 = 3$ and $y_2 = 2$.

To find the slope, use the formula

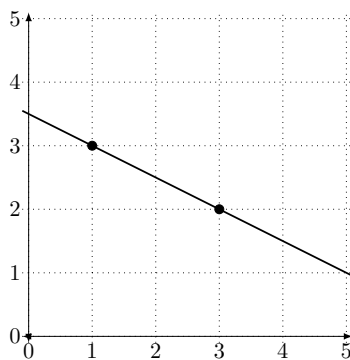
$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 3}{3 - 1} \\ &= \frac{-1}{2} = -\frac{1}{2} = -0.5 \end{aligned}$$

Plugging this into the point slope form, $y = m(x - x_1) + y_1$, you get the *point slope form*

$$y = -0.5(x - 1) + 3$$

If you are instead asked for the *slope-intercept form* $y = mx + b$, you just need to simplify the equation you found above

$$\begin{aligned} y &= -0.5(x - 1) + 3 \\ &= -0.5x + 0.5 + 3 \\ &= -0.5x + 3.5 \end{aligned}$$



Linear functions arise whenever you have a *constant rate of change*. We will examine rates in detail beginning in Chapter ?? . For now, a few examples will suffice.

1. Velocity is a rate measured in *miles per hour*.
2. Item Price is a rate measured in *dollars per item*.

Exploration 3.3 Suppose that you are driving down a state highway at a constant speed with a broken speedometer. At 3:00am, you drive by mile marker 5. Later, at 5:30 am, you drive by mile marker 152.5.

Find an equation for your position (mile marker) as a function of the number of *hours* since midnight.

Solution. Because we are asked to find the *equation* for a line, we first write down the point-slope form

$$y = m(x - x_1) + y_1.$$

To fill in the missing information (the red variables), we must find the slope m , and a point x_1 and y_1 .

First, we must convert the information we are given into two points (x, y) where x is a number of hours since midnight, and y is a mile marker.

Because 3:00 am is three hours after midnight, the first piece of information corresponds to the point $x_1 = 3$ hours and $y_1 = 5$ miles.

The second point is more tricky. 5:30 am is five *and a half* hours after midnight, so this corresponds to the point $x_2 = 5.5$ and $y_2 = 152.5$

To find the slope of the line, we can now use the formula

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{152.5 - 5}{5.5 - 3} \\ &= \frac{147.5}{2.5} = 59 \end{aligned}$$

In other words, our average velocity is 59 miles per hour.

To find the *linear equation* for our location, we now plug $m = 59$, $x_1 = 3$, and $y_1 = 5$ into our equation for point-slope form to get

$$y = 59(x - 3) + 5$$

- Use whenever there is a constant rate of change.
- Slope = $\frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$
- Slope Intercept Form: $f(x) = mx + b$ where m is the slope and b is the y -intercept.
- Point Slope form: $f(x) = m(x - x_1) + y_1$ where m is the slope and (x_1, y_1) is any point on the line.

3.2 Demand Price and Linear Functions

Objectives

1. Introduce the concept of *demand price* and *supply price*.
2. Be able to *find an equation for* the function describing the demand price and supply price based on a verbal description of the marketplace.
3. Find the *point of equilibrium* between a supply and demand function.

We want to use mathematics to study questions in economics and business. To do this, we first need to introduce the key ideas between each of these economics concepts, and then apply mathematical tools to study them.

In the sections that follow, each Application will introduce the concept from a more intuitive and economic perspective, and will connect these applications to the mathematical tools we have in this course. We will then *summarize* this connection, often with a mathematical Definition.

In the ebook, this gives you two different ways to study this material

1. Expand and read the Application before moving on to the mathematical Definition and Explorations.
2. Jump to the mathematical Definition and Explorations first to get a preview of the mathematics first. You'll notice that it hard to understand the question or follow many steps without an understanding of the economic background. Make sure to *go back* and study the Application to master the context of the question.

Application 3.2 Item Price, Scenarios, and Possible Universes

verses. In life, and particularly in business, you need to make specific, concrete decisions. You either hire a new employee or you don't. You take out a loan of one amount, rather than another.

Sometimes, you ask yourself *what would happen if I make this decision?* Or, once it's too late, you might ask yourself *what would have happened if I made that decision?*

When comparing mutually exclusive choices, people often use words like **scenario** or **possible universe** to identify the different possible options.

There are many ways of trying to decide the price at which you will sell an item. In an auction, buyers each decide the maximum price at which they will buy the item, and the individual who has largest maximum price wins the auction. In a going-out-of-business sale, the

company starts with high prices, and progressively lowers the price until everything in the store is sold. Each individual has a different maximum price at which they will purchase the item. The lower the price goes, the more items are sold.

In this class we always assume a company picks a *single item price*, and everyone pays that single price.

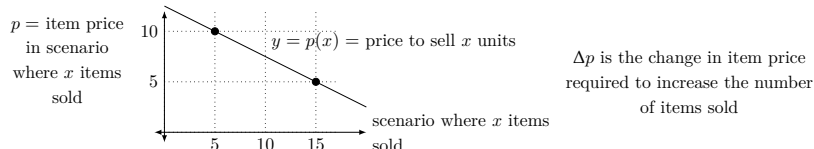
We also assume that you can change the price from day to day, holding all other things equal. That means that fewer items will be sold on a day when the price is high, and more items will be sold on a day when the price is low.

We will focus on how changing the the number of items to be sold impacts the revenue, cost, and profitability of a business. Because of that, we will generally think of the *item price* p as a function of the *number of items* x .

Definition 3.3 The **demand price** function is an equation $p = f(x)$. Given a number of items x that you wish to sell, this function tells you the price p that will result in exactly x items being sold. \diamond

If demand price is linear, we can use the techniques of Section ?? find an *equation* for the demand price function.

Exploration 3.4 A Demand Price Function. Suppose that 5 units are sold when the price is 10\$/item, and 15 units are sold when the price is 5 \$/item.



Suppose that demand is linear. Find the equation of the line $y = p(x)$ through these points.

Solution. From this problem, we are told that at the first quantity $x_1 = 5$, the item price is $p_1 = 10$. Furthermore, at the second quantity $x_2 = 15$, the item price is $p_2 = 5$. We need to find the equation of the line between these points.

The point slope form of a line is given by the equation $y = m(x - x_1) + y_1$. Because the dependent variable here is named p instead of y , we can write this as

$$p = m(x - x_1) + p_1.$$

For the slope of the line, we use

$$\begin{aligned} m &= \frac{\Delta p}{\Delta x} = \frac{p_2 - p_1}{x_2 - x_1} \\ &= \frac{5 - 10}{15 - 5} \end{aligned}$$

$$= \frac{-5}{10} = -0.5$$

Putting it all together, we get

$$\begin{aligned} p &= -0.5(x - 5) + 10 \\ &= -0.5x + 2.5 + 10 \\ &= -0.5x + 12.5 \end{aligned}$$

That function lets us know the price-per-item $p(x)$ sold in *every* possible universe x . In other words, the slope of $p(x)$ lets us know how much we need to lower the price to move to a possible universe where we sell one additional unit.

Exploration 3.5 Suppose you sell 15 items at 4 \$ per item, and 20 items at 2 \$ per item. Suppose also that demand price is linear.

- (a) Find an equation for price function $p = f(x)$, and simplify completely. Assume that the price function is linear.

Hint. You will need to use point-slope form to find the equation of the function.

Solution. We are given two points on the line. When the quantity is $x_1 = 15$, the price is $p_1 = 4$. When the quantity is $x_2 = 20$, the price is $p_2 = 2$.

To find the *equation for the line*, use the point-slope form

$$y = m(x - x_1) + y_1.$$

For the slope of the line, we use

$$\begin{aligned} m &= \frac{\Delta p}{\Delta x} = \frac{p_2 - p_1}{x_2 - x_1} \\ &= \frac{2 - 4}{20 - 15} \\ &= \frac{-2}{5} = -0.4 \end{aligned}$$

Putting it all together, we get

$$p(x) = -0.4(x - 15) + 4$$

Simplifying this, we get

$$p(x) = -0.4x + 10$$

- (b) What item price is required to sell 18 items?

Solution. We want to find the price $p(x) = -0.4x + 10$ when $x = 18$. In other words, substitute 18 for every occurrence of x .

$$p(18) = -0.4 \cdot 18 + 10 = -7.2 + 10 = 2.8$$

A price of $p = 2.8$ \$/item will result in 18 items being sold.

- (c) How many items will be sold when the price is 8.4 \$/item? What will the revenue be at this price?

Solution. We want to find the quantity x for which the price $p(x) = -0.4x + 10$ is equal to 8.4.

In other words, set the price function equal to 8.4.

$$8.4 = p(x) = -0.4x + 10$$

Solve for x .

$$\begin{array}{rcl} 8.4 & = & -0.4x + 10 \\ -10 & = & -10 \end{array}$$

$$\begin{array}{rcl} -1.6 & = & -0.4x \\ \div -0.4 & & \div -0.4 \end{array}$$

$$4 = x$$

Four items will be sold when each item costs 8.4 dollars.

Because Revenue equals price times quantity ($R(x) = x \cdot p(x)$),

$$R(x) = 4 \cdot 8.4 = 33.6$$

The total revenue will be \$33.60.

Application 3.4 Supply Price. In any economic transaction, the person selling a product and the person buying a product have very different perspectives.

We already talked about the *buyer's* perspective when we talked about the *demand* price. We now want to think about the *seller's* perspective by studying the *supply* price.

Any seller has a huge number of options available to them. On any given day, and in any given market, they might decide to create a new product, to cancel an existing product, to go out of business completely, etc.

Furthermore, making one product takes up time and energy that could otherwise be used to create another product. In other words, there are many tradeoffs.

The **supply price** of a particular number of items refers to the item price that the seller must be able to earn in order to make it worth their time to create that number of items.

For example, you might only be willing to increase production to $x = 100$ items if you are sure that you would be able to sell each item for $p = 15\$$ each. In this case, the supply price for 100 items is 15\$.

Note that a higher item price will encourage sellers to produce more. That means that supply functions are always increasing.

Definition 3.5 The **supply price** function is an equation $p = g(x)$. Given a number of items x , this function tells you the minimum item price p that will result in a manufacturer being willing to produce x items. \diamond

Exploration 3.6 Finding a supply price function. Suppose that a manufacturer is willing to supply 2 items when the item price is 5 \$ per item, and that they are willing to supply 10 items when the item price is 7 \$ per item. Suppose also that supply price is linear. Find an equation for the supply function.

Solution. Even though the application is new, the mathematics is identical to the demand price example.

From this problem, we are told the seller will supply the quantity $x_1 = 2$ at an item price of $p_1 = 5$. Furthermore, the seller will supply the quantity $x_2 = 10$ at an item price of $p_2 = 7$. We need to find the equation of the line between these points.

The point slope form of a line is given by the equation

$$p = m(x - x_1) + p_1.$$

For the slope of the line, we use

$$\begin{aligned} m &= \frac{\Delta p}{\Delta x} = \frac{p_2 - p_1}{x_2 - x_1} \\ &= \frac{6 - 5}{10 - 2} \\ &= \frac{2}{8} = 0.25 \end{aligned}$$

Putting it all together, we get

$$\begin{aligned} p &= 0.25(x - 2) + 5 \\ &= 0.25x - 0.5 + 5 \\ &= 0.25x + 4.5 \end{aligned}$$

Application 3.6 Equilibrium quantity and price. Supply price and demand price capture the two sides of the market, the seller (supply) side and the buyer (demand) side.

Because supply price is an increasing function, and demand price is a decreasing function, there will often be a single point where the two curves intersect.

If the curves do not intersect, then the demand price is always lower than the supply price. In other words, there is no quantity at which the market willing to pay the item price required by the seller. In other words, no items are sold in this case.

The price where this happens is called the **equilibrium point**, because at this quantity, the supply price and demand price are equal (in equilibrium).

Definition 3.7 Suppose you are given a demand price function $p = f(x)$ and a supply price function $p = g(x)$. The **equilibrium point** is the point (x, p) where the two curves intersect.

The value of x is called the equilibrium quantity, and is the quantity of items sold when supply and demand prices are equal. The value of p is the price that results in this equilibrium between supply and demand. \diamond

Exploration 3.7 Equilibrium Price and Quantity. In a certain market, 5 units will be sold when the price is 10 \$ per item, and 15 units will be sold when the price is 5 \$ per item.

Furthermore, the manufacturer is willing to supply 2 items when the item price is 5 \$ per item, and that they are willing to supply 10 items when the item price is 7 \$ per item.

Find the equilibrium quantity and price.

Answer. The equilibrium price is $p = 7.166\dots$ \$ per item, and the equilibrium quantity is $x = 10.666\dots$ items. That means that the equilibrium *point* is approximately $(10.666, 7.166)$.

Solution. We have already found in Exploration ?? above that the demand price function corresponding to $(5, 10)$ and $(15, 5)$ is given by the equation

$$p = -0.5x + 12.5$$

and found in Exploration ?? that supply price function corresponding to $(2, 5)$ and $(10, 7)$ is given by

$$p = 0.25x + 4.5.$$

Furthermore, we know that the equilibrium occurs at the intersection of these lines, which is the point at which the price from both of these equations is equal.

But if the price p in both equations is equal, that means that we can set the right hand sides to be equal, giving

$$0.25x + 4.5 = -0.5x + 12.5$$

Now we just need to solve this equation for x

$$\begin{array}{rcl} 0.25x + 4.5 & = & -0.5x + 12.5 \\ +0.5x & & +0.5x \\ \hline 0.75x + 4.5 & = & +12.5 \\ -4.5 & & -4.5 \end{array}$$

$$\begin{array}{rcl} 0.75x & = & 8 \\ \div 0.75 & & \div 0.75 \end{array}$$

$$x = 10.666\dots$$

In other words, the equilibrium quantity is $x = 10.666\dots$. To find the equilibrium price, plug this quantity into either the supply or demand price functions.

Using the supply price function, this gives

$$p = 0.25 \cdot 10.666 + 4.5 = 7.166$$

or equivalently using the demand price function gives

$$p = -0.5 \cdot 11.333 + 12.5 = 7.166$$

In either case, we get that the equilibrium price of $p = 7.166\dots$ \$ per item corresponds to an equilibrium quantity of about $x = 10.666\dots$ items. Putting it all together, we get that the equilibrium *point* is approximately $(10.666, 7.166)$

Demand Price.

- Economics studies different, incompatible **scenarios** or **possible universes**.

In this course, we often compare universes where different numbers of items x are sold.

- A price function $p = f(x)$ is a rule which when given a number x , tells you the *item price* in the universe where exactly x items are sold.
- In most of this course, we will take the perspective of a manufacturing company, and the price function represents their market research.

In a few problems, we will study the market from the outside. In these problems, we will have different price functions for supply and demand.

Each of the concepts we look at in this class will build on the previous ones. That means it is very important to understand *both* the mathematics *and* the economic meaning of each concept we study.

In this case, one of the best ways to remember what item price means is to think about it in connection to revenue. The **revenue** of a scenario is the total amount of money that you take in.

When the item price is fixed, there is a simple equation for this:

$$\text{Revenue} = (\text{Item Price}) \cdot (\text{Number of Items})$$

When the item price function is linear, the Revenue function is quadratic. In other words, to understand even basic questions about profitability, we will need more powerful mathematical tools. Because of this, we will discuss revenue in more detail in Section ??.

3.3 Item Cost, Total Cost, and Linear Functions

Objectives

- Introduce the concepts of **item cost**, **fixed cost**, **variable cost**, and **total cost**
- Use the fixed cost and item cost to find a formula for the total cost.
- Use the total cost function to find the item cost and fixed cost.

For the purposes of this course, we will generally focus on the perspective of a business that is selling a single product in a specific market. As discussed above, this business will need to understand the demand price function for their market. Multiplying the price times quantity will give the total income, or revenue, of any given scenario.

But in order to create an item for sale to others, the business will also have to purchase materials and labor of its own. We will describe the expense side of the business using the **total cost function**, usually abbreviated the **cost function**.

Definition 3.8

- The **fixed cost** is the amount of money that must be spent *before* any items can be produced. In other words, the fixed cost is the total cost of 0 items.
- The **variable cost** of an item is the cost of that item alone.
- The **total cost** of x items, written $C(x)$, is the *cumulative cost* of making items numbered 0 through x

◇

Exploration 3.8 Finding the total cost function. You are planning to manufacture a new plastic toy. It costs \$100 to create a mold, and \$5 in plastic, labor, and electricity per unit created.

Find the *total cost function*.

Solution. Once you have purchased the mold, but before you have created the first toy, you have manufactured $x = 0$ items, but you have already spent $C(0) = 100$ dollars. In other words, the cumulative cost

of 0 items is 100\$

After you have manufactured $x = 1$ item, you have spent $C(1) = 100 + 5 = 105$ \$.

After you have manufactured $x = 2$ items, you have spent $C(2) = 100 + 5 + 5 = 100 + 5 \cdot 2 = 110$ \$.

After you have manufactured $x = 30$ items, you have spent $C(30) = 100 + 5 \cdot 30 = 250$ \$.

Looking at the pattern in these equations, you can "see" the equation for the total cost function. For any number of items x , the total cost will be

$$C(x) = 100 + 5 \cdot x.$$

Note: There are several different ways to find a cost function. In this example, we followed an intuitive approach. In Exploration ?? and Exploration ?? (Exploration ??-??), we will use the point slope form of the line.

Exploration 3.9 Using the total cost function. Above, we found that if it costs \$100 to create a mold, and \$5 in plastic/labor/electricity per unit created, then the total cost to x items is $C(x) = 100 + 5 \cdot x$

If you (the toy making business) have \$300, how many toys can you make?

Solution. We want to find the number of items x such that the total cost $C(x)$ is equal to 300. In other words, we want to solve for x such that

$$300 = C(x).$$

But we also know that $C(x) = 100 + 5x$, so we can substitute that equation for $C(x)$.

$$300 = 100 + 5x$$

We can now solve this as normal

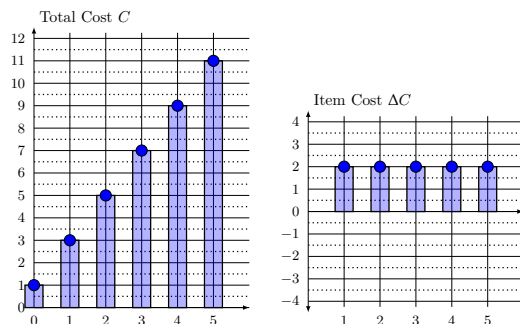
$$\begin{array}{rcl} 300 & = & 100 + 5x \\ -100 & & -100 \\ \hline 200 & = & 5x \\ \div 5 & & \div 5 \\ \hline 40 & = & x \end{array}$$

You are able to manufacture $x = 40$ items.

It is important to see and remember the difference between the concepts of *total cost*, *item cost*, and *fixed cost*. In the next section, we will use graphs of totals and changes to help make these distinctions

more vivid.

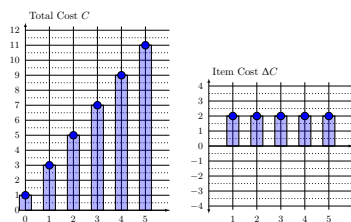
To get a broader and more intuitive picture of the relationship between **item cost** and **total cost**, it is helpful to think about total and change graphs, as in Section ?? . For example, suppose that $C(x)$ is a total cost function, and that its total and change graphs are given below.



In the total graph on the left, we are told the **total cost** of purchasing a given number of items. For example, purchasing 0 items cost \$1. This means that there is a **fixed cost** of \$1 (to set up the assembly line) that must be paid in addition to the cost for actually producing an item.

In the change graph on the right, we are told the **change** in cost due to each individual item. For simplicity, this is often called the **variable cost**. For example, items 1 through 5 each have a variable cost of \$2.

Exploration 3.10 Suppose that the total cost function is as described using the total and change graph below.



(a) What is the fixed cost? What is the variable cost?

Solution. Remember that in the graph above, we are given the *cumulative* (or total) cost C on the left, and the *item* cost ΔC on the right.

Using the graph of $C(x)$ on the left, we can see that the cumulative (total) cost of 0 items is \$1, the total cost of 1 item is \$3, the total cost of 2 items is 5, and so on.

First, this means that we have a *fixed cost* of \$1. Second, this suggests that each additional item increases the cost by \$2.

The graph on the right tells us the variable cost for each item. As expected, the cost increases by 2\$ for each item, so the *variable cost* is equal to 2.

- (b) Write down a formula for total cost C as a function of x .

Solution. Our strategy is to think through the meaning of the total cost as in Exploration ??.

We have already found above that the fixed cost is 1, and the variable cost is 2. Therefore, the total cost is equal to

$$C(x) = 1 + 2x$$

And indeed this matches exactly the line you would get by connecting every dot in the graph on the left.

- (c) Write down a formula for change in cost ΔC as a function of x .

Solution. Recall that ΔC is the graph on the right, which tells us the item cost.

Looking at the graph, we see that each item contributes the same amount to the total cost. In other words, no matter which item x you are asking about, the ΔC that item contributes will always be 2. In other words, ΔC is the constant function

$$\Delta C(x) = 2$$

And indeed connecting the dots in the graph on the right would indeed result in the line $y = 2$.

- (d) Explain how ΔC relates to the formula for $C(x)$.

Solution. In this example, ΔC is the *slope* of the line $y = C(x)$.

This will always be the case when the total cost function is *linear*, and the increment Δx is equal to 1.

The goal of calculus is to give a more complete connection between graphs of totals and changes for non-linear functions, and for situations where you are looking at changes with a different value of Δx .

Most total cost functions in this class will be given as word problems. But remembering the meaning of total and change graphs will often make it easier to interpret the different parts of the question correctly.

Exploration 3.11 A certain assembly line costs \$5 to set up, and that it costs a total of \$9 to make 4 items. Suppose that the cost to produce each item is the same.

- (a) Write down a formula for total cost $C(x)$ as a function of the number of items x .

Solution. Because the *cost per item* is constant, the *total cost* function is linear.

We must therefore use the point slope form of the line

$$C(x) = m(x - x_1) + y_1$$

The knowledge that no (0) items corresponds to a cost of 5 dollars tells us that the point $x_1 = 0$ and $y_1 = 5$ is on our line.

We are also told that $x_2 = 4$ items corresponds to a price of $y_2 = 9$ dollars, giving us the second point on our line.

To find m , use the slope formula $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 5}{4 - 0} = \frac{4}{4} = 1$

In other words, the slope is $m = \frac{1 \text{ \$}}{1 \text{ item}}$, or 1 \$ per item.

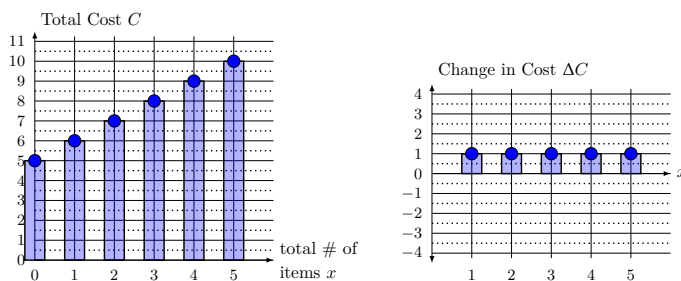
Putting it all together, we get the equation

$$C(x) = 1(x - 0) + 5$$

or equivalently $C(x) = 1(x - 4) + 9$.

- (b) Graph the total cost C and the change in cost ΔC

Solution.



Exploration 3.12 You are manufacturing custom dinner plates.

The total cost of manufacturing two items is \$6, and the total cost of manufacturing four items is \$11. The cost per item is constant.

- (a) Write down a formula for the total cost $C(x)$ as a function of the number of items x

Solution. Because the *cost per item* is constant, the *total cost* function is linear.

We must therefore use the point slope form of the line

$$C(x) = m(x - x_1) + y_1$$

The knowledge that 2 items corresponds to a cost of 6 dollars tells us that the point $x_1 = 2$ and $y_1 = 6$ is on our line.

We are also told that $x_2 = 4$ items corresponds to a price of $y_2 = 11$ dollars, giving us the second point on our line.

To find m , use the slope formula $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 6}{4 - 2} = \frac{5}{2} = 2.5$

In other words, the slope is $m = \frac{2.5 \text{ \$}}{1 \text{ item}}$, or 2.5 \$ per item.

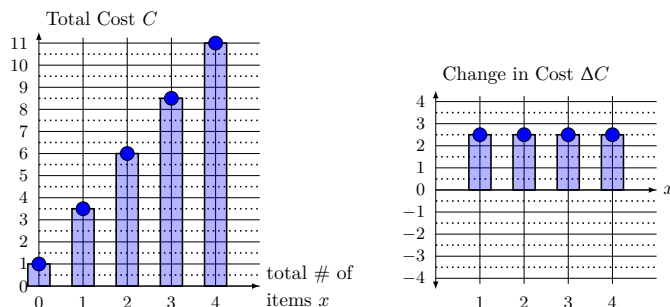
Putting it all together, we get the equation

$$C(x) = 2.5(x - 2) + 6$$

or equivalently $C(x) = 2.5(x - 4) + 11$.

(b) Graph the total cost C and the change in cost ΔC .

Solution.



For Reflection. Suppose that you are looking at a function of linear cost. Can you come up with a formula if you are given only a $C(x)$ graph? Why or why not? What if you are given only a ΔC graph? Why or why not?

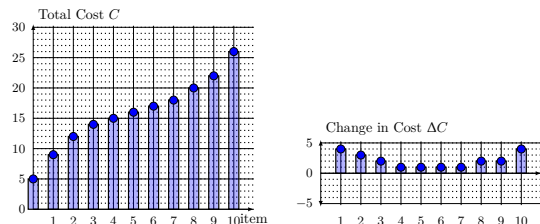
More Realistic Cost Functions are Not Linear. In the examples above, the change in cost, ΔC , of producing each additional item has been constant. In other words, in the examples above, creating an additional unit always costs the same amount. When this is the case, the total cost function $C(x)$ will be a linear function.

But life is often more complicated. Sometimes, you can get a bulk discount - if you manufacture enough items, the cost per item can come down. Other times, when there is only a limited amount of your raw materials, the item cost can increase as you produce more and more items. Often, you'll run into a more complex combination of these features.

In the remainder of the course, we will use the study of rates of change for nonlinear functions that capture these kinds of economic

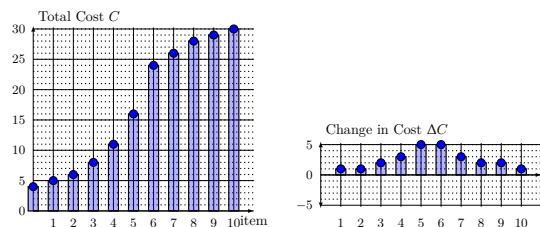
dynamics. But to give a more complete picture, we will close this section with two examples of non-linear cost functions.

Example 3.9 A Non-Linear Cost Function. A non-linear total cost function is given below.



Note that because the total graph C on the left is not a line, the change graph ΔC on the right is not constant. In fact, the 1st and 10th items cost the **most** to manufacture, and the 4th-7th items cost the **least** to manufacture. \square

Example 3.10 Another Non-Linear Cost Function. A second non-linear cost function is given below.



Note again that because the change ΔC is not constant, the total graph C is not linear. In this example, the 1st and 10th items cost the **least** to manufacture, and the 5th-6th items cost the **most** to manufacture. \square

Application 3.11 Think Economically with Units. Suppose that you are planning to publish a new book, and you know from experience that it will cost \$20,000 to edit and market the book, and \$5 to print each individual copy of the book.

These two costs are very different. The \$20,000 is a single cost that applies regardless of the number of books. To highlight this, we call this the **fixed cost**, and measure this in **dollars** (\$). On the other hand, the \$5 cost applies to each item, so we call this the **variable cost**, and measure this in **dollars per item** or $\frac{\$}{\text{item}}$.

Suppose that we wish to purchase 1,000 books. Then the total cost is $20,000 + 5 \cdot 1,000 = 25,000$. But there is actually something quite important going on here behind the scene. Here, each book is a separate item, so the units of the 1000 is going to be in **items**.

By using the letter x for the number of items, we get an equation that helps us think about the **total cost** of creating any number of

books.

Units of measure gives an easy trick for identifying the parts of the total cost function:

1. The fixed cost will be in \$.
2. The variable cost will be in $\frac{\$}{\text{item}}$.
3. Use the variable x for the number of items.

Units also help us make sense of equations once we've set them up. You can generally think of “units of measure” like variables, which means you can use algebra to interpret any equation.

$$\begin{aligned}
 \text{Total Cost} &= \$ = (\$) + \left(\text{item} \cdot \frac{\$}{\text{item}} \right) \\
 &= (\$) + \left(\text{item} \cdot \frac{\$}{\text{item}} \right) \\
 &= \$ + \$ \\
 &= \$
 \end{aligned}$$

Notice that you can *only combine like terms*. If you ever find yourself trying to add **number items** to **dollars**, then you know that there is a problem!

We can also use this idea to come up with a **function** for the cost of different numbers of books

Table 3.12

items	total cost C
5	$20000 + 1000 \cdot 5$
10	$20000 + 1000 \cdot 10$
x	$20000 + 1000 \cdot x$

For this reason, we often write the total cost function

$$C(x) = 20000 + 1000 \cdot x$$

3.4 Models and Applied Domains

In mathematics, functions are often defined by rules, and the *domain* of a function is usually set of all the inputs where the rule can be applied. For example, if $f(x) = \frac{1}{x}$, then I can apply the rule to any number *except* zero, because $\frac{1}{0}$ is *undefined* (or *DNE*). This was discussed in greater detail in Section ?? in general, and Exploration ?? in particular.

But some functions are intended to *model* realistic phenomena, such

as price, quantity, and so on. These realistic models impose additional constraints on when a model should be defined.

Definition 3.13 Suppose that $f(x)$ is some function intended to *model* some application, phenomena, etc.

The **applied domain** of the function is the set of inputs where the function is both defined *and* where all of the quantities involved make real, practical sense. \diamond

In this class, we will focus on price functions, and on two specific restrictions.

- The quantity cannot be negative. That is, in the implied domain of a price function, we must have $x \geq 0$
- The price cannot be negative. That is, for x to be in the implied domain of a price function, we must have $p(x) \geq 0$

Checkpoint 3.14 Suppose that $p(x) = -0.1x + 21$ defines an item price function. Find the applied (or implied) domain of $p(x)$.

Solution. The *implied domain* of a function is the set of inputs where its *applied meaning* makes sense.

In the case of item price, it doesn't make sense to look at the price of a negative number of items, and it doesn't make sense to talk about items with a negative prices. That means that the implied domain of price is the set of x where $x \geq 0$ and $p \geq 0$

But we can rewrite the equation $p \geq 0$ as follows:

$$\begin{aligned} p(x) &\geq 0 \\ -0.1x + 21 &\geq 0 \\ 21 &\geq 0.1x \\ 210 &\geq x \end{aligned}$$

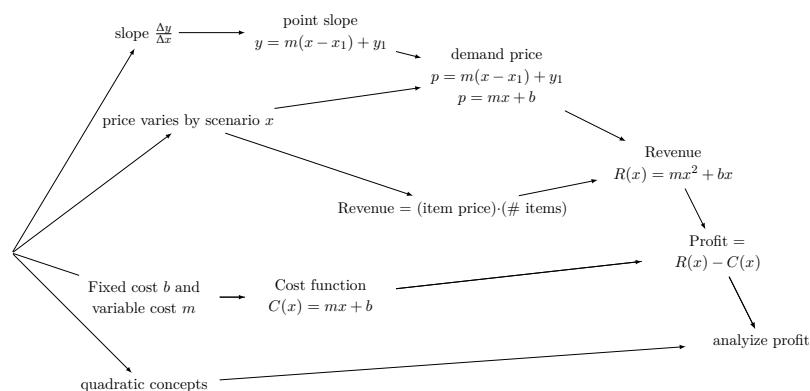
So the item price function makes sense when $0 \leq x$ and $x \leq 210$. In other words, the applied domain is the interval $[0, 210]$

4

Polynomial and Rational Functions

In the previous sections, we have discussed the mathematics of lines, and applications to item price and total cost functions. But linear functions are not powerful enough to model other important concepts, such as *revenue* and *profit*. Before we introduce these concepts, we first briefly review quadratic and polynoimal functions.

The concepts in this and the previous chapter are inter-related in complex ways. The following diagram summarizes the relationships between the main concepts in this section and the preceeding section. Refer back to this diagram from time to time to help organize your thoughts.



4.1 Quadratic and Polynomial Functions

Definition 4.1

- A **monomial** with variable x is some number times a power of x .
- A **polynomial** is a sum of monomials with positive integer powers.
The **degree** of a polynomial is the largest power that occurs.
- A **quadratic** function is a polynomial with maximum degree 2.

◇

Example 4.2 Basic polynomial terms.

1. $3x^2$ is a monomial of degree 2.
2. $0.1x^4$ is a monomial of degree 4.
3. $3x^7 - 2.5x^1 + 3$ is a polynomial of degree 7.
4. $-0.4x^2 + 10.2x + 15$ is a quadratic function.

□

Definition 4.3 Let $f(x)$ be any function. The **roots** of $f(x)$ are those inputs x such $f(x) = 0$. Graphically, the roots of a function are the x -values where the function intersects the x -axis, and are also called the function's **x -intercepts**.

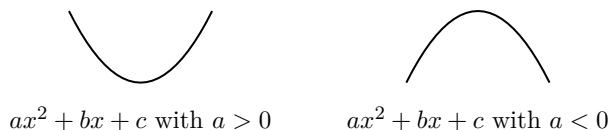
◇

Proposition 4.4 The Quadratic Formula. *If a, b, c are real numbers, then*

$$ax^2 + bx + c = 0 \text{ if and only if } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In particular, if $f(x) = ax^2 + bx + c$ is any quadratic function, the roots of f can be found using the quadratic formula.

The shape of the quadratic is determined by the leading coefficient a . The curve opens up if $a > 0$, and the curve opens down if $a < 0$.

Example 4.5 Quadratics can open up or down.

□

Definition 4.6 The **vertex** of a quadratic is the point in the middle of the curve about which the curve is symmetric. If $a > 0$, it is the point on the curve with the *smallest* y -value. If $a < 0$, it is the point on the curve with the *largest* y -value.

◇

If the quadratic function has vertex (h, k) , the biggest or smallest *output* will be $y = k$. This extreme output will occur when the *input* is $x = h$.

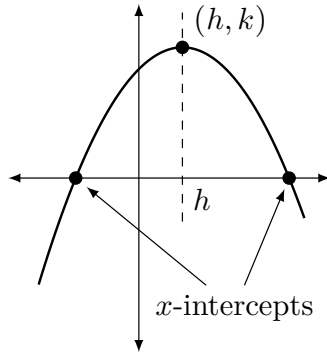
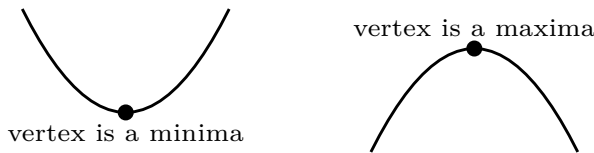


Figure 4.7 Features of a Quadratic Function

Example 4.8 Vertices can be maxima or minima.



□

Most quadratics that we see in this course will pass through the x -axis. In this case, there are two ways to find the vertex of a quadratic. One is easier to remember, and one is easier to use.

Notice that *quadratic* functions have a perfect reflection symmetry through the vertex (h, k) . That means that the vertex is *half way between* the two roots, or equivalently the vertex's x -value h is the *average of the roots* of the function (if they exist).

That means you can find the x -value of a vertex, denoted h , by first finding the roots (using the quadratic equation), and then by averaging them. To find the y -value of the vertex, plug h into the equation for the quadratic. In other words, $k = f(h)$.

There is also a shortcut for finding the x -value of a vertex.

Proposition 4.9 Suppose that $f(x) = ax^2 + bx + c$ is a quadratic function. Then the vertex is located at the point (h, k) with x -value $h = \frac{-b}{2a}$ is the average of the x -intercepts, and y -value $k = f(h)$, which is equal to $k = c - \frac{b^2}{4a}$.

Proof. We have already noticed that h is the average of the roots by the symmetry of the quadratic function.

Recall also that the *roots* of the function $f(x) = ax^2 + bx + c$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

To find the average of those roots, simply compute

$$\begin{aligned}
 h &= \frac{1}{2}(x_1 + x_2) \\
 &= \frac{1}{2} \left(\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} + \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right) \\
 &= \frac{1}{2} \left(\frac{-b}{2a} + \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right) \\
 &= \frac{1}{2} \left(\frac{-b}{2a} + \frac{-b}{2a} + 0 \right) \\
 &= \frac{1}{2} \left(\frac{-2b}{2a} \right) \\
 h &= \frac{-b}{2a}
 \end{aligned}$$

To find the y -value of the vertex, simply plug $h = \frac{-b}{2a}$ into the original function $f(x) = ax^2 + bx + c$. That gives

$$\begin{aligned}
 k &= f(h) = f\left(\frac{-b}{2a}\right) \\
 &= a \left(\frac{-b}{2a}\right)^2 + b \left(\frac{-b}{2a}\right) + c \\
 &= a \left(\frac{b^2}{4a^2}\right) + b \left(\frac{-b}{2a}\right) + c \\
 &= \frac{ab^2}{4a^2} + \frac{-b^2}{2a} + c \\
 &= \frac{b^2}{4a} + \frac{-b^2}{2a} + c \\
 &= \frac{b^2}{4a} + \frac{-2b^2}{4a} + c \\
 k &= \frac{-b^2}{4a} + c = c - \frac{b^2}{4a}
 \end{aligned}$$

■

Quadratic Functions.

- Have the form $f(x) = ax^2 + bx + c$
- The curve opens up if $a > 0$, and the curve opens down if $a < 0$.
- You can find the x -intercepts using the quadratic formula:

$$f(x) = 0 \text{ if and only if } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The vertex is located at (h, k) where $h = \frac{-b}{2a}$ is the average of the x -intercepts, and $k = f(h)$.

- The max/min *output* is k , and occurs when the *input* is h .
- You can visualize *any* quadratic as a cup that opens up (if $a > 0$) or opens down (if $a < 0$).
- You can find the y -intercept by evaluating f at 0. In other words, the y -intercept is $y = f(0)$

To solve polynomial inequalities, start with a sketch of the function. The key idea is that *any* quadratic looks like one of the two examples from Example ???. Then, use the formulas above to fill in x -intercepts and the coordinate of the vertex. Finally, use your sketch to answer the question.

Exploration 4.1 Let $f(x) = -x^2 + 3x - 2$.

(a) Solve $f(x) = 0$

Solution. We must solve the equation $-x^2 + 3x - 2 = 0$. We can do this by factoring, or by using the quadratic equation.

Using the quadratic equation $x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot (-1) \cdot (-2)}}{2 \cdot (-1)}$, we get that $x = 1$ or $x = 2$.

We will eventually need to solve equations by factoring, so this is a good time to start practicing. We will rewrite the equation

$$-x^2 + 3x - 2 = 0.$$

It is easiest to factor when the leading term is just x^2 . Here, the leading coefficient is -1 , so we should divide each term on the left by -1 .

$$(-1) \cdot \left(\frac{-1}{-1}x^2 + \frac{3}{-1}x - \frac{2}{-1} \right) = 0$$

Essentially, we are "un-distributing" the leading term of -1 . Simplifying the equation down, we now get

$$(-1) \cdot (x^2 - 3x + 2) = 0$$

Now we can factor the quadratic inside. We want to find some numbers d_1, d_2 such that

$$x^2 - 3x + 2 = (x + d_1)(x + d_2) = x^2 + (d_1 + d_2)x + d_1 \cdot d_2.$$

In other words, we want numbers that add to -3 and multiply to positive 2. Trying $d_1 = -1$ and $d_2 = -2$, we get $-1 + -2 = -3$ and $(-1)(-2) = 2$ as desired. That means we can rewrite our original equation as

$$(-1) \cdot (x - 1)(x - 2) = 0$$

But the product of two numbers can only equal zero if the individual numbers are zero. That means that the equation is equivalent to *either* having $x - 1 = 0$ *or* having $x - 2 = 0$.

In other words, this is equivalent to having $x = 1$ or $x = 2$, which are the two x -intercepts of our quadratic.

- (b) Find the vertex of $f(x)$.

Solution. Remember that there are two ways to find the x -value of the vertex.

Because the vertex is the average of the roots, and because we have already found that the roots are $x = 1$ and $x = 2$, the vertex has x -value

$$h = \frac{1 + 2}{2} = 1.5.$$

On the other hand, for *quadratics only*, we have a nice formula, that

$$h = \frac{-b}{2a} = \frac{-3}{2 \cdot (-1)} = 1.5$$

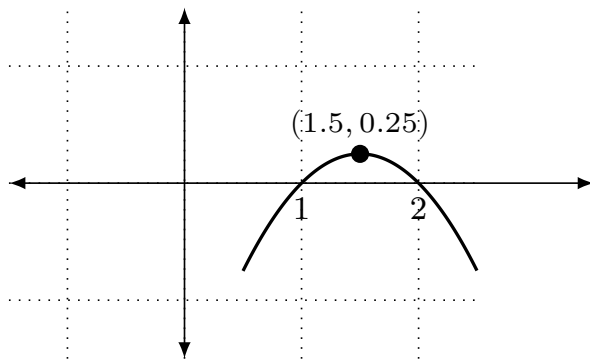
To find the y -value of the vertex k , just compute $k = f(h)$

$$k = -(1.5)^2 + 3(1.5) - 2 = 0.25$$

- (c) Sketch a graph of $f(x)$.

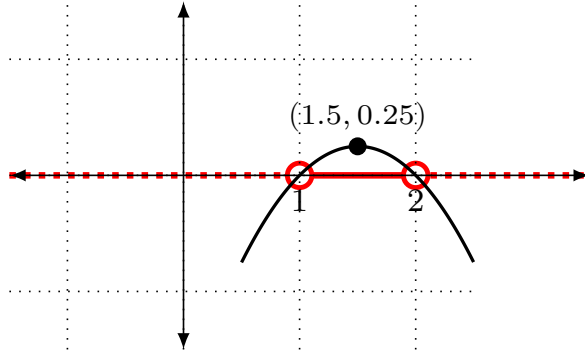
Solution. Recall that the shape of a quadratic depends only on the coefficient a of x^2 . Because $a = -1 < 0$, the graph of the function is a parabola that opens down.

We have also already found the x -intercepts $x = 1$ and $x = 2$ and the vertex $(1.5, 0.25)$. Putting this all together we get the following picture:



- (d) Solve the inequality $f(x) > 0$.

Solution. To solve this inequality, we first need the graph from the previous step. Notice that solving the equation $f(x) > 0$ is the same thing as looking for the *values of x which result in a height of greater than 0*. Let's look at what this means on the picture:



We can see that the function outputs values *strictly* greater than 0 if and only if $1 < x < 2$, which corresponds to the interval $(1, 2)$

- (e) Find the maximum y -value that the function achieves. At what x -value does the function achieve this output?

Solution. Again, the key is to look at the sketch of the function. Because the coefficient a of x^2 is negative, the curve opens down, and the largest y output occurs at the vertex. We have already shown that the vertex is $(1.5, 0.25)$.

That means a maximum output of $y = 0.25$ occurs at the x -value $x = 1.5$.

In the previous example, we have seen a technique that we will use in a variety of contexts during this course.

Solving Inequalities of Continuous Functions.

Suppose that $f(x)$ is *any* function.

To solve an inequality such as $f(x) > 0$, or $f(x) < 0$, follow the following steps:

1. Solve the inequality $f(x) = 0$.
2. Draw a number line, and label the x -values where the solutions occur.
3. Notice that you have divided the x -axis into several different intervals. Now, look at *each interval* separately.
For each interval, use a graph, a calculator, or some other technique to decide whether the inequality is true or false on that interval.
4. Write your final answer down in interval notation, *excluding* the endpoints (using parentheses).

To solve an inequality like $f(x) \geq 0$ or $f(x) \leq 0$, repeat the same steps but *include* the endpoints (using square brackets).

4.2 Revenue with Fair Pricing

Definition 4.10 Revenue is the total amount of money you take in from selling a number of items. In this class, we'll focus on fair pricing - where *everything* is sold at the same unit price. In this case, we have a simple equation:

$$\text{Revenue} = (\text{Item Price}) \cdot (\# \text{ Items})$$

◇

Exploration 4.2 In a certain market, you will sell 15 items when the item price is 4 \$/item, and you will sell 20 items when the item price is 2 \$/item.

- (a) Find the Revenue in the scenerio (possible universe) where you sell 15 items.

Solution. To find revenue, we need to apply Definition ???. That means we need to multiply quantity and price.

In the scenario (possible universe) where we sell 15 items at 4\$/item, our total revenue will be $R(15) = 15 \cdot 4 = 60\$$.

- (b) Find the Revenue in the scenerio (possible universe) where you sell 20 items.

Solution. Again, we must use Definition ???.

In the scenario where we sell 20 items at 2\$/item, our total revenue will be $R(20) = 20 \cdot 2 = 40\$$.

- (c) But there are infinitely many possible universes x that we need to compare! When the universe depends on the number of items x sold, we often call x the *quantity*.

Find a formula that you the revenue $R(x)$ as a function of the number of items x to be sold.

Solution. To find revenue, we need to apply Definition ???. That means we need to multiply quantity and price.

$$\text{Revenue} = (\text{Item Price}) \cdot (\# \text{ Items})$$

In this question, we are told to use x to represent the *quantity*, or number of items.

We must also find the *price* as a function of x . In fact, we have already done that for ***exactly these numbers*** in the solution to Exploration ??.

In that example, we used the point-slope form of the line to show that the *item price* function is $p = -0.4x + 10$.

But we need to find the *Revenue*, plugging in the above values into the equation

$$\text{Revenue} = (\text{Item Price}) \cdot (\# \text{ Items}).$$

That means that in any universe x , the total revenue will be given by

$$\begin{aligned} R(x) &= (\text{price}) \cdot (\text{quantity}) \\ &= (p(x)) \cdot (x) \\ &= (-0.4x + 10) \cdot (x) \\ &= -0.4x^2 + 10x \end{aligned}$$

Once you have found a revenue function, you can answer a range of questions by translating the question into a more familiar algebraic question.

Exploration 4.3 Suppose that the revenue of in the scenario where x items are sold is given by the function $R(x) = -0.4x^2 + 10x$

- (a) Find the scenarios (quantities x) where you make no revenue.

Solution. Solve

$$0 = R(x) = -0.4x^2 + 10x$$

There are several ways to solve this equation.

Quadratic Equation Use the quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = -0.4$, $b = 10$, and $c = 0$

Solving by Factoring We know that a product $a \cdot b = 0$ if and only if $a = 0$ or $b = 0$. So to solve the equation

$$-0.4x^2 + 10x = 0$$

since we have 0 on the right, we should try factoring the left side. Because both terms have an x , we can factor x out of both terms.

$$x \cdot (-0.4x + 10) = 0$$

This product can equal zero if and only if *either*

$$x = 0$$

or

$$-0.4x + 10 = 0.$$

There are two quantities where revenue equals zero. The first point is $x = 0$

To find the second point, solve the second equation for x

$$\begin{array}{rcl} -0.4x + 10 & & = 0 \\ -10 & & -10 \end{array}$$

$$\begin{array}{rcl} -0.4x & & = -10 \\ \div -0.4 & & \div -0.4 \end{array}$$

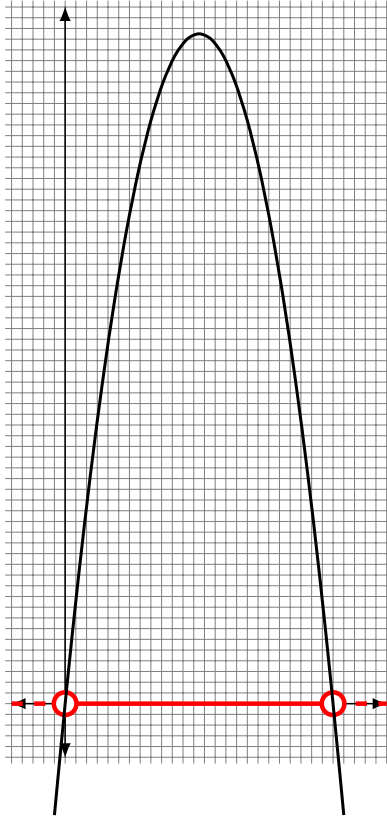
$$\begin{array}{rcl} x & & = 25 \end{array}$$

The revenue equals zero when $x = 0$ or $x = 25$.

- (b) For what scenarios (values of x) is your revenue positive? Give your answer in interval notation.

Solution. The revenue function is a quadratic with negative leading coefficient, so it is a parabola that opens down, which goes through the x -axis at the break-even points. This allows us to create a quick sketch of the function.

Using this sketch, we can solve the inequality $R(x) > 0$, and see that it is true for $0 < x < 25$. In interval notation, this is written $(0, 25)$.



- (c) For what scenario (value of x) is your revenue maximized?

Solution. Because revenue is a quadratic polynomial with negative leading coefficient, the maximum revenue occurs at the vertex (h, k) .

Recall also that the x -value of the vertex is located at the average of the two x -intercepts. In other words, revenue is maximized at $x = \frac{0 + 25}{2} = 12.5$ items.

We can now put together everything we have done so far in this class into a single example (exploration). Make sure that you are able to do problems like this on your own, since they are common quiz and exam questions.

Exploration 4.4 Suppose you sell 100 items at 12 \$/item and 500 items at 4 \$/item. Suppose also that price is linear.

- (a) Find an equation $p(x)$ for price as a function of x . Give the units of each number in your equation for your price function p .

Solution. To find the equation of a linear price function, use the point-slope form of the line

$$p(x) = m(x - x_1) + p_1$$

Note that we are told that $x_1 = 100$ items are sold when the item price is $p_1 = 12$, and that $x_2 = 500$ items are sold when the price is $p_2 = 4$.

That gives us a slope

$$m = \frac{\Delta p}{\Delta x} = \frac{p_2 - p_1}{x_2 - x_1} = \frac{4 - 12}{500 - 100} = \frac{-8}{400} = -0.02$$

Putting it all together we get

$$p(x) = -0.02(x - 100) + 12 = \dots = -0.02x + 14$$

- (b) Find an equation $R(x)$ for total revenue as a function of x . What are the units of your total revenue function R ?

Solution. To find revenue, we need to apply Definition ?? . That means we need to multiply quantity and price.

$$\text{Revenue} = (\text{Item Price}) \cdot (\# \text{ Items})$$

In this question, we are told to use x to represent the *quantity*, or number of items.

In the first part of this question, we have already found that the price that will result in x items being sold is $p(x) = -0.02x + 14$

To find the *total Revenue* of scenario x , plug the above values into the equation

$$\text{Revenue} = (\text{Item Price}) \cdot (\# \text{ Items}).$$

In any universe x , the total revenue will be given by

$$\begin{aligned} R(x) &= (\text{price}) \cdot (\text{quantity}) \\ &= (p(x)) \cdot (x) \\ &= (-0.02x + 14) \cdot (x) \\ &= -0.02x^2 + 14x \end{aligned}$$

- (c) Evaluate $p(600)$ and $R(600)$. What are the units of each of these quantities? What do they mean?

