Fréchet Distribution Parameter Estimation

Stephen Merrill
Brigham Young University

February 11, 2017

1 Introduction

Motivation The Fréchet distribution is used in modeling extreme events in engineering and finance such as snowfall, flooding, and market crashes. It is part of a family of distributions that model the distribution of maximum order statistics. In this study, three estimation techniques for the parameter values of the two parameter Fréchet distribution are presented and analyzed in terms of bias and mean squared error (MSE).

History The Fréchet Distribution was discovered by Maurice René Fréchet in 1927. However, his paper was published in a remote journal and received little attention. In 1928, Fisher and Tippett developed the Generalized Extreme Value Distribution (GEV), and incorporated the Fréchet Distribution as the type II GEV Distribution. Fréchet's work was also influential in the development of the Weibull (1951) and Gumbel (1958) Distributions.

2 Distribution Description

PDF

$$f(x|\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1} exp\left[-\left(\frac{\beta}{x}\right)^{\alpha}\right] \tag{1}$$

CDF

$$F(x|\alpha,\beta) = exp\left[-\left(\frac{\beta}{x}\right)^{\alpha}\right] \tag{2}$$

Support

$$x \in (0, \infty)$$

Parameters

- $\alpha \in (0, \infty)$ - Shape Parameter
- $\beta \in (0, \infty)$ - Scale Parameter

Mean

$$\beta\left(\Gamma\left(1 - \frac{1}{\alpha}\right)\right) \tag{3}$$

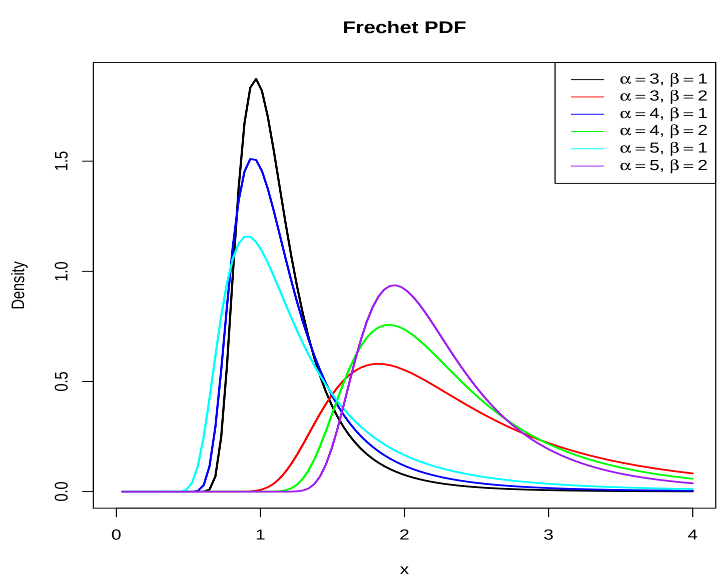
For $\alpha > 1$

Variance

$$\beta^2\left(\Gamma\left(1 - \frac{2}{\alpha}\right) - \left(\Gamma\left(1 - \frac{1}{\alpha}\right)\right)^2\right) \tag{4}$$

For $\alpha > 2$

PDF Graph



3 Description of Estimators

3.1 Maximum Likelihood Estimator

Rather than attempt a closed form MLE solution, the Newton-Raphson Algorithm is used to estimate the MLE estimators.

