Complex Eigenvectors

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1 Conjugate Eigenvalues and Eigenvectors

Let $\lambda = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix A in the homogeneous system $\vec{X}' = A\vec{X}$ and let $\vec{K}$ be a (complex) eigenvector associated with λ. Also let $\vec{u_1} = \text{Re}(\vec{K})$ and $\vec{u_2} = \text{Im}(\vec{K})$ such that $\vec{K} = \vec{u_1} + i\vec{u_2}$.

Also let $\overline{\lambda} = \alpha - i\beta$ be the ***conjugate*** eigenvalue of λ. We first prove that the eigenvector of $\overline{\lambda}$ is the conjugate of $\vec{K}$, or $\overline{\vec{K}} = \vec{u_1} - i\vec{u_2}$.

Because $\vec{K}$ is the eigenvector of λ by definition,

$$A\vec{K} = \lambda\vec{K}$$

Expanding on the left side,

$$\begin{aligned} A\vec{K} &= A(\vec{u_1} + i\vec{u_2}) \\ &= A\vec{u_1} + iA\vec{u_2} \end{aligned}$$

Expanding on the right side,

$$\begin{aligned} \lambda\vec{K} &= \lambda(\vec{u_1} + i\vec{u_2}) \\ &= (\alpha + i\beta)(\vec{u_1} + i\vec{u_2}) \\ &= (\alpha\vec{u_1} - \beta\vec{u_2}) + i(\beta\vec{u_1} + \alpha\vec{u_2}) \end{aligned}$$

Setting real and imaginary parts equal,

$$A\vec{u_1} = \alpha\vec{u_1} - \beta\vec{u_2} \text{ and } A\vec{u_2} = \beta\vec{u_1} + \alpha\vec{u_2}$$

We now compute $\overline{\lambda}\overline{\vec{K}}$ which should be equal to $A\overline{\vec{K}}$ if $\overline{\vec{K}}$ is an eigenvector of $\overline{\lambda}$.

$$\begin{aligned} \overline{\lambda}\overline{\vec{K}} &= (\alpha - i\beta)(\vec{u_1} - i\vec{u_2}) \\ &= (\alpha\vec{u_1} - \beta\vec{u_2}) + i(-\beta\vec{u_1} - \alpha\vec{u_2}) \end{aligned}$$

Using $A\vec{u_1}$ and $A\vec{u_2}$ computed earlier,

$$\begin{aligned} &= A\vec{u_1} - iA\vec{u_2} \\ &= A(\vec{u_1} - i\vec{u_2}) \\ &= A\overline{\vec{K}} \quad \square \end{aligned}$$

1.1 Real Solutions

We have two eigenvalues λ and its conjugate $\overline{\lambda}$ and we have the two associated eigenvectors $\vec{K}$ and its conjugate $\overline{\vec{K}}$.

Thus, the two (complex) solutions are $\vec{K}e^{\lambda t}$ and $\overline{\vec{K}}e^{\overline{\lambda}t}$. We want to write these two solutions in terms of reals, which is possible because we are allowed to arbitrarily linearly combine the two solutions by the superposition principle.

We first explicitly write out each solution in terms of $\vec{u_1}$ and $\vec{u_2}$.

$$\vec{K}e^{\lambda t} = (\vec{u_1} + i\vec{u_2})e^{(\alpha+i\beta)t}$$
$$= [\vec{u_1}e^{i\beta t} + i\vec{u_2}e^{i\beta t}]e^{\alpha t}$$

Using Euler's formula $e^{ix} = \cos x + i \sin x$,

$$= [\vec{u_1}(\cos\beta t + i\sin\beta t) + i\vec{u_2}(\cos\beta t + i\sin\beta t)]e^{\alpha t}$$

Seperating real and imaginary parts,

$$= [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) + i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$

We now do the same for the other solution.

$$\overline{\vec{K}}e^{\overline{\lambda}t} = (\vec{u_1} - i\vec{u_2})e^{(\alpha-i\beta)t}$$
$$= [\vec{u_1}e^{-i\beta t} - i\vec{u_2}e^{-i\beta t}]e^{\alpha t}$$

Because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$,

$$= [\vec{u_1}(\cos\beta t - i\sin\beta t) - i\vec{u_2}(\cos\beta t - i\sin\beta t)]e^{\alpha t}$$