Solution. Recall that $p(600)$ is the item price at which $x = 600$ items will be sold, and $R(600)$ will be the *total* revenue from that scenario.

In the scenario where $x = 600$ items are sold, $p(600) = -0.02x + 14 = 2$. In other words, each item will be sold for \$2

In this scenario, the total revenue will be 600 items times 2 dollars, which equals \$1200 of total revenue.

Equivalently, $R(600) = -0.02 \cdot (600)^2 + 14 \cdot (600) = \dots = 1200$ dollars.

- (d) Compare the total revenue you can earn from the scenario (possible universe) where you sell 100 units, with the scenario where you sell 500 units, and with the scenario where you sell 600 units. Which of these three scenarios yields the highest total revenue?

Solution. We have already seen that $R(600) = 1200$.

$$R(100) = -0.02 \cdot (100)^2 + 14 \cdot (100) = 1200$$

and

$$R(500) = -0.02 \cdot (500)^2 + 14 \cdot (500) = 2000$$

Note that you will earn the same amount of revenue from selling $x = 100$ items as you will from selling $x = 600$ items. However, you will earn even more revenue from selling $x = 500$ items.

- (e) Find the number (quantity) of items x that yields highest total revenue. What item price p will result in the highest total revenue?

Solution. Note that the total revenue function is a quadratic function with negative leading coefficient. That means that the largest revenue will occur at the vertex. To find the vertex, either find and average the roots, or use the formula $h = \frac{-b}{2a}$. In this case,

$$h = \frac{-14}{2 \cdot (-.02)} = 350$$

The maximum revenue will occur when 350 items are sold.

To find the price at which 350 items are sold, use the item price function.

$$p(350) = -0.02 \cdot (350) + 14 = 7$$

The maximum revenue will occur when the item price is 7 \$/item.

$$R(350) = \dots = 350 \cdot 7 = 2450$$

The maximum revenue achieved will be 2,450 dollars.

Application 4.11 What does the Revenue Function Teach

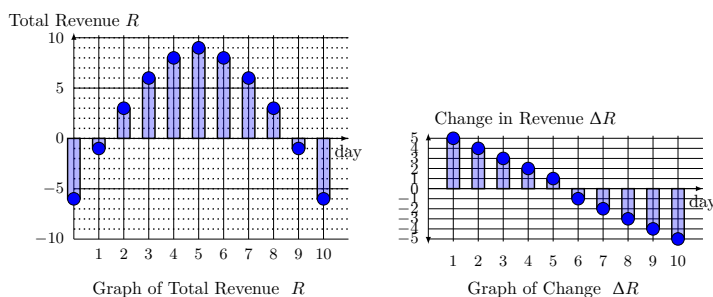
Us? As soon as you decide on how much you want to sell each item for, and print the price tag, you are locked into that price until you decide to print a new price tag. Because the quantity you sell will change depending on the price you choose, it is important to think through the implications of your choice **before** you make that decision. In practice, this usually means a multi-step process:

1. Do market research to decide how many units you are likely to sell at different item prices.

- Use this information to get a function for price as a function of the scenario x . This gives a model of how the price changes depending on which possible scenario you are in.
- The revenue for each scenario x is the product of that x by the scenario's item price.

You then compare the different values of Revenue R , and select the scenario (possible universe) that works best for you. In this case, the meaning of the change in revenue ΔR between different values of x is a bit complex. Here, ΔR represents how much revenue would have changed if you had chosen a different possible universe x , by deciding to sell additional units by lowering the fixed unit cost that everyone pays.

In other words, the maximum revenue occurs when increasing the quantity x stops increasing the total revenue and starts decreasing the total revenue, because ΔR stops being positive and starts being negative. This is an important feature, and we'll come back to this later in the semester.



Application 4.12 Graphs of Changes in Time vs Items. You may have noticed a little bit of a change in how we are thinking about price and revenue versus how we thought about gas prices earlier in the semester. The gas prices G is a function of time, so the change in gas price ΔG simply measures how much the price has changed between two days. But for price and revenue, the different values of x correspond to different and *incompatible* scenarios where you imagine selling a different number of units.

To see how these fit together, imagine you are running a sequence of experiments where on day 1, you set the price so you sell 1 unit. Then, on day 2, you set the price so you sell 2 units, and so on. In this experiment ΔR measures the change in revenue as you move between different days, which each correspond to a different scenario (possible universe).

Just remember - with revenue your goal isn't the experiment itself. You want to *use* this thought experiment to find the single scenario x where you have the best revenue.

4.3 Understanding Profit

The most important concept for the applications we will is the idea of different *possible universes*, which we often call *scenarios*.

The goal of planning is to figure out what the *best* possible world would look like, and how to achieve that possible world.

In a market economy, businesses must make a **profit**, by ensuring that all of the money they take (their total revenue) is greater than all of the money they spend (their total cost). In this case, finding the best possible world means finding the scenario where the profit is the greatest.

Definition 4.13 The **profit** of a particular possible world is the total revenue of that scenerio minus the total cost of that scenerio. In other words,

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

Recall that we write $R(x)$ to denote the total revenue in the scenario where x items are sold, and $C(x)$ to denote the total cost of the scenerio where x items are sold. Following that notation, we write $P(x)$ for the total profit of this scenerio, and have the equation

$$P(x) = R(x) - C(x)$$

You **break even** when your Revenue equals your Cost. In other words, you break even at those quanitites x which make the profit $P(x) = 0$. ◇

Application 4.14 Approaching Word Problems. When you read an economics problem, always try to translate it into mathematical language. For example, suppose that you have been asked to *find all quantities where you make a profit*. You might think through the question as follows:

- Remember that we use the variable x to represent the *quantity* of items sold in any scenerio.
- Remember that we write $P(x)$ for the total *profit* of the scenario.
- *Think carefully* and realize that you make a profit if and only if *the total profit is greater than 0*.
- That means the question is asking you to find an equation for the profit function $P(x)$, and to solve the equation $P(x) > 0$, using the techinques from Solving Inequalities of Continuous Functions.

Exploration 4.5 You are developing a new product line. You have already found equations for the total revenue and total cost in a variety of scenarios, based on the total number of items to be sold. Suppose

that

$$R(x) = -x^2 + 5x$$

and

$$C(x) = x + 2$$

- (a) Find an equation for the profit function.

Solution. Recall Definition ???. Write down the definition of profit

$$P(x) = R(x) - C(x)$$

Next, replace $R(x)$ with $(-x^2 + 5x)$ and $C(x)$ with $(x + 2)$.

$$P(x) = (-x^2 + 5x) - (x + 2)$$

Note that the parentheses are very important -- you must subtract both the x and the fixed cost 2.

Finally, simplify the expression

$$\begin{aligned} P(x) &= (-x^2 + 5x) - (x + 2) \\ &= -x^2 + 5x - x - 2 \\ &= -x^2 + 4x - 2 \end{aligned}$$

- (b) Find the break even points.

Solution. Recall that breaking even is the same thing as having $P(x) = 0$. Using the equation above, we want to solve

$$-x^2 + 4x - 2 = 0$$

Recall that we can often solve quadratic equations by factoring or using the quadratic equation. But this case, the function does not factor nicely, so we must use the quadratic equation.

You break even when the quantities are

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot (-1) \cdot (-2)}}{2(-1)}$$

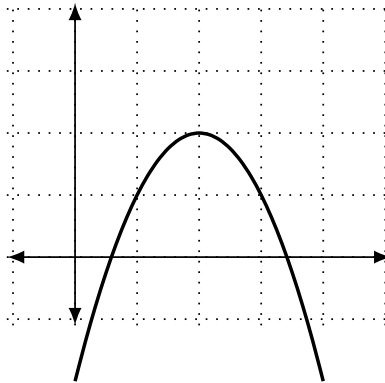
Using a calculator, we see that you break even when $x = 0.5857\dots$ or $x = 3.4142\dots$

- (c) Sketch a graph of the profit function

Solution. You can graph the function using a graphing calculator, or you can use your knowledge of quadratic functions and the x -intercepts you have found above.

Because the profit function is a quadratic with negative leading coefficient $a = -1$, it is a parabola that opens down. Furthermore, the parabola goes through the x -axis at $x = 0.5857\dots$ and

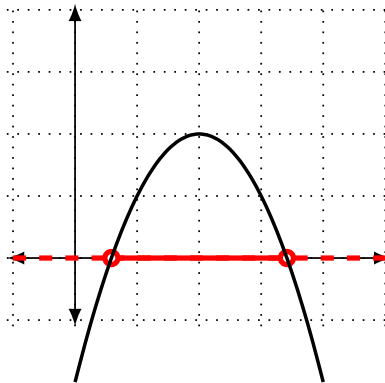
$x = 3.4142\dots$, and the parabola is symmetric around the vertex which is half way between those points. Taken together, we get the following image.



- (d) Find all quantities where you make a profit. Write your answer in interval notation.

Solution. Solve the inequality $P(x) > 0$ by looking at the break even points, and the graph of the profit function.

From this graph, we see that the profit is positive when x is between $x = 0.5857\dots$ and $x = 3.4142\dots$, which is equivalent to having x in the interval $(0.5857\dots, 3.4142\dots)$



Application 4.15 To Round or Not to Round? In the previous example, we found that the break even quantities were $x = 0.5857\dots$ or $x = 3.4142\dots$. But you cannot sell 0.587 units, so what does this answer mean practically?

Unfortunately, this depends a lot on the specific context of your specific application.

In the simple case of \$ profit as a result of x items given above, you will only be able to make a profit by selling $x = 1$ items, $x = 2$ items, or $x = 3$ items, since any integer larger or smaller than these will result in a negative profit.

But the power of mathematics is that the same tools can be used

to answer questions in very different contexts. For example, suppose instead that x was the number of *thousands* of items sold. In other words, suppose that $P(5)$ is the profit from selling *five thousand* items. In that case, you will break even when you sell $x = 0.5857\dots$ *thousand* items or $x = 3.4142\dots$ *thousand* items. In that case, you would need to sell between 586 items and 340 items to make a profit.

Other applications require even greater precision. For example, currency exchange rates are often given to at least four decimal places. But what if you are investing millions of dollars in a foreign currency? Then the fifth, sixth, etc decimal places will suddenly be more important.

In a *mathematics* class, we will generally try to keep as much precision as possible. In any specific *application*, you should think carefully and consult generally accepted practices related to rounding.

Exploration 4.6 You are planning to open a business selling ornamental pins. After a few weeks of experimenting with the price, you decide that during the average week, you can sell 100 pins at \$11 per pin, and 200 pins at \$1 per pin. Suppose also that it costs \$500 to rent the production line, and \$3 to make each pin.

Finally, after looking at a graph of some additional data, you decide that demand price is a linear function.

- (a) Find the item price $p(x)$ as a function of the quantity x to be sold.

Solution. To find the equation of a linear function, use the point-slope form of the line

$$p(x) = m(x - x_1) + p_1$$

Note that we are told that $x_1 = 100$ items are sold when the item price is $p_1 = 11$, and that $x_2 = 200$ items are sold when the price is $p_2 = 1$.

That gives us a slope

$$m = \frac{\Delta p}{\Delta x} = \frac{p_2 - p_1}{x_2 - x_1} = \frac{1 - 11}{200 - 100} = \frac{-10}{100} = -0.1$$

Putting it all together we get

$$p(x) = -0.1(x - 100) + 11 = \dots = -0.1x + 21$$

- (b) Find the total profit function $P(x)$ as a function of the total number of items to be sold.

Solution. Recall from Definition ?? that

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

That means to find $P(x)$, we must first find equations for $R(x)$ and $C(x)$.

To find the equation for Revenue, recall from Definition ?? that

$$\text{Revenue} = (\# \text{ Items}) \cdot (\text{Item Cost})$$

We write x for the number of items, and we have already found that the item price is $p(x) = -0.1x + 21$ in the scenerio where x items are sold. That means we just multiply x by $p(x)$ to get revenue.

$$R(x) = (x) \cdot (-0.1x + 21) = -0.1x^2 + 21x$$

To find the total cost, recall our approach from Exploration ??.

If you are going to sell x items, you must pay a fixed cost of 500 dollars, as well as 3\$ for each item x . That gives us a total cost function

$$C(x) = 500 + 3x$$

We can now combine the total revenue and total cost functions to get the profit function.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-0.1x^2 + 21x) - (500 + 3x) \\ &= -0.1x^2 + 21x - 500 - 3x \\ P(x) &= -0.1x^2 + 18x - 500 \end{aligned}$$

- (c) What is the total profit if you sell 100 units? What if you sell no units?

Solution. In the scenario (possible universe) where $x = 100$ items are sold, the *total profit* (which includes all revenue minus all cost) will be

$$P(100) = -0.1(100)^2 + 18(100) - 500 = 300$$

In the scenario (possible universe) where $x = 0$ items are sold, the *total profit* (which includes all revenue minus all cost) will be

$$P(0) = -0.1(0)^2 + 18(0) - 500 = -500$$

- (d) Find the number of units x where you will break even.

Solution. To break even is to have exactly zero profit $P(x) = 0$. That means solving the equation

$$-0.1x^2 + 18x - 500 = 0$$

You can solve this equation using the quadratic equation or by factoring. This function does not have a nice factorization, so we use the quadratic equation

$$x = \frac{-18 \pm \sqrt{18^2 - 4 \cdot (-0.1) \cdot (-500)}}{2 \cdot (-1)}$$

This gives us $x = 34.32 \dots$ or $x = 145.68$

- (e) What is the number of units x needed to maximize profit? What is the maximum profit?

Solution. Because profit is a quadratic that opens down, the maximum profit occurs at the vertex (h, k) , which is located half way between the two roots.

That means we can find h by averaging $x = 34.32 \dots$ and $x = 145.68$

$$h = \frac{34.32 + 145.68}{2} = \frac{180}{2} = 90$$

The maximum profit occurs when $h = 90$ items are sold.

To find the maximum profit, plug this into the function $P(x)$

$$P(90) = -0.1 \cdot (90)^2 + 18 \cdot (90) - 500 = 310$$

The maximum profit you can achieve in any possible world (number of quantities sold) is \$310. That total profit is achieved in the scenario where you set the price so that $x = 90$ items are sold.

- (f) What is the number of units x needed to maximize profit? What is the maximum profit?

4.4 Rational Functions

Definition 4.16 A function $f(x)$ is a **rational function** if $f(x)$ can be written as a fraction of polynomials. In other words, a rational function $f(x)$ can be written as a fraction

$$f(x) = \frac{p(x)}{q(x)}$$

where the numerator and denominator $p(x)$ and $q(x)$ are both polynomials.

Because *division by 0* is undefined, there are often values of x which make a function **undefined**. For such an input x , we say that $f(x)$ *DNE*. ◇

Every polynomial is a rational function (with denominator $q(x) = 1$), but not every rational function is a polynomial.

Solving Inequalities of Rational Functions.

Suppose that $f(x)$ is a rational function. In other words, suppose that $f(x) = \frac{p(x)}{q(x)}$ where the numerator and denominator $p(x)$ and $q(x)$ are both polynomials.

To solve an inequality such as $f(x) > 0$, $f(x) < 0$, $f(x) \geq 0$, or $f(x) \leq 0$ follow the following steps:

1. Solve the inequality $f(x) = 0$, by finding all inputs that make the *numerator* $p(x)$ equal to 0.

Draw a number line, and label the x -values where the solutions occur. Use an open circle if the inequality is $>$ or $<$, or a closed circle if the inequality is \geq or \leq .

2. Find any inputs x where the function is undefined (such that $f(x)$ *DNE*). In other words, find all values of x which make the *denominator* $q(x)$ equal 0.

Label the x -values where the function *DNE* on your number line using open circles.

3. For each interval, use a graph, a calculator, or some other technique to decide whether the inequality is true or false on that interval.
4. Write your final answer down in interval notation. *include* all endpoints with closed circles (use square brackets), and *exclude* all endpoints with open circles (use parentheses).

5

Compound Interest and Depreciation

In the last chapter, we have looked at polynomials, where the terms are monomials like $5 \cdot x^2$. In a monomial, the base is a variable x , and the coefficient and power are both constant numbers.

In this chapter, we will look at **exponential** functions like $5 \cdot 2^x$. In this example, the *exponent* is a variable x , and the coefficient and base are both constant numbers.

5.1 Exponential Functions

To see what this looks like, it is helpful to try out a few examples.

Exploration 5.1

- (a) Let $f(x) = 5 \cdot 2^x$. Compute $f(-1)$, $f(0)$, $f(1)$, and $f(2)$. Then, sketch the graph of $y = 5 \cdot 2^x$.

Solution. Be very careful to follow the order of operations PEM-DAS.

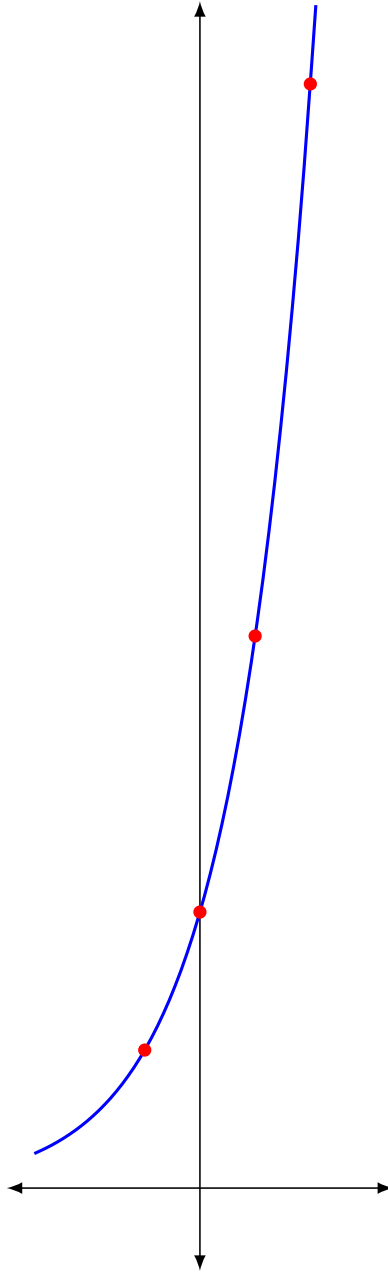
$$f(-1) = 5 \cdot 2^{-1} = 5 \cdot 0.5 = 2.5$$

$$f(0) = 5 \cdot 2^0 = 5 \cdot 1 = 5$$

$$f(1) = 5 \cdot 2^1 = 5 \cdot 2 = 10$$

$$f(2) = 5 \cdot 2^2 = 5 \cdot 4 = 20$$

Graphing these points and connecting them gives the following picture



- (b) Let $f(x) = 5 \cdot 1.1^x$. Compute $f(-1)$, $f(0)$, $f(1)$, and $f(2)$. Then, sketch the graph of $y = 5 \cdot 1.1^x$.

Solution. Be very careful to follow the order of operations PEMDAS.

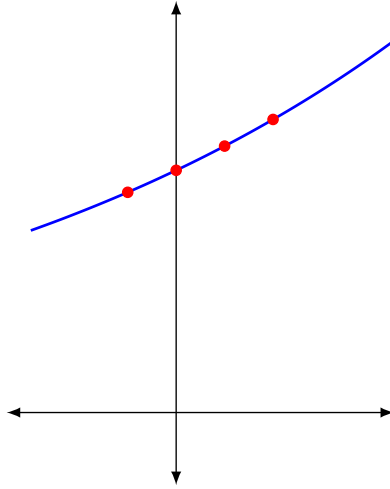
$$f(-1) = 5 \cdot 1.1^{-1} = 5 \cdot 0.9090 \dots = 4.5454 \dots$$

$$f(0) = 5 \cdot 1.1^0 = 5 \cdot 1 = 5$$

$$f(1) = 5 \cdot 1.1^1 = 5 \cdot 1.1 = 5.5$$

$$f(2) = 5 \cdot 1.1^2 = 5 \cdot 1.21 = 6.05$$

Graphing these points and connecting them gives the following picture



- (c) Let $f(x) = 5 \cdot (0.75)^x$. Compute $f(-1)$, $f(0)$, $f(1)$, and $f(2)$. Then, sketch the graph of $y = 5 \cdot (0.75)^x$

Solution. Be very careful to follow the order of operations PEM-DAS.

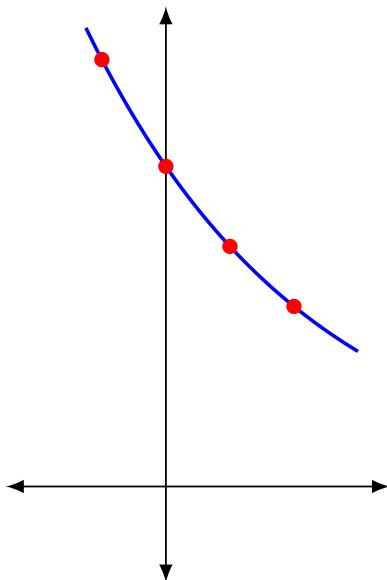
$$f(-1) = 5 \cdot 0.75^{-1} = 5 \cdot 1.333 \dots = 6.666 \dots$$

$$f(0) = 5 \cdot 0.75^0 = 5 \cdot 1 = 5$$

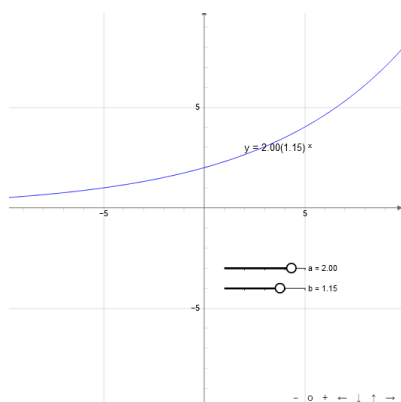
$$f(1) = 5 \cdot 0.75^1 = 5 \cdot 0.75 = 3.75$$

$$f(2) = 5 \cdot 0.75^2 = 5 \cdot 0.5625 = 2.8125$$

Graphing these points and connecting them gives the following picture



A picture is worth a thousand words. When you chose a different number for the coefficient a or base b , you get a different function. In the graphic below, use the sliders to change these two numbers, and get a sense of how fast the different functions grow or decay.



[Standalone](#)

[Embed](#)

Figure 5.1

In this chapter, we will make extensive use of the Properties of Exponential Functions. If you get stuck doing a computation, it is a good idea to see if there is a property that you may find useful.

Exploration 5.2 Use the laws of exponents to rewrite the following functions in the form $f(x) = a \cdot b^x$ where a, b are numbers.

- (a) Rewrite $f(x) = 2^{-2x}$ as an exponential function in standard form.

Solution.

$$\begin{aligned} f(x) &= 2^{-2x} \\ &= \frac{1}{2^{2x}} \end{aligned}$$

$$\text{Using } a^{-r} = \frac{1}{a^r}$$

$$\begin{aligned}
&= \frac{1}{(2^2)^x} && \text{Using } a^{r \cdot s} = (a^r)^s \\
&= \frac{1}{(4)^x} \\
&= \left(\frac{1}{4}\right)^x && \text{Using } \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \\
&= 1 \cdot (0.25)^x
\end{aligned}$$

Note that in the second to last step, we only had a base and a power. In the last step, we introduce a coefficient of 1, which does not change the value of the expression.

We have shown that $f(x)$ is equal to $a \cdot b^x$ with $a = 1$ and $b = 0.25$.

- (b) Rewrite $f(x) = \left(\frac{1}{2}\right)^{2x+1}$ as an exponential function in standard form.

Solution.

$$\begin{aligned}
f(x) &= \left(\frac{1}{2}\right)^{2x+1} \\
&= \left(\frac{1}{2}\right)^{2x} \cdot \left(\frac{1}{2}\right)^1 && \text{Using } a^{r+s} = a^r \cdot a^s \\
&= \left(\frac{1}{2}\right)^{2x} \cdot \frac{1}{2} \\
&= \left(\left(\frac{1}{2}\right)^2\right)^x \cdot \frac{1}{2} && \text{Using } (a^r)^s = a^{r \cdot s} \\
&= \left(\frac{1}{4}\right)^x \cdot \frac{1}{2} && \text{Using } \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \\
&= \frac{1}{2} \cdot \left(\frac{1}{4}\right)^x.
\end{aligned}$$

We have shown that $f(x)$ is equal to $a \cdot b^x$ with $a = \frac{1}{2}$ and $b = \frac{1}{4}$.

- (c) Rewrite $f(t) = 100 \cdot \left(1 + \frac{0.2}{4}\right)^{4t}$ as an exponential function in standard form.

Solution.

$$\begin{aligned}
f(x) &= 100 \cdot (1 + 0.05)^{4t} \\
f(x) &= 100 \cdot (1.05)^{4t} \\
f(x) &= 100 \cdot \left((1.05)^4\right)^t && \text{Using } (a^r)^s = a^{r \cdot s} \\
f(x) &= 100 \cdot (1.2155)^t
\end{aligned}$$

We have shown that $f(t)$ is equal to $a \cdot b^t$ with $a = 100$ and $b = 1.2155$.

5.2 Percents and Percent Change

You should already be familiar with the concept of percents. But because percents play an essential role in many topics in economics and business, we briefly review the definitions before moving on to our main applications.

The english word *cent* derives from the Latin word *centum* which means “hundred”. In other words, *20 percent* literally means *20 per hundred* or $\frac{20}{100} = 0.2$

Definition 5.2 Given any number A , the expression A percent (also written $A\%$) refers to the *decimal rate*

$$r = \frac{A}{100}$$

Given any numbers A, P , you can find $A\%$ of P by multiplying P by the decimal rate $r = \frac{A}{100}$ \diamond

Example 5.3 Basic Percentages.

- 15% is equal to the decimal $r = \frac{15}{100} = 0.15$
- 2% is equal to the decimal $r = \frac{2}{100} = 0.02$
- 0.12% is equal to the decimal $r = \frac{0.12}{100} = 0.0012$
- 15% of \$40 is equal to $40(0.15) = 6$
- 2% of \$500 is equal to $500(0.02) = 10$
- 0.12% of \$30 is equal to $30(0.0012) = 0.036$

□

Application 5.4 Understanding percent change. It is common to talk about an economic quantity by changing by a given percentage.

The term **percent** is an abbreviation of the expression *per cent*, which means *per one hundred*. In other words, 25% is the same thing as 25 hundreds, or a decimal rate of $\frac{25}{100} = 0.25$.

For example, 25% of 1000, is equal to $1000 \cdot 0.25 = 250$.

Now, suppose that you say that 1000 *increases* by 25 percent. How much do you have after this increase?

If you are *increasing* the value by 25%, you *keep* the original 1000, and *add* the new 250. In other words, you have

$$1000 + 1000 \cdot 0.25 = 1250.$$

Suppose that you increase 700 by 25%. You take the original 700, and add the $700 \cdot 0.25 = 175$ to get the final value

$$700 + 700 \cdot 0.25 = 875$$

There is a pattern here. If you take a principal value of P , and increase it by a decimal rate of r , the final value will be

$$P + P \cdot r$$

We can factor a P out of each term, to get that the result of increasing P by a decimal rate of r results in a final value of

$$P \cdot (1 + r)$$

To *decrease* 1000 by 25%, subtract $0.25 \cdot 1000 = 250$ from 1000, resulting in

$$1000 - 1000 \cdot 0.25 = 1000 - 250 = 750$$

To decrease a principal value of P by a decimal rate of r , compute

$$P - P \cdot r = P(1 - r)$$

Definition 5.5 Percent Change. Suppose that r is a decimal rate, and that P is any value. F is the result of *increasing* P by the rate r if $F = P + rP$, also written

$$F = P \cdot (1 + r).$$

If F is the result of *decreasing* P by the rate of r , we subtract rP instead, resulting in the formula

$$F = P \cdot (1 - r)$$

◇

Exploration 5.3 Suppose you begin with \$1000. How much do you have if the value is increased by 25%, and then decreased by 30%?

Solution. 25% corresponds to a decimal rate of $r = \frac{25}{100} = 0.25$. To increase \$1000 by 25%, add \$1000 to $1000(0.25)$

$$1000 + 1000(0.25) = 1000 + 250 = 1250$$

We could also just have multiplied 1000 by $1 + 0.25 = 1.25$

30% corresponds to a decimal rate of $r = 0.3$. To decrease \$1250 by 30%, take \$1250 and subtract $1250(0.3)$

$$1250 - 1250(0.3) = 875$$

We could also just have multiplied 1000 by $1 + (-0.3) = 0.7$

To *find* the percent change between F and P , set up the equation

$$F = P(1 + r)$$

and solve for r

Exploration 5.4

- (a) If the balance of your investment goes from \$300 to \$412, by what percent has it changed?

Solution. We want to find the percent r so that

$$412 = 300 \cdot (1 + r)$$

Solving for r gives

$$1 + r = \frac{412}{300} = 1.37333333333333$$

$$1 + r = 1.37333333333333$$

$$r = 0.37333333333333$$

The value has increased by 37.33%

- (b) If the balance of your investment goes from \$300 to \$234, by what percent has it changed?

Solution. We want to find the percent r so that

$$234 = 300 \cdot (1 + r)$$

Solving for r gives

$$1 + r = \frac{234}{300} = 0.78$$

$$1 + r = 0.78$$

$$r = -0.22$$

The value has decreased by 22%

Exploration 5.5 Let P be any principal value. For each of the following final values, find out whether the principal has been increased or decreased, and then find the corresponding decimal rate and percent of this change.

- (a) By what percent has P been changed, if you end up with a final value of $1.1P$?

Solution. We want to rewrite $1.1P$ in the form $P(1 + r)$ for some number r . Here we have

$$1.1P = P(1.1) = P(1 + 0.1)$$

In other words, the principal P has been increased by a decimal rate of 0.1, which corresponds to an increase of 10%.

- (b) By what percent has P been changed, if you end up with a final value of $0.85P$?

Solution. We want to rewrite $0.85P$ in the form $P(1 + r)$ for some number r . Here we have

$$0.85P = P(0.85) = P(1 - 0.15)$$

In other words, the principal P has been decreased by a decimal rate of 0.15, which corresponds to an decrease of 15%.

- (c) By what percent has P been changed, if you end up with a final value of $3P$?

Solution. We want to rewrite $3P$ in the form $P(1 + r)$ for some number r . Here we have

$$3P = P(3) = P(1 + 2)$$

In other words, the principal P has been increased by a decimal rate of 2, which corresponds to an increased of 200%.

We have focused above on understanding and applying the definition of percent change $F = P(1 + r)$, since this will be important to understand compound interest below. However, there are also a variety of other equivalent formulas that can be used to find percent change.

Corollary 5.6 *The percent change from P to F is given by $r = \frac{F}{P} - 1$, or equivalently by $r = \frac{F - P}{P}$*

Proof. To get both equations, solve $F = P(1 + r)$ for r . For the first,

$$\begin{aligned} F &= P(1 + r) \\ \frac{F}{P} &= 1 + r \\ \frac{F}{P} - 1 &= r \end{aligned}$$

For the second,

$$\begin{aligned} F &= P(1 + r) \\ F &= P + Pr \\ F - P &= Pr \\ \frac{F - P}{P} &= r \end{aligned}$$

■

5.3 Compound Interest and Depreciation

The power mathematics comes from the fact that the same basic tools apply in a surprisingly wide variety of contexts. For example, percent changes occur in many contexts including

- loan payments,
- inflation, and
- the change in value of physical objects.

In finance, your hope is that the value of your investments will increase over time. In other contexts (manufacturing, purchasing a car, etc.), you understand that the value of your investment will *decrease* over time. The process of an object losing value over time is called **depreciation**.

To describe the results of inflation at a fixed percent (say, 4%) over several years, or to describe an object that repeatedly decreases in value by a fixed percent over several years, we will need a formula for repeated percent changes.

Application 5.7 Repeated percent changes. Inflation refers to the process of prices increasing over time. Often, we talk about prices increasing at the same rate per year, over a period of several years.

For example, suppose that prices are increasing at 5% per year, which corresponds to a decimal rate $r = \frac{5}{100} = 0.05$. Suppose also that a certain product costs 100\$ in the first year.

One year later, the cost is the original amount $P = 100$ plus a change of $rP = 0.05 \cdot 100 = 5$ dollars. In other words, the value one year later is

$$V(1) = 100 + 0.05 \cdot 100 = 105.$$

In our notation for percent change, we can also write this as

$$V(1) = 100 \cdot (1 + 0.05) = 105.$$

How much will the value be after two years? In the second year, the value of 105 increases a second time, also by 5%, or a rate of $r = 0.05$. That means

$$V(2) = 105 + 105 \cdot 0.05 = 110.25$$

In our notation for percent change, we can write this as

$$V(2) = 105 \cdot (1 + 0.05) = 105 \cdot (1.05)$$

But the leading coefficient of 105 can also be rewritten, since $105 = 100 \cdot (1.05)$. In other words,

$$V(2) = 100 \cdot (1.05) \cdot (1.05) = 100 \cdot (1.05)^2$$

We can find the value after three years in the same way. We will take the value from the second year, and increase it by a rate of $r = 0.05$. Furthermore, that increase is equivalent to multiplying the value from the second year by another coefficient of $1 + 0.05 = 1.05$.

$$V(3) = 100 \cdot (1.05) \cdot (1.05) \cdot (1.05) = 100 \cdot (1.05)^3$$

This line of reasoning works for any positive period of time t , giving us a formula for the value of an object with original value 100, where the price has been increased 5% for t years.

$$V(t) = 100 \cdot (1.05)^t = 100 \cdot (1 + 0.05)^t$$

Proposition 5.8 *Suppose that an original principal value of P is repeatedly changed by a fixed decimal rate of r **one time per year**. The value after t years will be given by*

$$V(t) = P \cdot (1 + r)^t.$$

If the value is being increased, use a positive value for the rate r . If the value is being decreased, use a negative value for the rate r .

Accountants often keep track of the present value of a larged investment or fixed asset. The term **depreciation** refers to a decrease in the value of the asset. You can find a more complete introduction to the concept on Investopedia at <https://www.investopedia.com/terms/d/depreciation.asp>.

Application 5.9 Accounting and depreciation. Value can be measured in many ways. For example, suppose that you purchase a car and that you somehow know that it will last 200,000 miles, and that you will drive 20,000 miles each year. By how much has the value of the car decreased after 3 years?

The "correct" answer to this question depends on a surprising number of factors.

- If you are measuring value as "percent of lifetime miles remaining", there would be 160,000 of the total 200,000 miles remaining, so the value of the car would be $\frac{160,000}{200,000} = 0.8$, or 80%, of its original value. After four years, the value of the car would be $\frac{140,000}{200,000} = 0.7$, or 70%, of its original value. And after five years, the value of the car would be $\frac{120,000}{200,000} = 0.6$, or 60%, of its original value.

In this case, the value is decreasing at a constant amount per year. This is often called **straight line depreciation**, since connecting the dots yields a linear equation

$$(\text{value at year } t) = (\text{starting value}) - (\text{amount of value used per year})t$$

- On the other hand, cars are known to break more and more fre-

quently as they get older. If you knew which repairs would be required each year, you could make a table of values telling you the value of the vehicle at each point in time (or at each number of miles). For each year, you would take its value from the previous year, decrease the value based on the reduced mileage of the car, and subtract the expected maintenance from that point of time.

This way of tracking the decrease, or depreciation, is in some ways more accurate than straightline depreciation but is much more complicated to compute.

- From a different perspective, it is clear that new cars have many features that older cars do not. To oversimplify things, suppose that a new car always has 10% more features than an older car. If you are going to buy a car that is one year older, you will want the price to decrease enough to make up for that loss in additional features.

Said another way, many investments will lose a fixed percent of their value every year. This is called **declining balance** depreciation.

Suppose that the value of a different car decreases by 25% each year. After one year, 75% of the value is retained. After 2 years, 75% of the original 75% is retained, resulting in $0.75 \cdot 0.75 = 0.5625$, or 56.25% of the value remains. After 3 years, the value that remains is $0.75 \cdot 0.75 \cdot 0.75 = 0.421875$, or 42.19% of the original value.

More generally, the value after t years is defined by the exponential equation

$$(\text{value in year } t) = (\text{starting value}) \cdot (\text{percent of value retained after a year})^t$$

Deciding when to use each depreciation scheme is an important question in Accounting, and is well beyond the scope of this course. As a result, in this course, we will restrict our attention to declining balance depreciation, and we will usually state explicitly that a fixed percentage of value is lost each year.

Exploration 5.6 Your company has recently invested in a \$500,000 CNC machine for a high throughput production line. After some research, you have decided that each year, the machine will lose 20% of its remaining value.

- (a) What is the net value of the asset after one year? After two years?

Solution. After one year, the value is

$$\begin{aligned} V(1) &= 500,000 - 0.2 \cdot 500,000 \\ &= 500,000 - 100,000 \\ &= 400,000 \end{aligned}$$

After two years, the value is

$$\begin{aligned} V(2) &= 400,000 - 0.2 \cdot 400,000 \\ &= 400,000 - 80,000 \\ &= 320,000 \end{aligned}$$

- (b) Find a formula for the future value of the asset as a function of the number of years t in the future. Sketch a graph of this function.

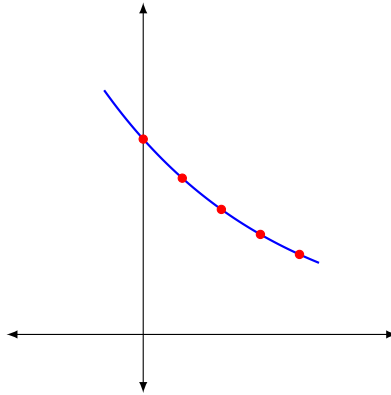
Solution. We are repeatedly reducing the principal (original value) $P = 500,000$ by 20%, or a decimal rate of $r = -0.2$. Using Proposition ??, we see that the value of the investment at time t is given by

$$\begin{aligned} V(t) &= P \cdot (1 + r)^t \\ &= 500,000 \cdot (1 + (-0.2))^t \\ &= 500,000 \cdot (0.8)^t \end{aligned}$$

In other words, 80% of the value is retained each year. We can now use this formula to answer the first part of the question:

$$V(1) = 500,000 \cdot (1 - 0.2)^1 = 500,000 \cdot 0.8 = 400,000$$

$$V(2) = 500,000 \cdot (1 - 0.2)^2 = 500,000 \cdot (0.8)^2 = 320,000$$



Application 5.10 Compound interest. The process of repeatedly increasing a value by a fixed percent is called **compounding**. If you increase the value every year, you are said to be compounding yearly. If you increase the value 12 times per year, you are said to be compounding monthly.

The term **interest** refers to a very specific kind of compounding. As Investopedia puts it, “Interest is the monetary charge for borrowing money—generally expressed as a percentage, such as an annual percentage rate (APR).” (<https://www.investopedia.com/terms/i/interest>.

asp)

In **compound interest**, you pay interest on both the principal and on the interest accrued so far.

The term **compounding** refers to the action of computing the new interest, and adding it back into the principal.

The simplest example (which we have already mentioned above) happens when the For example, suppose that you borrow \$100, at an interest rate of 3%, and that the interest is compounded (or added to the principal value) *onetime* each year.

Table 5.11 Interest Compounded Yearly

Years Elapsed	Previous Debt	New interest	New Debt
1	100	$100 \cdot 0.03 = 3$	$100 + 3 = 103$
2	100	$100 \cdot 0.03 = 3.09$	$103 + 3.09 = 106.09$

Often, interest will be compounded multiple times each year. Suppose, for example, that you borrow money at 3% interest, compounded *twice* each year.

Does this mean that you need to pay 3% interest after 6 months, and a *second* 3% interest after the next 6 months? Absolutely not - that would be like paying 6% interest over the year!

Instead, the convention is to divide the state **annual percentage rate** evenly among the number of compoundings per year. For example, if you borrow at 3% compounded twice per year, then the *interest per compounding* is $\frac{3\%}{2 \text{ compoundings}} = 1.5\%$ per compounding, or a decimal rate of 0.015 per compounding.

Table 5.12 Interest Compounded Biannually

Years Elapsed	Previous Debt	New interest	New Debt
0.5	100	$100 \cdot 0.015 = 1.5$	$100 + 1.5 = 101.5$
1.0	101.5	$101.5 \cdot 0.015 = 1.5225$	$101.5 + 1.5225 = 103.0225$
1.5	103.0225	$103.0225 \cdot 0.015 = 1.5453375$	$103.0225 + 1.5453375 = 104.5678375$
2.0	104.5678375	$104.5678375 \cdot 0.015 = 1.5685175625$	$104.5678375 + 1.5685175625 = 106.136355062$

Most banking instruments are compounded monthly, or more frequently. That means that you would compound $m = 12$ times each year, with an interest per compounding of $\frac{3\%}{12 \text{ compoundings}} = 0.25\%$ per compounding, or a decimal rate of $r = 0.0025$.

We could compute this using a table, but you can see that this rapidly becomes unworkable. Instead, we can modify the for repeatedly changing the principal by a fixed value.

Definition 5.13 Suppose that a principal P is invested at an annual decimal rate r , and compounded m times each year. The value of the

investment after t years is equal to

$$F(t) = P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}$$

The rate r is called the **annual percentage rate**, and the value at year t is called the **future value** of the investment at time t .

The *interest earned* is the difference between the principle and the future value. \diamond

To remember the formula, remember that you are increasing the principal by the decimal rate $\frac{r}{m}$ each compounding. Because you are computing m times per year, after t years, you will have repeated this process $m \cdot t$ times.

Exploration 5.7 Suppose that you have invested \$1000 at an APR of 3%, compounded monthly.

- (a) Find the future value of this instrument as a function of the number years t .

Solution. Use the compound interest formula with $P = 1000$, $r = 0.03$, $m = 12$, and time represented by the variable t .

$$F(t) = 1000 \cdot \left(1 + \frac{0.03}{12}\right)^{12t}$$

- (b) Find the future value after 10 years and six months.

Solution. 10 years and 6 months corresponds to 10.5 years. Using the compound interest formula,

$$F(10.5) = 1000 \cdot \left(1 + \frac{0.03}{12}\right)^{12(10.5)} = 136.972077476$$

The future value after 10.5 years is approximately \$1369.72

- (c) Rewrite the future value function in the form $a \cdot b^t$ where a, b are numbers.

Solution. In the expression

$$F(t) = 1000 \cdot \left(1 + \frac{0.03}{12}\right)^{12t}$$

the base $1 + \frac{0.03}{12} = 1.0025$ is a single number. That lets us rewrite the expression as

$$F(t) = 1000 \cdot (1.0025)^{12t}$$

Next, we can use the property of exponents $a^{r \cdot s} = (a^r)^s$ to rewrite the expression as

$$F(t) = 1000 \cdot (1.0025^{12})^t$$

Now we can again simplify the base $1.0025^{12} = 1.03041595691$, which gives the expression

$$F(t) = 1000 \cdot (1.03041595691)^t$$

We have now written $F(t) = a \cdot b^t$ with $a = 1000$ and $b = 1.03041595691$.

- (d) What is the **effective** annual interest rate? In other words, what is the *actual* percentage by which the balance increases every year?

Solution. Recall that $P \cdot (1 + r)^t$ gives the result of increasing the value P by a rate of r , repeated t times.

We have shown above that the future value is given by

$$F(t) = 1000 \cdot (1.03041595691)^t$$

Rewriting this we get

$$F(t) = 1000 \cdot (1 + 0.03041595691)^t$$

In other words, this is equivalent to increasing the value by an *actual*, or *effective*, rate of $3.041 \dots$ percent each year.

Definition 5.14 The **effective APR** of an investment is the rate r_{eff} such that

$$P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t} = P \cdot (1 + r_{eff})^t$$

◇

Application 5.15 Another way to find Effective APR. Given the definition of effective APR, it is generally easier to find the effective APR by using the laws of exponents to simplify the compound interest formula with your given values.

This approach may, however, make it a bit more difficult to see the actual meaning of r_{eff} . To make this meaning more clear, we will introduce another more numerical way to compute r_{eff} .

Recall that the *effective APR* of an investment tells you the *actual* percent by which the value grows each year. That means that you can also find the effective APR by *computing and comparing the value of the investment at any consecutive two years*.

For example, in the exercise above, we found that the future value was given by

$$F(t) = 1000 \cdot \left(1 + \frac{0.03}{12}\right)^{12t}.$$

That means that the value in year 0 is

$$F(0) = 1000 \cdot \left(1 + \frac{0.03}{12}\right)^{12(0)=1000}$$

and that the value in year 1 is

$$F(1) = 1000 \cdot \left(1 + \frac{0.03}{12}\right)^{12(1)} = 1030.41595691.$$

To find the percent change from years 0 to 1, we can compute the ratio

$$\frac{F(1)}{F(0)} = \frac{1030.41595691}{1000} = 1.03041595691$$

Between years 0 and 1, the principal has been increased by a multiple of

$$1.03041595691 = 1 + 0.03041595691$$

which corresponds to a decimal rate of $r_{eff} = 0.03041595691$, or a percent rate of 3.041595691%

Exploration 5.8 You invest in a company at 20%, compounded quarterly.

- (a) If you start with \$100, find the balance as a function of time t in years. What is the balance after 1 year?

Solution. Start by writing down the formula for Definition ??.

$$F(t) = P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}$$

Using the principal $P = 100$, decimal rate $r = 0.2$, and frequency of compounding $m = 4$, since there are four quarters in each year.

That gives the equation

$$F(t) = 100 \cdot \left(1 + \frac{0.2}{4}\right)^{4 \cdot t}$$

We could stop there, but it is often nice to clean this up by simplifying the base of the exponent to get a formula for the value after t years

$$F(t) = 100 \cdot (1.05)^{4 \cdot t}$$

To find the value after one year, plug in $t = 1$ and simplify,

$$F(1) = 100 \cdot (1.05)^{4 \cdot 1} = \dots = 100 \cdot 1.2155 = 121.55$$

- (b) How much should you start with to get \$1000 after 5 years?

Solution. Again, start by writing down the formula for Definition ??

$$F(t) = P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}.$$

Next, fill in the values given. In this problem, we want the *future value* $F = 1000$ at the *future time* $t = 5$. We are also given $r = 0.2$ and $m = 4$.

$$1000 = P \cdot \left(1 + \frac{0.2}{4}\right)^{4 \cdot 5}.$$

Note that now the unknown variable is the starting (or present) value P .

Simplifying the base and exponents gives

$$1000 = P \cdot (1.05)^{20}$$

Using a calculator, we obtain

$$1000 = P \cdot 2.6532$$

Solve for P by dividing both sides by 2.6532

$$P = \frac{1000}{2.632} = 376.89$$

You must start with a present value $P = 376.89$ to achieve the desired future value in 5 years.

- (c) Rewrite $F(t)$ in the form $a \cdot b^t$. What is the effective APR?

Solution. Again, start by writing down the formula for Definition ??

$$F(t) = P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}.$$

Next, fill in the values given.

$$F(t) = P \cdot \left(1 + \frac{0.2}{4}\right)^{4 \cdot t}.$$

No other values are given in this part, so we now simplify.

$$F(t) = P \cdot (1.05)^{4 \cdot t}$$

Using the law of algebra $a^{rs} = (a^r)^s$, we get

$$\begin{aligned} F(t) &= P \cdot (1.05^4)^t \\ &= P \cdot (1.2155 \dots)^t \\ &= P \cdot (1 + 0.2155 \dots)^t \end{aligned}$$

Recall Proposition ??. This reminds us that the expression above corresponds to the process of repeatedly increasing the principal P by 21.55...% each year.

In other words, the effective APR is 21.55...%.

Checkpoint 5.16 How much must you invest at 3% interest, compounded monthly, to have 10,000 after 5 years?

Solution. Again, start by writing down the formula for Definition ??

$$F(t) = P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}.$$

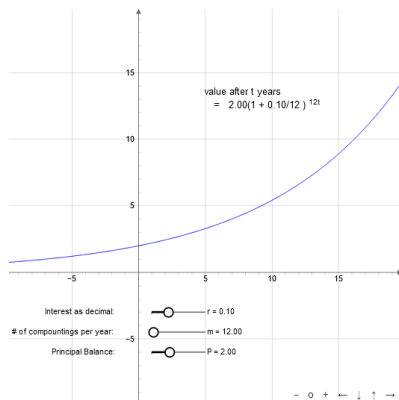
Next, fill in the values given.

$$10000 = P \cdot \left(1 + \frac{0.03}{12}\right)^{12 \cdot 5}.$$

We want to solve for the unknown variable P . We could simplify the expression using a calculator, or we could just use the fact that $\left(1 + \frac{0.03}{12}\right)^{12 \cdot 5}$ is a constant number and divide both sides by it to get.

$$P = \frac{10,000}{\left(1 + \frac{0.03}{12}\right)^{12 \cdot 5}} = 8,609.$$

A picture is worth a thousand words. The present value will grow more fast or slow based on the value of the principal P , APR r , and frequency of compounding m . In the graphic below, use the sliders to change these two numbers, and get a sense of how fast each of these parameters impact the growth of a balance compounded continuously.



Standalone
Embed

Figure 5.17 Compound interest as a function of time

The Natural Exponential for Calculus.

Application 5.18 Compounding Faster and Faster. In the interactive graphic above, note that once you increase the frequency m above about 20, the graph of compound interest appears to stop changing.

Actually, if you compute things out with a calculator, you will see that there *are* differences when you change m , but those changes get smaller and smaller as you increase m .

Fix an APR of $r = 1$ (a rate of 100%).

Frequency of compounding m	Future Value function $F(t)$	Effective APR r_{eff}
1	$F = P \cdot (1 + \frac{1}{1})^{1t} = P \cdot (2)^t$	$r_{eff} = 2 - 1 = 1$
10	$F = P \cdot (1 + \frac{1}{10})^{10t} = P \cdot (2.59 \dots)^t$	$r_{eff} \approx 2.59 - 1 = 1.59$
20	$F = P \cdot (1 + \frac{1}{20})^{20t} = P \cdot (2.59 \dots)^t$	$r_{eff} \approx 2.65 - 1 = 1.65$
100	$F = P \cdot (1 + \frac{1}{100})^{100t} = P \cdot (2.70 \dots)^t$	$r_{eff} \approx 2.70 - 1 = 1.70$
1000	$F = P \cdot (1 + \frac{1}{1000})^{1000t} = P \cdot (2.71 \dots)^t$	$r_{eff} \approx 2.71 - 1 = 1.71 \dots$

In other words, as you compound more and more frequently at an APR of 100%, the *effective* APR settles down to a specific number (171%).

Said another way, as you compound more and more frequently at 100%, the base settles down on a specific number 2.71.... Mathematicians use the letter e to denote this number.

Surprisingly, there is a very simple equation for what happens when you compound continuously.

Theorem 5.19 *If you compound **continuously** at a rate r , the value after t years is given by the function*

$$F(t) = P \cdot (e^r)^t$$

where e is a number that is approximately 2.71....

Surprisingly, exponential functions with base e play a very important role in calculus. Because of this, we call it the *natural* base to use for the exponential function. More precisely, we make the following definition.