Newton-Raphson Algorithm

- Let θ_0 = an initial guess of the MLE values
- Let i = 0
- While $|\nabla f(\theta_i)| > \epsilon$
  - i = i + 1.
  - $\theta_i = \theta_{i-1} - [D^2 f(\theta_{i-1})]^{-1}[\nabla f(\theta_{i-1})]$
- If $D^2 f(\theta_{i-1})$ is negative definate, $\hat{\theta}_{MLE} = \theta_i$

Where $D^2 f(\theta)$ is defined as the Hessian matrix for $f(\theta)$

Derivations

$$L(\alpha, \beta) = \alpha^n \beta^{n\alpha} \prod_{i=1}^{n} (x_i)^{-(\alpha+1)} \; exp\left[-\sum_{i=1}^{n} \left(\frac{\beta}{x_i}\right)^{\alpha} \right] \tag{5}$$

$$\ln L = n\ln(\alpha) + n\alpha\ln(\beta) \; - (\alpha+1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha} \tag{6}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + n\ln(\beta) \; - \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha}\ln\left(\frac{\beta}{x_i}\right) \tag{7}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{\alpha(n - \beta^{\alpha})}{\beta}\sum_{i=1}^{n} x_i^{-\alpha} \tag{8}$$

$$\frac{\partial^2 \ln L}{\partial^2 \alpha} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha}\ln\left(\frac{\beta}{x_i}\right)^2 \tag{9}$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} = \frac{\partial^2 \ln L}{\partial\beta\partial\alpha} = \frac{n}{\beta} - \beta^{\alpha-1}(\alpha\sum_{i=1}^{n} x_i^{-\alpha} \ln\left(\frac{\beta}{x_i}\right) + \sum_{i=1}^{n} x_i^{-\alpha}) \tag{10}$$

$$\frac{\partial^2 \ln L}{\partial^2\beta} = -\frac{\alpha n}{\beta^2} - \alpha(\alpha-1)\beta^{\alpha-2}\sum_{i=1}^{n} x_i^{-\alpha} \tag{11}$$

3.2 Method of Moments

Equate the sample moments to the theoretical moments and solve in order to find the Method of Moments estimators.

$$\bar{x} = \beta\left(\Gamma\left(1-\frac{1}{\alpha}\right)\right) \tag{12}$$

$$s^2 = \beta^2\Big(\Gamma\big(1-\frac{2}{\alpha}\big) - \big(\Gamma\big(1-\frac{1}{\alpha}\big)\big)^2\Big) \tag{13}$$

$$\hat{\beta} = \frac{\bar{x}}{\Gamma\left(1-\frac{1}{\hat{\alpha}}\right)} \tag{14}$$

$$\frac{s^2}{\bar{x}^2} + 1 = \frac{\Gamma\big(1-\frac{2}{\hat{\alpha}}\big)}{\big(\Gamma(1-\frac{1}{\hat{\alpha}})\big)^2} \tag{15}$$

Solve for $\hat{\alpha}$ in (15) using an iterative guess and check method. Use that result to solve for $\hat{\beta}$.

3.3 Bayes Estimator

Bayesian Methodology

- Prior Distributions

$$\alpha \sim \text{Gamma}(\alpha_0, \beta_0)$$
$$\beta \sim \text{Gamma}(\alpha_1, \beta_1)$$
$$\alpha_0 = \alpha_1 = 4,\ \beta_0 = \tfrac{\alpha}{\alpha_0},\ \beta_1 = \tfrac{\alpha}{\alpha_1}$$

- Posterior Distribution

$$f(\alpha, \beta | x_1, ..., x_n) \propto f(\alpha, \beta) f(x_1, ..., x_n | \alpha, \beta)$$

Where $f(\alpha, \beta)$ is the joint prior distribution and $f(x_1, ..., x_n|\alpha, \beta)$ is the likelihood given in (5)

Markov chain Monte Carlo Method

- n=10,000 steps of alternating proposals for α and β
- Proposal densities are $cUnif(0,1)$ where constant c varies with the parameter values
- $c_\alpha \in (0.2, 0.3, 0.4)$
- $c_\beta \in (0.0075, 0.015, 0.025, 0.03, 0.0375, 0.045, 0.0525, 0.06)$

The Bayes Estimator of interest is taken under squared error loss. That is, the posterior distributions for α and β are found using the MCMC methodology and $\hat{\alpha}$ and $\hat{\beta}$ are the respective posterior means.