Beware of the sign on the $\vec{u_2}\sin\beta t$ term!
The two negative signs from each conjugation cancel:

$$= [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) - i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$

To summarize,

$$\vec{K}e^{\lambda t} = [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) + i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$
$$\overline{\vec{K}}e^{\overline{\lambda}t} = [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) - i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$

We notice we can cancel the imaginary parts if we add them up:

$$\frac{1}{2}(\vec{K}e^{\lambda t} + \overline{\vec{K}}e^{\overline{\lambda}t}) = \frac{1}{2}[2(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t)]e^{\alpha t}$$
$$\vec{X_1} = [\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t]e^{\alpha t}$$

We notice we can cancel the real parts if we take the difference, and to make it real we can simply multiply it by a factor of i.

$$\frac{-i}{2}(\vec{K}e^{\lambda t} - \overline{\vec{K}}e^{\overline{\lambda}t}) = \frac{-i}{2}[2i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$
$$\vec{X_2} = [\vec{u_2}\cos\beta t + \vec{u_1}\sin\beta t]e^{\alpha t} \quad \square$$

2 Defective Eigenvalues

Let λ be an defective eigenvalue with multiplicity m, with a single eigenvector by definition. Then each solution is of the form:

$$\vec{X}_m = \vec{K}_1 \frac{t^{m-1}}{(m-1)!}e^{\lambda t} + \vec{K}_2 \frac{t^{m-2}}{(m-2)!}e^{\lambda t} + \cdots + \vec{K}_m e^{\lambda t}$$

Proof. Inductive sketch. For $\vec{X}_m$ to be a valid solution, it must satisfy $\vec{X}'_m = A\vec{X}$.

$$\begin{aligned}\vec{X}'_m = \vec{K}_1 \frac{t^{m-2}}{(m-2)!}e^{\lambda t} + \vec{K}_2 \frac{t^{m-3}}{(m-3)!}e^{\lambda t} + \cdots + \vec{0}\\ + \vec{K}_1 \frac{t^{m-1}}{(m-1)!}\lambda e^{\lambda t} + \vec{K}_2 \frac{t^{m-2}}{(m-2)!}\lambda e^{\lambda t} + \cdots + \vec{K}_m \lambda e^{\lambda t}\end{aligned}$$

By the inductive hypothesis, the top is $\vec{X}_{m-1}$:

$$= \vec{X}_{m-1} + \lambda \vec{X}_m$$

This must be equal to $A\vec{X}_m$, so

$$\vec{X}_{m-1} + \lambda \vec{X}_m = A\vec{X}_m$$

or, moving $\lambda \vec{X}_m$ to the right side,

$$(A - \lambda I)\vec{X}_m = \vec{X}_{m-1}$$

We need to show this system is solvable, and the easiest way to do that is to write it in terms of $\vec{K}_1 \ldots \vec{K}_m$ first and then reconstruct the solutions $\vec{X}_1 \ldots \vec{X}_m$.

$$\begin{aligned}(A - \lambda I)(\vec{K}_1 \frac{t^{m-1}}{(m-1)!}e^{\lambda t} + \vec{K}_2 \frac{t^{m-2}}{(m-2)!}e^{\lambda t} + \vec{K}_3 \frac{t^{m-3}}{(m-3)!}e^{\lambda t} + \cdots + \vec{K}_m e^{\lambda t}) =\\ \vec{K}_1 \frac{t^{m-2}}{(m-2)!}e^{\lambda t} + \vec{K}_2 \frac{t^{m-3}}{(m-3)!}e^{\lambda t} + \cdots + \vec{K}_{m-1} e^{\lambda t})\end{aligned}$$

Because $\vec{X}_m$ is one degree higher than $\vec{X}_{m-1}$, $\vec{K}_1$ has no correspondence, so we have

$$\begin{aligned}(A - \lambda I)\vec{K}_1 &= \vec{0}\\ (A - \lambda I)\vec{K}_2 &= \vec{K}_1\\ &\cdots\\ (A - \lambda I)\vec{K}_m &= \vec{K}_{m-1}\end{aligned}$$

By induction, we can assume $\vec{K}_1 \ldots \vec{K}_{m-1}$ exist because the solution $\vec{X}_{m-1}$ exists. We therefore just need to show that $(A - \lambda I)\vec{K}_m = \vec{K}_{m-1}$ has a solution, which is left as an exercise to the reader. □