Definition 5.20 The function $y = e^x$ is called the **natural exponential function**. Sometimes, we denote this function by writing

$$\exp(x) = e^x$$

◇

6

Logarithms and Solving for Time

In the previous section, we used exponential functions to compute the value of compound interest as a function of time. For example, Investing \$100 and 25% interest compounded yearly follows the function

$$F = 100 \cdot (1.25)^t$$

Often, you will be given the principal, rate, and frequency of compounding, and you will need to *find the amount of time* you will have to wait to achieve a certain future balance. If, in the example above, you want to find the time needed to achieve a balance of \$4000, you will need to solve the equation

$$4000 = 100 \cdot (1.25)^t.$$

This is equivalent to

$$40 = \cdot(1.25)^t,$$

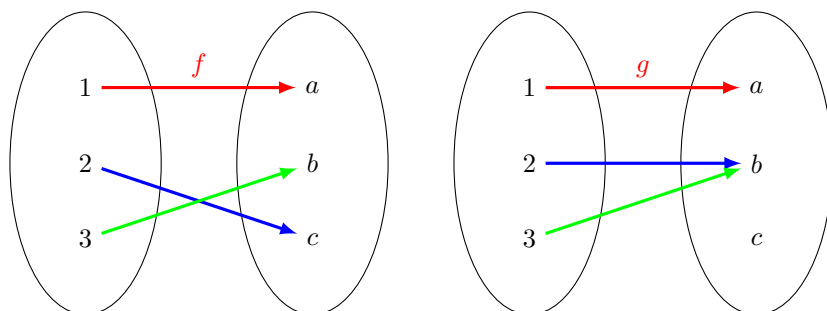
but now we are stuck, since the t is trapped inside the exponent.

To get further, we will need to "undo" our compound interest calculation. Mathematically, this means that we will need an *inverse* function.

6.1 Inverse Functions

Recall that a **one-to-one** function sends each input to a *different* output.

Example 6.1 Exploring the definition of one-to-one. This definition is easiest to apply when you are thinking about functions defined using arrow diagrams.



The function f is 1-1

The function g is *not* 1-1

□

If your function is one-to-one, then flipping the arrows gives you a *new* function that *undoes* your original function. That new function is called the **inverse** of the original function.

Definition 6.2 Let f be a one-to-one function. Its **inverse** is the function, written f^{-1} , which satisfies

$$\begin{aligned} f(x) &= y \\ \iff \\ x &= f^{-1}(y) \end{aligned}$$

◇

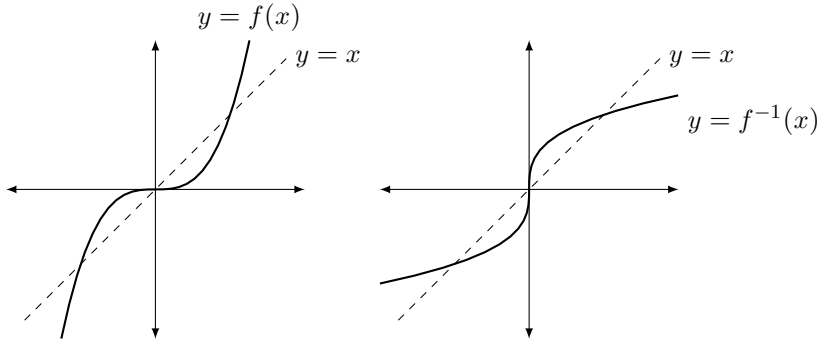
In Example ??, the function f has an inverse function f^{-1} , defined by flipping each of the arrows. For example, because $f(2) = c$, we have $f^{-1}(c) = 2$.

Danger Zone: In general, the inverse f^{-1} is **not** the same thing as the reciprocal. In other words, $f^{-1}(x)$ is **not** $\frac{1}{f(x)}$ in general.

Key Idea.

Inverse functions switch the roles of x and y (of input and output).

Example 6.3 Understanding Inverses Graphically. Graphically, reversing the roles of input and output is the same thing as reflecting the graph of the function across the diagonal line $y = x$.



□

Recall that the function f^{-1} *undoes* $f(x)$. Mathematically, this means

$$f\left(f^{-1}(x)\right) = x$$

and

$$f^{-1}\left(f(x)\right) = x$$

Finding inverses and solving equations are two sides of the same coin.

Exploration 6.1 Let $f(x) = 2x + 1$

(a) Find an equation for $f^{-1}(x)$

Solution. First, write down the equation $y = f(x)$. In this case,

$$y = 2x + 1$$

Next, solve this equation for x .

$$\begin{array}{rcl} y & = & 2x + 1 \\ & -1 & -1 \\ y - 1 & = & 2x \\ (y - 1) & = & (2x) \\ \times \frac{1}{2} & & \times \frac{1}{2} \\ \frac{1}{2}(y - 1) & = & x \end{array}$$

Finally, reverse the roles of x and y . This gives the equation

$$y = \frac{1}{2}(x - 1) = 0.5x - 0.5$$

The equation for our inverse is

$$f^{-1}(x) = 0.5x - 0.5$$

(b) Use your equation for f^{-1} to evaluate $f\left(f^{-1}(x)\right)$.

Solution. We are given that $f(x) = 2x + 1$. We have also found that $f^{-1}(x) = 0.5x - 0.5$.

To compute $f(f^{-1}(x))$, replace every x in the equation for $f(x)$ with the equation $f^{-1}(x) = 0.5x - 0.5$.

$$\begin{aligned} f(f^{-1}(x)) &= f(0.5x - 0.5) \\ &= 2 \cdot (0.5x - 0.5) + 1 \\ &= 2 \cdot 0.5x - 2 \cdot 0.5 + 1 \\ &= 2 \cdot 0.5x - 2 \cdot 0.5 + 1 \\ &= x - 1 + 1 \\ &= x \end{aligned}$$

We have shown in this case that $f(f^{-1}(x)) = x$. In other words, we have shown that the function and inverse undo each other.

6.2 Logarithms Undo Exponentials

Our main goal is to use logarithms to solve for time in a compound interest equation. Those equations involve many parts, so we will start with a few slightly simpler questions, in the form of a riddle.

Exploration 6.2

- (a) How many times do you need to double the number 10 to achieve an eight-fold increase?

Solution. Just start doubling 10, and see what works!

$$\begin{aligned} 10 \cdot 2 &= 20 \\ 10 \cdot 2 \cdot 2 &= 40 \\ 10 \cdot 2 \cdot 2 \cdot 2 &= 80 \end{aligned}$$

You need to double 10 *three times* to get from 10 to 80 (an eight-fold increase). In other words,

$$80 = 10 \cdot 2^3$$

- (b) How many times do you need to double any number P to achieve an eight fold increase of P ?

Solution. Our strategy for the first example works the same here:

$$\begin{aligned} P \cdot 2 &= 2P \\ P \cdot 2 \cdot 2 &= 4P \\ P \cdot 2 \cdot 2 \cdot 2 &= 8P \end{aligned}$$

You need to double any number *three times* to get an eight-fold increase. This is the same as saying that ,

$$8 = 2^3$$

- (c) How many times do you need to triple P to get an 81-fold increase?

Solution. Our strategy for the first example works the same here:

$$P \cdot 3 = 3P$$

$$P \cdot 3 \cdot 3 = 9P$$

$$P \cdot 3 \cdot 3 \cdot 3 = 27P$$

$$P \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 81P$$

You need to triple any number *four times* to get an eighty one-fold increase. This is the same as saying that ,

$$81 = 3^4$$

Logarithms are functions that answer the riddle "how many times do I need to multiply by a number b " to increase by a given factor. More precisely, we make the following definition.

Definition 6.4 Given any numbers $b, x > 0$,

$$\log_b(x) = r$$

is equal to the exponent r such that

$$b^r = x$$

◇

We can now rephrase Exploration ?? in logarithmic language. Because $8 = 2^3$, we say that $\log_2(8) = 3$. Similarly, because $81 = 3^4$, we say that $\log_3(81) = 4$. More precisely, we can now answer the following question.

Exploration 6.3

- (a) Compute $\log_2(8)$

Solution. Written in exponential form, finding $\log_2(8) = r$ is the same as finding the exponent r such that

$$2^r = 8$$

Here, because $8 = 2^3$, we say that $\log_2(8) = 3$

- (b) Compute $\log_3(81)$

Solution. Written in exponential form, finding $\log_3(81) = r$ is the

same as finding the exponent r such that

$$3^r = 81$$

Here, because $81 = 3^4$, we say that $\log_3(81) = 4$.

(c) Compute $\log_{10}(1,000,000,000)$

Solution. Written in exponential form, finding

$\log_{10}(1,000,000,000) = r$ is the same as finding the exponent r such that

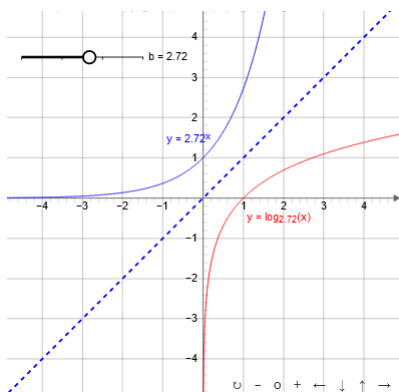
$$10^r = 1,000,000,000$$

To get the right hand side, we must multiply 10 nine times.

That means $1,000,000,000 = 10^9$. Therefore, we have that

$$\log_{10}(1,000,000,000) = 9.$$

Proposition 6.5 Fix any base $b > 1$. The function $y = \log_b(x)$ is the inverse of the the base b exponential function $\exp_b(x) = b^x$.



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The Natural Logarithm for Calculus. Recall that the function $y = e^x$ is called the *natural exponential function*. We call it's inverse the **natural logarithm**.

Definition 6.6 We write $\ln(x) = \log_e(x)$ where $e \approx 2.718\dots$

In other words, the **natural logarithm** $y = \ln(x)$ is the inverse of the natural exponential function $\exp(x) = e^x$. \diamond

Exploration 6.4

(a) Compute $\ln(\sqrt{e})$

Solution. There are two ways to answer this question. Algebraically, we know that \ln undoes base e exponentiation. So simply rewrite the argument of the function as a power of e

$$\ln(\sqrt{e}) = \ln(e^{1/2}) = \frac{1}{2}$$

Alternatively, you can answer the question by playing the riddle game: what does the exponent r of e need to be for e^r to equal \sqrt{e} ?

We know that $\sqrt{e} = e^{1/2}$. In other words, we need the exponent of e to be $\frac{1}{2}$. That tells us that $\ln(\sqrt{e}) = \frac{1}{2}$

- (b) Compute $\ln\left(\frac{1}{e}\right)$

Solution. Again, there are two ways to answer this question. Algebraically, we know that \ln undoes base e exponentiation. So simply rewrite the argument of the function as a power of e

$$\ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = -1$$

Alternatively, you can answer the question by playing the riddle game: what does the exponent r of e need to be for e^r to equal $\frac{1}{e}$?

We know that $\frac{1}{e} = e^{-1}$. In other words, we need the exponent of e to be -1 . That tells us that $\ln\left(\frac{1}{e}\right) = -1$

- (c) Let $f(x) = \ln(x)$ and $g(x) = e^x$. Simplify $f(g(3))$, $f(g(-2))$, $f(g(45))$, and $f(g(x))$

Solution. This is just a more complex way of asking us to simplify the natural log of e to various numbers. Reasoning as above gives

$$\begin{aligned} f(g(3)) &= \ln(e^3) = 3 \\ f(g(-2)) &= \ln(e^{-2}) = -2 \\ f(g(45)) &= \ln(e^{45}) = 45 \\ f(g(x)) &= \ln(e^x) = x \end{aligned}$$

Key Facts.

The functions $f(x) = \ln(x)$ and $g(x) = \exp(x) = e^x$ undo each other.

$$\begin{aligned} \ln(e^x) &= x \\ e^{\ln(x)} &= x \end{aligned}$$

Checkpoint 6.7 Simplify $e^{2 \cdot \ln(5)}$.

Solution.

$$\begin{aligned} e^{2 \cdot \ln(5)} &= e^{\ln(5) \cdot 2} && \text{Because } a^{rs} = a^{sr} \\ &= \left(e^{\ln(5)}\right)^2 && \text{Because } a^{rs} = (a^r)^s \\ &= (5)^2 && \text{Because } e^{\ln(5)} = 5 \\ &= 25 \end{aligned}$$

6.3 Laws of Logarithms

Logarithms have three important properties, which will play a central role in our computations.

Laws of Logarithms.

$$\ln(x^r) = r \cdot \ln(x)$$

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

Exploration 6.5 Expand the following completely using the laws of logarithms.

(a) $\ln(5^3)$

Solution. When you look at this expression, you *see* a power, so *write* down the first law of logarithms $\ln(x^r) = r \cdot \ln(x)$. Using the numbers we're given, we obtain

$$\ln(5^3) = 3 \cdot \ln(5).$$

(b) $\ln(10 \cdot 5^3)$

Solution. When you look at this expression, you *see* that the input is the product 10 and 5^3 , so *write* down the second law of logarithms $\ln(x \cdot y) = \ln(x) + \ln(y)$. Applying that in our example, we get

$$\ln(10 \cdot 5^3) = \ln(10) + \ln(5^3).$$

Recopy the first logarithm $\ln(10)$. For the second logarithm, *see* that the input is a power, so *write* down the first law of logarithms $\ln(x^r) = r \cdot \ln(x)$. That gives

$$\ln(10 \cdot 5^3) = \ln(10) + 3 \cdot \ln(5).$$

Warning: Note that our three laws of logarithms only apply when the exponential, product, or fraction is *inside* the logarithm.

If you try to divide logarithms, such as in $\frac{\ln(5)}{\ln(2)}$, you will get something very different than our log laws. This is because of the surprising fact that

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

For example, if you divide $\frac{\ln(5)}{\ln(2)}$, you actually get $\frac{\ln(5)}{\ln(2)} = \log_2(5) = 0.9162\dots$, which is **not** the same thing as $\ln\left(\frac{2}{5}\right) = 0.4306\dots$

Again, for the remainder of the course, we will almost always restrict our attention to natural logarithms. But this last property is useful in many applications, and may help you remember when you can (and when you can't) perform a simplification using the laws of logarithms.

6.4 Solving Compound Interest for Time

We can now use the laws of logarithms to help us solve equations where the variable is in the exponent.

As with everything else in algebra, if we have the equation $A = B$, we can replace it with the equation $\ln(A) = \ln(B)$.

Exploration 6.6

- (a) Solve for x satisfying $2^{3x} = 5$

Solution. The key idea of algebra is that if you have the equation $A = B$, you can do the same thing to both sides and have another true equation. Here, we are given that

$$2^{3x} = 5$$

That means that we also know that the equation $\ln(A) = \ln(B)$ is true, replacing $A = 2^{3x}$ and $B = 5$. This gives

$$\ln(2^{3x}) = \ln(5)$$

We have the property of logarithms that $\ln(x^r) = r \cdot \ln(x)$ on the left, to get the equation

$$3x \cdot \ln(2) = \ln(5)$$

Now, this looks complicated, but we can use a calculator to simplify $\ln(2) = 0.693\dots$ and $\ln(5) = 1.609\dots$. This gives us the equation

$$3x \cdot 0.693\dots = 1.609\dots$$

Rearranging the left side, we have

$$3 \cdot 0.693\dots \cdot x = 1.609\dots$$

Using a calculator, we see that $3 \cdot 0.693\dots = 2.079\dots$, which gives

$$2.079\dots x = 1.609\dots$$

Dividing both sides by $2.079\dots$, we get

$$x = \frac{1.609\dots}{2.079\dots} = 0.773\dots$$

You could also have done this with the original $3x \cdot \ln(2) = \ln(5)$ by dividing both sides by $3 \cdot \ln(2)$ to get

$$x = \frac{\ln(5)}{3 \cdot \ln(2)}$$

- (b) Solve for x satisfying $4 \cdot (1.95)^{2x} = 16$

Solution. There are two different approaches for solving an

equation like this, planning ahead or using brute force.

Planning Ahead We wish to solve the equation

$$4 \cdot (1.95)^{2x} = 16$$

We want to eventually introduce logarithms to allow us to apply Log Law 3, to pull the $2x$ out of the exponent. However, that law will not apply to $4 \cdot (1.95)^{2x}$ because of the coefficient of 4.

But we can divide both sides of the equation by 4. This gives us

$$\begin{array}{ccc} 4 \cdot (1.95)^{2x} & = & 16 \\ \times \frac{1}{4} & & \times \frac{1}{4} \\ (1.95)^{2x} & = & 4 \end{array}$$

Now, we can take the logarithm of both sides

$$\ln \left((1.95)^{2x} \right) = \ln (4)$$

Now we can apply the law of logs $\ln(x^r) = r \ln(x)$ on the left with $x = 1.95$ and $r = 2x$

$$2x \cdot \ln (1.95) = \ln (4)$$

We can now divide both sides by $2 \cdot \ln(1.95)$ to get

$$x = \frac{\ln (4)}{2 \ln (1.95)}$$

This example is exact, but it can be a bit overwhelming. You can also use your calculator to simplify

$$2x \cdot \ln (1.95) = \ln (4)$$

$$2x \cdot 0.667 \dots = 1.386 \dots$$

$$x \cdot 1.335 \dots = 1.386 \dots$$

$$x = \frac{1.386 \dots}{1.335 \dots} = 1.038 \dots$$

If you do use your calculator, try to use as many decimal places as possible. Rounding errors can grow if you do a lot of calculations. It is possible to understand how these errors grow, but that question is beyond the scope of this course.

Brute Force We have seen how to solve this equation with a bit of planning ahead. But what happens if you *start* by taking the natural log of both sides of the equation

$$4 \cdot (1.95)^{2x} = 16$$

You get

$$\ln(4 \cdot (1.95)^{2x}) = \ln(16)$$

On the left side, we have $\ln(4 \cdot (1.95)^{2x})$. We *cannot* use Log Law 1, since this is a product. Instead, we need to use Log Law $2 \ln(a \cdot b) = \ln(a) + \ln(b)$ with $a = 4$ and $b = (1.95)^{2x}$. This gives

$$\ln(4) + \ln((1.95)^{2x}) = \ln(16)$$

Now we can use Law Log 1 $\ln(a^r) = r \cdot \ln(a)$ with $a = 1.95$ and $r = 2x$ to get

$$\ln(4) + 2x \cdot \ln(1.95) = \ln(16)$$

You can simplify this using a calculator to get

$$1.386 \dots + 2 \cdot x \cdot 0.667 \dots = 2.772 \dots$$

You can solve this as usual to get

$$x = \frac{2.772 \dots - 1.386 \dots}{2 \cdot 0.667 \dots} = 1.038 \dots$$

Alternatively, you can solve the exact equation to get

$$x = \frac{\ln(16) - \ln(4)}{2 \cdot \ln(1.95)}$$

- (c) The balance of a certain investment is given by $F(t) = 10 \cdot 2^t$ as a function of the number of years t . How long must you wait until the balance reaches 80?

Solution. Note that *this specific function* is very simple, so it is possible to answer this numerically. After one year, the balance will be $F(1) = 20$, after two years the balance will be $F(2) = 40$, and after three years the balance will be $F(3) = 80$. It will take 3 years for the balance to reach 80.

For more complex equations, we will need to set up and solve an

exponential equation. In this case, we want to find a t such that $F(t) = 10 \cdot 2^t$ equals 80. This gives us the equation

$$\begin{aligned} 80 &= 10 \cdot 2^t \\ 8 &= 2^t && \text{Multiply both sides by } \frac{1}{10} \\ \ln(8) &= \ln(2^t) && \text{Take } \ln \text{ of both sides} \\ \ln(8) &= t \cdot \ln(2) && \text{Log Law 1 } \ln(a^r) = r \ln(a) \\ \frac{\ln(8)}{\ln(2)} &= t && \text{Divide both sides by } \ln(2) \end{aligned}$$

Alternatively, we could have simplified the natural log terms and solved as follows

$$\begin{aligned} \ln(8) &= t \cdot \ln(2) \\ 2.079\dots &= t \cdot 0.693\dots \\ t &= \frac{2.079\dots}{0.693\dots} = 3 \end{aligned}$$

We can now apply these techniques to solve the equations that arise in questions of compound interest.

Exploration 6.7 You invest \$1000 at 15% interest compounded monthly.

- (a) How many years are required until you double your money?

Solution. First, write down the compound interest formula from Definition ??.

$$F = P \cdot \left(1 + \frac{r}{m}\right)^{mt}$$

with $P = 1000$, $r = 0.15$, and $m = 12$. Furthermore, we want the *future value* to be twice 1000. In other words, we want $F = 2000$. This gives

$$2000 = 1000 \cdot \left(1 + \frac{0.15}{12}\right)^{12t}$$

Dividing both sides by 1000 gives

$$2 = \left(1 + \frac{0.15}{12}\right)^{12t}$$

Next, we can simplify this equation using a calculator and the properties of exponents ??

$$2 = \left(1 + \frac{0.15}{12}\right)^{12t}$$

$$\begin{aligned}
2 &= (1.0125)^{12t} \\
2 &= (1.0125^{12})^t && \text{Because } a^{rs} = (a^r)^s \\
2 &= (1.16075\dots)^t
\end{aligned}$$

Finally, finish solving the exponential equation as above using the Laws of Logs.

$$\begin{aligned}
\ln(2) &= \ln((1.16075\dots)^t) \\
\ln(2) &= t \cdot \ln(1.16075\dots) && \text{Using } \ln(a^r) = r \ln(a) \\
t &= \frac{\ln(2)}{\ln(1.16075\dots)} && \text{Dividing both sides by } \ln(1.16075\dots) \\
t &= 4.64\dots
\end{aligned}$$

With 15% interest, it will take about 4.64 years for the balance to double.

- (b) How long will it be until the balance reaches \$10,000?

Solution. First, write down the compound interest formula

$$F = P \cdot \left(1 + \frac{r}{m}\right)^{mt}$$

and plug in the values we are given. Note that we want the time when the future value $F = 10,000$. That gives

$$10,000 = 1,000 \cdot \left(1 + \frac{0.15}{12}\right)^{12t}$$

Simplifying this with a calculator and the laws of exponents gives

$$\begin{aligned}
10 &= \left(1 + \frac{0.15}{12}\right)^{12t} \\
10 &= (1.0125)^{12t} \\
10 &= (1.16075\dots)^t
\end{aligned}$$

Taking \ln of both sides and applying the laws of logarithms, we get

$$\begin{aligned}
\ln(10) &= \ln((1.16075\dots)^t) \\
\ln(10) &= t \cdot \ln(1.16075\dots) \\
\frac{\ln(10)}{\ln(1.16075\dots)} &= t \\
t &= 14.868\dots
\end{aligned}$$

The balance will increase to \$10,000 in about 15 years.

As an exercise, find how long it takes for a balance of \$1000 to double at 2% interest, at 5% interest, and at 30% interest.

For our last application, we will return to the concept of **depreciation**, discussed in more depth in Application ?? and Exploration ??

Exploration 6.8 Suppose that a certain asset loses 30% of its value each year. In other words, suppose that the value of the asset after t years is given by

$$V(t) = P \cdot (1 - 0.3)^t$$

- (a) Suppose that the asset starts with value $P = 1000$. Find the value after 12 years.

Solution. The future value function is

$$V(t) = 1000 \cdot (1 - 0.3)^t = 1000 \cdot 0.7^t$$

After 12 years,

$$V(12) = 1000 \cdot (1 - 0.3)^{12} = 13.841$$

After 12 years, the value of the asset will be approximately \$13.84

- (b) Suppose that the value after 12 years is \$50. Find the starting value P .

Solution. The future value function is

$$V = P \cdot (1 - 0.3)^t$$

We know that when $t = 12$, that $V = 50$. Plugging this in gives

$$\begin{aligned} 50 &= P \cdot (1 - 0.3)^{12} \\ 50 &= P \cdot 0.0138 \dots \\ \frac{50}{0.138} &= P \\ P &= 3612.380 \dots \end{aligned}$$

The starting value must have been about \$3,612.38

- (c) Suppose that the asset starts with a value $P = 1000$. Find the number of years t for the value to decrease to 100.

Solution. The future value function is

$$V = P \cdot (1 - 0.3)^t$$

We want the time t when the value reaches $V = 100$ when from a starting value of $P = 1,000$. Plugging this in gives

$$100 = 1000 \cdot (0.7)^t$$

$$\begin{aligned}
 0.1 &= (0.7)^t \\
 \ln(0.1) &= \ln((0.7)^t) \\
 \ln(0.1) &= t \cdot \ln(0.7) \\
 t &= \frac{\ln(0.1)}{\ln(0.7)} = 6.45\dots
 \end{aligned}$$

It will take about 6.45 years for the balance to decline from 1000 to 100.

Application 6.8 Depreciating Assets and Scrap Value. In this class, our focus is on the mathematical tools that underly economics, business, and accounting.

In accounting, you will learn about how to decide which mathematical tool to use (straight line versus declining balance depreciation), and to find a numerical value for the rate of depreciation.

Rates of depreciation can be found in many ways. For example, you can use the actual sales price for cars based on their mileage to empirically determine how much the value drops per mile.

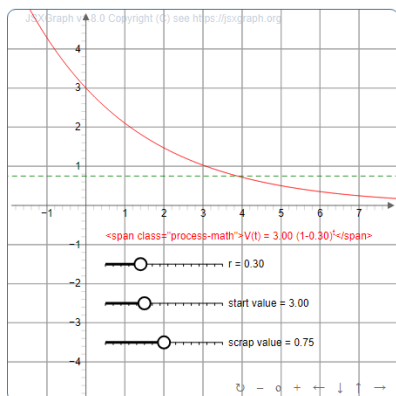
But there are also other ways of calculating the rate of depreciation. For example, some assets have a known or conventional "scrap value" and "expected lifespan."

Once you know the starting value, the scrap value, and the expected lifespan, you can actually compute the yearly depreciation you should use. Mathematically, that means solving

$$F = P(1 - r)^t$$

for the rate r to get

$$r = 1 - \sqrt[t]{\frac{F}{P}}$$



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6.5 Applications of Logarithms

Although logarithms are less familiar than lines, polynomials, and rational functions, they do play an important role in the physical and economic sciences. In this section we will introduce two examples of logarithms. These examples are somewhat simplified for mathematical clarity, but the interested reader is encouraged to continue their study to understand the relevant nuances.

Logarithmic models often arise when discussing quantities that are more meaningful when compared to a baseline than when viewed as isolated numbers.

For example, it is hard to say how happy it would make someone to win a \$1000 prize. Presumably this win would make someone with \$5,000 in their bank account would be much happier than a similar win would make someone with \$5,000,000 in wealth.

Similarly, a cell phone ringer might seem very loud in a quiet library, but the same ringer would sound very quiet at a packed concert.

Exploration 6.9 Sound Perception and Logarithms. Let x denote the magnitude of a sound, measured by the amplitude of the sound wave itself. The *percieved sound level* is given by

$$dB(x) = 10 \cdot \log_{10}(x)$$

This formula is a slight oversimplificaiton. See, for example, [Wikipedia](#)¹ for more details.

- (a) The sound waves of rustling leaves have power $x = 100$. Find $dB(100)$.

Solution. $dB(100) = 10 \cdot \log_{10}(100) = 10 \cdot \log_{10}(10^2) = 20$ decibels

- (b) The sound waves in a quiet library have power 10,000. Find $dB(10,000)$. How much louder does this sound compared to rustling leaves?

Solution. $dB(10,000) = 10 \cdot \log_{10}(10,000) = 10 \cdot \log_{10}(10^4) = 40$ decibels.

The decibel rating of a quiet library (40 dB) is twice the decibel rating of leaves (20 dB), so the library sounds twice as loud.

But surprisingly, this apparent doubling is actually the result of a $\frac{10,000}{100} = 100$ -fold increase in the power of the waves.

- (c) The sound waves from an alarm clock have power 100,000,000. Find $dB(100,000,000)$. How much louder does this sound compared to a quiet library?

Solution. $dB(100,000,000) = 10 \cdot \log_{10}(100,000,000) = 10 \cdot \log_{10}(10^8) = 80$ decibels.

The decibel rating of an alarm clock (80 dB) is twice the decibel rating of a quiet library (40 dB), so the library sounds twice as loud.

But surprisingly, this apparent doubling is actually the result of a $\frac{100,000,000}{10,000} = 10,000$ -fold increase in the power of the waves.

- (d) How much does doubling the power of the sound waves increase the decibel rating of a given frequency x ?

Solution. We want to understand the decibel rating of $2x$. Using the definition of dB gives

$$\begin{aligned} dB(2x) &= 10 \log_{10}(2x) \\ &= 10 \left(\log_{10}(2) + \log_{10}(x) \right) && \ln(ab) = \ln(a) + \ln(b) \\ &= 10 \left(0.301029 + \log_{10}(x) \right) && \text{calculator} \\ &= 3.01029 + 10 \log_{10}(x) \\ &= 3.01029 + dB(x) \end{aligned}$$

Doubling the power of the sound waves only increases the decibel rating by about $3dB$

- (e) How much does increasing the power of the sound waves by a factor of 100 change the decibel rating of a given frequency x ?

Solution. We want to understand the decibel rating of $100x$. Using the definition of dB gives

$$\begin{aligned} dB(100x) &= 100 \log_{10}(100x) \\ &= 10 \left(\log_{100}(10) + \log_{10}(x) \right) && \ln(ab) = \ln(a) + \ln(b) \\ &= 10 \left(2 + \log_{10}(x) \right) && \text{calculator} \\ &= 20 + 10 \log_{10}(x) \\ &= 20 + dB(x) \end{aligned}$$

Multiplying the power of sound waves by a factor of 100 only increases the decibel rating by the addition of 20dB.

¹ en.wikipedia.org/wiki/Decibel#Perception

Exploration 6.10 Utility and Logarithms. Some economists argue that the *perceived value of money* (often called the *utility* of money) is logarithmic. Of course, different people will have different utility functions. To see why, we will explore a specific utility function. You can find more on this in Chapter 25 of *Thinking Fast and Slow*.

Given *life savings* of money x in dollars, let the utility of $\$x$ of these savings be given by

$$U(x) = 21.7 \ln(x) + 150$$

- (a) Find the utility of \$0, \$10, \$100, \$1000, \$10,000, \$100,000, and \$1,000,000.

Explain the meaning of these numbers. Why are some negative? Why are some positive? How do they change as your life savings increases?

Solution.

dollars x	0	10	100	1,000	10,000	100,000	1,000,000
utility $U(x)$	<i>DNE</i>	-100.0	-50.06	-0.10	49.86	99.83	149.79

The first thing to note is that $U(0)$ is undefined. Mathematically, this is because $\ln(0)$ is undefined. Practically, this is because you would be *very* unhappy if you did not have any money to use to pay rent, buy food, etc.

Next, note that this person has negative utility until approximately $x = 1,000$. Perhaps this would mean that the individual needs at least \$1000 for basic necessities. Then a balance of less than \$1000 implies unmet necessities, which would make the individual unhappy indeed.

Finally, consider the values above \$1000. Note that increasing the balance by a factor of 10 *approximately* increases their happiness by 50 utility points. Often, this is because the individual will start with a list of very clear and very productive uses of the money. As those clear productive uses are taken care of, the individual needs to look for less clear or less productive uses for their money. Thus, each dollar does not have the same value to the individual!

Of course, this is an overly simplified example. In fact, utility functions are much more complex. But even these more complex functions will share many of the above properties with our logarithmic utility.

- (b) How does doubling a wealth of $\$x$ impact utility?

Solution. We are asked to understand the utility of $\$2x$ dollars. Using the utility function given, we have

$$\begin{aligned}
 U(2x) &= 21.7 \ln(2x) + 150 \\
 &= 21.7 \left(\ln(2) + \ln(x) \right) + 150 && \ln(ab) = \ln(a) + \ln(b) \\
 &= 21.7 \ln(2) + 21.7 \ln(x) + 150
 \end{aligned}$$

$$\begin{aligned}
&= 15.04 + 21.7 \ln(2x) + 150 \\
&= 15.04 + U(x)
\end{aligned}$$

Doubling the amount of money available *adds* 15.04 utility (or "happiness") points.

- (c) How does multiplying a wealth of $\$x$ by a factor of 10 impact utility?

Solution. We are asked to understand the utility of $\$10x$ dollars. Using the utility function given, we have

$$\begin{aligned}
U(10x) &= 21.7 \ln(10x) + 150 \\
&= 21.7 \left(\ln(10) + \ln(x) \right) + 150 \quad \ln(ab) = \ln(a) + \ln(b) \\
&= 21.7 \ln(10) + 21.7 \ln(x) + 150 \\
&= 49.97 + 21.7 \ln(x) + 150 \\
&= 49.97 + U(x)
\end{aligned}$$

Multiplying by 10 the amount of money available *adds* 49.97 utility (or "happiness") points.

Part II

Change and Rates

7

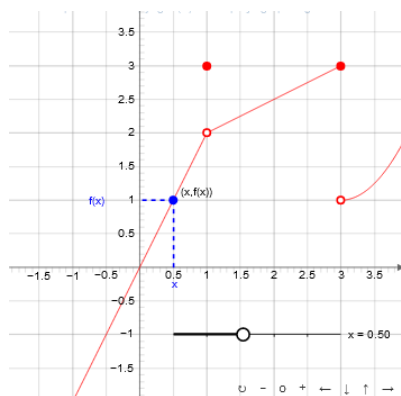
Limits

7.1 Reading Graphs Dynamically

Reading graphs *dynamically* will be an essential part of understanding the key concepts of calculus, and applying those concepts to your daily life.

To read any graph dynamically, start at a specific x value, and imagine "standing" on the function above that point on the x -axis. The y value corresponds with your height above the x -axis.

Next, imagine walking right (increasing x) and left (decreasing x) along the curve. As you walk, the curve *forces* your height to move up or down.



[Standalone](#)
[Embed](#)

Figure 7.1

Example 7.2 Exploring Figure ??. Let's explore what happens in Figure ??.

- To see what happens as you *approach* $x = 1$ *from the left*,
 - Move the x value to some number less than 1. For simplicity, start by moving to $x = 0$. At that point on the curve, your height will be $y = 0$.
 - Drag the slider to increase x towards 1, making sure never to

allow x to equal 1.

As you increased x (as you move it right), you will have seen the height increase as well. Even more, you will have seen the height getting closer and closer to $y = 2$.

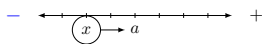
3. Because of this, we say that your height $y = f(x)$ gets closer and closer to $y = 2$ as x approaches 1 from the left.

Important: The fact that $f(1) = 3$ is unrelated to our question, because that tells us what happens *at that point*, not what happens *during the process of approaching that point*.

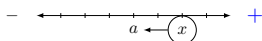
- To see what happens as you *approach* $x = 1$ *from the right*,
 1. Move the slider so that x is significantly bigger than 2. For simplicity, start by moving to $x = 2$. At that point on the curve, your height will be $y = 2.5$.
 2. Drag the slider to *decrease* x toward 1, making sure never to allow x to equal 1. As you do this, the height decreases, getting closer and closer to 2.
 3. Because of this, we say that the height $y = f(x)$ gets closer and closer to $y = 2$ as x approaches 1 from the right.
- As you approach $x = 3$ from the left, your height increases to $y = 3$.
As you approach $x = 3$ from the right, your height decreases to $y = 1$.

□

To answer these questions clearly and concisely, we need new mathematical language. To understand our notation, it will be helpful to remember the number line. If you (x) approach a number a **from the left**, you are approaching a from the side of the number line with all the negative numbers. Because of that, we write $x \rightarrow a^-$ as a shorthand for *approaches a from the left*.



If you (x) approach a number a **from the right**, you are approaching a from the side of the number line with all the positive numbers. Because of that, we write $x \rightarrow a^+$ as a shorthand for *approaches a from the right*.



Definition 7.3 Suppose that $f(x)$ is some function, and that a and L

are two numbers.

Left Suppose that is x approaching a from the left (that $x \rightarrow a^-$). If the heights $f(x)$ get closer and closer to a number L , then we say that the **left hand limit** exists, and write

$$\lim_{x \rightarrow a^-} f(x) = L$$

We can read this expression by saying "(1) the limit (2) as x approaches a from the left is (3) equal to "L".

Right Suppose that is x approaching a from the right (that $x \rightarrow a^+$). If the heights $f(x)$ get closer and closer to a number L , then we say that the **right hand limit** exists, and write

$$\lim_{x \rightarrow a^+} f(x) = L$$

We can read this expression by saying "(1) the limit (2) as x approaches a from the right is (3) equal to "L".

Undirected Sometimes, we talk about limits without any direction.

We say that the (undirected) **limit** of $f(x)$ as x approaches a is equal to L , written

$$\lim_{x \rightarrow a} f(x) = L$$

if as the *input* x gets *closer to* a , the *output* $f(x)$ gets *closer to* L .

◇

Exploration 7.1 Let $f(x)$ be the function defined by the graph in Figure ???. Compute the following limits.

(a) $\lim_{x \rightarrow 1^-} f(x)$

Answer. Following the reasoning from Example ??, we see that

$$\lim_{x \rightarrow 1^-} f(x) = 2$$

(b) $\lim_{x \rightarrow 1^+} f(x)$

Answer. Following the reasoning from Example ??, we see that

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

(c) $\lim_{x \rightarrow 1} f(x)$

Answer. We have seen that as x approaches 1, the height goes to 2 *regardless* of the direction. That tells us that the undirected limit exists and equals 2. In other words, $\lim_{x \rightarrow 1} f(x) = 2$

(d) $\lim_{x \rightarrow 3^-} f(x)$

Answer. Following the reasoning from Example ??, we see that $\lim_{x \rightarrow 3^-} f(x) = 3$

(e) $\lim_{x \rightarrow 3^+} f(x)$

Answer. Following the reasoning from Example ??, we see that $\lim_{x \rightarrow 3^+} f(x) = 1$

(f) $\lim_{x \rightarrow 3} f(x)$

Answer. We have seen that as x approaches 3, the height goes to *different values* depending on your direction of approach. That tells us that the undirected limit *does not exist*. In other words, $\lim_{x \rightarrow 3} f(x) = DNE$

In the graph above, we have already seen that as we move the input (slider) x closer to 1, that the height (output) gets closer to 2. In our new lanugage, that says

$$\lim_{x \rightarrow 1} f(x) = 2$$

Note 7.4 A little extra precision. In this class, we only need an informal understanding of limits, and the definitions above should be more than enough.

But there is still a little bit of ambiguity in expressions like "getting closer and closer to." It is natural to ask, how close is close enough? The answer to that question gives us the full, mathematical definition of a limit.

That definition says that $\lim_{x \rightarrow a} f(x) = L$ if and only if for *any* acceptable error $\varepsilon > 0$, there is an acceptable distance $d > 0$ such that the difference between the actual height $f(x)$ and the limit height of L is less than ε whenever the input x is no more than distance d from the point a that you are approaching.

Again, unless you find it helpful we don't need that more formal definition for this class. The main point is that it is possible to eliminate any ambiguity about what "close enough" really means.

The function in Figure Figure ?? is very unusual. Most mathematical functions can be drawn smoothly, with one or two strokes of a pen. But this function includes lots of jumping around, and we need to lift the pen every time there is a "jump". We can make this precise using the following definition.

Definition 7.5 We say that a function $f(x)$ is **continuous** on its domain if you can draw the graph of the function without lifting your pen.

A function is **continuous at the number a** if

$$\lim_{x \rightarrow a} f(x) = f(a),$$

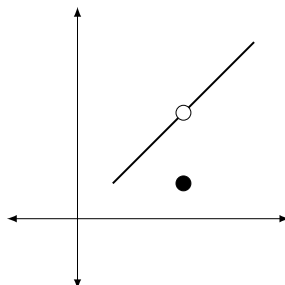
which means that both the left and right limits exist, and equal the value that f achieves at a . \diamond

The two definitions are really two ways of saying the same thing. If your function can be drawn without lifting your pen, then each point flows smoothly into the next point, and the limit $\lim_{x \rightarrow a} f(x)$ exists and equals $f(a)$.

On the other hand, if the function can't be drawn without lifting your pen, then there must be some point in drawing the curve where the function does *not* smoothly flow into the next point. In that case, the limit $\lim_{x \rightarrow a} f(x)$ either does not exist, or it does not equal $f(a)$.

Example 7.6 Three reasons a function might *not* be continuous..

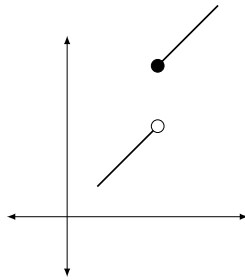
Removable Discontinuity



Imagine running along this graph. The discontinuity is a hole in the ground, like an open manhole cover. If you were to walk over that hole, you would fall down the hole, and hit the ground at the bottom. But it seems like someone should be able to **remove** this discontinuity by covering up the manhole cover (by redefining the function at that specific point).

We say that *any* function has a **removable discontinuity** if the left and right limits both exist, and both equal the same number, but if that number is not equal to $f(a)$.

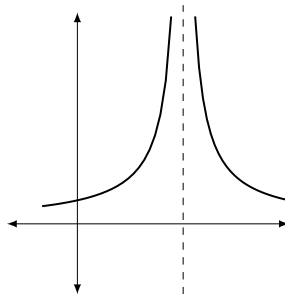
Jump Discontinuity



In this graph, the limit does not exist. If you imagine running along this curve, you would need to **jump** to get from one side of the discontinuity to the other.

We say that *any* function has a **jump discontinuity** if the left and right limits equal two different finite numbers.

Infinite Discontinuity



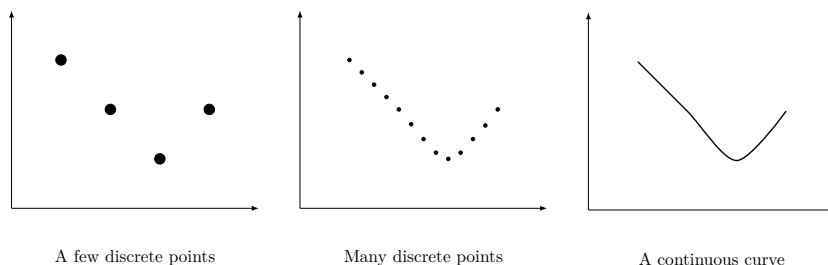
In this graph, the left and right limits both go off to infinity, and the graph of the function has a vertical asymptote. If you imagine running along this curve, you would need to be able to go to infinity in the process of getting from one side to the other.

We say that *any* function has an **infinite discontinuity** if either the left or right limit goes to infinity.

□

7.2 Macroeconomics and Mostly Continuous Functions

There is an important difference between histograms and scatter plots on the one hand, and continuous functions on the other.



In a discrete graph, you would need to "jump" to move between the discrete points. But as you add more and more points, the distance you would need to "jump" gets smaller and smaller. If you add enough discrete points, the graph becomes indistinguishable from a *continuous* curve, where you can walk smoothly along the curve.

In the words of [Investopedia](http://www.investopedia.com/terms/m/macroeconomics.asp)¹, "Macroeconomics is the branch of economics that deals with the structure, performance, behavior, and decision-making of the whole, or aggregate, economy".

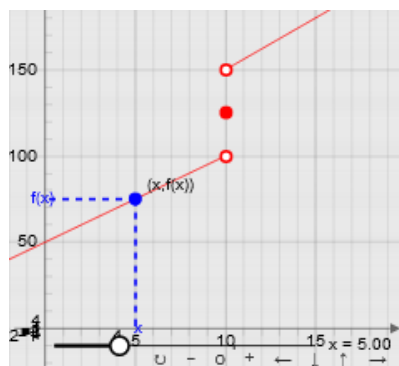
¹ www.investopedia.com/terms/m/macroeconomics.asp

There are a *lot* of data points in the overall economy, so many macroeconomic functions are essentially continuous. However, certain unique events can introduce jumps.

As a very simplified example, any given oil rig will be able to produce different amounts of oil, based on how many people are working on it, how much electricity is used for the pump, and other factors. These factors can generally be scaled smoothly, so the cost of producing oil should be roughly continuous as a function of the number of barrels to be produced.

But there are also limits to the amount of oil that a certain rig will be able to extract. To produce more than that amount of oil, an entirely new oil rig will need to be set up, causing a massive jump in the cost. Furthermore, additional refinement will probably be needed after the new rig is produced, possibly resulting in another jump in cost.

Exploration 7.2 Suppose that the following graph shows the cost C in millions of dollars of producing x million barrels of oil at a specific oil field. Use this graph to answer the following questions.



[Standalone](#)
[Embed](#)

- (a) Compute $\lim_{x \rightarrow 10^-} C(x)$. What does this tell us about oil production?

Solution. As the number of (millions of) gallons of oil increases to 10 from the *left*, the cost increases to \$100 (million) dollars. Therefore, $\lim_{x \rightarrow 10^-} C(x) = 100$.

- (b) Compute $\lim_{x \rightarrow 10^+} C(x)$. What does this tell us about oil production?

Solution. As the number of (millions of) gallons of oil decreases to 10 from the *right*, the cost decreases to \$150 (million) dollars. Therefore, $\lim_{x \rightarrow 10^+} C(x) = 150$.

- (c) Compute $C(10)$. What does this tell us about oil production?

Solution. To produce exactly 10 (million) barrels, the cost will be exactly \$125 (million) dollars. Therefore, $C(10) = 125$.

To define functions with jumps, we often use *piecewise functions*. The graph in Exploration ?? is made up of the following three functions:

- When $x < 10$, graph the line $50 + 5x$
- When $x = 10$, graph the point $50 + 5x$
- When $x > 10$, graph the line $90 + 6x$

Mathematically, this is often abbreviated as

$$f(x) = \begin{cases} 50 + 5x & \text{if } x < 10 \\ 125 & \text{if } x = 10 \\ 90 + 6x & \text{if } x > 10 \end{cases}$$

7.3 Computing Limits Numerically

You can also compute limits numerically. The process is straightforward, once you understand what the question is asking. For example, suppose that

$$f(x) = \begin{cases} 1 + x & \text{if } x < 2 \\ 1 & \text{if } x = 2 \\ -1 + 0.5x & \text{if } x > 2 \end{cases}$$

- You are asked to compute $\lim_{x \rightarrow 2^+} f(x)$
- Remember this means you want to see *what happens to your height* when your x value approaches $a = 2$ from the right.

- Think about some x values you would pass through when approaching 2 from the right. For example, you would pass through $x = 3$, $x = 2.5$, $x = 2.1$, $x = 2.01$ and so on.
- Compute the height at each of the x values we listed above.

x	$f(x)$
3	$f(3) = -1 + 0.5(3) = 0.5$
2.5	$f(2.5) = -1 + 0.5(2.5) = 0.25$
2.1	$f(2.1) = -1 + 0.5(2.1) = 0.05$
2.01	$f(2.01) = -1 + 0.5(2.01) = 0.005$

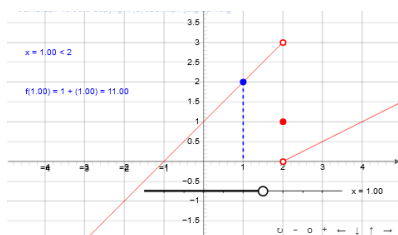
- Think about what is happening to your height as your x value is approaching 2 from the right.

Notice that the y -value is getting ***closer and closer to 0***. In other words,

$$\lim_{x \rightarrow 2^+} f(x) = 0$$

To get more comfortable with what this means, the graph of the piecewise function $f(x) = \begin{cases} 1 + x & \text{if } x < 2 \\ 1 & \text{if } x = 2 \\ -1 + 0.5x & \text{if } x > 2 \end{cases}$ is given below. As you change the value of x , the left side of the screen displays

1. The current value of x .
2. Whether x is greater, equal, or less than 2. This is used to decide which of the three equations to use when computing $f(x)$.
3. The result of plugging the current value of x into the equation for $f(x)$, which gives the current y value.



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[Embed](#)

We can use the equation

Exploration 7.3 Suppose that the cost of producing x barrels of oil is given by

$$C(x) = \begin{cases} 50 + 5x & \text{if } x < 10 \\ 125 & \text{if } x = 10 \\ 90 + 6x & \text{if } x > 10 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 10^+} C(x)$

Solution. Think about some x values you would pass through when approaching 10 from the right. For example, you would pass through $x = 11$, $x = 10.5$, $x = 10.1$, $x = 10.01$ and so on. Compute the height at each of the x values we listed above.

x	$C(x)$
11	$C(11) = 90 + 6(11) = 156$
10.5	$C(10.5) = 90 + 6(10.5) = 153$
10.1	$C(10.1) = 90 + 6(10.1) = 150.6$
10.01	$C(10.01) = 90 + 6(10.01) = 150.06$

As you move from the top row of the table to the bottom row of the table, the x values get closer and closer to $a = 10$, always from the right.

And as these x values approach 10, we see that the heights $C(x)$ get closer and closer to 150. That means

$$\lim_{x \rightarrow 10^+} C(x) = 150$$

- (b) Compute $\lim_{x \rightarrow 10^-} C(x)$

Solution. Think about some x values you would pass through when approaching 10 from the left. For example, you would pass through $x = 9$, $x = 9.5$, $x = 9.9$, $x = 9.99$ and so on. Compute the height at each of the x values we listed above.

x	$C(x)$
9	$C(9) = 90 + 6(11) = 95$
9.5	$C(9.5) = 90 + 6(10.5) = 97.5$
9.9	$C(9.9) = 90 + 6(10.1) = 99.5$
9.99	$C(9.99) = 90 + 6(10.01) = 99.95$

As you move from the top row of the table to the bottom row of the table, the x values get closer and closer to $a = 10$, always from the left.

And as these x values approach 10, we see that the heights $C(x)$ get closer and closer to 100. That means

$$\lim_{x \rightarrow 10^-} C(x) = 100$$

- (c) Is $C(x)$ continuous at $x = 10$? If not, classify the type of discontinuity.

Solution. We have seen above that the left and right hand limits exist, but do not equal each other. In other words, we have seen that

$$\lim_{x \rightarrow 10} C(x) DNE.$$

It follows that the function is not continuous at $a = 10$.

Because the directed limits exist and equal finite numbers, this is a *jump discontinuity*.

Checkpoint 7.7 Let $A(x) = \frac{10 + 4x}{x}$. Find $\lim_{x \rightarrow 0^+} A(x)$.

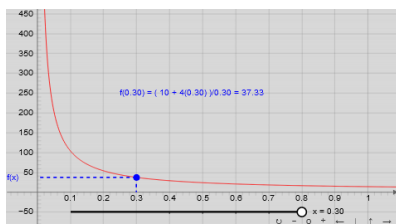
Solution. Think about some x values you would pass through when approaching 0 from the right. For example, you would pass through $x = 1$, $x = 0.5$, $x = 0.1$, $x = 0.01$ and so on. Compute the height at each of the x values we listed above.

x	$f(x)$
1	$f(1) = \frac{10 + 4(1)}{1} = 14$
0.5	$f(0.5) = \frac{10 + 4(0.5)}{0.5} = 24$
0.1	$f(0.1) = \frac{10 + 4(0.1)}{0.1} = 104$
0.01	$f(0.01) = \frac{10 + 4(0.01)}{0.01} = 1,004$
0.001	$f(0.001) = \frac{10 + 4(0.001)}{0.001} = 10,004$

As you move from the top row of the table to the bottom row of the table, the x values get closer and closer to $a = 0$, always from the right.

And as these x values approach 0, we see that the heights $f(x)$ are getting bigger and bigger, without bound. That means

$$\lim_{x \rightarrow 0^+} f(x) = \infty$$

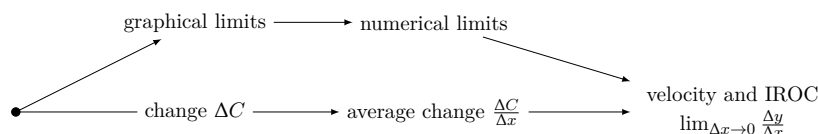


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8

Average Rates of Change

The concepts contained in Chapter ??–?? are all connected in important, but sometimes tricky to follow ways. As you go through this section, you may find it helpful to refer to the diagram below to see how the concepts we have seen so far are related.



8.1 Average Rates of Change

Let y be any function of x . That means that any change of input Δx produces a change in the output Δy . If we *average* the change in outputs Δy over the inputs Δx , we get the **average change** $\frac{\Delta y}{\Delta x}$.

For reasons we will see in a future section, we will use the terms **average change** and **average rate of change** interchangeably.

Definition 8.1 Given a curve $y = f(x)$ and two input values x_1 and x_2

- The **change in input** that results from moving from x_1 to x_2 is $\Delta x = x_2 - x_1$
- The resulting **change in output** that results from moving from x_1 to x_2 is

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$$

where $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

- The **average rate of change of** y as a result of changing the inputs from x_1 to x_2 is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

◇

Checkpoint 8.2 Recall that the total cost function $C(x)$ is a function of the total number of items x to be produced. Suppose that increasing your production by 100 items will increase the total cost by \$2000. Find the *average* (aka *average rate of*) change in the cost.

Solution. We are told that a change of input $\Delta x = 100$ will result in an increase in total cost of $\Delta C = 2000$. To find the *average* change, divide that \$2000 evenly across the 100 additional items

$$\frac{\Delta C}{\Delta x} = \frac{2000}{100} = 20$$

On average, each of the 100 additional items will increase the total cost by \$20.

Average rates are common throughout economics and daily life. When the number of items is large (in the context of macroeconomics), limits show up and calculus begins. But before we get there, we will first explore a number of examples that emphasize the connection between *totals*, *changes* and *averages/rates*.

Exploration 8.1 A total (cumulative) cost function is defined by $C(x) = 0.1x^2 + 50x + 1000$ in dollars as a function of the total number of items x to be produced.

- (a) Last year, you made 100 items. This year, you are considering increasing your production to 150 items. Find the *change* in cost and the *average rate of change* in the cost that would result from this change.

Solution. From the information given in the problem, we can compute the total cost of producing 100 items is

$$C_1 = C(100) = \$7,000$$

and that the total cost of producing 150 items would be

$$C_2 = C(150) = \$10,750$$

The change in cost between these quantities is

$$\Delta C = C(150) - C(100) = 10,750 - 7,000 = \$3,750$$

This change is a result of changing the quantity by

$$\Delta x = 150 - 100 = 50 \text{ items}$$

. Dividing the total cost ΔC evenly among the Δx items results in an *average* cost per item of

$$\frac{\Delta C}{\Delta x} = \frac{3,750}{50} = 75 \frac{\$}{\text{item}}$$

There were a lot of parts of this calculation. You can either use a the definition $\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$ which is a single formula that combines all of these steps, or you can organize the different parts separately using the table below. This is called an Average Rate of Change table, often abbreviated AROC.

x_1	x_2	Δx	y_1	y_2	Δy	$\Delta y / \Delta x$
100	150	50	7000	10750	3750	75

- (b) Use the cost function above to find the average change in cost between 100 and 125 items, between 100 and 110 items, and between 100 and 101 items.

Solution.

x_1	x_2	Δx	y_1	y_2	Δy	$\Delta y / \Delta x$
100	125	$125 - 100 = 25$	$C(100) = 7000$	$C(125) = 8812.5$	$C(125) - C(100) = 8812.5 - 7000 = 1812.5$	$\frac{1812.5}{25} = 72.5$
100	110	$110 - 100 = 10$	$C(100) = 7000$	$C(110) = 7710$	$C(110) - C(100) = 7710 - 7000 = 710$	$\frac{710}{10} = 71$
100	101	$101 - 100 = 1$	$C(100) = 7000$	$C(101) = 7070.1$	$C(101) - C(100) = 7070.1 - 7000 = 70.1$	$\frac{70.1}{1} = 70.1$

- (c) Suppose that last year you produced 100 items, and that you are planning to increasing production by 5 items this year. Find the average rate of change in cost that results from this change.

Solution. We are given the starting quantity $x_1 = 100$ and the change in quantity $\Delta x = 5$. Using this, we can reason backward to get the ending quantity $x_2 = 105$. We can use the AROC equation

$$\frac{\Delta C}{\Delta x} = \frac{C(105) - C(100)}{105 - 100} = \frac{7352.5 - 7000}{5} = 70.5$$

or create an AROC table

x_1	x_2	Δx	y_1	y_2	Δy	$\Delta y / \Delta x$
100	105	$105 - 100 = 5$	$C(100) = 7000$	$C(105) = 7352.5$	$C(105) - C(100) = 7352.5 - 7000 = 352.5$	$\frac{352.5}{5} = 70.5$

The 5 additional items will increase the *total cost* by an average of 70.50 *dollars per item*.

8.2 Graphs of Rates, Changes, and Totals

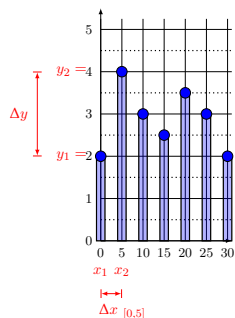
Application 8.3 Understanding and Applying "Average Rates of Change". We've spent a lot of time thinking about "total" and "change" graphs. In each of those examples, we knew the value of the total for each natural number $1, 2, 3, \dots$

But what happens if we only have information about *a few* values of x . For example, in the graph on the right, we only know the total value on day 0, 5, 10, etc.

What should we do if we want to know the change between day 0 and 1? We can't know that number for certain, but we *can* try to compute the *average* change.

Since the value went up 2 units over 5 days, if we divide that change evenly over the 5 days we get the **Average Rate Of Change**

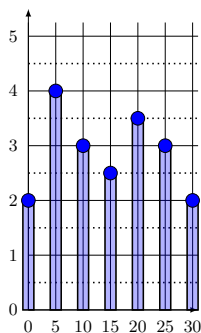
$$(\text{AROC}) = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{5 - 0} = \frac{2}{5} = 0.4$$



Summarizing this in a table, we get the row in an Average Rate Of Change (AROC) table:

x_1	x_2	Δx	y_1	y_2	Δy	$\Delta y / \Delta x$
0	5	$5 - 0 = 5$	$C(0) = 2$	$C(5) = 4$	$C(0) - C(5) = 2$	$\frac{2}{5} = 0.4$

Exploration 8.2 Suppose, as in Application ??, that a total cost function is given by the histogram below



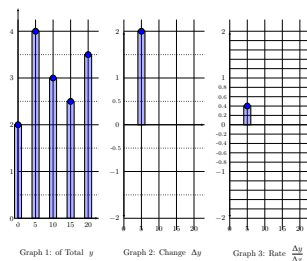
(a) Fill in the following AROC table

x_1	x_2	Δx	y_1	y_2	Δy	$\Delta y / \Delta x$
0	5					
5	10					
10	15					
15	20					

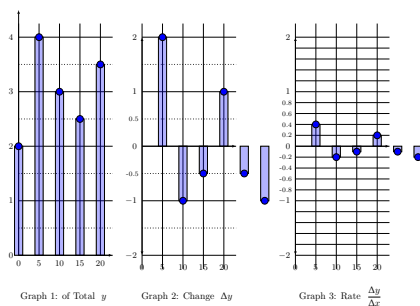
Solution.

x_1	x_2	Δx	y_1	y_2	Δy	$\Delta y / \Delta x$
0	5	$5 - 0 = 5$	2	4	$4 - 2 = 2$	$\frac{2}{5} = 0.4$
5	10	$10 - 5 = 5$	4	3	$3 - 4 = -1$	$\frac{-1}{5} = -0.2$
10	15	$15 - 10 = 5$	3	2.5	$2.5 - 3 = -0.5$	$\frac{-0.5}{5} = -0.1$
15	20	$20 - 15 = 5$	2.5	3.5	$3.5 - 2.5 = 1$	$\frac{1}{5} = 0.2$

(b) Use your AROC table to fill in the graphs of the change Δy and of the rate $\frac{\Delta y}{\Delta x}$ in the graph below.



Solution.

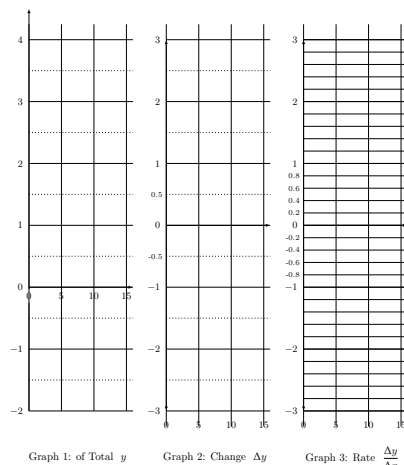


Exploration 8.3 Suppose that $y = -0.05x^2 + 0.8x - 1$. Find the average rate of change between $x = 0$ and 5, between $x = 5$ and 10, and between $x = 10$ and 15.

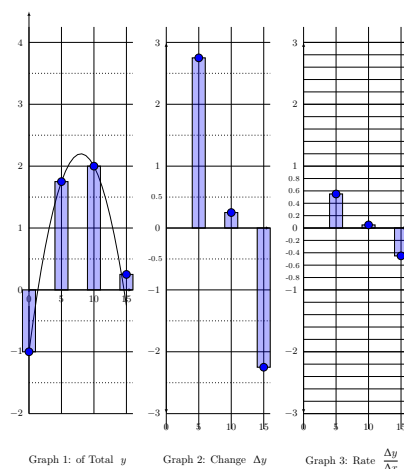
(a) Use an AROC table to find the average rate of change between $x =$

0 and $x = 5$, between $x = 5$ and $x = 10$, and between $x = 10$ and $x = 15$

- (b) Graph histograms representing the total function y , the change function Δy , and of the rate of change function $\frac{\Delta y}{\Delta x}$.



Solution. Here's a graph of the solution:

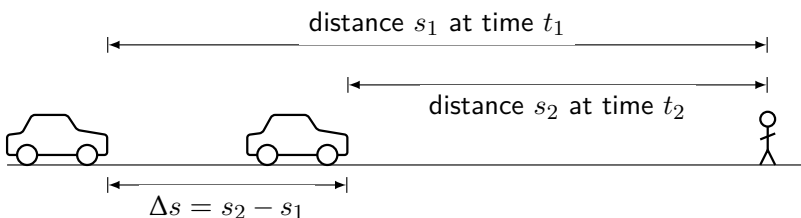


8.3 Velocity and Instantaneous Rates of Change

So far, we have seen many examples of finding and applying *average change*. But we have not yet seen why we would call $\frac{\Delta y}{\Delta x}$ the *average rate of change*. To understand that side of the story, we need to make a small detour into the physical world.

8.3.1 Measuring Speed with Radar

Imagine driving down the highway, coming around a corner at 75 miles per hour. Suddenly, you see a speed trap ahead, measuring your speed with radar. How does this work? The detector actually measures your *distance* at two points in time t_1 and t_2 .



Subtracting the distances s_1, s_2 gives your *change in distance* Δs in miles, and subtracting the times t_1, t_2 gives your *change in time* Δt in hours (or minutes, or seconds).

Your *estimated* velocity will be the **average rate of change** of your position

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$$

Better radar detectors use *smaller* Δt .

- $\Delta t = 1$ hour gives a bad approximation.
- $\Delta t = 0.016$ hours (1 minute) gives a better approximation.
- $\Delta t = 0.000000000001$ hours gives a **very good** approximation.
- By making Δt smaller and smaller, we get closer and closer to your **true speed**. Mathematically, that means

$$\text{true speed} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}$$

To find velocity *numerically*, compute the *average rate* (AROC) over smaller and smaller intervals ($\Delta t \rightarrow 0$) to see where $\frac{\Delta s}{\Delta t}$ ends up going.

Exploration 8.4 Suppose that a position function $s(t)$ corresponds to the AROC table below

t_1	t_2	Δt	s_1	s_2	Δs	$\Delta s / \Delta t$
5	5.1		10	16.98065		
5	5.01		10	10.74421		
5	5.001		10	10.07485		
5	5.0001		10	10.00749		

- (a) Complete the AROC table.

Solution.

t_1	t_2	Δt	s_1	s_2	Δs	$\Delta s / \Delta t$
5	5.1	0.1	10	16.98065	6.98065	69.8965
5	5.01	0.01	10	10.74421	0.74421	74.421
5	5.001	0.001	10	10.07485	0.007485	74.85
5	5.0001	0.0001	10	10.00749	0.00749	74.9

Important: Note that as Δt gets smaller, the decimal accuracy of your average gets *smaller and smaller*. To ensure that you had 4 decimal places of accuracy for $\Delta t = 0.0001$, you would need to start with *eight* decimal places of accuracy.

On the homework, you will need to make sure you leave as many decimal places as possible. Otherwise, your Δy and Δt may be marked correct, but your $\frac{\Delta y}{\Delta t}$ may be marked incorrect.

- (b) Use your table to find the exact (instantaneous) velocity at $t = 5$.

Solution. To find the exact velocity (IROC) at $t = 5$, read the table from top to bottom, since the smaller values of Δt are further down in the table.

As you read the $\frac{\Delta s}{\Delta t}$ column, note that it is getting closer and closer to 75. In other words, at $t = 5$,

$$\frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = 75 \frac{\text{miles}}{\text{hour}}$$

How do *instantaneous* rates of change connect to economics? Take a look at Exploration ??, and note that the AROC in the solution to that exercise is arranged in a very similar way to the table we used to compute the car's velocity at time $t = 5$ in the example above.

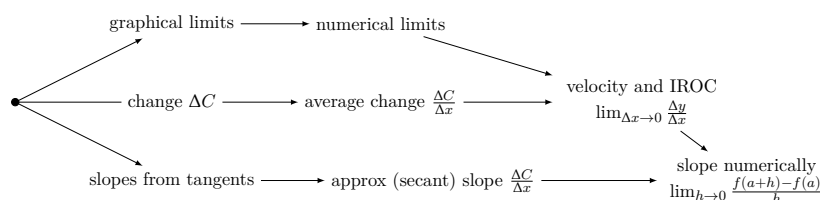
Using a similar argument, we can use the AROC table in Exploration ?? to see that the *instantaneous rate of change* of the total cost function $C(x) = 0.1x^2 + 50x + 1000$ at 100 items is equal to 70 dollars per item.

What does this mean? We will return to this in much greater detail in section Section ??.

9

Understanding Slope

The concepts contained in Chapter ??-?? are all connected in important, but sometimes tricky to follow ways. As you go through this section, you can refer to the diagram below to see how what we are covering here is related to, and also distinct from, what we have already covered.



9.1 Slopes and Rates of Change

In general, the **average rate of change** (AROC) of any function $f(x)$ over the interval $[a, b]$ is given by

$$\left(\frac{\text{change in output}}{\text{change in input}} \right) = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Note: the AROC formula is the *same formula* as the one for the slope of a line. In other words, the concepts of AROC and slope are closely related.

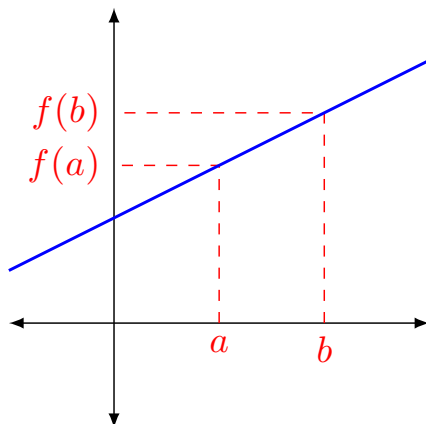
The concept of "slope" is easier to understand for some functions

than for others.

Linear Functions

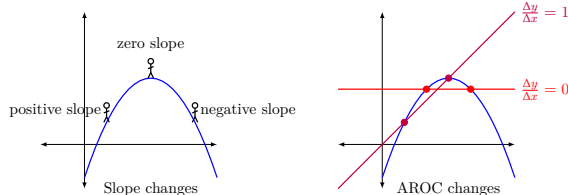
If $f(x)$ is a line, the following are always the same, regardless of the points a, b chosen

1. the slope of f
2. the average rate of change of f on $[a, b]$
3. the instantaneous rate of change of f at a



Quadratic Functions

Quadratic functions do not have a *constant slope*. In other words, if you imagine standing on the curve, the slope of the hill you are standing on will be steeper in some places, and flatter in other places.



As a consequence, quadratic functions do *not* have a constant average rate of change $= \frac{\Delta y}{\Delta x}$.

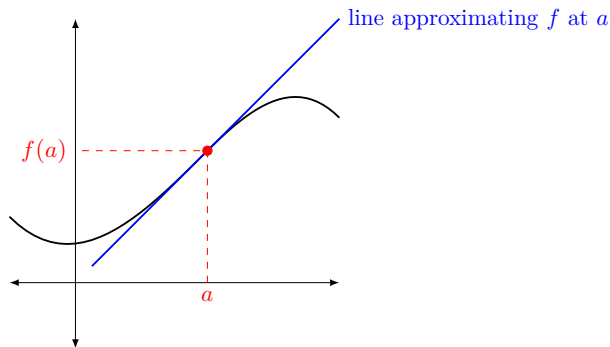
In fact, we have a more general result. A function has a constant slope *if and only if* the function is a line.

9.1.1 Understand Rates by Studying Slopes

Our goal in this class is to develop tools that will help us understand the *economic concept* of a "rate of change" by connecting it to the *graphical concept* of a slope.

Because we only really understand slopes in the context of "lines,"

the first step is to define a *line approximating* the function $f(x)$. We will call this the **tangent line**.



In the image above, the blue line perfectly lines up with $f(x)$ at the point $(a, f(a))$. We call the blue line the *line tangent to $f(x)$ at a* . If we can find the slope of the tangent line, then we will be able to find the slope of the function at a , and visa versa.

Proposition 9.1 Suppose that we know the slope m of the tangent line or, equivalently of f , at the point a . Then the **line tangent to $f(x)$ at a** has the equation

$$y = m \cdot (x - a) + f(a)$$

where m , a , and $f(a)$ are **constant numbers**.

Exploration 9.1 Suppose that you are told that at the point with $a = 5$, that $f(5) = 17$, and that the line tangent to f at 5 has slope -3 . Find an equation for the tangent line to f at $a = 5$.

Solution. We are asked for the equation of a line, so start by writing down General Point Slope Form.

$$y = m \cdot (x - x_1) + y_1$$

First, x_1 will always be the point that we are looking for the tangent to be "at". In this example, we want the tangent at $a = 5$, so we have $x_1 = 5$

Second, y_1 will always be the height of the tangent at x_1 . In other words,

$$y_1 = f(x_1) = f(5) = 17$$

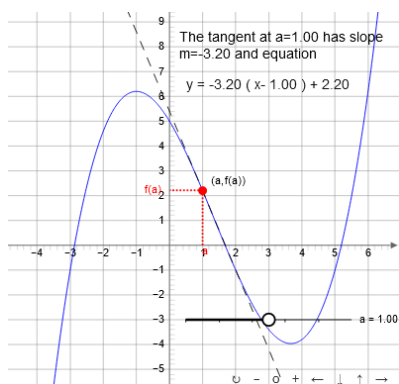
Finally, the slope of the tangent is always equal to the slope of the function. In this example, we are just told that both slopes are equal to

$$m = -3$$

Putting it all together, we get

$$y = -3 \cdot (x - 5) + 17$$

Because most functions have curves in them, looking at different points a will give you tangent lines with very different slopes.



[Standalone](#)
[Embed](#)

Said another way, the *slope of the tangent to $f(x)$* is a function of the point a where you are centering your approximation. For simplicity, we often denote this using the notation

$f'(a)$ = the slope of the line tangent to the function $f(x)$ at a

Connecting Slope and Tangent Lines.

1. Near a point $(a, f(a))$, the function $f(x)$ is **approximately equal to** the *line tangent* to f at that point. Mathematically, this means

$$(\text{height on function } f) \approx (\text{height on tangent line centered at } a)$$

for all values of x near to a

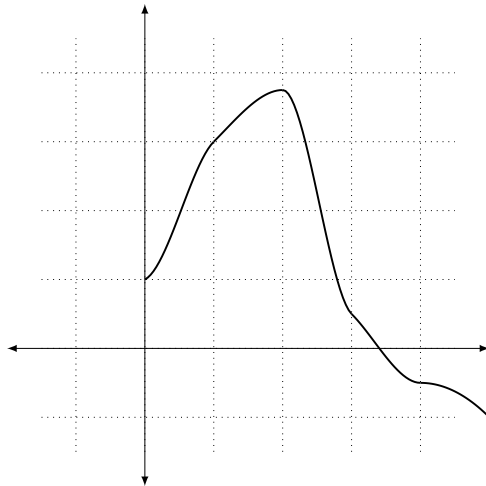
2. It follows that the **slope** of the function f at a is approximately equal to the **slope** of the tangent at the point. In other words,

$$(\text{slope of } y = f(x) \text{ at } x = a) \approx (\text{slope of the *tangent* to } f \text{ at } a)$$

This observation is very powerful, and it works in two directions. For now, we will see how to use tangent lines to help us to *approximate* the slope of a curve $y = f(x)$ at various values of a .

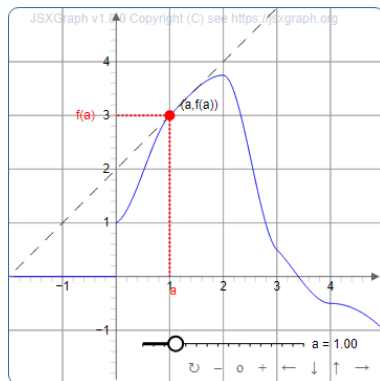
Later, we will see how to compute the slope of $f(x)$ *precisely* at some point a . As a result, we will also have found the slope of the *line tangent* to the function at that value of a .

Exploration 9.2 A function $y = f(x)$ is defined using the graph below.



- (a) Approximate the slope to the function $y = f(x)$ at the point $a = 1$

Solution. We will approximate the slope at a value of a by first drawing the tangent to the curve at that point.



Standalone
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We can then find two points *on the line* and use the slope formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that these points will not be on the original curve, since we are technically finding the slope of the *tangent line*, which we are only using to *approximate* the real curve.

To find the tangent to a curve at a point, take a ruler and "line it up" with the curve at that point. The interactive graphic below shows the line you would get at each point.

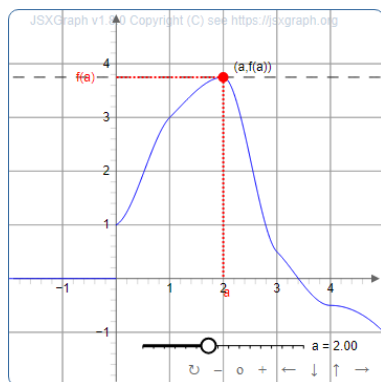
To find the slope of f at $a = 1$, find the tangent to f at $a = 1$. Find two points on this tangent line. One is $(1, 3)$, and another is $(0, 2)$. The slope of the line between these lines is

$$m = \frac{3 - 2}{1 - 0} = \frac{1}{1} = 1$$

Therefore, the slope of $f(x)$ at $a = 1$ is equal to 1.

- (b) Approximate the slope to the function $y = f(x)$ at the point $a = 2$

Solution. We will approximate the slope at a value of a by first drawing the tangent to the curve at that point.



Standalone
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We can then find two points *on the line* and use the slope formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that these points will not be on the original curve, since we are technically finding the slope of the *tangent line*, which we are only using to *approximate* the real curve.

To find the tangent to a curve at a point, take a ruler and "line it up" with the curve at that point. The interactive graphic below shows the line you would get at each point.

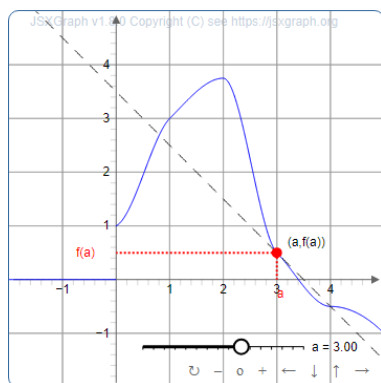
To find the slope of f at $a = 2$, first, draw the line tangent to $f(x)$ at $a = 2$, and find two points on that line. One point is $(2, 3.75)$, and another is $(0, 3.75)$. The slope of the line between these lines is

$$m = \frac{3.75 - 3.75}{2 - 0} = \frac{0}{2} = 0$$

Therefore, the slope of $f(x)$ at $a = 2$ is equal to 0.

- (c) Approximate the slope to the function $y = f(x)$ at the point $a = 3$

Solution. We will approximate the slope at a value of a by first drawing the tangent to the curve at that point.



Standalone
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We can then find two points *on the line* and use the slope formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that these points will not be on the original curve, since we are technically finding the slope of the *tangent line*, which we are only using to *approximate* the real curve.

To find the tangent to a curve at a point, take a ruler and "line it up" with the curve at that point. The interactive graphic below shows the line you would get at each point.

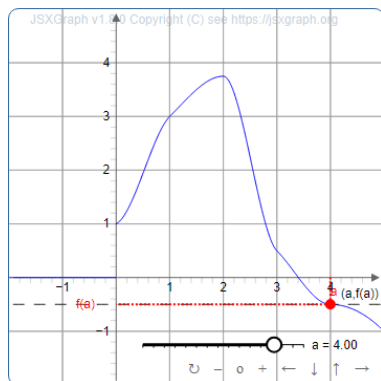
To find the slope of f at $a = 3$, first, draw the line tangent to $f(x)$ at $a = 3$, and find two points on that line. One point is $(3, 0.5)$, and another is $(0, 3.5)$. The slope of the line between these lines is

$$m = \frac{0.5 - 3.5}{3 - 0} = \frac{-3}{3} = -1$$

Therefore, the slope of $f(x)$ at $a = 3$ is equal to -1.

- (d) Approximate the slope to the function $y = f(x)$ at the point $a = 4$

Solution. We will approximate the slope at a value of a by first drawing the tangent to the curve at that point.



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We can then find two points *on the line* and use the slope formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that these points will not be on the original curve, since we are technically finding the slope of the *tangent line*, which we are only using to **approximate** the real curve.

To find the tangent to a curve at a point, take a ruler and "line it up" with the curve at that point. The interactive graphic below shows the line you would get at each point.

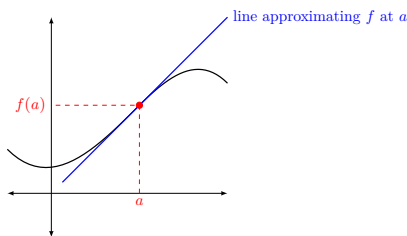
To find the slope of f at $a = 4$, first, draw the line tangent to $f(x)$ at $a = 4$, and find two points on that line. One point is $(4, -0.5)$, and another is $(0, -0.5)$. The slope of the line between these lines is

$$m = \frac{0.5 - (-0.5)}{4 - 0} = \frac{0}{4} = 0$$

Therefore, the slope of $f(x)$ at $a = 4$ is equal to 0.

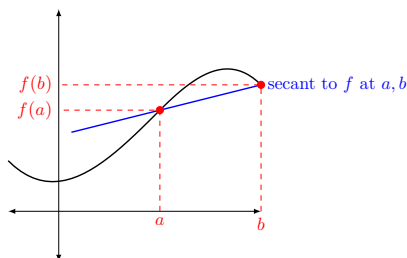
9.2 Approximating Slope with Secant Lines

Given a function $f(x)$, and an x value a , we want to find the slope of f at a . Unfortunately, you need *two points* to define a line, but a tangent line usually only touches the curve at *one point*. To help us define slope precisely, we will need a new concept.



Definition 9.2

The **secant line** to f at a, b is the line that passes through the graph of $y = f(x)$ at $x = a$ and $x = b$.

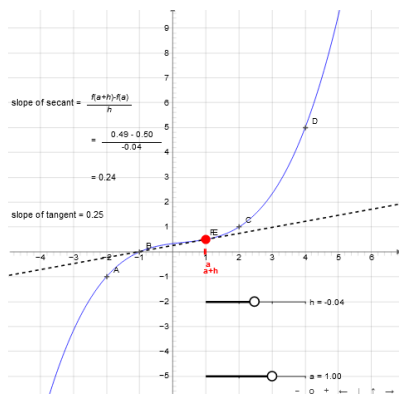


Furthermore, the *slope* of the secant to f at a, b is equal to

$$\frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x}$$

◇

The key idea thing to notice is that when b is *very close* to a , then the secant line is *very close* to the tangent line.



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Exploration 9.3 Let $f(x) = x^2 + 2x$. We want to study the function at $a = 3$.

- (a) Estimate the slope of f at $a = 3$ by computing the slope of the secant between $a = 3$ and a value b that is 0.1 units away, 0.01 units away, and 0.001 units away.

Solution. Compute the average rate of change $\frac{\Delta y}{\Delta x}$ over the intervals $[3, 3.1]$, $[3, 3.01]$, and $[3, 3.001]$. In other words, we have the x values, so we need to find the y values, the changes Δx and Δy , and then divide those changes to get the average change $\frac{\Delta y}{\Delta x}$.

x_1	x_2	Δx	y_1	y_2	Δy
3	3.1	0.1	$f(3) = 3^2 + 2 \cdot 3 = 15$	$f(3.1) = 3.1^2 + 2 \cdot 3.1 = 15.81$	$y_2 - y_1 = 15.81 - 15 = 0.81$
3	3.01	0.01	$f(3) = 3^2 + 2 \cdot 3 = 15$	$f(3.01) = 3.01^2 + 2 \cdot 3.01 = 15.0801$	$y_2 - y_1 = 15.81 - 15 = 0.0801$
3	3.001	0.001	$f(3) = 3^2 + 2 \cdot 3 = 15$	$f(3.001) = 3.001^2 + 2 \cdot 3.001 = 15.008001$	$y_2 - y_1 = 15.81 - 15 = 0.008001$

To interpret this table, note that the top row is an approximation of the slope where the points are reasonably far apart ($\Delta x = 0.1$). In the middle row the points are closer together ($\Delta x = 0.01$), and in the bottom row the points are very close together ($\Delta x = 0.001$).

In other words, as you move from the top to bottom, the approximating line gets closer and closer to the function.

As the approximation improves, the "estimated slope" decreases from 8.1 to 8.01 to 8.001. It looks like the slopes are getting *closer and closer* to a slope of exactly $m = 8$ at $a = 3$.

- (b) Use this slope to find an equation for the line tangent to f at $a = 3$.

Solution. To find the equation of a line, start by writing down the point-slope form

$$y = m(x - x_1) + y_1$$

We are given that the x -value is $x_1 = 3$. Plug this into the equation for the function $f(x) = x^2 - 2x$ to get $y_1 = f(3) = 3^2 - 2 \cdot 3 = 15$.

The only remaining number to find is the slope of the function m . But we have already found above that the slope of the function at $a = 3$ is 8. And the slope of the tangent equals the slope of the function, so we have that $m = 8$.

Putting this together, the slope of the line tangent to f at 3 is

$$y = 8(x - 3) + 15$$

This is the equation for the line that approximates f at the point $a = 3$.

Secant Lines, AROC, and Slope.

- Just as we will use *tangent lines* to approximate the original function, we will *secant lines* to approximate tangent lines.
- The *Average Rate of Change* (AROC) is defined to be $\frac{\Delta y}{\Delta x}$. To compute the average rate of change just compute the height of your function at the given x values, and divide by the difference between those x values.
- We often use a table to help us compute the average rate of change, and call these AROC tables. These are simply a way of organizing the different parts of computing $\frac{\Delta y}{\Delta x}$ for various values of x_1 and x_2 .

x_1	x_2	Δx	y_1	y_2	Δy	$\Delta y / \Delta x$
a	b	$b - a$	$f(a)$	$f(b)$	$f(b) - f(a)$	$\frac{f(b) - f(a)}{b - a}$
c	d	$d - c$	$f(c)$	$f(d)$	$f(d) - f(c)$	$\frac{f(d) - f(c)}{d - c}$
\dots	\dots	\dots	\dots	\dots	\dots	\dots

- Suppose you are *given* a function $f(x)$ and an x -value a . To *approximate the slope of $f(x)$ at a* , create an AROC table as follows:
 - Set $x_1 = a$ in each row. Pick several values for Δx that get smaller and smaller (think 0.1, 0.01, \dots). Put the largest Δx in the top row, and put smaller Δx in lower rows. For each row, set $x_2 = x_1 + \Delta x$
 - Fill in the table, computing $y_1, y_2, \Delta y$, and $\frac{\Delta y}{\Delta x}$.
 - Read the right hand of the table from top to bottom. Where is the average rate $\frac{\Delta y}{\Delta x}$ going as $\Delta x \rightarrow 0$?

The above process for approximating slope is the most accurate way of doing things, but it can be a bit involved. In specific circumstances, you can often *approximate* the slope using a single computation. Instead of repeatedly calculating $\Delta y/\Delta x$ for smaller and smaller values of Δx , you can sometimes pick a single very small value for Δx . For example, picking $\Delta x = 0.001$ gives us

$$\text{slope of } f \text{ at } a \approx \frac{f(a + 0.001) - f(a)}{0.001}$$

That value is only *approximate*, so it might be too large and it might be too small. By using an AROC table, we get a better sense of whether the exact slope is smaller or larger.

9.3 Defining Slope Algebraically

In the previous section, we developed a method that allows us to approximate slope by an iterative process of seeing where *average slope is going* as the change in input Δx *goes to zero*.

Using the language of Chapter ??, we can translate this into a limit definition of slope.

Definition 9.3 Given a function $f(x)$ and a number a . The **slope of f at a** , abbreviated $f'(a)$ is equal to

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

English keyboards do not have a button for Δx , so we can use the letter h to abbreviate the change in the input. This gives us an alternative form:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

◇

To *use* the limit definition,

1. Completely simplify the fraction using the specific function f and number a that you are given.
2. Compute the limit of the simplified function by letting h go to 0.
For the purposes of this course, this means plugging in $h = 0$ as long as this does not result in division by 0.

Exploration 9.4 Let $f(x) = x^2 + 2x$. Use the limit definition to find the slope of f at $a = 3$.

Solution. The limit definition of slope at 3 gives us

$$\text{slope at } 3 = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h}$$

This function includes several components, that we first simplify on their own. The function is defined by the rule $f(x) = x^2 + 2x$. That means that $f(3) = 3^2 + 2 \cdot 3 = \dots = 15$. Similarly, we can simplify the expression as follows:

$$\begin{aligned} f(3+h) &= (3+h)^2 + 2 \cdot (3+h) \\ &= (3+h) \cdot (3+h) + 2 \cdot (3+h) \\ &= 9 + 3h + 3h + h^2 + 6 + 2h \\ f(3+h) &= 15 + 8h + h^2 \end{aligned}$$

We can now plug these simplified expressions back into our limit definition

$$\begin{aligned} \text{slope at } 3 &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(15 + 8h + h^2) - (15)}{h} \end{aligned}$$

It is tempting to try to plug in $h = 0$, but if you do that on your calculator, you end up dividing by 0, which gives a value of *UNDEFINED* or *DNE*. Because of this, we need to keep simplifying as in Section ??:

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{8h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h \cdot (8 + h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(8 + h)}{1} \\ &= \lim_{h \rightarrow 0} (8 + h) \end{aligned}$$

As $h \rightarrow 0$, the expression $8 + h$ goes to $8 + 0 = 8$. In other words, the slope of f at $a = 3$ is exactly $m = 8$.

Our goal in this course is to use mathematics as a tool to deepen our understanding of economics and business concepts. One of the most important insights from the limit definition of slope is the following result.

Corollary 9.4 Suppose that $y = f(x)$ is a function, and that there are units for both the input x and output f . The following are **always equal**

- The slope of f at a
- The instantaneous rate of change (IROC) of f at a

Furthermore, the units of slope and the units of IROC are always

$$\frac{\text{units of } y}{\text{units of } x}$$

Application 9.5 Units and applied slope.

- If y gives the value of a stock in dollars as a function of the number of days x , then you can always find *how fast the stock is changing* by looking at the *slope of the value function*. Furthermore, that slope is measured in *dollars per day*.
- If y gives your milemarker on a highway as a function of the time x in hours, then you can always find *how fast your position is changing* by looking at the *slope of the position function*. Furthermore, that slope is measured in miles per hour.

Exploration 9.5 Suppose that the profit of a certain oil investment is given by $P(x) = -x^2 + 4x - 1$ in millions of \$ as a function of the number of gallons of oil x produced over the lifetime of the investment.

How fast is the profit changing when $x = 1$ million units?

Solution. We are asked for the instantaneous rate of change of profit at $a = 1$. Because IROC is the same thing as slope, we can use the limit definition of slope. In other words, we must compute $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ at $a = 1$ using the formula

$$\lim_{h \rightarrow 0} \frac{P(1+h) - P(1)}{h}$$

As before, we will first simplify the components of this expression separately, and only put them back together later on.

$$P(1) = -1^2 + 4 \cdot 1 - 1 = -1 + 4 - 1 = 2$$

$$\begin{aligned} P(1+h) &= -(1+h)^2 + 4(1+h) - 1 \\ &= -(1+h) \cdot (1+h) + 4(1+h) - 1 \\ &= -(1+2h+h^2) + 4(1+h) - 1 \\ &= -1-2h-h^2+4+4h-1 \\ &= -h^2+2h+2 \end{aligned}$$

Putting this back into the limit definition gives

$$\text{IROC at } 1 = \lim_{h \rightarrow 0} \frac{P(1+h) - P(1)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{(-h^2 + 2h + 2) - (2)}{h} \\
&= \lim_{h \rightarrow 0} \frac{-h^2 + 2h}{h} \\
&= \lim_{h \rightarrow 0} \frac{\cancel{h} \cdot (-h + 2)}{\cancel{h}} \\
&= \lim_{h \rightarrow 0} \frac{(-h + 2)}{1} \\
&= \lim_{h \rightarrow 0} (-h + 2) \\
&= (-0 + 2) = 2
\end{aligned}$$

The rate of change in profit is 2 (million \$) per (million gallons). Simplifying the units, this tells us that the profit is changing by 2 dollars per gallon when $x = 1$ million gallons.

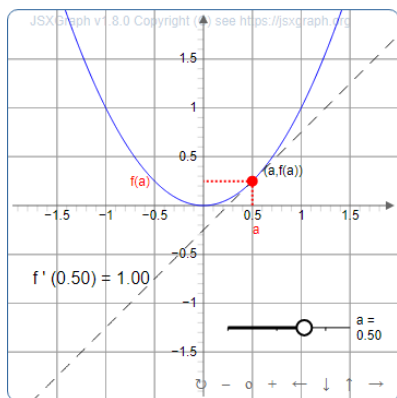
9.4 Defining a Slope Function

So far, we have seen how to understand, approximate, and define the slope of a function at a point a . If you want to look at the slope of the function at a different point, you can redo all of the computations, only with a different number.

As you recompute the slope at different values, you will notice that there is a lot of repeated effort that seems very similar. We can avoid that repeated effort by instead studying **slope functions**.

Definition 9.6 Let $f(x)$ be any function. The **derivative** of f , written $f'(x)$, is the function which, when given any number a , outputs the *slope* of the original function at a . \diamond

Exploration 9.6 The function $f(x) = x^2$ is graphed below.



Standalone
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- (a) Use the graph to approximate $f'(-1)$, $f'(0)$, $f'(1)$.

Solution. To find a slope from a graph, first draw the tangent line at that point. Then, find the slope of that tangent line.

In this interactive graphic, adjusting the slider draws the tangent line for you.

In either case, we see that

$$f'(-1) = -2,$$

$$f'(0) = 0,$$

$$f'(1) = 2$$

- (b) Use the interactive graphic to come up with a *conjecture* (educated guess) for an equation for the *slope function* $f'(x)$.

Solution. Try adjusting the value of x , and look at what the slope ends up equaling.

No matter what value of x you end up selecting, it looks like the slope is exactly twice the x value. This leads to a conjecture that

$$f'(x) = 2x$$

- (c) Use the limit definition of the derivative to find an equation for $f'(x)$

Solution. To apply the limit definition of slope to *any* value of x , we just need to write down the same formula, only using a variable x instead of a specific number a . In other words, we must compute

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

As always, we simplify the corresponding expressions separately.

First, by definition $f(x) = x^2$. This expression cannot be simplified further, so can now move onto the second expression

$$\begin{aligned} f(x+h) &= (x+h)^2 \\ &= (x+h) \cdot (x+h) \\ &= x^2 + 2xh + h^2 \end{aligned}$$

This cannot be simplified further, so we can now plug our expressions back into the limit definition of slope

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2) - (x^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{\textcolor{red}{h}(2x + h)}{\textcolor{red}{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h) \\
 f'(x) &= 2x
 \end{aligned}$$

- (d) Find an equation for the line tangent to $f(x) = x^2$ at 10.

Solution. To find the equation for the tangent line, first write down point slope form of the line

$$y = \textcolor{red}{m}(x - \textcolor{red}{x_1}) + \textcolor{red}{y_1}$$

We are given that $x_1 = 10$. To find y_1 , find plug 10 into the equation for the *original* function $f(x) = x^2$

$$y_1 = f(10) = 10^2 = 100$$

We can't use the graph to find the slope of the function at 10, so we either need to approximate the slope numerically, use the limit definition, *or* use the equation for the slope function $f'(x)$ that we have found above.

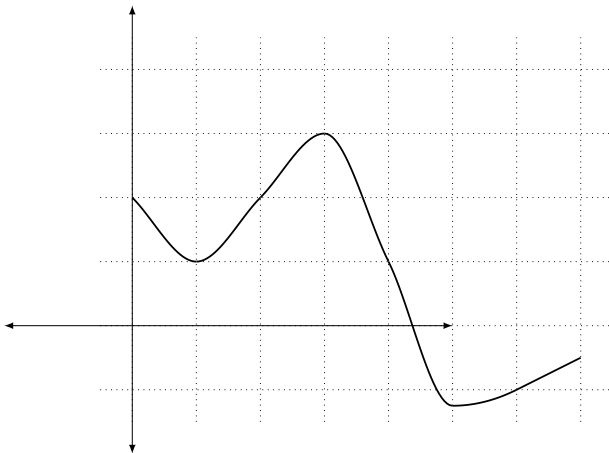
Since we know that the *slope of f at x* is $f'(x) = 2x$, we can find the slope at 10 by plugging 10 into the *slope* or *derivative* function $f'(x) = 2x$

$$m = f'(10) = 2 \cdot 10 = 20$$

Putting all of this into the point slope form, we get that the equation of the tangent line is

$$y = \textcolor{red}{20}(x - \textcolor{red}{10}) + \textcolor{red}{100} = 20x + 80$$

Exploration 9.7 A function $y = f(x)$ is defined using the graph below.

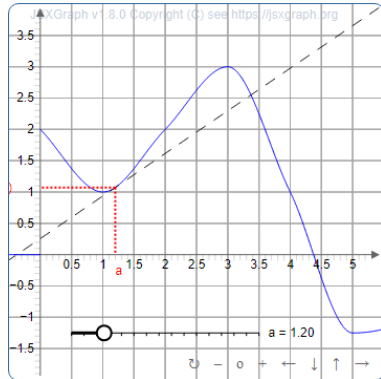


- (a) Compute $f'(0)$, $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$, $f'(5)$, $f'(6)$.

Solution. We will approximate the slope at a value of a by first drawing the tangent to the curve at that point, and using the fact that

$$\text{Slope of } f \text{ at } a = \text{Slope of line tangent to } f \text{ at } a$$

. To find the tangent to a curve at a point, take a ruler and "line it up" with the curve at that point. The interactive graphic below shows the line you would get at each point.



[Standalone](#)
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Using these tangent lines, you can determine that

$$f'(0) = -1$$

$$f'(1) = 0$$

$$f'(2) = 1$$

$$f'(3) = 0$$

$$f'(4) = -2$$

$$f'(5) = 0$$

$$f'(6) = 1/2$$

- (b) Find all points where the tangent is horizontal.

Solution. The tangent is horizontal at x if and only if the slope of the function is equal to 0 at x , if and only if $f'(x) = 0$.

This happens at the values $x = 1, 3, 5$, and at no other x -values.

- (c) Find the intervals where $f(x)$ is increasing.

Solution. The function is increasing if and only if the slope of f at x is positive if and only if $f'(x) > 0$.

Looking at the graph, the slope is positive at two different parts of the graph:

- between $x = 1$ and $x = 3$

- to the right of $x = 5$

In interval notation, f is increasing on the interval $(1, 3) \cup (5, \infty)$

- (d) Find the intervals where $f(x)' < 0$.

Solution. The function is decreasing *if and only if* the slope of f at x is negative *if and only if* $f'(x) < 0$.

Looking at the graph, the slope is negative at two different parts of the graph:

- to the left of $x = 1$
- between $x = 3$ and $x = 5$

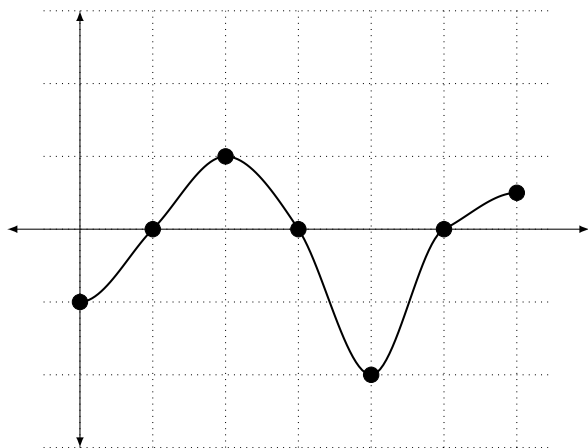
In interval notation, f is decreasing on the interval $(-\infty, 1) \cup (3, 5)$

- (e) Draw a sketch of the slope function $f'(x)$.

Solution. We have already computed the *slope function* f' at x -values 0, 1, 2, 3, 4, 5, 6. Summarizing that in a table, we have

x	0	1	2	3	4	5	6
$f'(x)$	-1	0	1	0	-2	-1.1	1/2

To sketch the graph of the *slope* as a function of x , plot each of these *slopes*, and then connect them with a reasonably smooth curve.



9.4.1 Interpreting the Meaning of $f'(x)$

So far, we have focused on the graphical interpretation of the derivative function by emphasizing that for a graph $y = f(x)$

$$f'(a) = (\text{the slope of } f \text{ at } x_1 = a)$$

But the true power of calculus comes from the *many* applications of derivatives to *other kinds of questions*. We have already seen this briefly in Corollary ??, but it will be helpful to expand on this in greater detail.

Perhaps the simplest way to understand the meaning of f' in different situations is to pay attention to the *units of measurement* for f and f' . For example, for a *height function* $y = f(x)$, the units of f might be meters or feet. On the other hand, for a *profit function* $P(x)$, the units of P might be dollars, or millions of dollars, or yuan.

The units of the input x also play an important part in the meaning of the function. There is a world of difference between moving 6 miles between *hours* $x_1 = 0$ and $x_2 = 1$ (an average velocity of 6 miles per hour) and moving 6 miles between *seconds* $x_1 = 0$ and $x_2 = 1$ (an average velocity of 360 miles per hour).

To help us remember the units of derivatives, it is helpful to consider an alternate notation for the derivative function f' . The key idea here is to realize that Δ is the greek letter D . Since we use the blocky symbol Δ to refer to approximate changes, we can instead use the smooth symbol d to refer to instantaneous changes. This gives us the following notation.

Definition 9.7 Let $y = f(x)$ be any function.

Because $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$, we often write

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx}$$

This notation helps us remember Corollary ??, which says

$$(\text{units of } f') = \frac{\text{units of } f}{\text{units of } x}$$

◇

For example, if $C(x)$ is the *total* (cumulative) cost in \$ of x items, then the *rate of change* of cost (how fast the cost is changing)

$$C'(x) = \frac{dC}{dx} = \frac{\$}{\text{item}}$$

In other words, if you have a graph of total cost, the *item cost* is equal to the *slope of the graph*. We will return to this example in greater detail in Section ??

Interpreting Derivatives.

Height Functions	Suppose that $y = f(x)$ gives the <i>height</i> of a hill in feet as a function of the horizontal position x in feet. Then $f'(x)$ gives the <i>slope</i> of the hill in $\frac{\text{meters}}{\text{meters}}$ (meters-per-meter).
Position Functions	Suppose that $f(t)$ gives the <i>position</i> of a vehicle in miles as a function of the current time t in hours. Then $f'(t)$ gives the <i>velocity</i> of the vehicle in $\frac{\text{miles}}{\text{hour}}$ (miles per hour).
Profit Functions	Suppose that $P(x)$ gives the <i>total profit</i> of a business in dollars as a function of the number of items x being sold. Then $P'(x)$ tells you <i>how fast Profit changes</i> in $\frac{\text{dollars}}{\text{item}}$ (dollars per item) when you increase the number of items being sold.

Exploration 9.8

- (a) Suppose that $y = f(x)$ gives the height of a hill in feet as a function of the horizontal position x in feet. Suppose also that $f(100) = 300$, and that $f'(100) = 2$. Explain what this tells you about f .

Solution. Because $f(100) = 300$, the height of the hill is 300 feet at the location $x = 100$. Because $f'(100) = 2$, the slope of the hill will be $2 \frac{\text{feet}}{\text{feet}}$ at this particular location. In other words, if you are standing above $x = 100$, moving one foot forward will increase your height by about 2 feet.

- (b) Suppose that $f(t)$ gives the position of a car on a highway in miles as a function of the time t in hours. Suppose also that $f(0.5) = 37$, and that $f'(0.5) = 62$. Explain what this tells you about f .

Solution. Because $f(0.5) = 37$, the position of the car is mile marker 37 after $x = 0.5$ hours of travel. Because $f'(0.5) = 62$, the velocity of the car will be $62 \frac{\text{miles}}{\text{hour}}$ at this particular point in time. In other words, assuming you don't change your speed, you would travel an additional 62 miles per additional hour of travel.

- (c) Suppose that $P(x)$ is a function that tells you the profit in dollars as a function of x in items. Suppose also that $P(100) = 30,000$, and that $P'(100) = 2,000$. Explain what this tells you about $P(x)$.

Solution. Because $P(100) = 30,000$, we know that the total profit that results from selling items 0-100 will be \$30,000.

Because $P'(100) = 2,000$, we know that when you are selling 100 items, the profit is changing at $2000 \frac{\text{dollars}}{\text{item}}$. In other words, if you are selling 100 items, and then increase your sales x , then the profit will go up by approximately 2,000 dollars-per-additional-item.

The goal of this course is to understand the application of derivatives, particularly to questions of economics, accounting, and management.

We now have two functions to think about in every problem, the *original* (height) function $f(x)$ and the *derivative* (slope) function $f'(x)$. In the last example, we even found the graph of the derivative function.

The next example makes the relationship between these concepts very concrete, and shows its important connection to economic and business applications.

Application 9.8 Derivative Graphs Are (Related to) Change Graphs. Previously, we have used histograms to understand the connection between graphs of total functions C , change functions ΔC , and average rates of change $\frac{\Delta C}{\Delta x}$

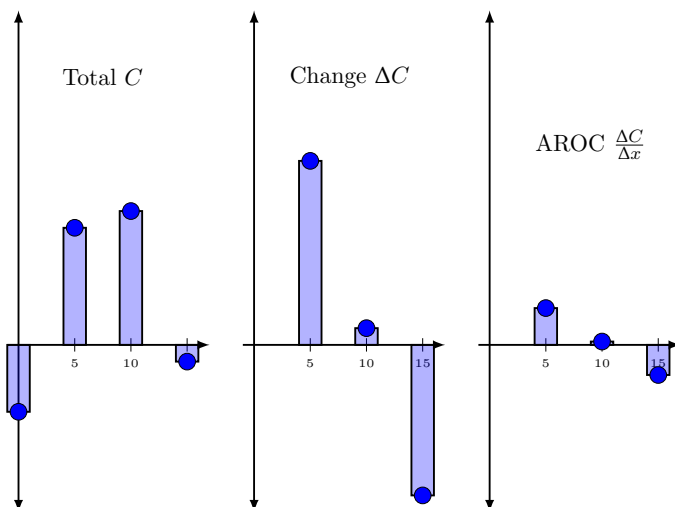
- In Section ??, we carefully studied economic functions such as the total (cumulative) cost function $C(x)$.
- In Section ??, we studied the relationship between the graph of a total C and the change graph ΔC .
- And in Section ?? we seen that the average change graph $\frac{\Delta C}{\Delta x}$ is just a scaled version of the graph of the change ΔC .

In this example, we will see that these concrete applications are very closely related to the continuous total function f and derivatives (slope) functions f' that we have studied in this section.

To make this concrete, we will look at a specific total (cumulative) cost function. Let

$$C(x) = -0.05x^2 + 0.8x - 1.$$

As in previous sections, we can create the graphs of C , ΔC , and $\frac{\Delta C}{\Delta x}$.

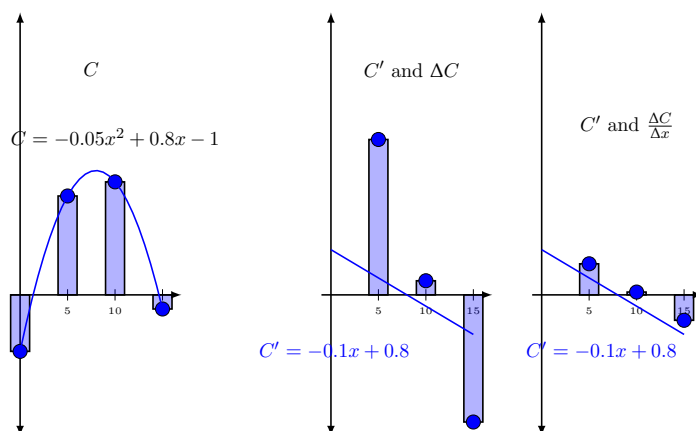


How do the functions $C(x)$ and $C'(x)$ fit into this picture? The graph of the total is a discrete version of $C(x)$.

To find an equation for the derivative (slope) function, you can use the techniques from this section to find that

$$C'(x) = -0.1x + 0.8$$

Now that we have formulas for $y = C(x)$ and its derivative $C'(x)$, let's see how the derivative relates to the graphs of Δy and $\frac{\Delta y}{\Delta x}$. The total $C = -0.05x^2 + 0.8x - 1$ is graphed on the left, and the derivative $C' = -0.1x + 0.8$ is graphed on the same axes as ΔC in the middle, and as $\frac{\Delta C}{\Delta x}$ on the right. What is the relationship between the derivative C' , the change ΔC , and the AROC $\frac{\Delta C}{\Delta x}$?



From the graphs above, it looks like the derivative C' is very close to the average rate of change $\frac{\Delta C}{\Delta x}$, and is a scaled version of the graph of the change ΔC . For that reason, the derivative C' is often abbreviated $C' = \frac{dC}{dx}$. To see why, note that Δ is Greek for letter "D," so the

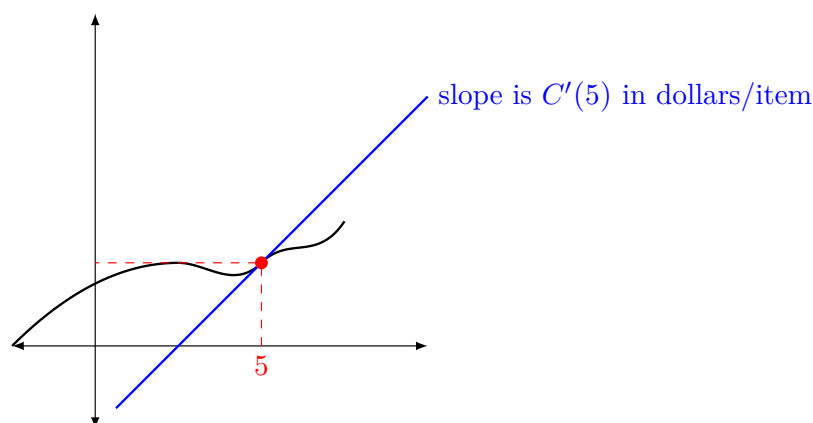
notation $\frac{dC}{dx}$ is intended to help you remember that it's very closely related to $\frac{\Delta C}{\Delta x}$.

Slope Shortcuts for Basic Functions

Before introducing the shortcut rules for finding slope functions, let's review where we have been so far. If $C(x)$ = **total** cost of x items, then C' is the derivative function.

More precisely, $C(5)$ refers to the total cost of making items 1-5 in *dollars*, and $C'(5)$ refers to *how fast* the total cost is changing when the number of items is 5 in *dollars per item*, which is equal to the *slope* of the cost function at $x = 5$.

Graphically,



Algebraically, $C'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

Our goal in this section will be to *find* or *take* the derivative of a given function. We will use the following notation

$$f'(x) = \frac{d}{dx} \left[f(x) \right]$$

as a shorthand for the sentence "Find the derivative of $f(x)$ with respect to x "

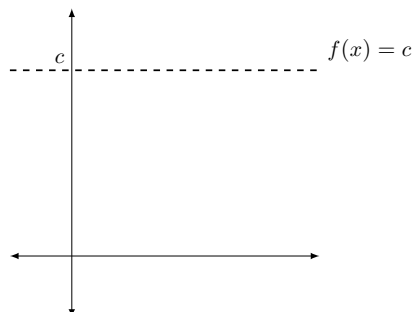
Some students find the $\frac{d}{dx} [\dots]$ notation difficult to read because of the large number of symbols involved. In that case, you can use an alternate notation:

$$f'(x) = \left[f(x) \right]'$$

In this notation, we use square brackets with an apostrophe on the right to say "take the derivative of the function inside the brackets. "

10.1 Basic Derivative Rules

Example 10.1 Slope of a Constant Function. Some functions always output the same number c , regardless of the input x . These functions have the form $f(x) = c$, and their graphs are horizontal lines.



This function is horizontal everywhere, so has a slope of 0 for all x . In other words, we have

$$f'(x) = \frac{d}{dx} [c] = 0$$

which is the same as saying

$$[c]' = 0$$

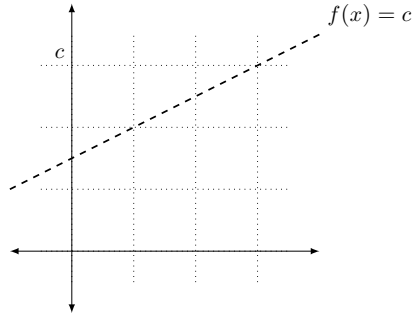
□

Exploration 10.1 Let $f(x) = 3$. Find $f'(x)$.

Solution. $f'(x) = \frac{d}{dx} [3] = 0$

Equivalently, we could use our bracket notation to write $[3]' = 0$

Example 10.2 Slope of a Linear Function. Some functions are lines. These functions have the form $f(x) = mx + b$ for some number m, b .



This function has the same slope everywhere, which is equal to the slope m of the line. In other words, we have

$$f'(x) = \frac{d}{dx}[mx + b] = m$$

which is the same as saying

$$[mx + b]' = m$$

□

Exploration 10.2 Let $f(x) = 3x + 2$. Find $f'(x)$.

Solution. $f'(x) = \frac{d}{dx}[3x + 2] = 3$

Equivalently, we could use our bracket notation to write $[3x + 2]' = 3$

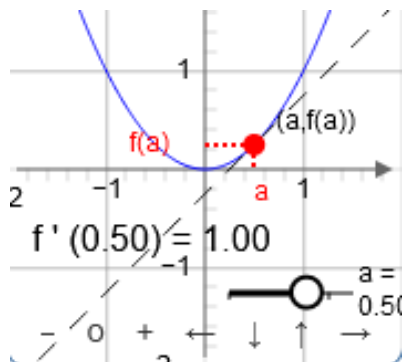
Slopes of Lines and Constant Functions. Suppose that c, m, b are constant numbers. We have the following derivative shortcuts

- $\frac{d}{dx}[c] = 0$
- $\frac{d}{dx}[m \cdot x + b] = m$
- $\frac{d}{dx}[x] = 1$

In bracket notation, this is

- $[c]' = 0$
- $[m \cdot x + b]' = m$
- $[x]' = 1$

More complex functions need derivative *templates* or *shortcut rules*. For example, the function $f(x) = x^2$ has *different slopes* at different values of x .



[Standalone](#)

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The same thing is true for other power functions like x^3 , x^4 , $x^{1/2}$, x^{-5} , and so on.

In Exploration ??, we showed that when $f(x) = x^2$, then $f'(x) = 2x$ using the limit definition of the derivative.

In theory, you could use the limit definition of the derivative to find f' for *every* power function $f(x) = x^m$ for any constant power m . Fortunately, there is a pattern in these derivatives which gives a much easier formula to remember.

The Power Rule. If the exponent m is a constant number,

$$\frac{d}{dx} [x^m] = m \cdot x^{m-1}$$

In bracket notation, this is

$$[x^m]' = m \cdot x^{m-1}$$

Exploration 10.3 Find the slope of Let $f(x) = x^4$. Find the slope of f at $x = 10$, at $x = -10$, and at $x = 0$

Solution. First, we must find the slope function using the power rule

$$f'(x) = \frac{d}{dx} [x^4] = 4 \cdot x^{4-1} = 4x^3$$

In bracket notation, this is

$$f'(x) = [x^4]' = 4x^{4-1} = 4x^3$$

Next, we can use the *slope function* to find the slope at each given x value.

Slope at $x = 10$ $f'(10) = 4 \cdot 10^3 = 4000$

Slope at $x = -10$ $f'(-10) = 4 \cdot (-10)^3 = -4000$

Slope at $x = 0$ $f'(0) = 4 \cdot (0)^3 = 0$

Exploration 10.4 Let $f(x) = \sqrt[4]{x^3}$. Find $f'(x)$

Solution. We always start by *setting up the right notation*

$$f'(x) = \frac{d}{dx} \left[\sqrt[4]{x^3} \right]$$

Now, we need to find a derivative rule to apply. So far, the rule we have are $\frac{d}{dx}[c] = 0$, $\frac{d}{dx}[mx + b] = m$, and $\frac{d}{dx}[x^m] = m \cdot x^{m-1}$. None of those rules have a root in them. Fortunately, we can use our knowledge of algebra to rewrite the *original function* (without taking the derivative) to see that $\sqrt[4]{x^3} = x^{3/4}$. That means we can rewrite the derivative as follows

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[\sqrt[4]{x^3} \right] \\ &= \frac{d}{dx} \left[x^{3/4} \right] \end{aligned}$$

Now we can use the power rule $\frac{d}{dx}[x^m] = m \cdot x^{m-1}$

$$\begin{aligned} &= \frac{3}{4} \cdot x^{3/4-1} \\ &= \frac{3}{4} \cdot x^{-1/4} \end{aligned}$$

This is an acceptable answer, but you can also rewrite if you like

$$= \frac{3}{4} \cdot \frac{1}{\sqrt[4]{x}}$$

Square roots are particularly common, so it may be helpful to remember a specific derivative rule for square roots.

Corollary 10.3 Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2}x^{-1/2}$

Proof. Recall from *algebra* that roots correspond to fractional powers.

In particular $\sqrt{x} = x^{1/2}$. Using this,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[x^{1/2} \right] \\ &= \frac{1}{2} x^{1/2-1} \\ &= \frac{1}{2} x^{-1/2} \end{aligned}$$

■

After enough practice, you should be able to use the derivative shortcut rules in this section off the top of your head.

But we will eventually learn much more complicated derivative rules, and need to apply them to more complicated functions. To help build the skills you will need to do this, it is important to *learn how to use the shortcuts using $\frac{d}{dx}[\dots]$ or $[\dots]'$ notation.*

Recall from section Section ?? that can break more complex functions into combinations of simpler functions. Mathematically, we say that we build functions *recursively* (layer by layer) from the inside out.

As a simple example, $f(x) = 3x^2$ is built by taking x^2 , and then tripling it. In other words, $f(x) = \text{triple}(x^2)$.

Our derivative rules will allow us to compute derivatives from the outside in (removing one layer at a time).

Derivatives and Constant Coefficients. If c is a constant number, and $f(x)$ is any differentiable function,

$$\frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} [f(x)]$$

In bracket notation, this is

$$[c \cdot f(x)]' = c \cdot [f(x)]'$$

Exploration 10.5 Let $f(x) = 3x^2$. Find $\frac{df}{dx}$

Solution. Recall that $\frac{df}{dx}$ is just another way of saying $f'(x)$. So we can start by writing

$$f'(x) = \frac{d}{dx} [3x^2]$$

Now, we must use one of the four rules we have so far: $\frac{d}{dx} [c] = 0$, $\frac{d}{dx} [mx + b] = m$, $\frac{d}{dx} [x^m] = m \cdot x^{m-1}$, and $\frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} [f(x)]$.

The only rule that can handle the coefficient of 3 is the last one: $\frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} [f(x)]$. That means we must interpret the problem as follows, first using the coefficient rule, and then using the power rule`

$$\begin{aligned} f'(x) &= \frac{d}{dx} [\textcolor{red}{3} \cdot \textcolor{blue}{x}^2] \\ &= \textcolor{red}{3} \cdot \frac{d}{dx} [\textcolor{blue}{x}^2] \\ &= 3 \cdot 2 \cdot x^{2-1} \\ &= 6 \cdot x^1 \\ &= 6 \cdot x \end{aligned}$$

In bracket notation, the computation is exactly the same

$$[3x^2]' = 3[x^2]' = 3 \cdot 2x^1 = 6x$$

In words, the constant coefficient rule says that you can *pull constant coefficients through the derivative*.

Exploration 10.6 Let $f(x) = \frac{-2}{x}$. Find $f'(x)$.

Solution. The first step is *always* to set up the problem in $\frac{d}{dx}[\dots]$ or $[\dots]'$.

$$f'(x) = \frac{d}{dx} \left[\frac{-2}{x} \right]$$

Next, we look at our list of derivative rules $\frac{d}{dx}[c] = 0$, $\frac{d}{dx}[mx + b] = m$, $\frac{d}{dx}[x^m] = mx^{m-1}$, and $\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]$.

Unfortunately, none of these apply as written. That means that you need to **rewrite** the original function *before* we can apply any rules.

Here, we can use the basic property of algebra that $\frac{1}{x} = x^{-1}$

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[\frac{-2}{x} \right] \\ &= \frac{d}{dx} [-2 \cdot x^{-1}] \end{aligned}$$

Now we can apply the derivative rule $\frac{d}{dx}[c]f(x) = c \frac{d}{dx}[f(x)]$

$$\begin{aligned} &= \frac{d}{dx} [-2 \cdot x^{-1}] \\ &= -2 \cdot \frac{d}{dx} [x^{-1}] \end{aligned}$$

Now we can use the power rule $\frac{d}{dx}[x^m] = m \cdot x^{m-1}$

$$\begin{aligned} &= -2 \cdot \frac{d}{dx} [x^{-1}] \\ &= -2 \cdot -1 x^{-1-1} \\ &= -2 \cdot -1 x^{-2} \\ &= 2x^{-2} = 2 \frac{1}{x^2} = \frac{2}{x^2} \end{aligned}$$

The key idea of each of the computation above is that derivative rules attack functions from the outside in.

Derivatives of Sums and Differences. Suppose that $f(x)$ and $g(x)$ are differentiable functions. Then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

and

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

In bracket notation, this is written $[f(x) + g(x)]' = [f(x)]' + [g(x)]'$

and $[f(x) - g(x)]' = [f(x)]' - [g(x)]'$

Exploration 10.7 Let $f(x) = 15x^4 + 2x^{-1} + 13$. Find $f'(x)$, and find the slope of f at $x = 2$

Solution. As always, we need to set up

$$f'(x) = \frac{d}{dx} [15x^4 + 2x^{-1} + 13]$$

and we look at our list of derivative rules $\frac{d}{dx} [c] = 0$, $\frac{d}{dx} [mx + b] = m$, $\frac{d}{dx} [x^m] = mx^{m-1}$, $\frac{d}{dx} \left[\frac{c}{f}(x) \right] = c \frac{d}{dx} \left[\frac{1}{f}(x) \right]$, and $\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)]$.

How do we decide which rule we can use? We use the order of operations to figure out the "outermost" or "weakest" operation. Following the acronym PEMDAS, subtraction and addition are the *weakest* operations, so we can think of the expression as

$$f'(x) = \frac{d}{dx} [\textcolor{red}{15x^4} + \textcolor{red}{2x^{-1}} + \textcolor{red}{13}]$$

Now we have the derivative of a sum, which is equal to the sum of the derivatives

$$f'(x) = \frac{d}{dx} [\textcolor{red}{15x^4}] + \frac{d}{dx} [\textcolor{red}{2x^{-1}}] + \frac{d}{dx} [\textcolor{red}{13}]$$

Using the other, more familiar rules we can now compute

$$\begin{aligned} &= 15 \frac{d}{dx} [x^4] + 2 \frac{d}{dx} [x^{-1}] + \frac{d}{dx} [13] \\ &= 15 \cdot 4x^3 + 2 \cdot -1x^{-2} + 0 \\ &= 60x^3 - 2x^{-2} = 60x^3 - \frac{2}{x^2} \end{aligned}$$

To find the slope at $x = 2$, just plug 2 into the formula for $f'(x)$.

$$f'(2) = 60 \cdot 2^3 - 2 \cdot \frac{1}{2^2} = 480 - 0.5 = 479.5$$

Basic Derivative Rules.

Suppose that f, g are differentiable functions of x , that m, c are constant numbers. Then

- $\frac{d}{dx} [c] = 0$
- $\frac{d}{dx} [m \cdot x + b] = m$ and $\frac{d}{dx} [x] = 1$
- $\frac{d}{dx} [x^m] = m \cdot x^{m-1}$
- $\frac{d}{dx} [\sqrt{x}] = \frac{1}{2} \cdot x^{-1/2}$

- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Danger! The derivative rules for \times and \div are *much* more complicated. We will address those rules in a later section.

In bracket notation, these rules are written

- $[c]' = 0$
- $[m \cdot x + b]' = m$ and $[x]' = 1$
- $[x^m]' = m \cdot x^{m-1}$
- $[\sqrt{x}]' = \frac{1}{2} \cdot x^{-1/2}$
- $[c \cdot f(x)]' = c \cdot [f(x)]'$
- $[f(x) \pm g(x)]' = [f(x)]' \pm [g(x)]'$

Exploration 10.8 Suppose that the total profit in the scenario where you sell x items is equal to $P(x) = -0.1x^2 + 30x - 10$. Find how fast the profit is changing when $x = 100$ items, and when $x = 200$ items.

Solution. We will study many applications of derivatives, beginning in the next section. For now, recall that if $y = P(x)$ gives the *total profit* in \$ in the scenario when you sell x items, then the *slope of profit* $P'(x)$ is in \$/item, and tells you approximately how much a single additional item will change the total profit.

The first step is to find the slope function by writing

$$\begin{aligned}
 P'(x) &= \frac{d}{dx}[-0.1x^2 + 30x - 10] \\
 &= \frac{d}{dx}[-0.1x^2 + 30x - 10] \\
 &= \frac{d}{dx}[-0.1x^2] + \frac{d}{dx}[30x] - \frac{d}{dx}[10] \\
 &= \frac{d}{dx}[-0.1x^2] + \frac{d}{dx}[30x] - \frac{d}{dx}[10] \\
 &= -0.1 \frac{d}{dx}[x^2] + 30 \frac{d}{dx}[x] - \frac{d}{dx}[10] \\
 &= -0.1 \frac{d}{dx}[x^2] + 30 \frac{d}{dx}[x] - \frac{d}{dx}[10] \\
 &= -0.1 \cdot 2x^{2-1} + 30 \cdot 1 - \frac{d}{dx}[10] \\
 &= -0.1 \cdot 2x^1 + 30 \cdot 1 - 0
 \end{aligned}$$

$$= -0.2x + 30$$

In the scenario where you are selling $x = 100$ items, we have

$$P'(100) = -0.2(100) + 30 = 10$$

in \$/item. In other words, selling an additional item (going from 100 to 101 items) will *increase* total profit by approximately 10\$ for that single item.

In the scenario where you are selling $x = 200$ items, we have

$$P'(200) = -0.2(200) + 30 = -10$$

in \$/item. In other words, selling an additional item (going from 200 to 201 items) will *decrease* total profit by approximately 10\$ for that single item.

We will frequently have questions that combine *several* of the strategies and concepts that we have learned in this course. These are often called *synthesis problems*.

Finding the equation for tangent lines is a classic kind of synthesis problem.

Exploration 10.9 Let $f(x) = 3x^2 - 2x + 4$. Find the equation for the line tangent to $f(x)$ at $x = 1$.

Solution. We want to find the equation of a *line*. That means we will need to use point slope form

$$y = m(x - x_1) + y_1$$

Furthermore, we want the line to pass through our function at the point $x = 1$, *and* we want the line to have the same slope as our function at this point.

To make sure the line passes through our curve, we set

$$x_1 = 1$$

and

$$y_1 = f(1) = 3 \cdot 1^2 - 2 \cdot 1 + 4 = 5$$

To make sure the slope of this line matches the slope of the curve, we want to set

$$m = f'(1)$$

That means to find m , we first need an equation for $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} [3x^2 - 2x + 4] \\ &= \frac{d}{dx} [3x^2] - \frac{d}{dx} [2x] + \frac{d}{dx} [4] \end{aligned}$$

$$\begin{aligned}
&= 3 \frac{d}{dx} [x^2] - 2 \frac{d}{dx} [x] + \frac{d}{dx} [4] \\
&= 3 \cdot 2x - 2 \cdot 1 + 0 \\
&= 6x - 2
\end{aligned}$$

That means that

$$m = f'(1) = 6 \cdot 1 - 2 = 4$$

Putting this all together, the equation for the line tangent to f at $x = 1$ is

$$y = 4(x - 1) + 5$$

If you wanted, you could simplify this to get

$$y = 4x + 1$$

Exploration 10.10 Suppose that under current market conditions, your revenue is going up at 5\$/unit and that your costs are going up at 12 \$/unit. How fast is your cost changing?

Solution. The key concept here is to remember that

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

Mathematically, this is

$$P(x) = R(x) - C(x)$$

To understand *how fast profit is changing*, we must take the *derivative* of profit

$$P'(x) = \frac{d}{dx} [R(x) - C(x)]$$

Using our derivative rules,

$$\begin{aligned}
P'(x) &= \frac{d}{dx} [R(x)] - \frac{d}{dx} [C(x)] \\
&= R'(x) - C'(x)
\end{aligned}$$

In the text of the problem, we are told the values of R' and C' that "under current market conditions". In other words, we are told $R'(x) = 5$ and $C'(x) = 12$ for whatever the "current" value of x is. That means we can plug this information into our formula for P'

$$P'(x) = 5 - 12 = -7$$

Under current market conditions (for the current value of x), profit is decreasing by 7 \$/item.

Exploration 10.11

- (a) Suppose that $f(x) = 17x - 3$. Find $f'(x)$

Solution. There are two ways to find this derivative. First, note that $f(x) = 17x - 3 = mx + b$ where $m = 17$ and $b = -3$. Using the shortcut $[mx + b]' = m$, we get

$$[mx + b]' = 17$$

You can also find this derivative using the formula for derivatives of sums and differences, for constant coefficients $[m \cdot f(x)]' = m \cdot [f(x)]'$ and the formulas $[x]' = 1$ and $[m]' = 0$ for any number m .

$$\begin{aligned} [17x - 3] &= [17x]' - [3]' && \text{sum rule} \\ &= 17[x]' - [3]' && \text{coefficient rule} \\ &= 17 \cdot 1 - 0 && \text{basic rules} \\ &= 17 \end{aligned}$$

(b) Let $f(x) = 6x^5$. Find $f'(x)$.

Solution.

$$\begin{aligned} [6x^5]' &= 6 \cdot [x^5]' && \text{coefficient rule} \\ &= 6 \cdot 5x^4 && \text{power rule} \\ &= 30x^4 \end{aligned}$$

(c) Let $f(x) = 5 - 2x^7 + 3x$. Find $f'(x)$

Solution.

$$\begin{aligned} [5 - 2x^5 + 3x]' &= [5]' - [2x^7]' + [3x]' && \text{sum rule} \\ &= [5]' - 2 \cdot [x^7]' + 3 \cdot [x]' && \text{coefficient rule} \\ &= [5]' - 2 \cdot 7x^6 + 3 \cdot [x]' && \text{power rule} \\ &= 0 - 14x^6 + 3 \cdot 1 && \text{basic rules} \\ &= -14x^6 + 3 \end{aligned}$$

(d) Let $f(x) = \frac{3}{x} + \frac{5}{x^2}$. Find $f'(x)$

Solution. Recall that $\frac{3}{x} = 3\frac{1}{x} = 3 \cdot x^{-1}$ and that $\frac{5}{x^2} = 5 \cdot \frac{1}{x^2} = 5 \cdot x^{-2}$. That means we are being asked to find

$$\begin{aligned} [3x^{-1} + 5x^{-2}]' &= [3x^{-1}]' + [5x^{-2}]' \\ &= 3[x^{-1}]' + 5[x^{-2}]' \\ &= 3(-1)x^{-1-1} + 5(-2)x^{-2-1} \\ &= -3x^{-2} - 10x^{-3} \\ &= -\frac{3}{x^2} - \frac{10}{x^3} \end{aligned}$$

- (e) Let $f(x) = 3\sqrt{x} - 2\sqrt[4]{x}$. Find $f'(x)$

Solution. Recall that $\sqrt{x} = x^{1/2}$ and that $\sqrt[4]{x} = x^{1/4}$. That means that we are asked to compute

$$\begin{aligned} [3x^{1/2} - 2x^{1/4}]' &= [3x^{0.5} - 2x^{0.25}]' \\ &= [3x^{0.5}]' - [2x^{0.25}]' \\ &= 3[x^{0.5}]' - 2[x^{0.25}]' \\ &= 3 \cdot 0.5x^{0.5-1} - 2 \cdot 0.25x^{0.25-1} \\ &= \\ &= 1.5x^{-1/2} - 0.5x^{-3/4} \\ &= 1.5 \frac{1}{\sqrt{x}} - 0.5 \frac{1}{\sqrt[4]{x^3}} \end{aligned}$$

In most problems, you get to choose how to write your answer. For example, you don't need to do all of the simplifications above. They are shown here to remind you of some algebra that you might need to do in other problems.

A perfectly acceptable answer to this question would be something like

$$f'(x) = 1.5x^{-0.5} - 0.5x^{-0.75}$$

- (f) Let $f(x) = \frac{5}{\sqrt[3]{x}}$. Find the equation for the line tangent to $f(x)$ at $x = 1$.

Solution. Recall that the equation of the line tangent to $f(x)$ at $x = 1$ will have the form

$$y = m(x - x_1) + y_1$$

where $x_1 = 1$, $y_1 = f(1)$, and $m = f'(1)$. We can immediately compute

$$y_1 = f(1) = \frac{5}{\sqrt[3]{1}} = 5$$

To find m , we first need an equation for $f'(x)$.

$$\begin{aligned} f'(x) &= \left[\frac{5}{\sqrt[3]{x}} \right]' \\ &= [5x^{-1/3}]' \\ &= [5x^{-0.333...}]' \\ &= 5[x^{-0.333...}]' \\ &= 5 \cdot (-0.333)x^{-0.333-1} \\ &= 5 \cdot (-0.333)x^{-1.333} \end{aligned}$$

Using this, we can find that

$$m = f'(1) = 5 \cdot (-0.333) \cdot 1^{-1.333} = -1.666 \dots$$

Putting it all together, we get that the equation for the tangent line to $y = f(x)$ at $x = 1$ is

$$y = -1.666(x - 1) + 5$$

Tips for Taking Derivatives.

1. Always start by writing down

$$f'(x) = \frac{d}{dx} [\dots],$$

filling in the original function you are given.

2. Write down the derivative rules every time you use them. To identify which rules you are *allowed*, think about Order of Operations (PEMDAS). You will be able to use the rule that goes along with the *weakest* operation that you see in your expression.

3. If you cannot use any derivative rule with the function as written, use algebra to rewrite the *original* function.

- For example, rewriting $\frac{5}{x^2} = 5\frac{1}{x^2} = 5x^{-2}$ can help you find the derivative of

$$\frac{d}{dx} \left[\frac{5}{x^2} \right] = \frac{d}{dx} [5x^{-2}]$$

- Rewriting $4 \cdot \sqrt[3]{x^2} = 4 \cdot (x^{2/3})$ can help you find the derivative of

$$\frac{d}{dx} [4 \cdot \sqrt[3]{x^2}] = \frac{d}{dx} [4 \cdot (x^{2/3})]$$

- Rewriting $\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$ can help you find the derivative of

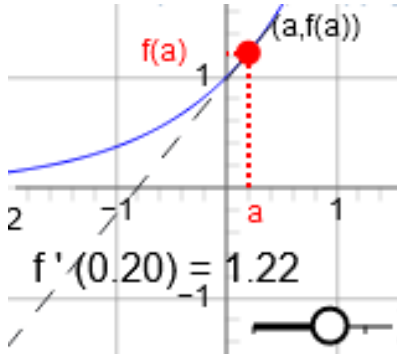
$$\frac{d}{dx} \left[\frac{1}{\sqrt{x}} \right] = \frac{d}{dx} [x^{-1/2}]$$

10.2 Derivatives of Exponentials and Logarithms

In a [previous section](#), we have already noted that the number $e \approx 2.714\dots$ will play an important role in calculus.

More precisely, we said that $f(x) = e^x = \exp(x) = (2.714\dots)^x$ is the *natural* exponential for *calculus*, so we will call it the **natural exponential** function.

Just like polynomials, the *slope* of the natural exponential function is different at different values of x



Standalone
Embed

Is there a nice *formula* that gives us the slope as a function of x ? Surprisingly, the answer is yes.

The Derivative of e^x .

$$\frac{d}{dx}[e^x] = e^x$$

In bracket notation, this is

$$[e^x]' = e^x$$

In other words, the *slope* of any point on the curve is equal to the *height* of that point on the curve.

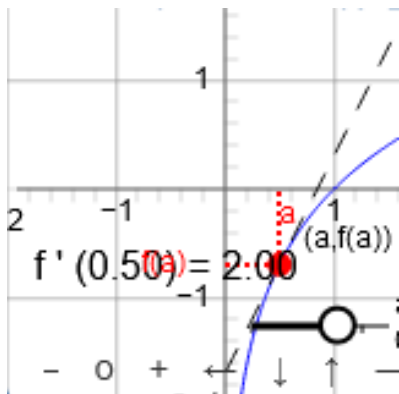
Exploration 10.12 Let $f(x) = 6e^x - 2$. Find $f'(x)$.

Solution.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[6e^x - 2] \\ &= \frac{d}{dx}[6e^x] - \frac{d}{dx}[2] \\ &= 6 \frac{d}{dx}[e^x] - 0 \\ &= 6e^x \end{aligned}$$

Note that the only new thing in this computation was the last step, where we used our new formula that we can replace $\frac{d}{dx}[e^x]$ with e^x

Recall also that $\ln(x)$, called the **natural logarithm** is the inverse of e^x . This function also has different slopes at different values of x



Standalone

Embed

Is there a nice *formula* that gives us the slope as a function of x ? Surprisingly, the answer is *also* yes.

The Derivative of $\ln(x)$.

$$\frac{d}{dx} [\ln(x)] = \frac{1}{x}$$

In bracket notation, this is

$$[\ln(x)]' = \frac{1}{x}$$

Exploration 10.13 Let $f(x) = 3 \ln(x) - x^{1/3}$. Find $f'(x)$.

Solution.

$$\begin{aligned} f'(x) &= \frac{d}{dx} [3 \ln(x) - x^{1/3}] \\ &= \frac{d}{dx} [3 \ln(x)] - \frac{d}{dx} [x^{1/3}] \\ &= 3 \frac{d}{dx} [\ln(x)] - \frac{1}{3} \cdot x^{\frac{1}{3}-1} \\ &= 3 \cdot \frac{1}{x} - \frac{1}{3} \cdot x^{-2/3} \end{aligned}$$

Derivatives of Exponentials and Logarithms.

- $\frac{d}{dx} [e^x] = e^x$
- $\frac{d}{dx} [\ln(x)] = \frac{1}{x}$

Exploration 10.14

(a) Let $f(x) = 12e^x - 2$. Find $f'(x)$.

Solution.

$$\begin{aligned} [12e^x - 2]' &= [12e^x]' - [2]' \\ &= 12[e^x]' - 0 \end{aligned}$$

$$12e^x$$

- (b) Let $f(x) = 7\ln(x) + 5$. Find $f'(x)$

Solution.

$$\begin{aligned} [7\ln(x) + 5]' &= [7\ln(x)]' + [5]' \\ &= 7[\ln(x)]' + 0 \\ &= 7 \cdot \frac{1}{x} \\ &= \frac{7}{x} \end{aligned}$$

- (c) Let $f(x) = 3x^8 + 2e^x - 4\ln(x)$. Find $f'(x)$.

Solution.

$$\begin{aligned} [3x^8 + 2e^x - 4\ln(x)]' &= [3x^8]' + [2e^x]' - [4\ln(x)]' \\ &= 3[x^8]' + 2[e^x]' - 4[\ln(x)]' \\ &= 3 \cdot 8x^7 + 2e^x - 4 \cdot \frac{1}{x} \\ &= 24x^7 + 2e^x - \frac{4}{x} \end{aligned}$$

- (d) Let $f(x) = e^x + x^e$. Find $f'(x)$.

Solution. This problem requires *very careful reading*. The key is to remember that our *only variable is x* . The other characters e and 2 both refer to constant numbers. This is where $\frac{d}{dx}$ notation shines, since it reminds us that the variable is x .

$$\begin{aligned} f'(x) &= \frac{d}{dx} [e^x + x^e] \\ &= \frac{d}{dx} [e^x] + \frac{d}{dx} [x^e] \end{aligned}$$

Look very carefully about where the variable is in each term. In the first term, x is in the exponent, so use the exponential rule

$$\frac{d}{dx} [e^x] = e^x$$

In the second term, x is in the base, so use the power rule with $m = e = 2.718\dots$

$$\frac{d}{dx} [x^m] = mx^{m-1}$$

Putting this all together gives the following

$$\begin{aligned} f'(x) &= \frac{d}{dx} [e^x + x^e] \\ &= \frac{d}{dx} [e^x] + \frac{d}{dx} [x^e] \\ &= e^x + ex^{e-1} \end{aligned}$$

(e) Let $f(x) = e^2 + 2^e$. Find $f'(x)$.

Solution. This problem requires *very careful reading*. The key is to remember that our *only variable is x* . The other characters e and 2 both refer to constant numbers. This is where $\frac{d}{dx}$ notation shines, since it reminds us that the variable is x .

$$f'(x) = \frac{d}{dx} [e^2 + 2^e]$$

Note that there are no x 's in either term. That means we can type them into our calculator and replace them with numbers:
 $e^2 = 7.389\dots$ and $2^e = 6.580\dots$. Putting this in and using our derivative rules gives the following

$$\begin{aligned} f'(x) &= \frac{d}{dx} [7.389 + 6.580] \\ &= \frac{d}{dx} [7.389] + \frac{d}{dx} [6.580] \\ &= 0 + 0 \end{aligned}$$

11

Derivatives and Business

11.1 *Marginal Cost*

Derivatives provide a powerful tool for studying many concepts in economics and business. In this section, we will introduce a central application in the context of cost, revenue, and profit functions. Recall that

- $C(x)$ = *total* cost in \$. For example, $C(10)$ is the *cumulative* cost of items 1-10.
- $R(x)$ = *total* revenue in \$. For example, $R(10)$ is the *total* revenue of items 1-10.
- $P(x)$ = *total* profit in \$. For example, $P(10)$ is the *total* revenue of items 1-10.

Definition 11.1 Let $C(x)$ be a total cost function. The **marginal cost function** is the derivative of total cost. Furthermore,

$$C'(x) = \text{the APPROXIMATE cost of the NEXT item in \$/item}$$

◇

Exploration 11.1 Suppose that $C'(200) = 5$ \$/item. Estimate the cost of the 201st item *only*.

Solution. By Definition ??, $C'(200)$ = the cost of the next item only.

The next item after 200 is the *201st* item. That means that The approximate cost of the 201st item is 5\$.

Exploration 11.2 Suppose that $C(x) = 120 + 5x - 0.1x^2$

(a) What is the *approximate* cost of the 11th item only?

Solution. By Definition ??, when we are asked to *approximate* the cost of the next item, we must find and use the marginal cost function.

To find the marginal cost function, use the derivative shortcuts

$$\begin{aligned}
 C'(x) &= \frac{d}{dx} [120 + 5x - 0.1x^2] \\
 &= \frac{d}{dx} [120] + \frac{d}{dx} [5x] - \frac{d}{dx} [0.1x^2] \\
 &= 0 + 5 - 0.1 \cdot \frac{d}{dx} [x^2] \\
 &= 5 - 0.1 \cdot 2x \\
 &= 5 - 0.2x
 \end{aligned}$$

To use the marginal cost function, recall that

$$C'(x) = \text{the APPROXIMATE cost of the NEXT item in \$/item}$$

We want to find the approximate cost of the 11th item. Item 11 is the item that is *immediately after item 10*.

In other words, compute $C'(10)$. That will tell us the approximate cost of the item *after* 10, which is the 11th item.

$$C'(10) = 5 - 0.2 \cdot 10 = 3$$

The 11th item will cost approximately 3\$.

(b) What is the *exact* cost of the 11th item only?

Solution. By definition, the total cost function $C(x)$ gives the *cumulative* cost of items 1 through x .

In other words, $C(11)$ tells you the *cumulative* cost of items 1-11, and $C(10)$ tells you the *cumulative* cost of items 1-10.

$$\begin{array}{r}
 C(11) = \text{total cost of items 1-11} \\
 - \left(\begin{array}{l} C(10) = \text{total cost of items 1-10} \end{array} \right) \\
 \hline
 \text{cost of 11th item only}
 \end{array}$$

In other words, the cost of the 11th item only is equal to

$$\begin{aligned}
 &= C(11) - C(10) \\
 &= (120 + 5 \cdot 11 - 0.1(11)^2) - (120 + 5 \cdot 10 - 0.1(10)^2) \\
 &= 162.9 - 160 \\
 &= 2.9
 \end{aligned}$$

The exact cost of the 11th item only is \$2.90

Exploration 11.3 You are making 100 trailers, and the total cost function is given by $C(x) = 25,000 + 5x^3$

- (a) What is the *exact* cost of the 100th trailer?

Solution. Recall that $C(100)$ = total/cumulative cost of items 1 to 100. Using the equation given,

$$C(100) = 25,000 + 5 \cdot 100^3 = 5,025,000$$

We don't want to know the cost of *all* of these items, just the last item. To get rid of items 1-99, we compute their cost separately

$$C(99) = 25,000 + 5 \cdot 99^3 = 4,876,495$$

Subtracting out the items we don't care about, we get

$$\begin{aligned} \text{cost of 100th only} &= C(100) - C(99) \\ &= 5,025,000 - 4,876,495 \\ &= 148,505 \end{aligned}$$

The 100th trailer only costs exactly \$148,505.

- (b) What is the *approximate* cost of the 100th trailer?

Solution. By Definition ??, when we are asked to *approximate* the cost of the next item, we must find and use the marginal cost function.

To find the marginal cost function, use the derivative shortcuts

$$\begin{aligned} C'(x) &= \frac{d}{dx} [25,000 + 5x^3] \\ &= \frac{d}{dx} [25,000] + \frac{d}{dx} [5x^3] \\ &= 0 + 5 \cdot \frac{d}{dx} [x^3] \\ &= 5 \cdot 3x^2 \\ &= 15x^2 \end{aligned}$$

To use the marginal cost function, recall that

$$C'(x) = \text{the APPROXIMATE cost of the NEXT item in \$/item}$$

We want to find the approximate cost of the 100th item only. Item 100 is the item that is *immediately after item 99*.

In other words, compute $C'(99)$.

$$C'(99) = 15 \cdot 99^2 = 147,015$$

The 11th item will cost approximately \$147,015.

The example above helps explain part of the value of using the marginal cost function, instead of always finding exact cost. Computing the exact meant talking about costs in the millions, even though the individual item's value was only in the hundreds of thousands. Even better, once we have computed the equation for marginal cost function we can quickly compare the item cost for a range of numbers of items.

Application 11.2 Why Derivatives Approximate the Cost of the Next Item. What is the connection between derivatives and approximate item cost?

Looking at the units of measurement provides one way of understanding the connection. If total cost $C(x)$ is measured in dollars, and quantity x is measured in items, then

$$C'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta C}{\Delta x} \frac{\$}{\text{item}}$$

In other words, because the derivative of cost has units \$/item, it tells you approximately how many \$ an individual item will cost.

Why does the marginal cost tell us the cost of the *next* item?

Remember that limits arise in business when quantities are large. For example, suppose that x is 5000 items.

Expanding out the limit definition of slope/IROC, we get

$$\begin{aligned} C'(5000) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta C}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{C_2 - C_1}{x_2 - x_1} \\ &= \lim_{\Delta x \rightarrow 0} \frac{C(5000 + \Delta x) - C(5000)}{\Delta x} \end{aligned}$$

Because $x = 5000$ is large, then a change of $\Delta x = 1$ more item is (relatively) small. This means that

$$C'(5000) \approx \frac{C(5000 + 1) - C(5000)}{\Delta x}$$

simplifying this gives

$$C'(5000) \approx C(5001) - C(5000)$$

which is equal to the 5001st item, which is the *next* item after 5000.

Understanding Change, AROC, and IROC. We now have several tools for analyzing economic questions: *change* ΔC , average rate $\frac{\Delta C}{\Delta x}$, and marginal cost $C'(x) = \frac{dC}{dx}$. Always make sure you understand what the question is asking for, and know how to apply the correct

tool for that question.

**Average and
Change look
Backward**

If you *have moved* from x_1 to x_2 items, then you can compute the change

$$\Delta C = C_2 - C_1$$

and the average rate (AROC)

$$\frac{\Delta C}{\Delta x} = \frac{C_2 - C_1}{x_2 - x_1}$$

Finding the exact cost of a single item is a special case of finding the change in cost.

For example, to find the exact cost of item 105, compute $C(105) - C(104)$. This can be thought of as a change ΔC , or an average change $\frac{\Delta C}{\Delta x}$ with $\Delta x = 1$

Change and AROC are used for *accounting* for changes in the *past*.

**Marginal Cost
and
Derivatives
look Forward**

$$\begin{aligned} C'(x) &\approx \text{cost of the NEXT ITEM after } x \\ &\approx \text{cost of the } (x+1)^{\text{st}} \text{ item} \end{aligned}$$

Marginal cost and derivatives are used to make *decisions* about the *future*

The power of mathematics is that the same tools apply to a range of contexts.

Definition 11.3 Let $F(x)$ be any *total* or cumulative function, such as cost, revenue, or profit. The *marginal function* is

$$F'(x) = \text{the APPROXIMATE value of the NEXT ITEM ONLY}$$

◇

In other words, if $R(x)$ = total revenue and $P(x)$ = total profit, then $R'(500)$ = approximate revenue of the next (501st) item and $P'(500)$ = approximate revenue of the next (501st) item

Beautiful Consequence. The slope of **total** profit at $x = a$ items is approximately equal to the *profit*

Product and Quotient Rule

Previously, we have seen rules for taking derivatives of basic functions.

Basic Derivative Rules (So Far).

Suppose that f, g are differentiable functions of x , that m, c are constant numbers. Then

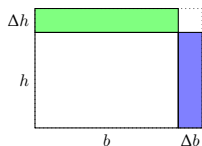
- $\frac{d}{dx}[c] = 0$
- $\frac{d}{dx}[m \cdot x + b] = m$ and $\frac{d}{dx}[x] = 1$
- $\frac{d}{dx}[x^m] = m \cdot x^{m-1}$
- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$
- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$
- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

Previously, we noted that the derivative rules for \times and \div are *much* more complicated. We will address these rules in Section ?? and Section ??.

12.1 Derivatives of Products

Remember from elementary school that the product $h \cdot b$ is equal to the *area* of the rectangle with base b and height h .

Suppose now that we want the base or height to *change* over time. How does that change the area $A = h \cdot b$ as a function of time? In other words, we want to know $\frac{dA}{dt}$.



Looking at the picture, there are two main parts of how the area is changing, as a result of the changing base (blue) and as a result of the changing height (green). There is also a little area in the upper corner, but it seems negligible compared to the other changes.

More precisely you can *prove* that there are two parts to the rate of change in area

- $\frac{dA}{dt}$ due to changing base is equal to $h \cdot \frac{d}{dt}[b]$
- $\frac{dA}{dt}$ due to changing height is equal to $b \cdot \frac{d}{dt}[h]$

Putting these together, we get the product rule:

$$\frac{d}{dt}[h \cdot b] = h \cdot \frac{d}{dt}[b] + b \cdot \frac{d}{dt}[h]$$

Theorem 12.1 The Product Rule. Suppose that $f(x)$ and $g(x)$ are differentiable functions. Then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$$

This is easier to remember in bracket notation:

$$[f \cdot g]' = f \cdot [g]' + g \cdot [f]' = f \cdot g' + g \cdot f'$$

Exploration 12.1 Let $f(x) = \sqrt{x} \cdot (9 + 7x)$. Find $f'(x)$.

Solution. As always, start by setting up the derivative

$$f'(x) = \frac{d}{dx}[\sqrt{x} \cdot (9 + 7x)]$$

We don't have any derivative rule for square roots, but we can always rewrite $\sqrt[n]{x^n} = x^{n/m}$. In particular, using $\sqrt{x} = x^{1/2}$ gives us

$$f'(x) = \frac{d}{dx}[x^{1/2} \cdot (9 + 7x)]$$

To apply our derivative rules, you need to use order of operations (PEMDAS) to identify the "outermost" operation. In this case, our function is the **product** of $x^{1/2}$ with $(9 + 7x)$. That means we will use the *product rule* $[f \cdot g]' = f \cdot [g]' + g \cdot [f]'$

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^{1/2} \cdot (9 + 7x)] \\ f'(x) &= x^{1/2} \cdot \frac{d}{dx}[(9 + 7x)] + (9 + 7x) \cdot \frac{d}{dx}[x^{1/2}] \\ f'(x) &= x^{1/2} \cdot \frac{d}{dx}[(9 + 7x)] + (9 + 7x) \cdot \frac{d}{dx}[x^{1/2}] \end{aligned}$$

$$\begin{aligned}
f'(x) &= x^{1/2} \cdot \frac{d}{dx}[(9 + 7x)] + (9 + 7x) \cdot \frac{d}{dx}[x^{1/2}] \\
f'(x) &= x^{1/2} \cdot \left(\frac{d}{dx}[9] + \frac{d}{dx}[7x]\right) + (9 + 7x) \cdot \frac{1}{2} \cdot x^{1/2-1} \\
f'(x) &= x^{1/2} \cdot (0 + 7) + (9 + 7x) \cdot \frac{1}{2} \cdot x^{-1/2} \\
f'(x) &= 7\sqrt{x} + (9 + 7x) \cdot \frac{1}{2} \cdot x^{-1/2} \\
f'(x) &= 7\sqrt{x} + (9 + 7x) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}}
\end{aligned}$$

You could keep "simplifying" this derivative, but we might as well stop here.

Exploration 12.2 Let $f(x) = 3x \ln(x) - 4x$. Find $f'(x)$.

Solution. As always, start by setting up the derivative

$$f'(x) = \frac{d}{dx}[3x \ln(x) - 4x]$$

To apply our derivative rules, you need to use order of operations (PEMDAS) to identify the "outermost" operation. In this case, our function is the **sum** of $3x \ln(x)$ and $-4x$. That means we will use the *sum rule* $[f + g]' = [f]' + [g]'$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[3x \ln(x) - 4x] \\
&= \frac{d}{dx}[3x \ln(x)] - \frac{d}{dx}[4x] \\
&= \frac{d}{dx}[3x \ln(x)] - \frac{d}{dx}[4x]
\end{aligned}$$

Now we have *two separate* derivatives to compute: $\frac{d}{dx}[3x \ln(x)]$ and $\frac{d}{dx}[4x]$. The second derivative is easy: $\frac{d}{dx}[4x] = 4$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[3x \ln(x)] - 4 \\
&= \frac{d}{dx}[3x \cdot \ln(x)] - 4
\end{aligned}$$

For the remaining derivative, we need to use PEMDAS to identify the "outermost" operation, which is a product, so we will use the product rule $[f \cdot g]' = f \cdot [g]' + g \cdot [f]'$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[3x \cdot \ln(x)] - 4 \\
&= 3x \cdot \frac{d}{dx}[\ln(x)] + \ln(x) \cdot \frac{d}{dx}[3x] - 4 \\
&= 3x \cdot \frac{d}{dx}[\ln(x)] + \ln(x) \cdot \frac{d}{dx}[3x] - 4
\end{aligned}$$

The remaining derivatives are basic rules: $\frac{d}{dx}[\ln(x)] = 1/x$ and $\frac{d}{dx}[cx] = c$.

$$\begin{aligned} f'(x) &= 3x \cdot \frac{1}{x} + \ln(x) \cdot 3 - 4 \\ &= \frac{3x}{x} + 3 \ln(x) - 4 \\ &= 3 + 3 \ln(x) - 4 \\ &= 3 \ln(x) - 1 \end{aligned}$$

Exploration 12.3 Let $f(x) = 5x^3 e^x \ln(x)$. Find $f'(x)$.

Solution. This problem requires several steps. First, begin by setting up the question

$$f'(x) = [5x^3 \cdot e^x \cdot \ln(x)]'$$

We want to take the derivative of a product, so we must use the product rule $[f \cdot g]' = f' \cdot g + f \cdot g'$. The problem is that here are only *two* functions in our formula, but there are multiplied terms in our definition of f . Fortunately, because of the “associativity” of multiplication $a \cdot b \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c)$, so we can group the terms any way we like.

$$\begin{aligned} f'(x) &= [(5x^3 e^x) \cdot \ln(x)]' \\ &= (5x^3 e^x) \cdot [\ln(x)]' + \ln(x) \cdot (5x^3 e^x)' \end{aligned}$$

Now we have two derivatives remaining: a simple derivative $[\ln(x)]' = \frac{1}{x}$ and a product rule $[(5x^3 \cdot e^x)]' = 5x^3 \cdot [e^x]' + e^x \cdot [5x^3]' = 5x^3 e^x + e^x 15x^2$. Putting it all together we get

$$\begin{aligned} &= (5x^3 e^x) \cdot \frac{1}{x} + \ln(x) \cdot (5x^3 e^x + e^x 15x^2) \\ &= 5x^3 e^x \frac{1}{x} + 5x^3 e^x \ln(x) + 15x^2 e^x \ln(x) \end{aligned}$$

Any time a product is changing, the product rule comes into play. In business, products play an important role in the study of revenue and profit, since

$$\text{Revenue} = \text{price} \cdot \text{quantity} = p \cdot q$$

Previously, we have only looked at the impact of changing quantity. But in *real economies*, things like marketing can increase the quantity sold over time *without* decreasing the price.

Exploration 12.4 Thanks to a word of mouth marketing campaign, business is booming. Suppose that

$$\text{quantity on day } t = q(t) = 0.4e^t$$

$$\text{price on day } t = p(t) = t^2$$

How fast is Revenue changing on day 3?

Solution. Before we can find any rates of change with respect to time, we first need to know what the revenue function is as a function of t . Using the formula

$$\text{Revenue} = \text{price} \cdot \text{quantity}$$

we have

$$R(t) = p(t) \cdot q(t) = t^2 \cdot 0.4e^t$$

Now, we can find the derivative with respect to t

$$\begin{aligned} R'(t) &= \frac{d}{dt} [t^2 \cdot 0.4e^t] \\ &= \frac{d}{dt} [\textcolor{red}{t}^2 \cdot \textcolor{blue}{0.4}e^t] \\ &= \textcolor{red}{t}^2 \cdot \frac{d}{dt} [\textcolor{blue}{0.4}e^t] + \textcolor{blue}{0.4}e^t \cdot \frac{d}{dt} [\textcolor{red}{t}^2] \\ &= t^2 \cdot \frac{d}{dt} [\textcolor{red}{0.4}\textcolor{blue}{e}^t] + 0.4e^t \cdot \frac{d}{dt} [\textcolor{blue}{t}^{\textcolor{red}{2}}] \end{aligned}$$

For the first derivative, we have a constant coefficient times a function, so we can use the rule $\frac{d}{dx} [\textcolor{red}{c}f(x)] = \textcolor{red}{c} \frac{d}{dx} [f(x)]$. For the second derivative, we can use the power rule $\frac{d}{dx} [\textcolor{red}{x}^{\textcolor{blue}{m}}] = \textcolor{red}{m}x^{\textcolor{blue}{m}-1}$

$$\begin{aligned} R'(t) &= t^2 \cdot \textcolor{red}{0.4} \frac{d}{dt} [\textcolor{blue}{e}^t] + 0.4e^t \cdot \textcolor{red}{2}t^{2-1} \\ R'(t) &= t^2 \cdot 0.4 \frac{d}{dt} [e^t] + 0.4e^t \cdot 2t^1 \\ R'(t) &= t^2 \cdot 0.4e^t + 0.4e^t \cdot 2t \\ R'(t) &= 0.4t^2e^t + 0.8te^t \end{aligned}$$

On day 3, the rate of change of revenue is

$$R'(3) = 0.4 \cdot 3^2e^3 + 0.8 \cdot 3 \cdot e^3 = 192.82$$

On the third day, the price is going up approximately 192.82 \$/day.

In general, if price $p(t)$ and quantity $q(t)$ are both functions of t , then because the revenue function is

$$R = q \cdot p$$

then the rate of change of revenue is

$$\frac{dR}{dt} = q \cdot \frac{dp}{dt} + p \cdot \frac{dq}{dt}$$

In other words, the rate of revenue change has two parts: the change in R due to changing price (the first term), and the change in R due to changing quantity (the second term).

12.2 Derivatives of Fractions

There is also a special rule for derivatives of fractions. Another name for a fraction is a *quotient*, so this is often called the **quotient rule**.

Theorem 12.2 Suppose that $f(x)$ and $g(x)$ are differentiable functions. Then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx} [f(x)] - f(x) \cdot \frac{d}{dx} [g(x)]}{(g(x))^2}$$

It is often easier to memorize this formula in an abbreviated form:

- $\left[\frac{t}{b} \right]' = \frac{b \cdot t' - t \cdot b'}{(b)^2}$
- $\left[\frac{hi}{lo} \right] = \frac{lo \ D \ hi - hi \ D \ lo}{(lo)^2}$

Exploration 12.5 Let $f(x) = \frac{e^x}{x}$. Find $f'(10)$.

Solution.

$$f'(x) = \frac{d}{dx} \left[\frac{e^x}{x} \right]$$

To apply our derivative rules, you need to use order of operations (PEMDAS) to identify the "outermost" operation. In this case, our function is the **fraction** of e^x and x . That means we will use the

quotient rule $\left[\frac{t}{b} \right]' = \frac{b \cdot t' - t \cdot b'}{(b)^2}$

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[\frac{e^x}{x} \right] \\ &= \frac{x \cdot \frac{d}{dx} [e^x] - e^x \frac{d}{dx} [x]}{(x)^2} \\ &= \frac{x \cdot \frac{d}{dx} [e^x] - e^x \frac{d}{dx} [x]}{(x)^2} \\ &= \frac{x \cdot e^x - e^x \cdot 1}{(x)^2} \\ &= \frac{x \cdot e^x - e^x}{(x)^2} \end{aligned}$$

At $x = 10$,

$$f'(10) = \frac{10 \cdot e^{10} - e^{10}}{(10)^2} = 1982.38$$

You can "show your work" in several ways. We have already seen the $\frac{d}{dx}[\dots]$ strategy. This strategy is good because it is very robust, and almost always works. Unfortunately, this strategy also uses a *lot* of symbols.

An alternative technique makes a much greater use of scrap work. We will illustrate this strategy using the following exercise.

Exploration 12.6 Let $f(x) = \frac{x^2 + x + 1}{x^2 - 1}$. Find $\frac{df}{dx}$.

Solution. As always, start by setting up the derivative.

$$f'(x) = \frac{d}{dx} \left[\frac{x^2 + x + 1}{x^2 - 1} \right]$$

The outermost operation is a fraction, so we must use the quotient

rule $\left[\frac{t}{b} \right]' = \frac{b \cdot t' - t \cdot b'}{(b)^2}$ with

$$t = x^2 + x + 1$$

and

$$b = x^2 - 1.$$

For the quotient rule, we need to substitute expressions for t , b , t' , and b' . We already have expressions for t and b . We next do some "scrap work" to find t' and b'

$$\begin{aligned} t' &= \frac{d}{dx} [x^2 + x + 1] \\ &= \frac{d}{dx} [x^2] + \frac{d}{dx} [x] + \frac{d}{dx} [1] \\ t' &= 2x + 1 + 0 \\ t' &= 2x + 1 \end{aligned}$$

Similarly, we can compute

$$\begin{aligned} b' &= \frac{d}{dx} [x^2 - 1] \\ &= \frac{d}{dx} [x^2] - \frac{d}{dx} [1] \\ b' &= 2x - 0 \\ b' &= 2x \end{aligned}$$

We now can go back to our original question, and apply the quotient rule

$$f'(x) = \frac{d}{dx} \left[\frac{x^2 + x + 1}{x^2 - 1} \right]$$

$$\begin{aligned}
&= \frac{b \cdot t' - t \cdot b'}{(b)^2} \\
&= \frac{(x^2 - 1) \cdot (2x + 1) - (x^2 + x + 1) \cdot 2x}{(x^2 - 1)^2}
\end{aligned}$$

Exploration 12.7 Let $f(x) = \frac{x^2}{\ln(x)}$. Find $f'(x)$.

Solution.

$$f'(x) = \frac{d}{dx} \left[\frac{x^2}{\ln(x)} \right]$$

We must use the quotient rule $\left[\frac{t}{b} \right]' = \frac{b \cdot t' - t \cdot b'}{(b)^2}$

$$\begin{aligned}
f'(x) &= \frac{d}{dx} \left[\frac{x^2}{\ln(x)} \right] \\
&= \frac{\ln(x) \cdot \frac{d}{dx} [x^2] - x^2 \cdot \frac{d}{dx} [\ln(x)]}{(\ln(x))^2} \\
&= \frac{\ln(x) \cdot 2x - x^2 \cdot \frac{1}{x}}{(\ln(x))^2} \\
&= \frac{2x \ln(x) - x}{(\ln(x))^2}
\end{aligned}$$

Exploration 12.8 Let $y = \frac{x \ln(x)}{e^x}$. Find $\frac{dy}{dx}$

Solution.

$$f'(x) = \frac{d}{dx} \left[\frac{x \ln(x)}{e^x} \right]$$

The most important part of any derivative is to find the "outer-most" operation. Here, we have a *fraction* where the numerator is $x \ln(x)$ and the denominator is e^x . We must use the quotient rule.

$$\left[\frac{t}{b} \right]' = \frac{b \cdot t' - t \cdot b'}{(b)^2}$$

$$f'(x) = \frac{e^x \frac{d}{dx} [x \ln(x)] - x \ln(x) \frac{d}{dx} [e^x]}{(e^x)^2}$$

There are two derivatives in the expression above. The second expression is straightforward: $\frac{d}{dx} [e^x] = e^x$. The first derivative is more complicated is a product, so will require the product rule

$$\frac{d}{dx} [x \ln(x)] = x \frac{d}{dx} [\ln(x)] + \ln(x) \frac{d}{dx} [x] = x \frac{1}{x} + \ln(x) \cdot 1 = 1 + \ln(x)$$

$$f'(x) = \frac{e^x \frac{d}{dx} [x \cdot \ln(x)] - x \ln(x) e^x}{(e^x)^2}$$

$$= \frac{e^x(1 + \ln(x)) - x \ln(x)e^x}{(e^x)^2}$$

The Chain Rule

So far, we have seen rules for sums, products, and quotients

Basic Derivative Rules (So Far).

Suppose that f, g are differentiable functions of x , that m, c are constant numbers. Then

- $[c]' = 0$
- $[m \cdot x + b]' = m$ and $[x]' = 1$
- $[x^m]' = m \cdot x^{m-1}$
- $[c \cdot f(x)]' = c \cdot [f(x)]'$
- $[f(x) \pm g(x)]' = [f(x)]' \pm [g(x)]'$
- $[e^x]' = e^x$
- $[\ln(x)]' = \frac{1}{x}$
- $[f \cdot g]' = f \cdot [g]' + g \cdot [f]' = f \cdot g' + g \cdot f'$
- $\left[\frac{t}{b}\right]' = \frac{b \cdot t' - t \cdot b'}{(b)^2}$

But what happens if you have a function *inside* of another function?

For example, in Excel you will often see an expression like

$$\ln(x^2)$$

Here, the function x^2 is *inside* the function $\ln(u)$. In this case, we call

- the inside function $g(x) = x^2$ the **argument** or **inner function**
- the outside function $f(u) = \ln(u)$ is the **outer function**
- the original function $f(g(x)) = \ln(x^2)$ is the **composition** of the outside function and the argument.

In this chapter, we will introduce one final rule (or set of rules) that will let us compute derivatives of compositions of functions.

13.1 The Generalized Power Rule

Compositions where the outside function is a power is the simplest case to work with.

Checkpoint 13.1 Let $h(x) = (3x + 2)^2$. Write this function in the form $f(g(x))$

Solution. The easiest way to interpret compositions is to try to *circle the inside function*.

$$h(x) = \left(\boxed{3x+2} \right)^2$$

- The **argument** is the circled inside term $g(x) = \boxed{3x+2}$.
- To find the **outside function**, replace the circled box with the letter u .

$$\begin{aligned} h(u) &= \left(\boxed{3x+2} \right)^2 \\ &= \left(\boxed{u} \right)^2 \\ h(u) &= u^2 \end{aligned}$$

If $f(u) = u^2$ and $g(x) = 3x + 2$, then

$$f(g(x)) = f(3x + 2) = (3x + 2)^2$$

as desired.

In very simple cases, we can compute the derivative of a composition with the rules we already have.

Checkpoint 13.2 Let $f(x) = (3x + 2)^2$. Find $f'(x)$ using only the rules from the previous sections.

Solution.

$$f'(x) = \frac{d}{dx} \left[(3x + 2)^2 \right]$$

No rule applies, but we can rewrite the function using algebra first.

$$f'(x) = \frac{d}{dx} \left[(3x + 2)^2 \right]$$

$$\begin{aligned}
&= \frac{d}{dx} [(3x+2) \cdot (3x+2)] \\
&= \frac{d}{dx} [9x^2 + 12x + 4] \\
&= \frac{d}{dx} [9x^2] + \frac{d}{dx} [12x] + \frac{d}{dx} [4] \\
&= 9 \frac{d}{dx} [x^2] + 12 + 0 \\
&= 9 \cdot 2x + 12 \\
&= 18x + 12
\end{aligned}$$

On the face of it, there is no clear connection between the equations for $f(x)$ and $f'(x)$. We need a new shortcut.

Theorem 13.3 The Generalized Power Rule. *Suppose that $g(x)$ is a differentiable function, and that m is a constant number.*

$$\frac{d}{dx} [(g(x))^m] = m \cdot (g(x))^{m-1} \cdot \frac{d}{dx} [g(x)]$$

In other words, if you have a composition $h(x) = (g(x))^m$, the derivative is the same thing as following the power rule and multiplying by the derivative of the argument.

In bracket notation, this rule is written

$$[(g(x))^m]' = m \cdot (g(x))^{m-1} \cdot g'(x)$$

Checkpoint 13.4 Let $h(x) = (3x+2)^2$ as before. Find $h'(x)$ using the generalized power rule.

Solution. As always, start by writing

$$h'(x) = \frac{d}{dx} [(3x+2)^2]$$

As in Checkpoint ??, this is a composition with argument $g(x) = 3x+2$ and outside function $f(u) = u^2$. We must use the generalized power rule $\frac{d}{dx} [(g(x))^m] = m \cdot (g(x))^{m-1} \cdot \frac{d}{dx} [g(x)]$

$$\begin{aligned}
h'(x) &= \frac{d}{dx} [(3x+2)^2] \\
&= \frac{d}{dx} [(g(x))^2] \\
&= 2 \cdot (g(x))^1 \cdot \frac{d}{dx} [g(x)] \\
&= 2 \cdot (3x+2) \cdot \frac{d}{dx} [3x+2] \\
&= 2 \cdot (3x+2) \cdot (3+0) \\
&= 18x + 12
\end{aligned}$$

Checkpoint 13.5 Let $h = (3 - \ln(x))^{10}$. Find $\frac{dh}{dx}$.

Solution.

$$\frac{dh}{dx} = \frac{d}{dx} \left[(3 - \ln(x))^{10} \right]$$

This is a composition of two functions, with the inside function (argument) $g(x) = 3 - \ln(x)$ and outside function $f(u) = u^{10}$

$$\begin{aligned} \frac{dh}{dx} &= \frac{d}{dx} \left[(3 - \ln(x))^{10} \right] \\ &= \frac{d}{dx} \left[(g(x))^{10} \right] \\ &= 10 \cdot (g(x))^9 \cdot \frac{d}{dx} [g(x)] \\ &= 10 \cdot (3 - \ln(x))^9 \cdot \frac{d}{dx} [3 - \ln(x)] \\ &= 10 \cdot (3 - \ln(x))^9 \cdot \left(0 - \frac{1}{x}\right) \\ &= -10(3 - \ln(x))^9 \cdot \frac{1}{x} \end{aligned}$$

Checkpoint 13.6 Let $y = \frac{1}{x^2 - 3x}$. Find $\frac{dy}{dx}$.

Solution.

$$\frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d}{dx} \left[\frac{1}{x^2 - 3x} \right]$$

You can compute this derivative using the quotient rule, as in Section ?? . You can also rewrite this as a power, and use the generalized power rule.

$$\frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d}{dx} \left[(x^2 - 3x)^{-1} \right]$$

You are now taking the derivative of a composition with argument $g(x) = x^2 - 3x$ and outside function $f(u) = u^{-1}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[(x^2 - 3x)^{-1} \right] \\ &= \frac{d}{dx} \left[(g(x))^{-1} \right] \\ &= -1 \cdot (g(x))^{-1-1} \cdot \frac{d}{dx} [g(x)] \\ &= -1 \cdot (x^2 - 3x)^{-2} \cdot \frac{d}{dx} [x^2 - 3x] \\ &= -1 \cdot (x^2 - 3x)^{-2} \cdot (2x - 3) \\ &= -1 \cdot \frac{1}{(x^2 - 3x)^2} \cdot (2x - 3) \end{aligned}$$

Checkpoint 13.7 Use the product and chain rules to derive equation for the quotient rule.

Solution. Suppose that $f(x)$ and $g(x)$ are differentiable functions.

We wish to compute

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right]$$

To use the product and generalized power rule, we can get rid of the fraction by using an exponent of -1 .

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{d}{dx} \left[f(x) \cdot (g(x))^{-1} \right]$$

Using the product rule, we get

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = f(x) \cdot \frac{d}{dx} \left[(g(x))^{-1} \right] + (g(x))^{-1} \cdot \frac{d}{dx} [f(x)]$$

There are two $\frac{d}{dx} [\dots]$ that remain. The second one is simply notation: $\frac{d}{dx} [f(x)] = f'(x)$. The first requires the chain rule

$$\begin{aligned} \frac{d}{dx} \left[(g(x))^{-1} \right] &= -1 (g(x))^{-1-1} \cdot \frac{d}{dx} [g(x)] \\ &= -1 (g(x))^{-2} \cdot g'(x) \end{aligned}$$

Putting this back into the broader derivative

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] &= f(x) \cdot -1 (g(x))^{-2} \cdot g'(x) + (g(x))^{-1} \cdot f'(x) \\ &= \frac{-f(x) \cdot g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)} \end{aligned}$$

To combine these fractions, we need a common denominator

$$\begin{aligned} &= \frac{-f(x) \cdot g'(x)}{(g(x))^2} + \frac{f'(x) \cdot g(x)}{(g(x))^2} \\ &= \frac{-f(x) \cdot g'(x) + f'(x) \cdot g(x)}{(g(x))^2} \end{aligned}$$

Rearranging the top gives

$$= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

This is exactly the quotient rule from Theorem ??.

Checkpoint 13.8 Let $f(x) = \sqrt{5 - 3x + x^5} + 10$. Find $f'(x)$

Solution. Recall that $\sqrt{g(x)} = (g(x))^{1/2} = (g(x))^{0.5}$. That means

we can rephrase the question given as follows.

$$\begin{aligned}
 f'(x) &= \left[(5 - 3x + x^5)^{0.5} + 10 \right]' \\
 &= \left[(5 - 3x + x^5)^{0.5} \right]' + [10]' && \text{sum rule} \\
 &= \left[(5 - 3x + x^5)^{0.5} \right]' + 0 && \text{constant rule} \\
 &= 0.5 (5 - 3x + x^5)^{0.5-1} \cdot [5 - 3x + x^5]' && \text{chain rule} \\
 &= 0.5 (5 - 3x + x^5)^{-0.5} \cdot (0 - 3 + 5x^4)
 \end{aligned}$$

Note that both sets of parentheses in the last term are required for a correct answer.

Recall that for root functions, we have an optional derivative shortcut $\frac{d}{dx} [\sqrt{x}] = \frac{1}{2}x^{-1/2}$. Combined with the generalized power rule, this gives an optional shortcut.

Corollary 13.9 Let $f(x) = \sqrt{g(x)}$. Then $f'(x) = \frac{1}{2} (g(x))^{-1/2} \cdot g'(x)$

Proof. Recall that roots correspond to fractional powers. In particular, $\sqrt{u} = u^{1/2}$. This allows us to rewrite the function before taking the derivative.

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} [\sqrt{g(x)}] \\
 &= \frac{d}{dx} [(g(x))^{1/2}] \\
 &= \frac{1}{2} (g(x))^{-1/2} \cdot \frac{d}{dx} [g(x)] \\
 &= \frac{1}{2} (g(x))^{-1/2} \cdot g'(x)
 \end{aligned}$$

■

There is a consistent strategy when using the generalized power rule in all of the examples above.

1. See a composition $h(x) = f(g(x))$.
2. Identify the argument $g(x)$ and the outside function $f(u) = u^m$.
3. Find the derivative of the outside

$$f'(u) = m \cdot u^{m-1}$$

4. Apply the rule

$$\frac{d}{dx} [(g(x))^m] = m \cdot (g(x))^{m-1} \cdot \frac{d}{dx} [g(x)]$$

An easy way to summarize this is to write

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

But is this more general formula true for *any* outside function $g(x)$?

13.2 The Chain Rule

Theorem 13.10 Suppose that $f(x)$ and $g(x)$ are differentiable functions.

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

To memorize this formula, use one of the abbreviations

$$[f(g)]' = f'(g) \cdot g'$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Because there are so many moving pieces in the chain rule, we will generally compute $\frac{d}{dx} [f(g(x))]$ using the following steps

1. Identify the argument $g(x)$ and the outer function $f(u)$
2. Using scrap work, separately compute $f'(u)$ and $g'(x)$.
3. Use your scrap work above to answer the original question using the Chain Rule formula

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

Checkpoint 13.11 Let $y = e^{3x^2+x}$. Find $\frac{dy}{dx}$.

Solution. We wish to compute

$$\frac{dy}{dx} = \frac{d}{dx} [e^{(3x^2+x)}]$$

This is a composition with argument $g(x) = 3x^2 + x$ and outside function $f(u) = e^u$.

Using our regular derivative rules, we know that $g'(x) = 6x + 1$ and that $f'(u) = \frac{d}{du} [e^u] = e^u$

Now that we have done our scrap work, we can return to our original question

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [e^{(3x^2+x)}] \\ &= \frac{d}{dx} [f(g(x))] \\ &= f'(g(x)) \cdot g'(x) \end{aligned}$$

Substituting the values first for f' , and then for g and g' , into the above equation gives

$$\frac{dy}{dx} = e^{(g(x))} \cdot g'(x)$$

$$= e^{(3x^2+x)} \cdot (6x+1)$$

Checkpoint 13.12 Let $h(x) = e^{5-x^2}$. Find $h'(x)$.

Solution.

$$h'(x) = \frac{d}{dx} \left[e^{(5-x^2)} \right]$$

This is a composition with argument $g(x) = 5 - x^2$ and outside function $f(u) = e^u$.

Using our regular derivative rules, we know that $g'(x) = \frac{d}{dx} [5 - x^2] = -2x$ and that $f'(u) = \frac{d}{du} [e^u] = e^u$

Now that we have done our scrap work, we can return to our original question

$$\begin{aligned} h'(x) &= \frac{d}{dx} \left[e^{(5-x^2)} \right] \\ &= \frac{d}{dx} \left[f(g(x)) \right] \\ &= f'(g(x)) \cdot g'(x) \end{aligned}$$

Substituting the values first for f' , and then for g and g' , into the above equation gives

$$\begin{aligned} h'(x) &= e^{(g(x))} \cdot g'(x) \\ &= e^{(5-x^2)} \cdot (-2x) \end{aligned}$$

In the previous two examples, there was a pattern that arose because outside function was $f(u) = e^u$ in both cases. This gives a new "optional shortcut" derivative rule.

$$\frac{d}{dx} \left[e^{g(x)} \right] = e^{g(x)} \cdot g'(x)$$

The one last function we have studied in this course is the natural logarithm.

Checkpoint 13.13 Let $F(x) = \ln(1 - 2x^3)$. Find $F'(x)$.

Solution.

$$F'(x) = \frac{d}{dx} \left[\ln(1 - 2x^3) \right]$$

This is a composition with argument $g(x) = 1 - 2x^3$ and outside function $f(u) = \ln(u)$.

Using our regular derivative rules, we know that $g'(x) = \frac{d}{dx} [1 - 2x^3] = -6x^2$ and that $f'(u) = \frac{d}{du} [\ln(u)] = \frac{1}{u}$

Now that we have done our scrap work, we can return to our original question

$$F'(x) = \frac{d}{dx} \left[\ln(1 - 2x^3) \right]$$

$$\begin{aligned}
&= \frac{d}{dx} \left[f(g(x)) \right] \\
&= f'(g(x)) \cdot g'(x)
\end{aligned}$$

Substituting the values first for $f' = \frac{1}{u}$, and then for g and g' , into the above equation gives

$$\begin{aligned}
F'(x) &= \frac{1}{g(x)} \cdot g'(x) \\
&= \frac{1}{1-2x^3} \cdot (-6x^2)
\end{aligned}$$

Checkpoint 13.14 Let $y = \ln(x^2 - 2x)$. Find y' .

Solution.

$$y'(x) = \frac{d}{dx} \left[\ln(x^2 - 2x) \right]$$

This is a composition with argument $g(x) = x^2 - 2x$ and outside function $f(u) = \ln(u)$.

Using our regular derivative rules, we know that $g'(x) = 2x - 2$ and that $f'(u) = \frac{d}{du} [\ln(u)] = \frac{1}{u}$.

Now that we have done our scrap work, we can return to our original question

$$\begin{aligned}
y'(x) &= \frac{d}{dx} \left[\ln(x^2 - 2x) \right] \\
&= \frac{d}{dx} \left[f(g(x)) \right] \\
&= f'(g(x)) \cdot g'(x)
\end{aligned}$$

Substituting the values first for $f' = \frac{1}{u}$, and then for g and g' , into the above equation gives

$$\begin{aligned}
y'(x) &= \frac{1}{g(x)} \cdot g'(x) \\
&= \frac{1}{x^2 - 2x} \cdot (2x - 2)
\end{aligned}$$

There is a pattern in the previous two examples, which results from the fact that in both cases, the outside function is $f(u) = \ln(u)$. This gives another "optional shortcut" derivative rule

$$\frac{d}{dx} \left[\ln(g(x)) \right] = \frac{1}{g(x)} \cdot g'(x)$$

Checkpoint 13.15 Suppose that b is a constant base (not necessarily $e = 2.714\dots$). Find a formula for $\frac{d}{dx} [b^x]$

Solution. In this derivative, the variable x is in the exponent. The only related rule we have is the formula $\frac{d}{dx} [e^x] = e^x$, but this only applies when the base is $e = 2.714\dots$

The trick is to use properties of exponentials and logarithms to rewrite the number b as $e^{\ln(b)}$. With that, we can find the derivative as follows

$$\begin{aligned}\frac{d}{dx}[b^x] &= \frac{d}{dx}\left[\left(e^{\ln(b)}\right)^x\right] \\ &= \frac{d}{dx}\left[e^{(\ln(b) \cdot x)}\right]\end{aligned}$$

This is a chain rule problem, with outside function $f(u) = e^u$ and inside function $g(x) = \ln(b) \cdot x$. Clearly, $f' = e^u$. And because b and $\ln(b)$ are constants, $g' = \frac{d}{dx}[\ln(b) \cdot x] = \ln(b)$

Now that we have done our scrap work, we can return to our original question

$$\begin{aligned}h'(x) &= \frac{d}{dx}\left[e^{(\ln(b) \cdot x)}\right] \\ &= \frac{d}{dx}\left[f(g(x))\right] \\ &= f'(g(x)) \cdot g'(x)\end{aligned}$$

Substituting the values first for f' , and then for g and g' , into the above equation gives

$$\begin{aligned}h'(x) &= e^{(g(x))} \cdot g'(x) \\ &= e^{(\ln(b) \cdot x)} \cdot (\ln(b)) \\ &= b^x \cdot \ln(b)\end{aligned}$$

The previous exercise gives us one last derivative rule. If b is any constant number,

$$\frac{d}{dx}[b^x] = b^x \cdot \ln(b)$$

For example,

$$\frac{d}{dx}[5^x] = 5^x \cdot \ln(5)$$

In this chapter, we have learned many different derivative rules. Make sure that you are

1. comfortable using the rules correctly and efficiently, and
2. able to accurately identify which rule to use in any given situation.

Exploration 13.1 Be Careful When Deciding Which Rule To Use.

(a) Let $f(x) = 15(\ln(x))^2$

Solution. Always start by setting up the derivative

$$f'(x) = \frac{d}{dx}\left[15(\ln(x))^2\right]$$

Use order of operations to identify the outermost operation. In case, after you pull the constant coefficient of 15 through the derivative, the outermost operation is the fact that $g(x) = \ln(x)$ is inside the power of 2 function $f(u) = u^2$

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} \left[15 \left(\ln(x) \right)^2 \right] \\
 &= 15 \frac{d}{dx} \left[\left(\ln(x) \right)^2 \right] && \text{constant coefficient} \\
 &= 15 \cdot 2 \cdot \left(\ln(x) \right)^{2-1} \frac{d}{dx} \left[\ln(x) \right] && \text{chain rule} \\
 &= 30 \cdot \left(\ln(x) \right) \frac{1}{x} \\
 &= \frac{30 \ln(x)}{x}
 \end{aligned}$$

(b) Let $f(x) = \ln(x)15x^2$. Find $f'(x)$.

Solution. Always start by setting up the derivative

$$f'(x) = \frac{d}{dx} \left[\ln(x)15x^2 \right]$$

Use order of operations to identify the outermost operation. In this case, the function is the **product** of $\ln(x)$ and $15x^2$, so we must use the product rule $[f \cdot g]' = f \cdot g' + g \cdot f'$.

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} \left[\ln(x)15x^2 \right] \\
 &= \ln(x) \frac{d}{dx} \left[15x^2 \right] + 15x^2 \frac{d}{dx} \left[\ln(x) \right] \\
 &= \ln(x) \cdot 15 \cdot 2x + 15x^2 \cdot \frac{1}{x} \\
 &= 30x \ln(x) + 15x
 \end{aligned}$$

(c) Let $f(x) = \ln(15x^2)$. Find $f'(x)$.

Solution. Always start by setting up the derivative

$$f'(x) = \frac{d}{dx} \left[\ln(15x^2) \right]$$

Use order of operations to identify the outermost operation. In this case the outermost operation is the **logarithmic function** $f(u) = \ln(u)$, and the argument is $g(x) = 15x^2$. That means we can either use the full chain rule $[f(g)]' = f'(g) \cdot g'$, or we can use the shortcut $[\ln(g(x))]' = \frac{1}{g(x)} \cdot g'(x)$. In either case, we get

$$f'(x) = \frac{d}{dx} \left[\ln(15x^2) \right]$$

$$\begin{aligned}
&= f'(g(x)) \cdot \frac{d}{dx} [g(x)] \\
&= \frac{1}{15x^2} \cdot (30x) \\
&= \frac{30x}{15x^2} \\
&= \frac{2}{x}
\end{aligned}$$

(d) Let $f(x) = e^{(\ln(x))^2}$. Find $f'(x)$.

Solution. Always start by setting up the derivative

$$f'(x) = \frac{d}{dx} [e^{(\ln(x))^2}]$$

Use order of operations to identify the outermost operation. In this case the outermost operation is the **exponential function** $f(u) = e^u$, and the argument is $g(x) = (\ln(x))^2$. That means we can either use the full chain rule $[f(g)]' = f'(g) \cdot g'$, or we can use the shortcut $[e^{g(x)}]' = e^{g(x)} \cdot g'(x)$. In either case, we get

$$\begin{aligned}
f'(x) &= \frac{d}{dx} [e^{(\ln(x))^2}] \\
&= e^{(\ln(x))^2} \frac{d}{dx} [(\ln(x))^2]
\end{aligned}$$

Now we must take the derivative of the inside function $(\ln(x))^2$. This is also a composition with outside function $f(u) = u^2$ and inside function $g(x) = \ln(x)$.

$$\begin{aligned}
&= e^{(\ln(x))^2} 2 \cdot (\ln(x))^{2-1} \cdot \frac{d}{dx} [\ln(x)] \\
&= e^{(\ln(x))^2} 2 \cdot (\ln(x)) \cdot \frac{1}{x}
\end{aligned}$$

13.3 List of All Derivative Rules

Complete List of Derivative Rules.

Suppose that f, g are differentiable functions of x , that m, c are constant numbers. Then

- $[c]' = 0$
- $[m \cdot x + b]' = m$ and $[x]' = 1$
- $[x^m]' = m \cdot x^{m-1}$

- $\left[c \cdot f(x) \right]' = c \cdot \left[f(x) \right]'$
- $\left[f(x) \pm g(x) \right]' = \left[f(x) \right]' \pm \left[g(x) \right]'$
- $\left[e^x \right]' = e^x$
- $\left[\ln(x) \right]' = \frac{1}{x}$
- $\left[f \cdot g \right]' = f \cdot \left[g \right]' + g \cdot \left[f \right]' = f \cdot g' + g \cdot f'$
- $\left[\frac{t}{b} \right]' = \frac{b \cdot t' - t \cdot b'}{(b)^2}$
- $\left[f(g) \right]' = f'(g) \cdot g'$
 - $\left[(g(x))^m \right]' = m(g(x))^{m-1} \cdot g'$
 - $\left[\sqrt{g(x)} \right]' = \frac{1}{2}(g(x))^{-1/2} \cdot g'$
 - $\left[e^{g(x)} \right]' = e^{g(x)} \cdot g'$
 - $\left[\ln(g(x)) \right]' = \frac{1}{g(x)} \cdot g'$
- If b is any constant number, $\left[b^x \right]' = b^x \cdot \ln(b)$

Part III

Applying Marginals and Slopes

14

Finding Maxima and Minima

14.1 Finding Local Extrema

In any economic system, and in any business, there are *many* different choices you can make. Of these choices, some are better than others.

Ideally, you will want to make the *best* choice. In practice, limits on resources and time often limit the amount of change that is feasible. In these cases, you want to make the *locally optimal* choice -- the choice that is better than all the alternatives that are "nearby" to your current location.

Part of the complexity of economic systems are the large number of variables and different ways of comparing which options are "better." In this course, we will focus on functions of a single variable, and we will always compare the magnitude of the outputs.

To help define our terms precisely, we will start with a more concrete, physical example.

Exploration 14.1 The example below shows a cross section of a path through a set of hills in the forest. In other words, the *height* of the trail at mile marker x is give by $y = f(x)$.

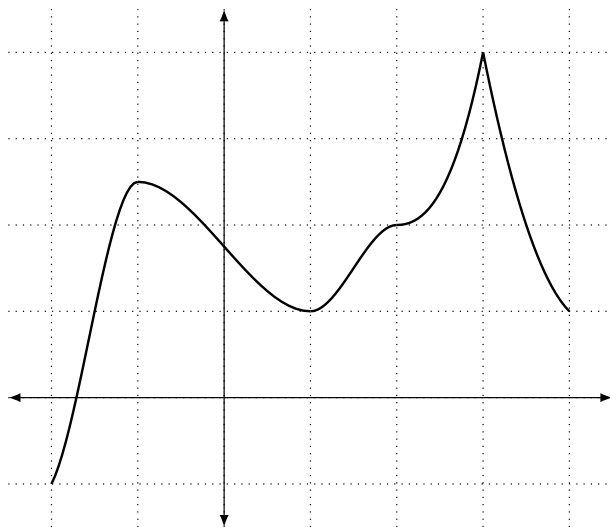


Figure 14.1

- (a) At which values does f have a *local/relative maximum*?

Solution. The graph has local maxima at $x = -1$ and $x = 3$, because f has

These are local maxima because if you imagine standing on the graph at those points, moving left or right will *decrease* your height.

Said another way, there is a local maxima because when you read the graph from left to right, these are the x -values where the graph looks like $\nearrow \searrow$.

- (b) At which values does f have a *local/relative minimum*?

Solution. The graph has a local minimum at $x = -2$, $x = 1$, or $x = 4$

These are local minima because if you imagine standing on the graph at those points, moving left or right will *increase* your height.

Said another way, there is a local minima because when you read the graph from left to right, these are the x -values where the graph looks like $\searrow \nearrow$.

- (c) At which values does f have a *absolute/global maximum*?

Solution. The graph has a global maximum of $y = 4$ at $x = 3$

These is a global maximum because if you imagine standing on the graph at those points, there is *no way place you could to move to to increase your height*.

- (d) At which values does f have a *absolute/global minimum*?

Solution. The graph has a global minimum of $y = -1$ at $x = -2$

These is a global maximum because if you imagine standing on the graph at those points, there is *no way place you could to move to to decrease your height*.

In the exercise above, there are three special kinds of points that mostly (but do not exactly) correspond to locations of maxima and minima:

1. The curve $y = f$ is *horizontal* at $x = -1, 1, 2$.

This is because $f'(x) = 0$ at these points.

2. The curve $y = f$ is has a *singularity* at $x = 3$.

This is because $f'(x) = DNE$ at this points.

3. The curve $y = f$ is has an *endpoint* at $x = -2$ and $x = 4$.

Theorem 14.2 *A function f can only have a maxima or minima at $x = a$ if one of the following are true*

1. f is horizontal at a . This happens when $f' = 0$.
2. f has a singularity at a . This happens when $f' DNE$.
3. a is an endpoint of the height function f .

The values of x where $f'(x) = 0$ or $f'(x) DNE$ are called the **critical points** of f . This is because finding these points is one of the most important steps when you are looking for the maxima and minima of the function f .

Warning: Not every critical point is a maxima or minima. In Figure ??, 2 is a critical point of f because $f'(2) = 0$, and so the function is horizontal at $x = 2$. However, $x = 2$ is *not* a local maxima or minima, since you can increase your height by moving right and decrease it by moving left.

To be able to check if a point is indeed a maxima or minima, we need a more precise definition.

Definition 14.3 Let f be a function which is continuous at $x = a$.

1. f has a **local maxima** at $x = a$ if the graph goes from increasing to decreasing at a (the function looks like $\nearrow \searrow$ at a).
2. f has a **local minima** at $x = a$ if the graph goes from decreasing to increasing at a (the function looks like $\searrow \nearrow$ at a).
3. f has an **extrema** at $x = a$ if there is either a maxima or minima at $x = a$.

◇

Theorem 14.4 *Suppose that f is a differentiable function at x*

1. *height f is increasing (\nearrow) at x if and only if the slope $f'(x)$ is*

positive (+)

2. height f is decreasing (\searrow) at x if and only if the slope $f'(x)$ is negative (-)

In Exploration ??, the function f is increasing on the interval $(-2, -1) \cup (1, 2) \cup (2, 3)$ because f' is positive on that interval.

Exploration 14.2 Let $f(x) = \frac{-x^3}{3} + x + 1$

- (a) Find the critical points of f .

Solution. Recall that the *critical points* of f are the x values where $f'(x) = 0$ or $f'(x)$ DNE.

To find when $f' = 0$ or f' DNE, we must first find an equation for the derivative f' .

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[\frac{-1}{3}x^3 + x + 1 \right] \\ &= \frac{d}{dx} \left[\frac{-1}{3}x^3 \right] + \frac{d}{dx} [x] + \frac{d}{dx} [1] \\ &= \frac{-1}{3} \frac{d}{dx} [x^3] + 1 + 0 \\ &= \frac{-1}{3} \cdot 3x^2 + 1 \\ &= -x^2 + 1 \end{aligned}$$

To find when $f' = 0$, simply solve the equation

$$-x^2 + 1 = 0$$

Solve the equation by factoring. It is easier to factor equations when the highest power term has a factor of 1. To do this, multiply both sides of the equation by -1, simplifying, and factoring

$$\begin{aligned} (-1) \cdot (-x^2 + 1) &= (-1) \cdot 0 \\ x^2 - 1 &= 0 \\ (x + 1)(x - 1) &= 0 \end{aligned}$$

The only way this can happen is if either the first or second terms equal 0. In other words, solve

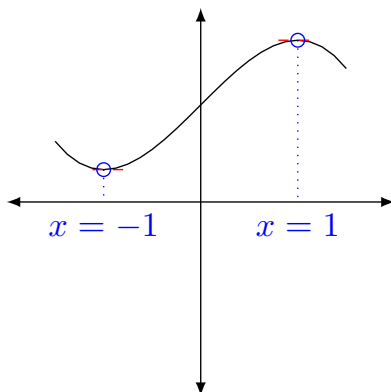
$$\begin{aligned} x + 1 &= 0 \text{ or} & x - 1 &= 0 \\ x &= -1 \text{ or} & x &= 1 \end{aligned}$$

We have a singularity when f' DNE. Looking at the graph of f , it does not look like there are any singularities.

Looking at the function, the derivative $f'(x) = -x^2 + 1$ is always defined. This confirms that this function f does not have any singularities.

- (b) Use a calculator to find the local extrema for f

Solution. Graphing the function we see



Looking at the graph, we can see that there is a local maxima at $x = 1$ and a local minima at $x = -1$. We also know that there are no other locations where there *might* be a maxima or minima, even off the screen of the calculator.

In particular, this tells us that there are *no* absolute maxima or minima.

- (c) Use the first derivative to find the local extrema for f . Find the intervals where f is increasing and when it is decreasing.

Solution. We are asked to locate the maxima and minima without looking at the graph of the function.

The *key idea* is that we can use the derivative function f' to make a very rough sketch of the function by using Theorem ??.

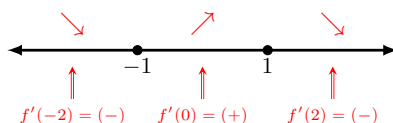
We have already used f' to find all values of x where $f' = 0$ or f' DNE. To make our sketch ***plot these points on a number line***.



Next, we must compute the derivative f' to the left and right of each of the points above. There are three intervals: to the left of -1, between -1 and 1, and to the right of 1. You can choose any number in those intervals.

left of -1	$f'(-2) = -(-2)^2 + 1 = -3 = (-)$
between -1 and 1	$f'(0) = -(0)^2 + 1 = 1 = (+)$
right of -1	$f'(2) = -(2)^2 + 1 = -3 = (-)$

Mark the magnitude of the derivative below the interval, and draw an \nearrow or \searrow above the line to indicate the behavior of the function.



This sign chart is a very rough sketch of the function, but it is complete enough to show that there is a local minima at $x = -1$ (due to the $\searrow \nearrow$) and that there is a local maxima at $x = 1$ (due to the $\nearrow \searrow$).

It is also clear from the sign chart that the function is increasing on the interval between -1 and 1. In interval notation, it is increasing on the interval $(-1, 1)$.

The function is decreasing on two intervals: to the left of -1 and to the right of 1. In interval notation, f is decreasing on the interval $(-\infty, -1) \cup (1, \infty)$.

A Strategy for Finding Extrema.

Let f be any function which is continuous on an open interval. To find the local extrema,

1. Use the derivative function f' to find where $f' = 0$ and when f' DNE.

- *Polynomials*: factor to find the 0's.

- *Fractions*: $\frac{A}{B}$ is equal to zero if when $A = 0$ and $B \neq 0$.
 $\frac{A}{B}$ is undefined when $B = 0$.

2. Plot these points on a number line. This cuts the real numbers into several intervals.

Use a closed dot for $f' = 0$ and an open circle for f' DNE.

3. Compute f' on each of the intervals you have found above. Mark whether f' is $(+)$ or $(-)$ below the interval, and draw \nearrow or \searrow above the interval.

4. Read off the answer to the question, using your sign chart as a very rough sketch of the function.

Exploration 14.3 Let $f(x) = \frac{-x^4}{4} + \frac{8}{3}x^3 + 4$

- (a) Find the intervals where f is increasing and decreasing..

Solution. First, we must find the critical points. In other words, find the x such that $f'(x) = 0$ or $f'(x)$ DNE.

First, find an equation for the derivative f' , and factor it.

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[\frac{-1}{4}x^4 + \frac{8}{3}x^3 + 4 \right] \\ &= \frac{-1}{4} \cdot 4x^3 + \frac{8}{3} \cdot 3x^2 + 0 \\ &= -x^3 + 8x^2 \end{aligned}$$

To find when $f' = 0$, simply solve the equation

$$-x^3 + 8x^2 = 0$$

Solve the equation by factoring. It is easier to factor equations when the highest power term has a factor of 1. To do this, factor out a $-x^2$ from both terms

$$(-x^2) \cdot (x - 8) = 0$$

The only way this can happen is if either the first or second terms equal 0. In other words, solve

$$\begin{array}{ll} x = 0 \text{ or} & x - 8 = 0 \\ x = 0 \text{ or} & x = 8 \end{array}$$

We have a singularity when f' DNE. Looking at the graph of f , it does not look like there are any singularities.

Looking at the function, the derivative $f'(x) = -x^2(x - 8)$ is always defined. This confirms that this function f does not have any singularities.

To find when the function is increasing and decreasing without looking at the graph of the function, we must use the derivative function f' to make a very rough sketch of the function by using Theorem ???. We will do this by making the sign chart of f' .



Next, we must compute the derivative f' to the left and right of each of the points above. There are three intervals: to the left of 0, between 0 and 8, and to the right of 8. You can choose any number in those intervals.

left of 0	$f'(-5) = (-5)^3 - 8(-5)^2 = \dots = (-)$
between 0 and 8	$f'(1) = (1)^3 - 8(1)^2 = \dots = (-)$
right of 8	$f'(10) = (10)^3 - 8(10)^2 = \dots = (+)$

Mark the magnitude of the derivative below the interval, and draw an \nearrow or \searrow above the line to indicate the behavior of the function.



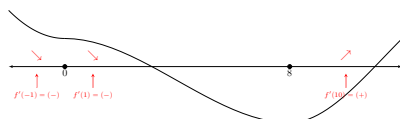
Using the sign chart, we can read off the behavior of the function.

f is *increasing* on the interval $(8, \infty)$, and f is *decreasing* on the interval $(-\infty, 0) \cup (0, 8)$

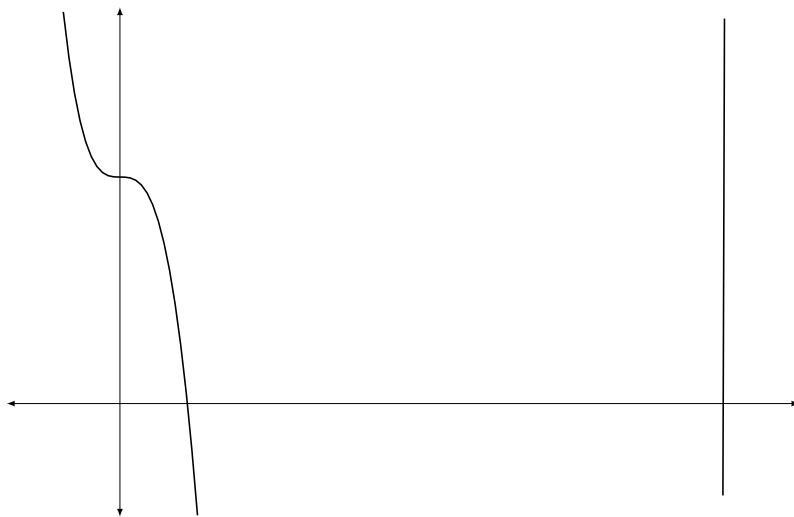
We can also see that f has a local maxima at $x = 8$, but f has *no* local minima (the local minima *DNE*).

- (b) Use your answers above to sketch the graph of f .

Solution.



Why would you want a sketch when you could just use a graphing calculator? In this case, trying to graph the function will give you the following image, which is harder to interpret, since it doesn't tell you what happens in the middle of the function, or off of the screen.



We now have a precise, repeatable strategy for finding maxima and minima. The following definition and theorem make this strategy precise.

Definition 14.5 The **critical numbers** of f are the numbers c such that

Both $f(c)$ is defined

And

$$\begin{array}{ll} \text{either} & f'(c) = 0 \\ \text{or} & f'(c) \text{ DNE} \end{array}$$

◇

Theorem 14.6 The First Derivative Test. Suppose that c is a critical number of a function f .

1. If f' goes from $(+)$ to $(-)$ at c (f goes from \nearrow to \searrow) then f has a local maximum at c .
2. If f' goes from $(-)$ to $(+)$ at c (f goes from \searrow to \nearrow) then f has a local minimum at c .
3. If f' does not change sign at c (if f goes $\searrow \searrow$ or $\nearrow \nearrow$), then f does not have a local maximum or minimum at c .

In some cases, we will have more knowledge about the derivative f' than we have about the height function. The key thing to notice is that we have been using the *slope function* f' to teach us about the *height function* f .

Exploration 14.4 Suppose that $f(x)$ is an unknown (or mystery) position function, with known derivative function

$$f'(x) = \frac{(x+3)e^x}{(x-2)^2}$$

- (a) Find the intervals where f is increasing and decreasing.

Solution. In this example, we are *given* a factored version of the derivative f' , so we do not need to use any of our rules.

Next, we must find the points where $f' = 0$ or f' DNE. Recall that a fraction $\frac{A}{B} = 0$ if and only if $A = 0$, and $\frac{A}{B}$ DNE if and only if $B = 0$. We must therefore solve two equations.

For the first,

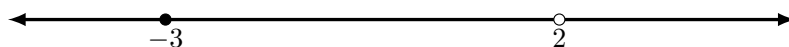
$$\begin{aligned} f' &= 0 \\ (x+3)e^x &= 0 \end{aligned}$$

Because $e^x \neq 0$ for all x , this can only happen if $x+3 = 0$, which means when $x = -3$.

For the second,

$$\begin{aligned} f' &\text{DNE} \\ (x-2)^2 &= 0 \\ x-2 &= 0 \\ x &= 2 \end{aligned}$$

Plotting these on a number line, we get



Recall that we were *given* the derivative function, which we now use to check the sign of each interval. We can either use a calculator as before

$$f'(-4) = \frac{(-4+3)e^{-4}}{(-4-2)^2} = \dots = (-)$$

$$f'(0) = \frac{(0+3)e^0}{(0-2)^2} = \dots = (+)$$

$$f'(3) = \frac{(3+3)e^3}{(3-2)^2} = \dots = (+)$$

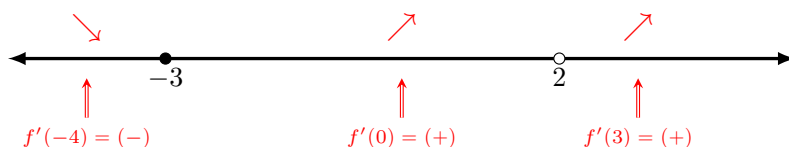
or we can do basic "sign arithmetic" in our head

$$f'(-4) = \frac{(-)(+)}{(-)^2} = \frac{(-)}{(+)} = (-)$$

$$f'(0) = \frac{(+)(+)}{(-)^2} = \frac{(+)}{(+)} = (+)$$

$$f'(3) = \frac{(+)(+)}{(+)^2} = \frac{(+)}{(+)} = (+)$$

Either way, we obtain



Reading off of the sign chart for f' , we see that the *original* (mystery) function f is increasing on the intervals $(-3, 2) \cup (2, \infty)$, and the *original* (mystery) function f is decreasing on the interval $(-\infty, -3)$.

- (b) Find the x values of all relative extrema.

Solution. Using the sign chart for f' , we see that there is a local minima at $x = -3$, and there is no local maxima.

Aside: Note that the something very strange happens at $x = -2$. With a little more information about the original function, we would learn that f is undefined at $x = -2$ and therefore could not be an extrema regardless of the sign chart. However, these details are beyond the scope of this course, and won't come up in any of our examples.

Checkpoint 14.7 Find all maxima and minima for the function

$$f(x) = \frac{x^2 + 1}{x}$$

Solution. First, we must find an equation for f' .

$$f'(x) = \frac{d}{dx} \left[\frac{x^2 + 1}{x} \right]$$

We are taking the derivative of a fraction, so we must use the quotient rule $\left[\frac{t}{b} \right]' = \frac{b \cdot t' - t \cdot b'}{b^2}$

$$\begin{aligned} f'(x) &= \frac{x[x^2 + 1]' - (x^2 + 1)[x]'}{(x)^2} \\ &= \frac{x(2x + 0) - (x^2 + 1)1}{(x)^2} \\ &= \frac{2x^2 - x^2 - 1}{x^2} \\ &= \frac{x^2 - 1}{x^2} \\ &= \frac{(x + 1)(x - 1)}{x^2} \end{aligned}$$

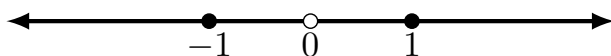
Next, we must find all values of x such that $f' = 0$ and f' DNE.

$$\begin{aligned} f' &= 0 \\ (x + 1)(x - 1) &= 0 \\ x &= 1, -1 \end{aligned}$$

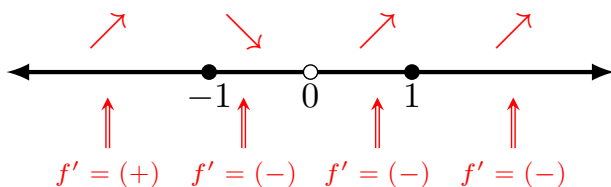
similarly

$$\begin{aligned} f' &\text{DNE} \\ x &= 0 \end{aligned}$$

Plotting these on the number line gives



Evaluating $f' = \frac{x^2 - 1}{x^2}$ on each of these intervals gives



The function has a local *maxima* at $x = -1$ and a local *minima* at $x = 1$

Note again that the original function is undefined (DNE) at $x = 0$, so $x = 0$ cannot be a max or a min regardless of the sign chart.

As we have seen, the steps in every max/min problem are the same

1. Find an equation for f' (if you are not already given it)
2. Solve $f' = 0$
3. Solve f' DNE
4. Make the sign chart for f'
5. Read off the answer.

14.2 Finding Absolute Extrema

In this class, we focus on local maxima and minima because of their close connections to calculus and their applications to marginal profit. But in many cases, global (also called absolute) maxima and minima are often of much greater interest. To see why, we begin with one more example.

Exploration 14.5 Suppose that $P(x) = x^3 - 6x^2 + 9x + 10$ is a total profit function.

- (a) Find the quantities where all local maximum and minimum profits occur.

Solution. First, find the marginal profit function

$$\begin{aligned} P'(x) &= \frac{d}{dx} [x^3 - 6x^2 + 9x + 10] \\ &= \frac{d}{dx} [x^3] - \frac{d}{dx} [6x^2] + \frac{d}{dx} [9x] + \frac{d}{dx} [10] \\ &= 3x^2 - 6 \cdot 2x^1 + 9 + 0 \\ &= 3x^2 - 12x + 9 \end{aligned}$$

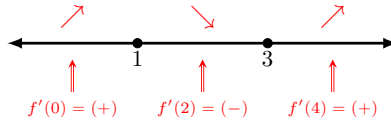
Next, find when $P' = 0$ or P' DNE

$$\begin{aligned} P' &= 0 \\ 3x^2 - 12x + 9 &= 0 \\ 3(x^2 - 4x + 3) &= 0 \\ 3(x - 3)(x - 1) &= 0 \end{aligned}$$

This happens if and only if $x = 3$ or $x = 1$, so these are the locations of the horizontal tangents.

On the other hand, because P' is a polynomial, it is always defined. Therefore, P' does not have any singular points.

Making a sign chart as usual gives



According to this analysis, a local maximum of profit occurs at $x = 1$, and a local minimum occurs at $x = 3$.

- (b) If you were able to decide how many items to sell, should you sell 1 item, 3 items, 10 items, or 100 items?

Solution. The easiest way to answer this question is to compute the total profit from each of the number of items given:

$$P(1) = (1)^3 - 6(1)^2 + 9(1) + 10 = 14$$

$$P(3) = (3)^3 - 6(3)^2 + 9(3) + 10 = 10$$

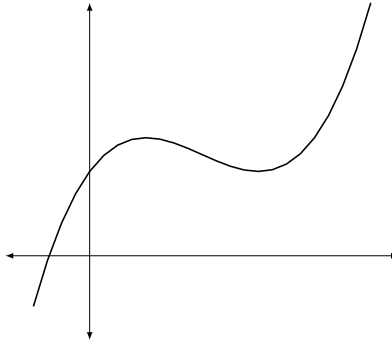
$$P(10) = (10)^3 - 6(10)^2 + 9(10) + 10 = 500$$

$$P(100) = (100)^3 - 6(100)^2 + 9(100) + 10 = 940,910$$

Clearly, if you are given these options you should sell 100 items and make a total profit of \$940,910

- (c) Use your work above to sketch the total profit function. Use this graph to explain how to decide how many items you should sell.

Solution. Graphing total profit shows that although the function decreases between $x = 1$ and $x = 3$, the function rapidly increases after $x = 3$ and then proceeds to grow without bound. In other words, you should always sell as many items as possible.

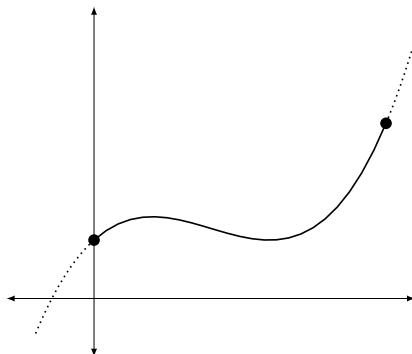


The example above helps illustrate the way that some mathematical models oversimplify reality. Mathematically, the function $P(x) = x^3 - 6x^2 + 9x + 10$ can grow without bound.

In real life, however, there are always limits on the quantities involved. In fact, there are only a finite number of atoms in the universe!

Definition 14.8 Real Life Restricts Quantities. Let f be any function and let l, r be any numbers.

The **restriction of $f(x)$ to the interval $[l, r]$** is the function that you get if you cut off the parts of the graph of the function that are less than l and greater than r . The image below shows the result of restricting $P(x) = x^3 - 6x^2 + 9x + 10$ to the interval $[0, 5]$. The restricted function is shown as solid, and the parts of the function that were removed are shown as dotted.



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Theorem 14.9 Suppose that f is a function restricted to the closed interval $[l, r]$. This restriction of f can have an absolute maxima or minima only at

- one of the endpoints (l or r)
- a horizontal tangent (a point c where $f'(c) = 0$)
- a singularity (a point c where $f'(c)$ DNE).

That means that to find *all* of the absolute extrema for f on a closed interval, it suffices to find the three kinds of *possible* locations listed above, and then to compare the *height* of f at each of these points.

Finding Absolute Extrema.

Suppose you are given a height function f . To find the absolute maximum height and minimum height of f on the interval $[l, r]$,

1. Find the derivative (slope) function $f'(x)$
2. Find all critical points c where $f'(c) = 0$ (horizontal tangents) or $f'(c)$ DNE (singular points).
3. Find the height of the function at each of the critical points c

4. Find the height of the function at the two endpoints (find $f(l)$ and $f(r)$).

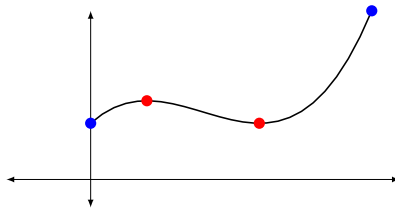
The above computation will give you things like $P(c_1) = y_1$, $P(c_2) = y_2, \dots$. If y_2 were the largest of all the y values, it would be the **absolute maximum** profit, and it would occur at quantity x_2 . If y_3 were the least of all the y values, it would be the **absolute minimum** profit, and it would occur at quantity x_3 .

Exploration 14.6 Let $P(x) = x^3 - 6x^2 + 9x + 10$

- (a) Find the absolute maximum and minimum profit for $P(x)$ restricted to the interval $[0, 5]$

Solution. In Exploration ??, we already found that the critical numbers for $P(x) = x^3 - 6x^2 + 9x + 10$ are $x = 1$ and $x = 3$

If we were to graph the function on the restricted domain, and to label the critical points, we would have



From our picture, it looks like the point at $x = 5$ will give us the largest profit, and the points at $x = 0$ and $x = 3$ will give us the same smallest profit

To confirm this, we evaluate the *original*, total profit function $P(x)$ at these four points:

$$P(1) = (1)^3 - 6(1)^2 + 9(1) + 10 = 14$$

$$P(3) = (3)^3 - 6(3)^2 + 9(3) + 10 = 10$$

$$P(0) = (0)^3 - 6(0)^2 + 9(0) + 10 = 10$$

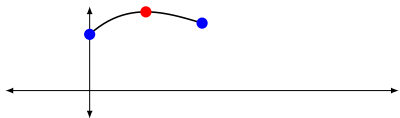
$$P(5) = (5)^3 - 6(5)^2 + 9(5) + 10 = 30$$

This confirms that on this interval, an absolute maximum profit of \$30 occurs when $x = 5$, and that an absolute minimum profit of \$10 occurs when either $x = 0$ or $x = 3$.

- (b) Find the absolute maximum and minimum profit for $P(x)$ restricted to the interval $[0, 2]$

Solution. In Exploration ??, we already found that the critical numbers for $P(x) = x^3 - 6x^2 + 9x + 10$ are $x = 1$ and $x = 3$

If we were to graph the function on the restricted domain, and to label the critical points, we would discover that the critical number $x = 3$ is *not* in our restricted domain, so we must leave it out.



From our picture, it looks like the point at $x = 1$ will give us the largest profit. It is not entirely clear which points will give us the same smallest profit

Therefore this, we evaluate the *original*, total profit function $P(x)$ at these three points:

$$P(1) = (1)^3 - 6(1)^2 + 9(1) + 10 = 14$$

$$P(0) = (0)^3 - 6(0)^2 + 9(0) + 10 = 10$$

$$P(2) = (2)^3 - 6(2)^2 + 9(2) + 10 = 12$$

We now see that on this interval, an absolute maximum profit of \$14 occurs when $x = 1$, and that an absolute minimum profit of \$10 occurs when $x = 0$.

14.3 Maxima, Minima, and Change Graphs

Why are the techniques in this chapter important for the study of economics and business?

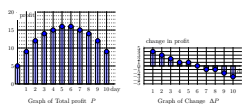
Beautiful Fact. The total profit function has a local maximum or minimum at the quantity c if and only if the marginal profit function P' goes between $(+)$ and $(-)$ at c .

Recall that the derivative P' and change function ΔP are closely related:

- **Total profit is increasing** when the **change in profit is positive**
- **Total profit is decreasing** when the **change in profit is negative**

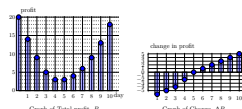
This means we can illustrate our method for finding maxima and minima using total and change graphs.

Recall that the graph on the left gives the total profit from selling a certain number of units, while the graph on the right shows the change in profit due to each unit.



In the graph above right, the change in profit goes from positive to negative at $x = 6$. That means that the total profit has a maximum at $x = 6$ units. This matches our graph of total profit in the graph above left exactly.

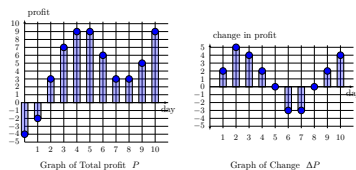
In the graph above right, the **change in profit** goes from **positive to negative** at $x = 6$. That means that the **total profit** has a **maximum** at $x = 6$ units. This matches our graph of total profit in the graph above left exactly.



In the graph above right, the change in profit goes from negative to positive at $x = 5$. That means that the total profit has a minimum at $x = 5$ units. This also matches exactly our graph of total profit, above left.

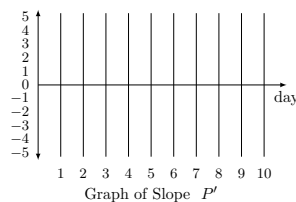
In the graph above right, the **change in profit** goes from **negative to positive** at $x = 5$. That means that the **total profit** has a **minimum** at $x = 5$ units. This also matches exactly our graph of total profit, above left.

Exploration 14.7



- (a) Use your knowledge of calculus to make a rough sketch of the derivative of profit.

Solution.



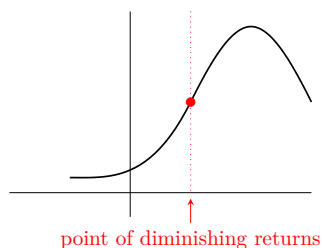
- (b) Use the graph you are given to fill in the table defining P and ΔP
- (c) Describe the maxima and minima for the profit function P

Diminishing Returns and Changing Rates

15.1 The Second Derivative

Recall that if $P(x)$ is total profit, then its slope function $P'(x)$ is used to approximate the profit from the *next* item.

The concept of **diminishing returns** plays an important role in economics.



- To the left of the point of diminishing returns, the *slope is increasing*.

That means that each item contributes more and more to the total profit. In other words, *returns are accelerating*.

- To the right of the point of diminishing returns, the *slope is decreasing*.

That means that each item contributes less and less to the total profit. In other words, *returns are decelerating*. At first, the items increase the profit, but by less and less. Eventually, the items cause the profit to decrease.

Most descriptions in economics of diminishing returns are very wordy, and somewhat vague. To make this concept precise, we need an additional mathematical concept.

Definition 15.1 Suppose that f is a differentiable function. The **second derivative** of f , written f'' , is a function that gives the *slope*

of f' . In symbols,

$$f''(x) = \frac{d}{dx} [f'(x)] = \frac{d}{dx} \left[\frac{df}{dx} \right] = \frac{d^2 f}{dx^2}$$

In practice, find the second derivative by finding the derivative of the derivative. \diamond

In physics,

- If $s(t)$ = position
- then $s'(t)$ = velocity
- and $s''(t)$ = acceleration.

Exploration 15.1 Suppose that $P(x) = -0.1x^2 + 30x - 1000$.

- (a) Find the marginal profit from items #10, #11, and #12.

Solution. The marginal profit function is $P'(x)$

$$\begin{aligned} P'(x) &= \frac{d}{dx} [-0.1x^2 + 30x - 1000] \\ &= -0.1 \cdot 2x + 30 - 0 \\ &= -0.2x + 30 \end{aligned}$$

The marginal profit function is equal to $P' = -0.2x + 30$. This tells us how much the *next item* contributes to the total profit.

$$\begin{aligned} P'(10) &= 28\$/item \\ P'(11) &= 27.8\$/item \\ P'(12) &= 27.6\$/item \end{aligned}$$

Notice that the 10th, 11th, and 12th items all increase the total profit. However, they contribute less and less to the total profit. In fact, the additional profit is decreasing by \$0.20 every additional item.

- (b) Find $P''(x)$

Solution. We have already found $P'(x)$.

We can now find the second derivative

$$\begin{aligned} P''(x) &= \frac{d}{dx} [P'(x)] \\ &= \frac{d}{dx} [-0.2x + 30] \\ &= -0.2 + 0 \\ &= -0.2 \$/item \text{ per item} \end{aligned}$$

This tells us that the *unit profit* is decreasing by \$0.20 \$/item for each additional item sold.

Exploration 15.2 Let $f(x) = \frac{3}{2x+1}$ is the position function. Find the acceleration $\frac{d^2f}{dx^2}$.

Solution. We must take the derivative twice.

$$f' = \frac{d}{dx} \left[\frac{3}{2x+1} \right]$$

You can find this derivative as written using the quotient rule. You can also rewrite the derivative as $\frac{d}{dx} [3 \cdot (2x+1)^{-1}]$ and use the chain rule.

We will use the quotient rule $\left[\frac{t}{b} \right]' = \frac{bt' - tb'}{b^2}$

$$\begin{aligned} f' &= \frac{(2x+1)0 - 3(2+0)}{(2x+1)^2} \\ &= \frac{-6}{(2x+1)^2} \\ &= \frac{-6}{4x^2 + 4x + 1} \end{aligned}$$

We can now take the derivative the second time, also using the quotient rule

$$\begin{aligned} f'' &= \frac{d}{dx} [f'] \\ &= \frac{d}{dx} \left[\frac{-6}{4x^2 + 4x + 1} \right] \\ &= \frac{(4x^2 + 4x + 1)0 - (-6)(4 \cdot 2x + 4 + 0)}{(4x^2 + 4x + 1)^2} \\ f'' &= \frac{48x + 16}{(4x^2 + 4x + 1)^2} \end{aligned}$$

Understanding Acceleration. A physical perspective provides one of the most straightforward ways to think about the connection between position, velocity, and acceleration. To make things concrete, suppose that you are driving on a highway, and that

- If $s(t)$ = your mile marker at time t
- then $s'(t)$ = is your velocity at time t
- and $s''(t)$ = your acceleration at time t . This tells us *how fast* the velocity s' is changing.

In other words,

Positive acceleration s'' means that velocity is *increasing*. This happens when you put your foot on the gas, as long as the car is in drive.

Negative acceleration s'' means that your velocity is *decreasing*. This happens when you put your foot on the break, as long as you are moving forward.

Exploration 15.3 Suppose that your *velocity* is given by the following table

t	0	1	2	3	4	5
$s'(t)$	60	65	85	30	10	0

- (a) On what interval is your acceleration s'' positive? Justify your answer

Solution. Positive acceleration means increasing velocity. The velocity increases between $t = 0$ and $t = 2$ (the velocity increases from 60 to 65 to 85).

That means that acceleration is positive between $t = 0$ and $t = 2$, which corresponds to the interval $(0, 2)$

- (b) On what interval is your acceleration s'' negative? Justify your answer

Solution. Negative acceleration means decreasing velocity. The velocity decreases between $t = 2$ and $t = 5$ (the velocity decreases from 85 to 30, to 10, to 0).

That means that acceleration is negative between $t = 2$ and $t = 5$, which corresponds to the interval $(2, 5)$.

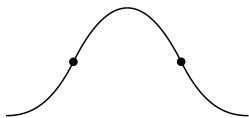
Example 15.2 Finding Maximum and Minimum Velocity.

The above examples show that the *velocity* function f' has a maximum or minimum value when the *acceleration* goes from $f'' = (+)$ to $f'' = (-)$ (for a maximum velocity f') or from $f'' = (-)$ to $f'' = (+)$ (for a minimum velocity f'). \square

15.2 Concavity and Graphs

In Chapter ??, we used the derivative function f' to give a very rough sketch of a height function f . However, this sketch is only a partial description of the original function.

For example, the following graph is more than just the sketch $\nearrow \searrow$



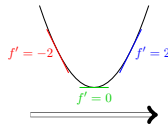
In particular, the graph first curves up, then curves down, and then curves up again.

Definition 15.3 The term **concave** means "curved like a bowl." In our context, there are two kinds of bowls: ones that open up, and ones that open down. Graphically, you can tell if a curve is concave up by putting eyes over the curve, and seeing if it is \smile or \frown \diamond

Concave Up The curve *opens up*



The slope f' is *increasing*

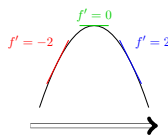


Because f' is increasing, the second derivative f'' is *positive*.

Concave Down The curve *opens down*



The slope f' is *decreasing*



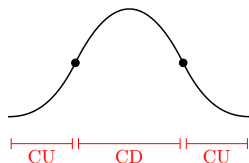
Because f' is decreasing, the second derivative f'' is *negative*.

We can summarize this as the following result.

Theorem 15.4 Suppose that f is a function with second derivative f'' .

- f is **concave up** on an interval if and only if the second derivative $f'' = (+)$ on that interval.
- f is **concave down** on an interval if and only if the second derivative $f'' = (-)$ on that interval.

Something interesting happens when concavity changes. Suppose the bell curve below represents the number of people ill with a new illness. In the bell curve below, the curve starts out concave up (CU), becomes concave down (CD), and ends concave up (CU).



The curve is *bent* at the two transitions.

- At the first "bending point," the number of new cases continues to rise. However, the daily change in cases (i.e. the slope f') stops getting bigger and starts getting smaller.

In other words, because the second derivative goes from $f'' = (+)$ to $f'' = (-)$, the fastest daily change in cases occurs at the first bend in the curve.

- At the second "bending point," the number of new cases continues to decrease. However, the daily change in cases (i.e. the slope f') starts to decrease less rapidly.

This is an important mathematical feature, but "bending point" does not a mathematical term. The word **inflect** means to bend, to change pitch, or to alter noticeably.

Definition 15.5 Suppose that f is a function with second derivative f'' . We say that f has an **inflection point** at x if and only if f changes concavity at x .

In other words, f has an inflection point at x if f'' goes between $(+)$ and $(-)$ at x . \diamond

To Find Intervals of Concavity.

1. Find an equation for f'' .

If you are given an equation for f , take the derivative twice.

If you are given an equation for f' , take the derivative once.

If you are given an equation for f'' , no derivative rules are needed.

2. Find all x such that $f''(x) = 0$.

3. Find all x such that $f''(x)$ DNE

4. Make a sign chart for f''

5. Read off your answer. If $f'' = (+)$, then f is \smile . If $f'' = (-)$, then f is \frown .

Exploration 15.4 Let $f(x) = x^3 - 9x^2 + 24x - 3$ be a function that gives the height of a mountain.

- (a) Find where f is concave up, where it is concave down, and find all

inflection points.

Solution. To find concavity and inflection points, we must work with the *second* derivative f'' . First, we must find f'' .

$$f'(x) = \frac{d}{dx} [x^3 - 9x^2 + 24x - 3] = 3x^2 - 18x + 24$$

$$f''(x) = \frac{d}{dx} [3x^2 - 18x + 24] = 6x - 18$$

Next, find when $f'' = 0$ by solving

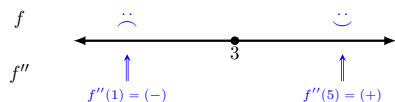
$$6x - 18 = 0$$

$$6x = 18$$

$$x = 3$$

Because f'' is a polynomial, it is defined on all x . In other words, there are no x that make f'' DNE

Next, make a sign chart for f'' as before, checking whether f'' is positive or negative on each interval.



Using our graph, we see that the function is concave up on the interval $(3, \infty)$, it is concave down on the interval $(-\infty, 3)$, and has a point of inflection at $x = 3$.

- (b) Find all local maxima and minima.

Solution. To find maxima and minima, we must work with the *first* derivative f' . We have already found that $f'(x) = \frac{d}{dx} [x^3 - 9x^2 + 24x - 3] = 3x^2 - 18x + 24$.

Next, we must find when $f' = 0$ by solving the equation

$$3x^2 - 18x + 24 = 0$$

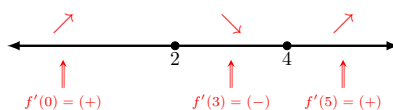
$$3(x^2 - 6x + 8) = 0$$

$$3(x - 4)(x - 2) = 0$$

$f' = 0$ if and only if $x = 4$ or $x = 2$.

Because f' is a polynomial, it is defined for all x . In other words, there are no places where f' DNE.

We now make a sign chart for f' as usual.



The function has a local maximum at $x = 2$, and a local minimum at $x = 4$.

Exploration 15.5 Suppose that f is an unknown (mystery) function with known *acceleration* function

$$f'' = \frac{(x+2)(x-7)}{x^2}$$

- (a) Find where the original function f is concave up or down.

Solution. You are *given* the formula for the derivative f'' , so you do *not* need to do any calculus.

$$f'' = \frac{(x+2)(x-7)}{x^2} = \frac{A}{B}$$

Recall that for fractions, $\frac{A}{B} = 0$ if and only if $A = 0$, and $\frac{A}{B}$ DNE if and only if $B = 0$

To find when $f'' = 0$, solve the equation

$$(x+2)(x-7) = 0$$

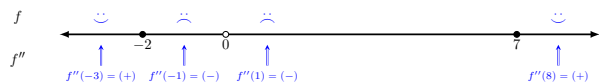
which happens when $x = -2$ or $x = 7$.

To find when f'' DNE, solve the equation

$$x^2 = 0$$

which happens when $x = 0$

Using these values, we can make the sign chart as before



Using the sign chart, we see that the *original/mystery* function f is concave up on the interval $(-\infty, -2) \cup (7, \infty)$. f is concave down on the interval $(-2, 0) \cup (0, 7)$.

There are inflection points (the curve bends) at $x = -2$ and $x = 7$.

We can combine all of the techniques we have used in the past section into a single problem.

Exploration 15.6 Let $f(x) = \frac{1}{5}x^5 - x^4$.

- (a) Find all maxima and minima for f .

Solution.

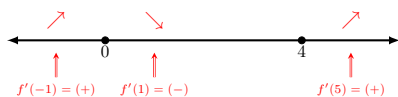
$$\begin{aligned} f' &= \frac{d}{dx} \left[\frac{1}{5}x^5 - x^4 \right] \\ &= \frac{1}{5}5x^4 - 4x^3 \\ &= x^4 - 4x^3 \end{aligned}$$

$f' = 0$ if and only if

$$\begin{aligned}x^4 - 4x^3 &= 0 \\x^3(x - 4) &= 0\end{aligned}$$

$f' = 0$ if $x = 0$ or $x = 4$.

f' is a polynomial so there are no values where f' DNE.



f has a local maximum at $x = 0$, and a local minimum at $x = 4$.

(b) Find all intervals of concavity for f .

Solution.

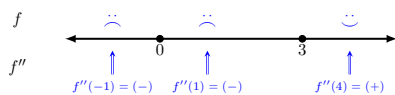
$$\begin{aligned}f'' &= \frac{d}{dx}[f'] = \frac{d}{dx}[x^4 - 4x^3] \\&= 4x^3 - 12x^2\end{aligned}$$

$f'' = 0$ when

$$\begin{aligned}4x^3 - 12x^2 &= 0 \\4x^2(x - 3) &= 0\end{aligned}$$

$f'' = 0$ when $x = 0$ or $x = 3$

f'' is a polynomial, so it is always defined. Making the sign chart for f'' gives.



The original function f is concave up on the interval $(3, \infty)$. The function f is concave down on the interval $(-\infty, 0) \cup (0, 3)$.

15.3 The Second Derivative and Change Graphs

So far, we've seen that if $f(x)$ is a function that represents a total amount of something, then the derivative function $f'(x)$ tells us how fast that total is changing.

We have also seen that simply computing the change function Δy gives a nice discrete analog of the derivative.

We have recently begun studying the second derivative. Recall that $f''(x)$ tells us how fast the slope of our function is changing.

As it turns out, we can get a discrete analog of the second derivative by computing the change in Δy . We might call the “change in the

change” function Δ^2y , since it involves finding two different changes.

Exploration 15.7

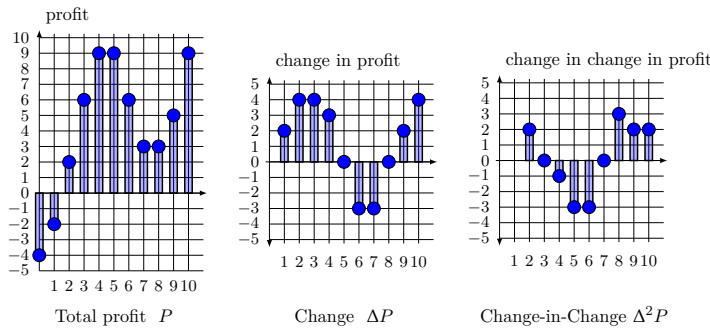


Table 15.6

day (x)	0	1	2	3	4	5	6	7	8	9	10
P											
ΔP		N/A									
$\Delta^2 P$		N/A	N/A								

1. What is the change in value ΔP due to the 2nd, 4th, and 6th items?
2. When is the function changing the fastest and slowest?
3. Which items contribute the maximum value? Which items contribute the minimum value?
4. How fast is the value changing between items 1 and 2? What about between items 2 and 3?
5. Fill in the remainder of the table above.

Exercise: Let $P(x) = 0.1x^3 - 1.4x^2 + 4.5x - 1$ be a function that describes the total profit as a function of the number of units x to be sold.

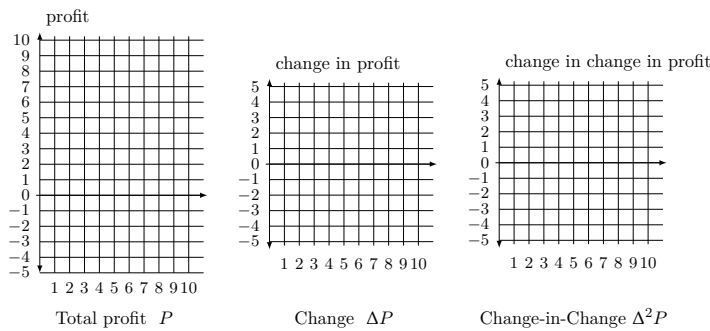


Table 15.7

day (x)	0	1	2	3	4	5	6	7	8	9	10
P											
ΔP		N/A									
$\Delta^2 P$		N/A	N/A								

1. Fill in the table above describing the total profit P , the change in profit ΔP , and the change in the change of profit $\Delta^2 P$. Then sketch graphs of all three functions.
2. For what intervals of x is the **change** in profit ΔP **increasing**? For what intervals of x is ΔP decreasing?
3. When is ΔP changing the fastest? How is this related to $\Delta^2 P$?

Exercise: Let $P(x) = -0.6x^2 + 3x - 1.4$ be a function that describes the total profit as a function of the number of units x to be sold.

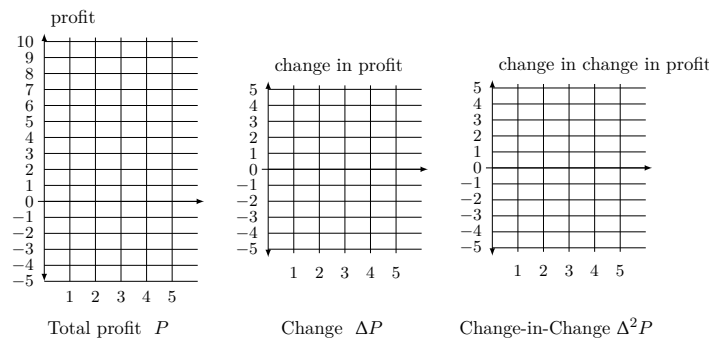


Table 15.8

day (x)	0	1	2	3	4	5
P						
ΔP		N/A				
$\Delta^2 P$		N/A	N/A			

1. Fill in the table above describing the total profit P , the change in profit ΔP , and the change in the change of profit $\Delta^2 P$. Then sketch graphs of all three functions.
2. For what intervals of x is the **change** in profit ΔP **increasing**? For what intervals of x is ΔP decreasing?
3. What does the fact that $\Delta^2 P$ is constant tell us about the graph of P ? What does this say about the relationship between the total profit function P and the various possible numbers of items x that you might intend to sell?

Part IV

Integral Calculus

16

Antiderivatives

16.1 *Undoing Derivatives*

In Chapter ??-??, we developed tools which let us turn a position function f into a slope function f' by taking the derivative.

But sometimes, we are *given* the slope function, such as the "mystery function" examples in as in Chapter ??.

In other words, sometimes f gives the slope of some *other* function. In this case, we want to know what the function was before taking the derivative.

Definition 16.1 We say that $F(x)$ is **an antiderivative** of $f(x)$ if

$$F'(x) = \frac{d}{dx}[F(x)] = f(x)$$

The prefix "ante" means "before." So if F is what you have *after* the derivative, f is what you have before (ante) the derivative. \diamond

We will see several methods for finding antiderivatives. The first and most foundational is we will call *guess, check, correct* (GCC).

Exploration 16.1 Finding antiderivatives using GCC.

- (a) Let $f(x) = x^5$. Find an antiderivative for $f(x)$

Solution. We want to find a function whose derivative is x^5 .

Guess 1 *Guess:* The derivative of any power function is a power function. So our first guess might as well be something like

$$F = x^5.$$

Check:

$$F'(x) = \frac{d}{dx} [x^5] = 5x^4 \neq x^5$$

Taking the derivative reduces the power by one, so if we want to end up with x^5 , we must start out with a x^6 term.

Guess 2: *Guess:* Now let's try guessing $F(x) = x^6$

Check:

$$F'(x) = \frac{d}{dx} [x^6] = 6x^5 \neq x^5$$

We now have the correct power, but the wrong coefficient. To fix our guess, we should start out with a coefficient that will *cancel out* the coefficient of 6.

Guess 3: *Guess:* $F(x) = \frac{1}{6}x^6$

Check:

$$\begin{aligned} F'(x) &= \frac{d}{dx} \left[\frac{1}{6}x^6 \right] \\ &= \frac{1}{6} \frac{d}{dx} [x^6] \\ &= \frac{1}{6} 6x^5 \\ &= x^5 \end{aligned}$$

That is the function we wanted *after* the derivative. Therefore $F(x) = \frac{1}{6}x^6$ is the *ante* derivative.

Therefore $\frac{1}{6}x^6$ is an antiderivative of x^5

(b) Let $f(x) = x^{11}$. Find an antiderivative for $f(x)$.

Solution. This is similar the last example, so your first guess might be the right one, but it is also OK to take a few guesses.

A general strategy is first to get the power of the antiderivative, and

then get the coefficient.

Guess 1 *Guess:* $F(x) = x^{12}$

Check:

$$F'(x) = \frac{d}{dx} [x^{12}] = 12x^{11} \neq x^{11}$$

Guess 2 *Guess:* $F(x) = \frac{1}{12}x^{12}$

Check:

$$F'(x) = \frac{d}{dx} \left[\frac{1}{12}x^{12} \right] = \frac{1}{12}12x^{11} = x^{11}$$

Therefore $\frac{1}{12}x^{12}$ is an antiderivative of x^{11}

- (c) Let $f(x) = x^n$ for some constant power $n \neq -1$. Find an antiderivative of $f(x)$.

Solution. We could do a multi-step guess-check-correct (GCC) strategy, or we could notice the pattern in the above examples:

- Increase the power by one
- Divide by the new power

Guess: $F(x) = \frac{x^{n+1}}{n+1}$ *Check:*

$$\begin{aligned} F'(x) &= \frac{d}{dx} \left[\frac{x^{n+1}}{n+1} \right] \\ &= \frac{1}{n+1} (n+1) \cdot x^n \\ &= x^n \end{aligned}$$

Therefore, $\frac{1}{n+1}x^{n+1}$ is an antiderivative of x^n

- (d) Let $f(x) = \frac{1}{x}$. Find an antiderivative of $f(x)$.

Solution. We want to have $\frac{1}{x}$ after taking a derivative.

Look at the Complete List of Derivative Rules. Make a guess, check it, and correct as needed.

Guess: $F(x) = \ln(x)$

Check: $\frac{d}{dx} [\ln(x)] = \frac{1}{x}$

Important: For technical reasons beyond the scope of this course, we actually need to place absolute values around the input to $\ln(x)$

Therefore $F = \ln(|x|)$ is an antiderivative of $f = \frac{1}{x}$.

- (e) Let $f(x) = e^x$. Find an antiderivative of $f(x)$

Solution. We want to have e^x after taking a derivative.

Look at the Complete List of Derivative Rules. Make a guess, check it, and correct as needed.

Guess: $F(x) = e^x$

Check: $\frac{d}{dx}[e^x] = e^x$

Therefore $F = e^x$ is an antiderivative of $f = e^x$.

- (f) Find the antiderivative of -2 .

Solution. We proceed by Guess, Check, Correct. There is no "correct" order of guesses, but once you get enough practice you will usually be able to get the right answer on your first guess.

Guess 1: *Guess:* $F(x) = -2$

Check: $F'(x) = \frac{d}{dx}[-2] = 0$

The derivative was zero because there was no x term at all.

Guess 2: *Guess:* $F(x) = -2x$

Check: $F'(x) = \frac{d}{dx}[-2x] = -2$

An antiderivative of $f(x) = -2$ is $F(x) = -2x$

Why do we keep asking for *an* antiderivative, instead of asking for *the* antiderivative?

Every f has multiple antiderivatives. Consider the function $f(x) = x^2$. There are *many* functions whose derivatives are x^2 :

$$\begin{aligned}\frac{d}{dx}\left[\frac{x^3}{3}\right] &= \frac{1}{3}3 \cdot x^2 &&= x^2 \\ \frac{d}{dx}\left[\frac{x^3}{3} + 5\right] &= \frac{1}{3}3 \cdot x^2 + 0 &&= x^2 \\ \frac{d}{dx}\left[\frac{x^3}{3} - 2\right] &= \frac{1}{3}3 \cdot x^2 + 0 &&= x^2\end{aligned}$$

Following this pattern, notice that if $F(x)$ is *one* antiderivative of $f(x)$, then $F(x) + C$ is *also* an antiderivative.

We have seen that for any C , $\frac{1}{3}x^3 + C$ is an antiderivative of x^2 . Are there any *other* antiderivatives of x^2 ? Fortunately, there are not.

Theorem 16.2 *If you find one antiderivative $F(x)$ to $f(x)$, then all other antiderivatives have the form*

$$F(x) + C$$

for some constant number C .

We call $F(x) + C$ the **general antiderivative** of f , abbreviated **the antiderivative** of f .

This gives us two kinds of antiderivative questions, based on Theorem ??.

- | | |
|--------------------------------|---|
| Find an antiderivative | Pick a specific number for the $+C$ |
| Find the antiderivative | Use a <i>letter</i> for the $+C$ to emphasize that <i>any</i> number works. |

Definition 16.3

- If $n \neq -1$, the general antiderivative of $f(x) = x^n$ is $F(x) = \frac{1}{n+1}x^{n+1} + C$
- The general antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln(|x|) + C$
- The general antiderivative of $f(x) = e^x$ is $F(x) = e^x + C$
- Suppose that a is a constant number. The general antiderivative of $f(x) = a$ is $F(x) = ax + C$.

◇

You can find many antiderivatives using *Guess, Check, Correct*, and working very carefully.

Exploration 16.2

- (a) Find the antiderivative of $f(x) = 14x^{-3}$

Solution. We know that the antiderivative of x^{-3} is $\frac{1}{-3+1}x^{-3+1} = \frac{1}{-2}x^{-2}$. This gives us the following guess for the antiderivative of $14x^{-3}$:

Guess: $F = 14 \cdot \frac{1}{-2}x^{-2} = -7x^{-2}$.

Check: $\frac{d}{dx}[-7x^{-2}] = -7 \frac{d}{dx}[x^{-2}] = -7(-2)x^{-2-1} = 14x^{-3}$

We have seen that $-7x^{-2}$ is *an* antiderivative of $14x^{-3}$. *The* antiderivative is

$$F(x) = -7x^{-2} + C$$

- (b) Find the antiderivative of $f(x) = 7 \cdot \frac{1}{x}$

Solution. To find the antiderivative of $7 \cdot \frac{1}{x}$, there are three steps

- Find an antiderivative of $\frac{1}{x}$
- Multiply the antiderivative of $\frac{1}{x}$ by 7
- Add $+C$ to get *the* antiderivative.

This means that if $f(x) = 7\frac{1}{x}$, the antiderivative is

$$F(x) = 7 \cdot \left(\ln(|x|) \right) + C$$

(c) Find the antiderivative of

$$f(x) = 6x^2 + 7e^x - 2$$

Solution. This function is the sum of three terms. Fortunately, the derivative of the sum is the sum of the derivatives. That means that we can separately find the antiderivative of each term.

From before,

- An antiderivative of $6x^2$ is

$$6 \frac{1}{2+1} x^{2+1} = 6 \cdot \frac{1}{3} x^3$$

- An antiderivative of $7e^x$ is

$$7e^x$$

- An antiderivative of -2 is

$$-2x$$

Putting these together, the antiderivative of

$$f(x) = 6x^2 + 7e^x - 2$$

is equal to

$$\begin{aligned} F(x) &= 6 \cdot \frac{1}{3} x^3 + 7e^x - 2x + C \\ &= 2x^3 + 7e^x - 2x + C \end{aligned}$$

Once you get some practice, you should get a good intuition for how to take antiderivatives "term by term" when dealing with sums and constant coefficients. We can also make this precise, as in the following theorem.

Theorem 16.4

- Suppose that one antiderivative of $f(x)$ is $F(x)$ and that c is a constant number. Then one antiderivative of $c \cdot f(x)$ is given by $c \cdot F(x)$.
- Suppose that one antiderivative of $f(x)$ is $F(x)$ and that one antiderivative of $g(x)$ is $G(x)$. Then one antiderivative of $f(x) + g(x)$ is given by $F(x) + G(x)$.

16.2 Indefinite Integrals

In the previous section, we developed a very intuitive approach to finding antiderivatives. In many situations, this will be the most efficient technique to use.

In this section, we will introduce a new notation for antiderivatives. This notation will make it easier to precisely state our anti-derivative formulas. Eventually, this notation will also connect to one of the most important applications of antiderivatives.

Definition 16.5 Indefinite Integral Notation. If the general antiderivative of $f(x)$ is $F(x) + C$, then we write

$$\int f(x) \, dx = F(x) + C$$

and say that the **indefinite integral** of $f(x)$ is $F(x) + C$

For now, treat the expression $\int f(x) \, dx$ as an abbreviation for the sentence “take the antiderivative (\int) of $f(x)$ with respect to the variable x (dx)”. \diamond

Indefinite Integral Formula List.

- If $m \neq -1$, $\int x^m \, dx = \frac{1}{m+1} x^{m+1} + C$
- $\int \frac{1}{x} \, dx = \ln(|x|) + C$
- $\int e^x \, dx = e^x + C$
- If a is a constant, $\int a \, dx = ax + C$
- If a is a constant, then $\int a \cdot f(x) \, dx = a \cdot \int f(x) \, dx$
- $\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$

Exploration 16.3

(a) Compute $\int x^{55} \, dx$.

Solution. We are asked to take the antiderivative of x^{55} . We could do that using our methods from before (including guess/check/correct) to say that the antiderivative is $F(x) = \frac{1}{56} x^{56}$

Because $55 \neq -1$, we can also use the indefinite integral formula $\int x^m \, dx = \frac{1}{m+1} x^{m+1} + C$

$$\begin{aligned} \int x^{55} \, dx &= \frac{1}{55+1} x^{55+1} \\ &= \frac{1}{56} x^{56} \end{aligned}$$

In this example, the notation is the only difference between guess/check/correct and $\int \dots dx$ notation. In more complex examples, $\int \dots dx$ notation will be much more helpful.

- (b) Compute $\int \frac{1}{\sqrt[3]{x}} dx$.

Solution. It is *extremely* hard to guess anything close to the antiderivative of the function $f(x) = \frac{1}{\sqrt[3]{x}}$. We also don't have any antiderivative formulas that relate to this situation. Fortunately, we can rewrite the function as $f(x) = x^{-1/3}$. It is now possible, but still difficult, to guess the right antiderivative.

Using indefinite integral notation, we can proceed more directly

$$\int \frac{1}{\sqrt[3]{x}} dx = \int x^{-1/3} dx$$

Because $\frac{-1}{3} \neq -1$, we can use the indefinite integral formula $\int x^m dx = \frac{1}{m+1}x^{m+1} + C$

$$\begin{aligned} &= \frac{1}{-1/3+1} x^{-1/3+1} \\ &= \frac{1}{2/3} x^{2/3} + C \\ &= \frac{1}{0.66\dots} x^{0.66\dots} \end{aligned}$$

- (c) Compute $\int \frac{5}{x} dx$.

Solution. Again, we can either find the antiderivative of $f(x) = \frac{5}{x} = 5\frac{1}{x}$ as before using our guess/check/correct approach, or we can use our indefinite integral formulas.

$$\begin{aligned} \int 5\frac{1}{x} dx &= 5 \int \frac{1}{x} dx \\ &= 5 \cdot \ln(|x|) + C \end{aligned}$$

We now have two different ways of describing the same concepts and techniques: antiderivatives and indefinite integrals. You can answer these questions using either approach.

Exploration 16.4 Let $f(x) = 2x^{-1} - 6\sqrt{x} + 10$

- (a) Compute $\int 2x^{-1} - 6\sqrt{x} + 10 dx$ using indefinite integral rules.

Solution.

$$\begin{aligned} \int 2x^{-1} - 6\sqrt{x} + 10 dx &= \int 2x^{-1} dx - \int 6\sqrt{x} dx + \int 10 dx \\ &= 2 \int x^{-1} dx - 6 \int \sqrt{x} dx + \int 10 dx \end{aligned}$$

For the first integral, because the exponent is -1 , we *cannot* use the

power rule. Instead, we must rewrite it as a fraction $x^{-1} = \frac{1}{x}$. For the second integral, we do not have a rule for square roots, so we *must* rewrite it as a power $\sqrt{x} = x^{1/2}$

$$\begin{aligned} &= 2 \int \frac{1}{x} dx - 6 \int x^{1/2} dx + \int 10 dx \\ &= 2 \ln(|x|) - 6 \cdot \frac{1}{1/2+1} x^{1/2+1} + 10x + C \\ &= 2 \ln(|x|) - 6 \cdot \frac{1}{3/2} x^{3/2} + 10x + C \\ &= 2 \ln(|x|) - 4x^{3/2} + 10x + C \end{aligned}$$

(b) Find the antiderivative of $f(x) = 2x^{-1} - 6\sqrt{x} + 10$.

Solution. We can also find the antiderivative of $f(x) = 2x^{-1} - 6\sqrt{x} + 10 = 2\frac{1}{x} - 6x^{1/2} + 10$ term-by-term. Then the antiderivative is

$$\begin{aligned} F(x) &= 2 \cdot \ln(|x|) - 6 \frac{1}{1/2+1} x^{1/2+1} + 10x + C \\ &= 2 \cdot \ln(|x|) - 6 \frac{1}{3/2} x^{3/2} + 10x + C \\ &= 2 \cdot \ln(|x|) - 4x^{3/2} + 10x + C \end{aligned}$$

Notice that this is *exactly the same* answer, and even the exact same computations as in computing $\int 2x^{-1} - 6\sqrt{x} + 10 dx$. The only difference between antiderivatives and indefinite integrals is the notation.

Exploration 16.5 Compute $\int 6x^2 + 3e^x + 1 dx$

Solution.

$$\begin{aligned} \int 6x^2 + 3e^x + 1 dx &= \int 6x^2 dx + \int 3e^x dx + \int 1 dx \\ &= 6 \int x^2 dx + 3 \int e^x dx + \int 1 dx \\ &= 6 \frac{1}{2+1} x^{2+1} + 3e^x + x + C \\ &= 6 \frac{1}{3} x^3 + 3e^x + x + C \\ &= 2x^3 + 3e^x + x + C \end{aligned}$$

16.3 Applying Antiderivatives

Recall from Section ??,

- If $s(t)$ = position
- then $s'(t)$ = velocity

- and $s''(t)$ = acceleration.

Application 16.6 Measuring position is hard, measuring acceleration is easier. It is actually very hard to measure position directly. Although GPS is widely used, the system itself is incredibly complex. From [Wikipedia](#)¹,

GPS ... is a satellite-based radionavigation system owned by the United States government and operated by the United States Space Force. It ... provides geolocation ... anywhere on or near the Earth where there is an unobstructed line of sight to four or more GPS satellites.

Special and general relativity predicted that the clocks on GPS satellites, as observed by those on Earth, run 38 microseconds faster per day than those on the Earth. The design of GPS corrects for this difference; because without doing so, GPS calculated positions would accumulate errors of up to 10 kilometers per day (6 mi/d).

On the other hand, it is much simpler to measure acceleration directly, and most modern electronics such as cell phones have built in self-contained accelerometers. By working backwards from acceleration s'' to velocity s' and then to position s , it is possible to use a sensor that measures s'' to model your position s .

For example, [Wikipedia](#)² observes that “Some pedometers use an accelerometer to more accurately measure the number of steps taken and distance traveled than a mechanical sensor can provide.”

Exploration 16.6 Suppose that $f'(x) = x^2 + x + 1$ gives your velocity in miles per hour, as a function of the time x in hours.

- (a) Find the antiderivative f of f' . What does this tell you about your position as a function of time?

Solution. The antiderivative of $f'(x) = x^2 + x + 1$ is

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$$

Your position function is *one* of the antiderivatives of your velocity. Notice that from your velocity only, we do *not* know what the constant C is. This prevents navigation based on velocity only, since knowing your position "plus or minus several thousand miles" is not very useful.

- (b) Suppose you also know that $f(1) = 5$. Find your exact position as a function of x .

Solution. Your position function is *one* of the antiderivatives of your velocity. We have already found that *the* (general) antiderivative f of f' is

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + C$$

¹ en.wikipedia.org/wiki/Global_Positioning_System

² en.wikipedia.org/wiki/Accelerometer#Consumer_electronics

We are now given an additional piece of information: that $f(1) = 5$, which says that your position at time $x = 1$ hours was 5 miles. Plugging 1 into our equation for $f(x)$ and setting this equal to 5 gives

$$\begin{aligned} f(1) = 5 &= \frac{1^3}{3} + \frac{1^2}{2} + 1 + C \\ 5 &= 1.833\ldots + C \\ 3.166\ldots &= C \end{aligned}$$

Putting the above information together, we can conclude that

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + 3.166\ldots$$

This gives your *exact* position at every time.

To work from acceleration to position, we need to take the antiderivative twice. If you have *two* additional pieces of information, you can use acceleration s'' to find an exact equation for position s .

Exploration 16.7 Suppose that $f''(x) = 6x + \frac{1}{x^2}$ gives your acceleration in miles per hour per hour, as a function of the time x in hours.

- (a) Find an equation for f using only the equation for f'' .

Solution. You are given that $f''(x) = 6x + \frac{1}{x^2}$.

$$\begin{aligned} f''(x) &= 6x + \frac{1}{x^2} \\ &= 6x^1 + x^{-2} \end{aligned}$$

The finding the antiderivative of f'' gives

$$\begin{aligned} f'(x) &= 6 \cdot \left(\frac{1}{1+1} x^{1+1} \right) + \left(\frac{1}{-2+1} x^{-2+1} \right) + C \\ &= 6 \cdot \left(\frac{1}{2} x^2 \right) + \frac{1}{-1} x^{-1} + C \\ &= 3x^2 - x^{-1} + C \\ &= 3x^2 - \frac{1}{x} + C \end{aligned}$$

We must find the antiderivative again.

First, note that the second term has exponent -1 , so we cannot use the power rule, and must use $\int \frac{1}{x} dx = \ln(|x|) + C$. Second, note that although it is unknown, C is a constant. Third, we must use a new variable for the unknown constant, since we have already used C .

Finding the antiderivative again f' gives

$$f(x) = 3 \left(\frac{1}{2+1} x^{2+1} \right) - \ln(|x|) + Cx + D$$

$$= x^3 - \ln(|x|) + Cx + D$$

Without knowing more, all we can say about the position function is that it has the form

$$f(x) = x^3 - \ln(|x|) + Cx + D$$

for some unknown constants C, D

- (b) Suppose that you also know that your velocity at time 1 is 4 miles per hour (that $f'(1) = 4$) and that your position at time 1 is 6 miles (that $f(1) = 6$). Find an exact equation for your position function f .

Solution. We are given that $f''(x) = 6x + \frac{1}{x^2}$.

Finding the antiderivative of f'' gives

$$\begin{aligned} &= 6 \cdot \left(\frac{1}{2}x^2\right) + \frac{1}{-1}x^{-1} + C \\ &= 3x^2 - \frac{1}{x} + C \end{aligned}$$

We are also given that $f'(1) = 4$. Plugging 1 into the above equation gives

$$\begin{aligned} f'(1) &= 4 = 3(1)^2 - \frac{1}{1} + C \\ 4 &= 3 - 1 + C \\ 2 &= C \end{aligned}$$

Plugging this into our equation for f' gives

$$f'(x) = 3x^2 - \frac{1}{x} + 2$$

Finding the antiderivative again gives

$$\begin{aligned} f(x) &= 3\left(\frac{1}{2+1}x^{2+1}\right) - \ln(|x|) + 2x + D \\ &= x^3 - \ln(|x|) + 2x + D \end{aligned}$$

We are also given that $f(1) = 6$. Plugging 1 into the above equation gives

$$\begin{aligned} f(1) &= 6 = (1)^3 - \ln(1) + 2(1) + D \\ 6 &= 1 - 0 + 2 + D \\ 3 &= D \end{aligned}$$

Plugging this into our equation for f gives

$$f(x) = x^3 - \ln(|x|) + 2x + 3$$

Total Cost is an antiderivative of Marginal Cost. Recall that if $C(x)$ is a total (cumulative) cost function, then the marginal cost function $C'(x)$ is a good approximation of the individual cost of the *next item only*.

In Section ??, we saw how to use the total function C to find the change function C' .

If we are instead given the change function C' , we can use antiderivatives to recover the original total cost C .

Exploration 16.8 Suppose that a product has *marginal cost* function

$$C'(x) = 36x + \frac{42}{x}$$

and the *total cost* of making 1 item is \$200.

- (a) Find the total cost function $C(x)$.

Solution. We are given that $C'(x) = 36x + \frac{42}{x}$

Finding the (general) antiderivative gives total cost function

$$\begin{aligned} C(x) &= 36 \cdot \frac{1}{2}x^2 + 42 \cdot \ln(x) + C \\ &= 18x^2 + 42 \ln(|x|) + C \end{aligned}$$

We are also told that 1 item costs \$200, which says $C(1) = 200$.

Plugging 1 into the equation above gives

$$\begin{aligned} C(1) &= 200 = 18(1)^2 + 42 \ln(|1|) + C \\ 200 &= 18 + 0 + C \\ 182 &= C \end{aligned}$$

Plugging this into the total cost function gives us that the total cost of x items is

$$C(x) = 18x^2 + 42 \cdot \ln(|x|) + 182$$

- (b) Find the total cost of manufacturing 100 items.

Solution. We have already found that the total cost function is $= 18x^2 + 42 \cdot \ln(|x|) + 182$.

The cost of manufacturing items 1 to 100 is given by

$$C(100) = 18(100)^2 + 42 \cdot \ln(|100|) + 182 = 180,375.41$$

The total cost is \$18,0375.41

Application 16.7 Understanding \int notation (Part I). To understand the meaning of \int notation, let's think of the *meaning* of cumulative and marginal cost.

We have already seen that if

- marginal cost = $C'(x) \approx$ item cost and
- total cost = $C(x)$, then
- $\int C'(x) dx = C(x) + a$ for some constant a .

Note: we have switched notation here to avoid confusion. In this case, since $C(x)$ refers to a function, we use a lower case a to denote our unknown constant.

Suppose you want to find the cost of items 10 through 100 only. There are two ways to find this

1. By *adding up* the *item costs* of items 10 to 100.

Because C' approximates the cost of the *next* item, this is approximated by computing

$$C'(9) + C'(10) + C'(11) + \cdots + C'(99)$$

2. By *subtracting* finding the *total cost* of items 1-9 from the *total cost* of items 1-100.

$$C(100) - C(9)$$

Condensing this notation, we get

$$\begin{array}{c} \text{to 99} \\ \text{Sum} \\ \text{from 9} \end{array} C'(x) \approx C(100) - C(9)$$

The art of approximating well with calculus is the subject of several courses (including *analysis* and *numerical analysis*).

For our purposes here, it is enough to say that 99 and 100 are "approximately equal," as are 9 and 10, so we can really say

$$\begin{array}{c} \text{to 100} \\ \text{Sum} \\ \text{from 10} \end{array} C'(x) \approx C(100) - C(10)$$

If you further simplify the word "Sum" to just the first letter S , you get

$$\begin{array}{c} \text{to 100} \\ S \\ \text{from 10} \end{array} C'(x) \approx C(100) - C(10)$$

which looks remarkably similar to

$$\int_{10}^{100} C'(x) dx = C(100) - C(10)$$

This similarity is *not* accidental. In fact, this is the entire reason we use the symbol \int for indefinite integrals and antiderivatives.

Understanding Areas

17.1 Indefinite Integrals

Recall from Section ?? that

$$\int f(x) \, dx \text{ means "find the antiderivative of" } f(x)$$

We called this the *indefinite integral* of $f(x)$.

In Application ??, we saw that this notation arises from the connection between marginal cost and item cost, and the fact that *summing* the item cost over a range gives you the *change* in total cost.

In this section, we will study this and related applications. We begin with the following definition.

Definition 17.1 Suppose that $F(x)$ is *any* antiderivative of $f(x)$, and that a, b are constant numbers. Then we define

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

the expression on the right is often abbreviated $\left[F(x)\right]_a^b$ \diamond

Exploration 17.1 Suppose that $f(x) = 36x + \frac{42}{x}$ is a marginal cost function.

(a) Evaluate $\int_{10}^{100} 36x + \frac{42}{x} \, dx$

Solution. To compute the integral

$$\int_{10}^{100} 36x + \frac{42}{x} \, dx$$

we must first find an antiderivative of. It is not difficult to see that *one* antiderivative of $f(x) = 36x + 42\frac{1}{x}$ is $F(x) = 36\frac{1}{2}x^2 + 42\ln(|x|)$.

By Definition ??, we can now compute

$$\int_{10}^{100} 36x + \frac{42}{x} \, dx = F(100) - F(10)$$

$$=[18x^2 + 42 \cdot \ln(|x|)]_{10}^{100}$$

Using a calculator, you can compute separately

$$F(100) = 18(100)^2 + 42 \ln(100) = 180,193.41$$

$$F(10) = 18(10)^2 + 42 \ln(10) = 1,896.70$$

We can now find the definite integral

$$\begin{aligned} \int_{10}^{100} 36x + \frac{42}{x} dx &= F(100) - F(10) \\ &= [18x^2 + 42 \cdot \ln(|x|)]_{10}^{100} \\ &= 178,296.71 \end{aligned}$$

- (b) What is the economic meaning of the expression

$$\int_{10}^{100} 36x + \frac{42}{x} dx?$$

Solution. Recall that

$$\int_{10}^{100} 36x + \frac{42}{x} dx = F(100) - F(10)$$

where $F(100)$ is the total cost of 100 items, and $F(10)$ is the total cost of 10 items.

That means that $\int_{10}^{100} 36x + \frac{42}{x} dx$ is the cost of items numbered 11 through 100.

- (c) Use an indefinite integral to find the *exact* exact cost of the 100th item only.

Solution. This question requires concepts and computations from Section ???. Take a moment to refresh your memory of this section.

In that section, to find the exact cost of the 100th item only, we need a total cost function $F(x)$.

Then $F(100)$ is the cumulative cost of items 0-100, and $F(99)$ is the cumulative cost of items 0-99.

That means that

$$\text{Cost of item 100} = F(100) - F(99) = \int_{99}^{100} 36x + \frac{42}{x} dx$$

We have already found that the cumulative cost function is $F(x) = 18x^2 + 42 \cdot \ln(|x|)$ Using this function we can compute

$$\begin{aligned} \int_{99}^{100} 36x + \frac{42}{x} dx &= F(100) - F(99) \\ &= 180,193.41 - 176,610.99 \\ &= 3,582.42 \end{aligned}$$

The next exploration explores what happens if we leave the function we are integrating the same, but change the limits of integration.

Exploration 17.2

- (a) Compute $\int_1^2 12x^2 - 2 \, dx$

Solution. One antiderivative of $f(x) = 12x^2 - 2$ is

$$\begin{aligned} F(x) &= 12 \cdot \frac{1}{3}x^3 - 2 \cdot x \\ F(x) &= 4x^3 - 2x \end{aligned}$$

We can now use Definition ??

$$\begin{aligned} \int_1^2 12x^2 - 2 \, dx &= F(2) - F(1) \\ &= [4x^3 - 2x]_1^2 \\ &= 28 - 2 \\ &= 26 \end{aligned}$$

- (b) Compute $\int_0^5 12x^2 - 2 \, dx$

Solution. One antiderivative of $f(x) = 12x^2 - 2$ is $F(x) = 4x^3 - 2x$. We can now use Definition ??

$$\begin{aligned} \int_0^5 12x^2 - 2 \, dx &= F(5) - F(0) \\ &= [4x^3 - 2x]_0^5 \\ &= 490 - 0 \\ &= 490 \end{aligned}$$

We have seen that integrals play an important role in connecting marginal and total cost functions.

But that is not the only, or even the most, important application of the derivative. The next is often called the *most important theorem in calculus*.

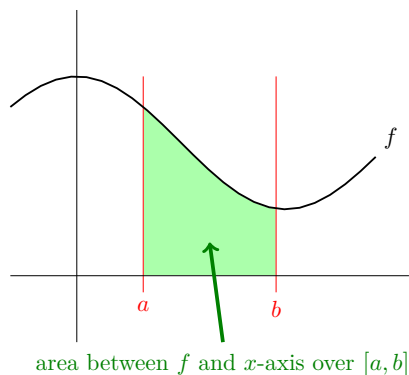
Theorem 17.2 Suppose that f is a height function, and that F is an antiderivative of f

Then the net area between f and the x -axis over the interval $[a, b]$ is equal to

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

In Section ??, we will discuss the exact meaning and intricacies of the phrase *the net area between f and the x axis on the interval $[a, b]$* . For now, the following illustration should suffice to show its general

meaning.



We will talk about *why* this theorem is true in Section ?? . For now, we will focus on computations and some preliminary applications.

Exploration 17.3

- (a) Let $f(x) = 15x^2 + 2$ describe the height of a mountain.

Find the net area between f and the x -axis over the interval $[1, 2]$.

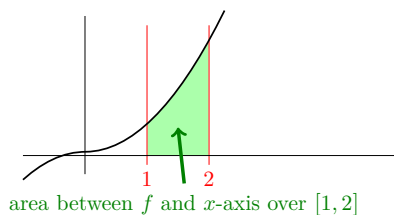
Solution. By Theorem ??, this means that we must compute

$$\int_1^2 12x^2 + 2 \, dx$$

One antiderivative of $f = 12x^2 + 2$ is $F = 6x^3 + 2x$

$$\begin{aligned} \int_1^2 12x^2 + 2 \, dx &= F(2) - F(1) \\ &= 52 - 8 \\ &= 44 \end{aligned}$$

Graphically, we have computed the area of the shaded region below:



- (b) Find the net area between $f(x) = \sqrt{x}$ and the x -axis between $x = 4$ and $x = 9$.

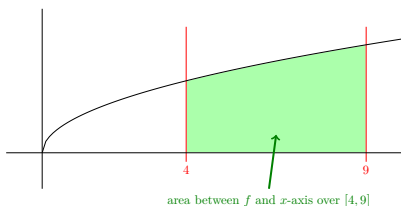
Solution. By Theorem ??, this means that we must compute

$$\int_4^9 \sqrt{x} \, dx = \int_4^9 x^{1/2} \, dx$$

One antiderivative of $f = x^{0.5}$ is $F = \frac{1}{0.5+1}x^{0.5+1} = \frac{x^{1.5}}{1.5}$

$$\begin{aligned}\int_4^9 x^{0.5} dx &= F(9) - F(4) \\ &= \left(\frac{9^{1.5}}{1.5}\right) - \left(\frac{4^{1.5}}{1.5}\right) \\ &= 18 - 5.33\dots \\ &= 12.66\dots\end{aligned}$$

Graphically, we have computed the area of the shaded region below:



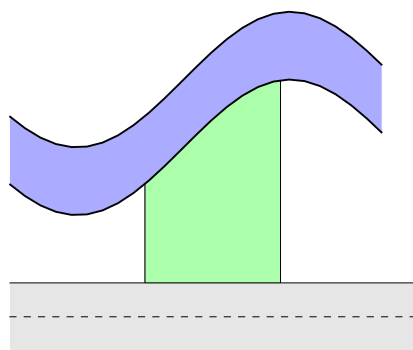
17.2 Understanding Geometrically of Area

In this course, our primary applications have come from macroeconomics, accounting and finance, and management. But the tools of calculus apply to a much broader set of applications.

For example, suppose that you are selling a plot of land enclosed between a stream and a road. From comparable recent sales, you know that the value of the land is \$12,800,000 \$ per square mile. You want to know how much your land is worth. You already know that

$$(\text{value}) = (\text{area}) \times (\text{price per unit area}).$$

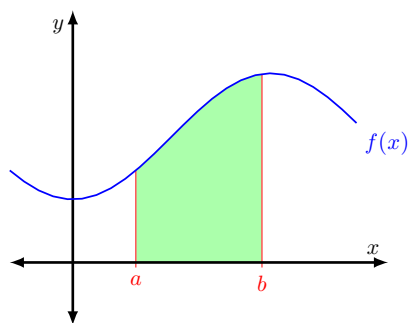
To get a numerical value, you need to be able to compute the *area enclosed*.



For triangles and squares, computing area is easy.

1. The area of a triangle is $\frac{1}{2} \cdot b \cdot h$
2. The area of a rectangle is $b \cdot h$

For other shapes, computing area is much more complex, and often requires more powerful mathematical tools. Translating our real estate question into mathematical language gives the following picture:



In this new language,

- The *river* is modeled by the function $y = f(x)$.
- The *road* is modeled by the x -axis.
- The *lot lines* are represented by the lines $x = a$ and $x = b$

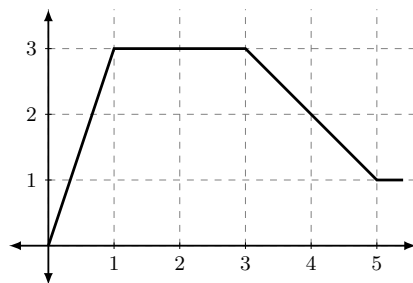
In this language, we want the *area enclosed between $f(x)$ and the x -axis on the interval $[a, b]$* .

We have already learned in Theorem ?? that

$$\int_a^b f(x) \, dx = \text{net area enclosed between } f(x) \text{ and } x\text{-axis over } [a, b]$$

But here, since we do not have an equation for the function, we must compute the area directly using geometry.

Exploration 17.4 Let $f(x)$ be defined by the graph below

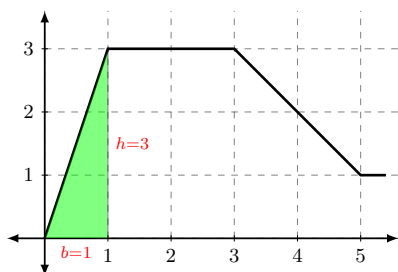


- (a) Compute the net area between f and the x -axis on the interval $[0, 1]$

Solution. The first step is to identify and shade the region described. Moving between $x = 0$ and $x = 1$, the area region enclosed on the top by $f(x)$ and the bottom by the x -axis is a triangle with base $b = 1$ and height $h = 3$.

This region has height

$$\text{area of triangle} = \frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot 3 = 1.5$$

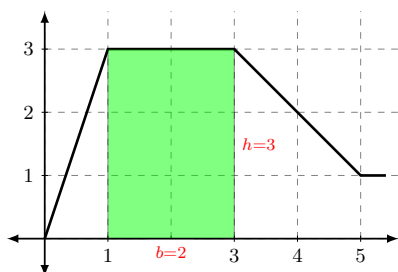


- (b) Compute the net area between f and the x -axis on the interval $[1, 3]$

Solution. The first step is to identify and shade the region described. Moving between $x = 1$ and $x = 3$, the area region enclosed on the top by $f(x)$ and the bottom by the x -axis is a rectangle with base $b = 2$ and height $h = 3$.

This region has area

$$\text{area of rectangle} = bh = 2 \cdot 3 = 6$$



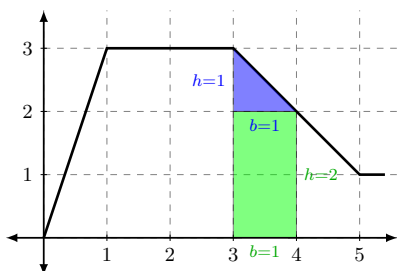
- (c) Compute the net area between f and the x -axis on the interval $[3, 4]$

Solution. The first step is to identify and shade the region described. Moving between $x = 1$ and $x = 3$, the area region enclosed on the top by $f(x)$ and the bottom by the x -axis is a *trapezoid*.

There is a formula for the area of a trapezoid, but it is easier and more flexible to just break up the shaded region into *two* parts: a triangle and a rectangle.

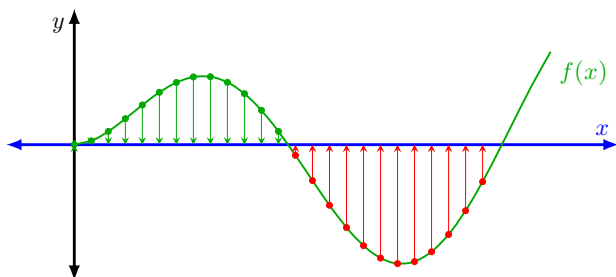
- The triangular piece has height $h = \Delta y = 3 - 2 = 1$ and base $b = \Delta x = 4 - 3 = 1$. Thus, the triangular piece has area $\frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot 1 = 0.5$
- The rectangular piece has height $h = \Delta y = 2 - 0 = 2$ and base $b = \Delta x = 4 - 3 = 1$. Thus, the rectangular piece has area $b \cdot h = 2 \cdot 1 = 2$

Therefore, the area of the entire enclosed region is $0.5 + 2 = 2.5$



We have now seen several examples that help explain what it means to talk about *area enclosed* between a function and the x -axis. But what does *net area* mean? To explore this, we need to switch to a new metaphor.

In the image below, suppose that $f(x)$ is the height of land at position x , and that the x -axis is the *water level*.



There are two parts of this graph

1. When f is *positive*, then the ground is above the water. In this case, *dry land* is the region enclosed between f and the x -axis

In this case, we say that f encloses **positive net area**.

2. When f is *negative*, then the water is above the ground. In this case *water* is the region enclosed between f and the x -axis

In this case, we say that f encloses **negative net area**.

More precisely, we make the following definition

Definition 17.3 Suppose that you are given any function $y = f(x)$, and two x values a, b . Let R represent the *region enclosed between f and the x -axis on the interval $[a, b]$*

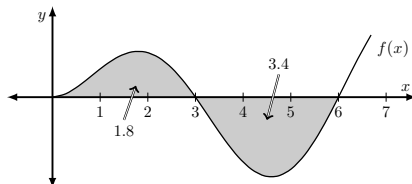
1. The **absolute** area of the region R is the area you treat the region as a geometric figure, and ignore the calculus setting of the question.
2. If the region R is *entirely above* the x -axis, then the region has **positive net area** equal to its absolute area.
3. If the region R is *entirely below* the x -axis, the region has **negative net area** equal to the negation of its absolute area.

4. If the region R is sometimes above and sometimes below the x -axis, you must divide it into separate components, find their net area, and then add them together.

◇

This definition is wordy, but fairly simple to apply in practice.

Exploration 17.5 Suppose that $f(x)$ is given by the graph below. In addition to the function, the absolute areas of several regions are indicated.



- (a) Find $\int_0^3 f(x) \, dx$

Solution. Between $x = 0$ and $x = 3$, the function $f(x)$ is entirely *above* the x -axis. That means that the *net* area of the region that it encloses is *positive*.

We are given that the absolute area of this first region is 1.8. It follows that

$$\int_0^3 f(x) \, dx = (+)1.8 = 1.8$$

- (b) Find $\int_3^6 f(x) \, dx$

Solution. Between $x = 3$ and $x = 6$, the function $f(x)$ is entirely *below* the x -axis. That means that the *net* area of the region that it encloses is *negative*.

We are given that the absolute area of this second region is 3.4. It follows that

$$\int_3^6 f(x) \, dx = (-)3.4 = -3.4$$

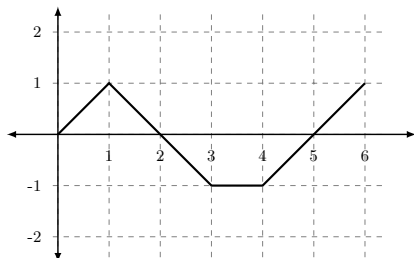
- (c) Find $\int_0^6 f(x) \, dx$

Solution. f crosses the x axis between $x = 0$ and $x = 6$. To find the net area enclosed on this interval, we must divide the function into the region between $0 < x < 3$, where f is entirely positive, and the region $3 < x < 6$, where f is entirely negative. We can then compute

$$\int_0^6 f(x) \, dx = \int_0^3 f(x) \, dx + \int_3^6 f(x) \, dx$$

$$\begin{aligned}
 &= (+)1.8 + (-)3.4 \\
 &= 1.8 - 3.4 \\
 &= -1.6
 \end{aligned}$$

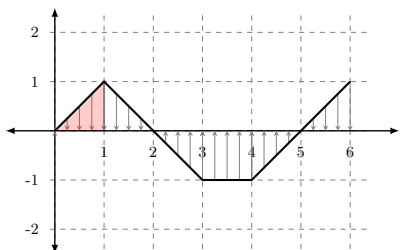
Exploration 17.6 Let $f(x)$ be defined by the following graph



(a) Find $\int_0^1 f(x) dx$

Solution. We want to find the area enclosed between the curve and the x -axis.

First, *shade* the area enclosed by following the curve, and drawing lines toward the x -axis. Then, identify the part of the enclosed area that corresponds to $0 < x < 1$ as in the following image.



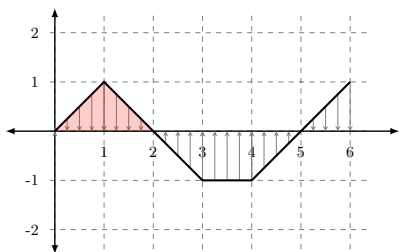
The shaded area is a triangle with base $b = 1$ and height $h = 1$. The region enclosed is *above* the x -axis so it has *positive* net area. Putting this together,

$$\int_0^1 f(x) dx = (+)\frac{1}{2} \cdot 1 \cdot 1 = 0.5$$

(b) Find $\int_0^2 f(x) dx$

Solution. We want to find the area enclosed between the curve and the x -axis.

First, *shade* the area enclosed by following the curve, and drawing lines toward the x -axis. Then, identify the part of the enclosed area that corresponds to $0 < x < 2$ as in the following image.



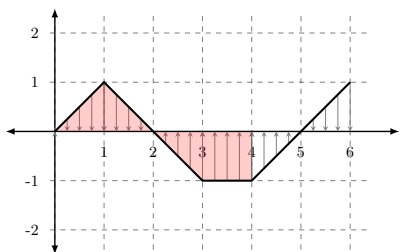
The shaded area is triangle with base $b = 2$ and height $h = 1$.
 The region enclosed is *above* the x -axis so it has *positive* net area.
 Putting this together,

$$\int_0^2 f(x) \, dx = (+) \frac{1}{2} \cdot 2 \cdot 1 = 0.5$$

(c) Find $\int_0^4 f(x) \, dx$

Solution. We want to find the area enclosed between the curve and the x -axis.

First, *shade* the area enclosed by following the curve, and drawing lines toward the x -axis. Then, identify the part of the enclosed area that corresponds to $0 < x < 4$ as in the following image.



The shaded area can be divided into *three* regions:

- (a) A triangle with base $0 < x < 2$ ($b = 2$) and height $h = 1$ *above* the x -axis.
- (b) A triangle with base $2 < x < 3$ ($b = 1$) and height $h = 1$ *below* the x -axis.
- (c) A rectangle with base $3 < x < 4$ ($b = 1$) and height $h = 1$ *below* the x -axis.

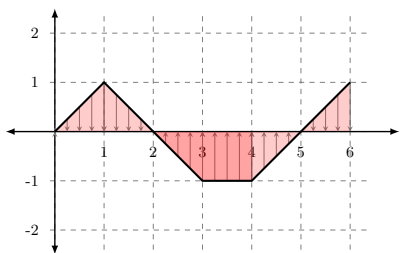
Putting this together,

$$\int_0^4 f(x) \, dx = (+) \frac{1}{2} \cdot 2 \cdot 1 + (-) \frac{1}{2} \cdot 1 \cdot 1 + (-) 1 \cdot 1 = -0.5$$

(d) Find $\int_0^6 f(x) \, dx$

Solution. We want to find the area enclosed between the curve and the x -axis.

First, *shade* the area enclosed by following the curve, and drawing lines toward the x -axis. Then, identify the part of the enclosed area that corresponds to $0 < x < 4$ as in the following image.



The shaded area can be divided into *five* regions:

- (a) A triangle with base $0 < x < 2$ ($b = 2$) and height $h = 1$ *above* the x -axis.
- (b) A triangle with base $2 < x < 3$ ($b = 1$) and height $h = 1$ *below* the x -axis.
- (c) A rectangle with base $3 < x < 4$ ($b = 1$) and height $h = 1$ *below* the x -axis.
- (d) A triangle with base $4 < x < 5$ ($b = 1$) and height $h = 1$ *below* the x -axis.
- (e) A triangle with base $5 < x < 6$ ($b = 1$) and height $h = 1$ *above* the x -axis.

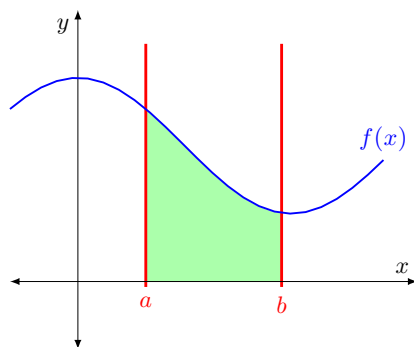
Putting this together,

$$\int_0^6 f(x) \, dx = (+)\frac{1}{2} \cdot 2 \cdot 1 + (-)\frac{1}{2} \cdot 1 \cdot 1 + (-)1 \cdot 1 + (-)\frac{1}{2} \cdot 1 \cdot 1 + (+)\frac{1}{2} \cdot 1 \cdot 1 = -0.5$$

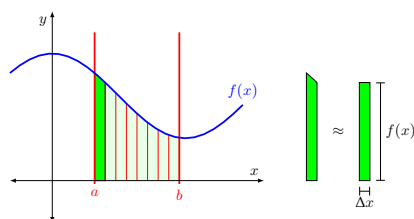
Application 17.4 Understanding \int notation (Part II). So far, we have seen the notation $\int_a^b f(x) \, dx$ in two very different contexts: antiderivatives and geometric area. But why are these concepts connected?

We have already seen the foundational idea: given a complex area we have *sliced* it into *small regions*, where we can find the area of each slice more easily.

Suppose you are given any curved function f an enclosed area you want to compute as below.



Slicing the shaded region into thin, vertical strips gives the image below on the left.



Notice that if the function is continuous (doesn't jump) and if the strips are thin enough, each strip will *approximately* be a rectangle with height $h = f(x)$ and base $b = \Delta x$. This is illustrated in the image above on the right.

- The area of *one* strip will be approximately $h \cdot b = f(x) \cdot \Delta x$.
- The area of the *whole* region is *approximately* the Sum of all slices from $x = a$ to $x = b$. Mathematically, this is written

$$\begin{aligned} \text{area of whole} &\approx \text{Sum of slices from } a \text{ to } b \\ &\approx \text{Sum of } f(x) \cdot \Delta x \text{ from } a \text{ to } b \end{aligned}$$

To emphasize that the slices are very thin, we replace the blocky Δx with a smooth dx , and we summarize the word "Sum" with the symbol \int

$$\begin{aligned} &\approx \text{Sum of } f(x) \cdot dx \text{ from } a \text{ to } b \\ &\approx \int_a^b f(x) \, dx \end{aligned}$$

17.3 Area, Antiderivatives, and Graphs of Change

Theorem 17.5 The Fundamental Theorem of Calculus (FTC).

Suppose that $f(x)$ is any continuous function, and that $F(x)$ is any antiderivative of $f(x)$. Then the net area enclosed between f and the

x -axis on the interval $[a, b]$ is given by

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

In other words, *antiderivatives* give *net area*, and *net area* gives *antiderivatives*. Given the many connections between derivatives and concepts in economic and business, this becomes a powerful set of tools.

1. The derivative of total cost C is the marginal cost C' .

Therefore, the total cost C is an *antiderivative* of marginal (item) cost C' .

It follows that the area under marginal (item) cost gives change in total cost.

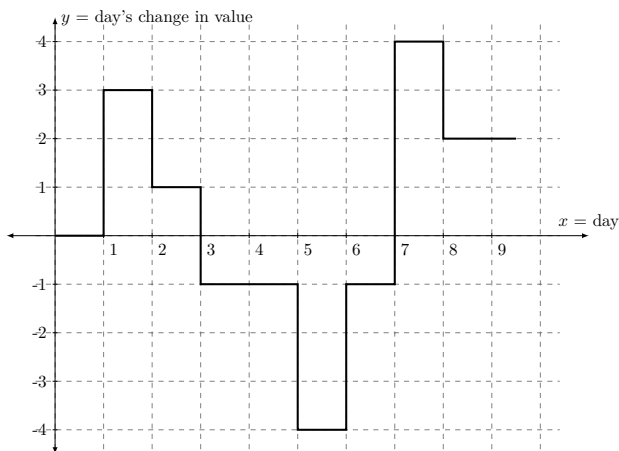
2. Let $f(x)$ give the value of a stock on day x . Then the *derivative* of value f' tells you how fast the value is changing on each day.

Therefore, the value function f is an *antiderivative* of the change in value function f' .

It follows that the area under the change in value function cost gives the change in actual value.

This line of reasoning extends to *any and all* graphs of rates of change.

Exploration 17.7 The graph of the *daily change* $V'(x)$ (in \$/day) in the price of a certain stock is given below.



- (a) Find the change in stock price V between days 1 and 5.

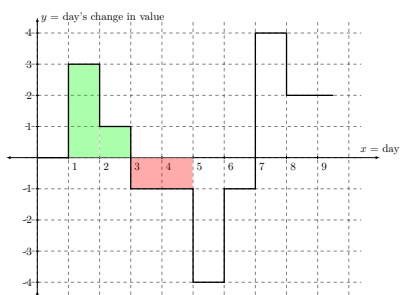
Solution. Recall that the total value V is the antiderivative of the change in value function V' . By Theorem ??, the area under V' is

equal to the net change in V . Looking at the graph of V' , we see

- Day 1** Between day 1 and 2, the area enclosed by V' is a rectangle with height $h = 3$ and base $b = 1$. That means the value increases by $3 \cdot 1 = 3\$$ between day 1 and 2.
- Day 2** Between day 2 and 3, the area enclosed by V' is a rectangle with height $h = 1$ and base $b = 1$. That means the value increases by $1 \cdot 1 = 1\$$ between day 2 and 3.
- Day 3 & 4** Between day 3 and 5, the area enclosed by V' is a rectangle with height $h = -1$ and base $b = 2$. That means the value increases by $(-1) \cdot 2 = -2\$$ between day 3 and 5.

Putting it together, the change in value between day 1 and 5 equals

$$V(5) - V(1) = \int_1^5 V'(x) dx = 3 + 1 - 2 = 2$$



- (b) Find the change in stock price between days 1 and 7.

Solution. Recall that the total value V is the antiderivative of the change in value function V' . By Theorem ??, the area under V' is equal to the net change in V . Looking at the graph of V' , we see

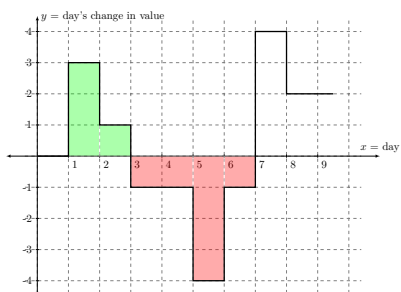
- Day 1** Between day 1 and 2, the area enclosed by V' is a rectangle with height $h = 3$ and base $b = 1$. That means the value increases by $3 \cdot 1 = 3\$$ between day 1 and 2.
- Day 2** Between day 2 and 3, the area enclosed by V' is a rectangle with height $h = 1$ and base $b = 1$. That means the value increases by $1 \cdot 1 = 1\$$ between day 2 and 3.
- Day 3 & 4** Between day 3 and 5, the area enclosed by V' is a rectangle with height $h = -1$ and base $b = 2$. That means the value increases by $(-1) \cdot 2 = -2\$$ between day 3 and 5.

Day 5 Between day 5 and 6, the area enclosed by V' is a rectangle with height $h = -4$ and base $b = 1$. That means the value increases by $(-4) \cdot 1 = -4\$$ between day 5 and 6.

Day 6 Between day 6 and 7, the area enclosed by V' is a rectangle with height $h = -1$ and base $b = 1$. That means the value increases by $(-1) \cdot 1 = -1\$$ between day 6 and 7.

Putting it together, the change in value between day 1 and 7 equals

$$V(7) - V(1) = \int_1^7 V'(x) \, dx = 3 + 1 - 2 - 4 - 1 = -3$$



(c) Find the change in stock price between days 5 and 8.

Solution. Recall that the total value V is the antiderivative of the change in value function V' . By Theorem ??, the area under V' is equal to the net change in V . Looking at the graph of V' , we see

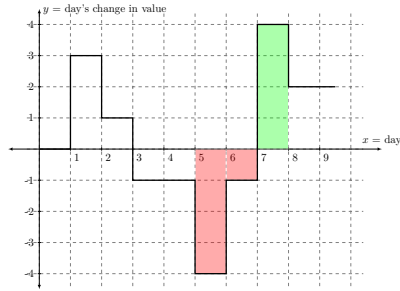
Day 5 Between day 5 and 6, the area enclosed by V' is a rectangle with height $h = -4$ and base $b = 1$. That means the value increases by $(-4) \cdot 1 = -4\$$ between day 5 and 6.

Day 6 Between day 6 and 7, the area enclosed by V' is a rectangle with height $h = -1$ and base $b = 1$. That means the value increases by $(-1) \cdot 1 = -1\$$ between day 6 and 7.

Day 7 Between day 7 and 8, the area enclosed by V' is a rectangle with height $h = 4$ and base $b = 1$. That means the value increases by $(+)4 \cdot 1 = 4\$$ between day 7 and 8.

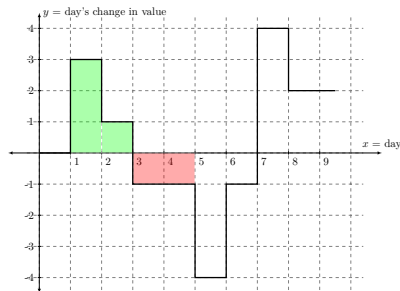
Putting it together, the change in value between day 5 and 8 equals

$$V(8) - V(5) = \int_5^8 V'(x) \, dx = -4 + -1 + 4 = -1$$



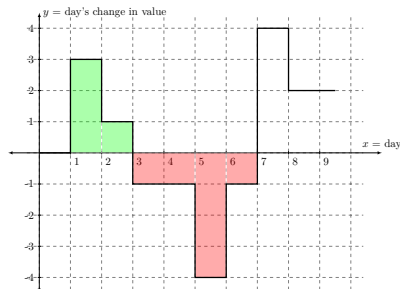
- (d) Suppose the stock price on day 1 was \$10. Find the stock price at day 5.

Solution. We have already seen that the value of the stock changes by \$+2 between days 1 and 5. If the value on day 1 was \$10, then the value on day 5 is $\$10 + (+2) = \12 .



- (e) Suppose the stock price on day 1 was \$10. Find the stock price at day 7.

Solution. We have already seen that the value of the stock changes by \$-3 between days 1 and 7. If the value on day 1 was \$10, then the value on day 5 is $\$10 + (-3) = \7 .



Appendices

A

Algebra Review

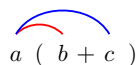
A.1 Laws of Algebra

Distributing and FOIL-ing.

Distribute a to all terms:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

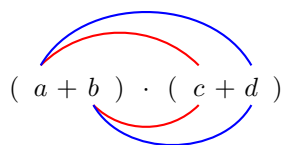
The best way to remember this is by drawing arcs from a to b and c


$$a \ (\ b + c \)$$

FOIL the expression:

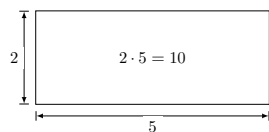
$$(a + b) \cdot (c + d) = (a \cdot c) + (a \cdot d) + (b \cdot c) + (b \cdot d)$$

The best way to remember this formula is by drawing arcs from a and b to c and d .

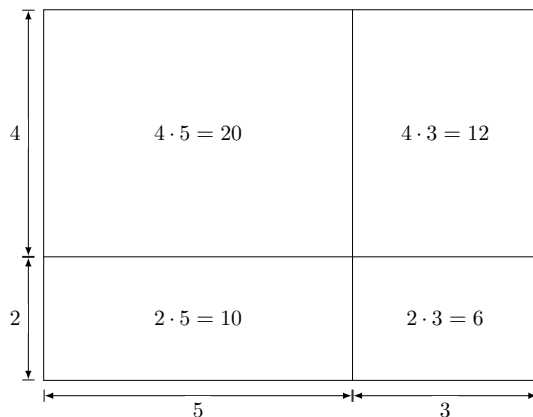

$$(\ a + b \) \cdot (\ c + d \)$$

Example A.1 Understanding FOIL-ing. The mathematical expression for simplifying $(a + b) \cdot (c + d)$ can look a bit mysterious. To understand why it looks the way it does, it is helpful to remember the connection between multiplication and area.

For example, if you have a rectangle of side lengths 5 and 2, then the rectangle has area $5 \cdot 2 = 10$. Visually, we have the following picture.



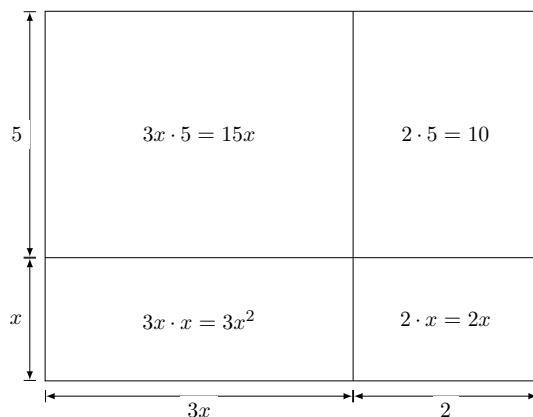
Suppose instead we increase the lengths of the sides to $8 = 5 + 3$ and $6 = 2 + 4$. Let's see how this changes our area.



The area given by $(5 + 3) \cdot (2 + 4)$ has four regions, each with their own area

$$5 \cdot 2 + 5 \cdot 4 + 3 \cdot 2 + 3 \cdot 4$$

The same mathematical pattern occurs when you have variables in your side lengths. Suppose you have a rectangle with one side $3x + 2$ and another side $x + 5$. Then the area of the rectangle still has four regions



Adding up the areas of those regions gives us the same thing as FOIL-ing

$$(3x + 2)(x + 5) = (3x \cdot x) + (3x \cdot 5) + (2 \cdot x) + (2 \cdot 5)$$

Simplifying that down we get

$$(3x + 2)(x + 5) = 3x^2 + 17x + 10$$

Of course, with practice, it is faster and easier to learn to use the formula for FOIL-ing than would be to go through the area derivation each time. \square

Exploration A.1 Use the laws of algebra listed above to simplify the following expressions completely.

(a) $5(x^2 - 3x + 2)$

Solution. We must distribute the 5 to every term of the expression

$$5(x^2 - 3x + 2) = 5x^2 - 15x + 10$$

(b) $(-1)(x - 5)$

Solution. We must distribute a -1 to each term

$$\begin{aligned} (-1)(x - 5) &= (-1)x - (-1)5 \\ &= -x + 5 \end{aligned}$$

(c) $(x + 6)(x - 2)$

Solution. We must foil the expression using the formula

$$(a + b) \cdot (c + d) = (a \cdot c) + (a \cdot d) + (b \cdot c) + (b \cdot d)$$

with $a = x$, $b = 6$, $c = x$, and $d = -2$. In other words, we have

$$\begin{aligned} (x + 6)(x - 2) &= (x \cdot x) + (x \cdot -2) + (6 \cdot x) + (6 \cdot -2) \\ &= x^2 - 2x + 6x - 12 \\ &= x^2 + 4x - 12 \end{aligned}$$

(d) $(5x - 1)(2x + 1)$

Solution. We must foil the expression using the formula

$$(a + b) \cdot (c + d) = (a \cdot c) + (a \cdot d) + (b \cdot c) + (b \cdot d)$$

with $a = 5x$, $b = -1$, $c = 2x$, and $d = 1$. In other words, we have

$$\begin{aligned} (5x - 1)(2x + 1) &= (5x \cdot 2x) + (5x \cdot 1) + (-1 \cdot 2x) + (-1 \cdot 1) \\ &= 10x^2 + 5x - 2x - 1 \\ &= 10x^2 + 3x - 1 \end{aligned}$$

(e) $(2x + 3)^2$

Solution. Squaring an expression means multiplying it by itself twice. In other words,

$$\begin{aligned} (2x + 3)^2 &= (2x + 3) \cdot (2x + 3) \\ &= (2x \cdot 2x) + (2x \cdot 3) + (3 \cdot 2x) + (3 \cdot 3) \\ &= 4x^2 + 6x + 6x + 9 \\ &= 4x^2 + 12x + 9 \end{aligned}$$

(f) $(-3)(2x + 3)^2$

Solution. We must use order of operations to decide which operation to perform first. Remember that PEMDAS (parentheses, exponents, multiplication/division, addition/subtraction) tells us which operations attach more "strongly", and must be performed first.

- (a) Here, the parentheses group the expression into two parts: $A \cdot B^2$ with $A = (-3)$ and $B = (2x + 3)$.
- (b) Exponentiation is stronger than multiplication, so we must first square $(2x + 3)$. As in the previous problem, we have

$$(-3)(2x + 3)^2 = (-3) \cdot (4x^2 + 12x + 9)$$

- (c) Now, we can distribute -3 to each term.

$$\begin{aligned} &= (-3) \cdot (4x^2 + 12x + 9) \\ &= (-3 \cdot 4)x^2 + (-3 \cdot 12)x + (-3 \cdot 9) \\ &= -12x^2 - 36x - 27 \end{aligned}$$

- (g) Factor the following expression as much as possible $9x - 15$

Solution. First, factor each term in the expression. There is a 3 in each term, so we can factor it out.

$$\begin{aligned} 9x - 15 &= (3 \cdot 3)x - (3 \cdot 5) \\ &= 3(3x - 5) \end{aligned}$$

There are no remaining common terms, so we are done factoring.

- (h) Factor the following expression as much as possible $x^2 + x - 6$

Solution. We want to write $x^2 + x - 6$ as an expression of the form $(ax + b)(cx + d)$

$$(ax + b)(cx + d) = (ab)x^2 + (ad + cb)x + bd$$

We want to pick a, b, c, d to get the right hand side to equal $x^2 + x - 6$. There are three terms to consider

- We need $(ab)x^2 = x^2$, so set $a = b = 1$. Plugging that into our factored form, we get

$$(x + b)(x + d) = x^2 + (d + b)x + bd$$

- We need the third term to be $db = -6$. Factoring gives $db = (-1) \cdot 2 \cdot 3$. We must divide the factors $-1, 2, 3$ between b, d .

- To pick exact values for b, d , recall that we need $b + d = 1$. Setting $b = 3$ and $d = (-1)2$, we can confirm

$$\begin{aligned}bd &= (3)(-1)(2) = -6 \\b + d &= 3 + (-1)(2) = 1\end{aligned}$$

as desired.

Putting it together, we get

$$x^2 + x - 6 = (x + 3)(x - 2)$$

- (i) Factor the following expression as much as possible $2x^7 + 5x^3$

Solution. First, factor each term in the expression. There are three occurrences of x in each term, so we can factor them out.

$$\begin{aligned}2x^7 + 5x^3 &= (2 \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) + (5 \cdot x \cdot x \cdot x) \\&= x \cdot x \cdot x(2 \cdot x \cdot x \cdot x \cdot x + 5) \\&= x^3(2x^4 + 5)\end{aligned}$$

There are no remaining common terms, so we are done factoring.

Axiom A.2 Laws of Fractions. Suppose that $b, e \neq 0$

$$\begin{array}{ll}\circ \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} & \circ \frac{a}{b} - \frac{c}{b} = \frac{a - c}{b} \\ \circ \frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} & \circ \frac{1}{\frac{b}{e}} = \frac{e}{b} \\ \circ \frac{a}{b} \cdot \frac{c}{e} = \frac{a \cdot c}{b \cdot e} & \circ a \cdot \frac{c}{b} = \frac{a \cdot c}{b} \\ \circ \frac{a \cdot e}{b \cdot e} = \frac{a}{b}\end{array}$$

Exploration A.2 Use the laws of algebra listed above to simplify the following expressions completely.

(a) $\frac{ab - a}{b - 1}$

Solution.

$$\frac{ab - a}{b - 1} = \frac{a(b - 1)}{1 \cdot (b - 1)} = \frac{a}{1}$$

(b) Simplify $\frac{a - b}{b - a}$

Solution.

$$\begin{aligned}\frac{a - b}{b - a} &= \frac{a - b}{-a + b} \\&= \frac{a - b}{(-1)(a - b)}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{-1} \\
&= -1
\end{aligned}$$

Axiom A.3 Laws of Exponents. Let $a, b > 0$ in the following.

$$\begin{array}{ll}
\circ \sqrt[r]{a} = a^{1/r} & \circ a^{r-s} = \frac{a^r}{a^s} \\
\circ a^{r+s} = a^r \cdot a^s & \circ a^{-s} = \frac{1}{a^s} \\
\circ (a^r)^s = a^{r \cdot s} & \circ a^s = \frac{1}{a^{-s}} \\
\circ (a \cdot b)^r = a^r \cdot b^r & \circ \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \\
\circ \sqrt[r]{a \cdot b} = \sqrt[r]{a} \cdot \sqrt[r]{b} &
\end{array}$$

Example A.4 Fractions which do NOT simplify easily.

Doesn't simplify at all: $\frac{3}{x+1}$

Must find a common denominator before you can combine terms:

$$\frac{x+1}{x-1} + \frac{1}{x} = \frac{x \cdot (x+1) + (x-1) \cdot 1}{x \cdot (x-1)} = \frac{x^2 + 2x - 1}{x^2 - x}$$

□

Example A.5 Exponents and Products that do NOT simplify easily. Doesn't simplify at all: $\sqrt{x+1} = (x+1)^{1/2}$

You must FOIL to simplify: $(x+1)^2 = (x+1)(x+1) = x^2 + 2x + 1$

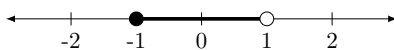
□

A.2 Intervals and Graphs

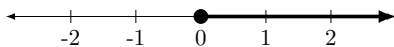
A.2.1 Interval Notation

When using *interval notation* to represent sets, square brackets include the endpoint and parentheses exclude the endpoint. Multiple intervals can be combined using \cup (or \cup).

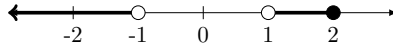
- $[-1, 1)$ represents the numbers x such that $-1 \leq x < 1$.



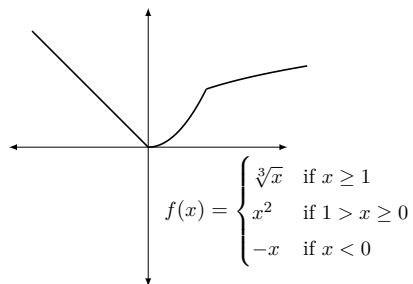
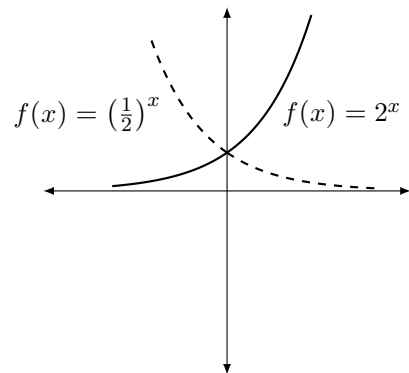
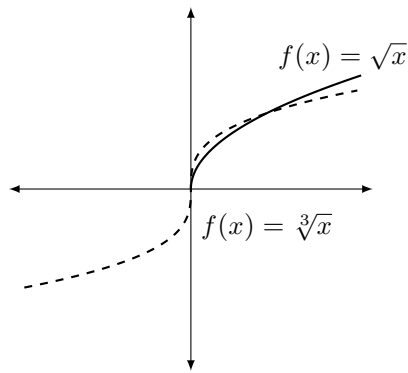
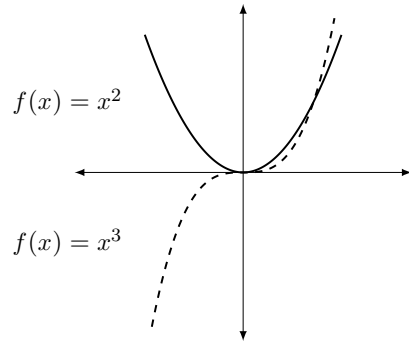
- $[0, \infty)$ represents the numbers x such that $0 \leq x$.



- $(-\infty, -1) \cup (1, 2]$ represents the numbers x such that *either* $x < -1$ *or* $1 < x \leq 2$.



A.2.2 Graphs of Common Functions



Colophon

This book was authored in PreTeXt.