4 Simulation Study

4.1 Structure

- Estimation for two parameters: α and β
- $\alpha \in (3, 4, 5)$ – In order to have a finite mean and variance
- $\beta \in (0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2)$ – In order to show a trend in the results
- Simulate n=100 and n=500 estimators and find long-run bias and MSE

Fréchet Distribution Sampling (Probability Integral Transformation)

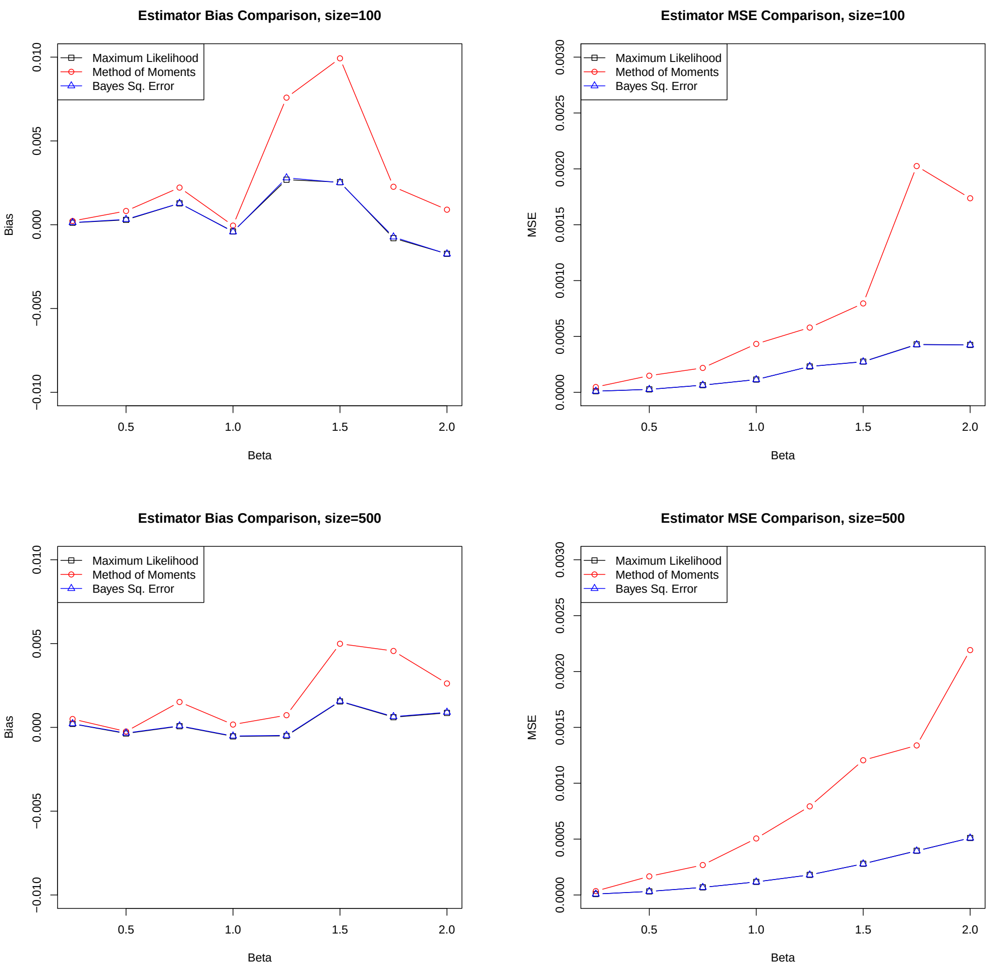
Let $U_1, ..., U_n \sim Unif(0,1)$

$$Y_i = F^{-1}(U_i|\alpha, \beta) = \beta\Big(\frac{-1}{\ln(U_i)}\Big)^{\frac{1}{\alpha}} \tag{16}$$

With $n = 1000$

4.2 Results

For brevity, the simulation results are presented with β varying over $\alpha = 3$.

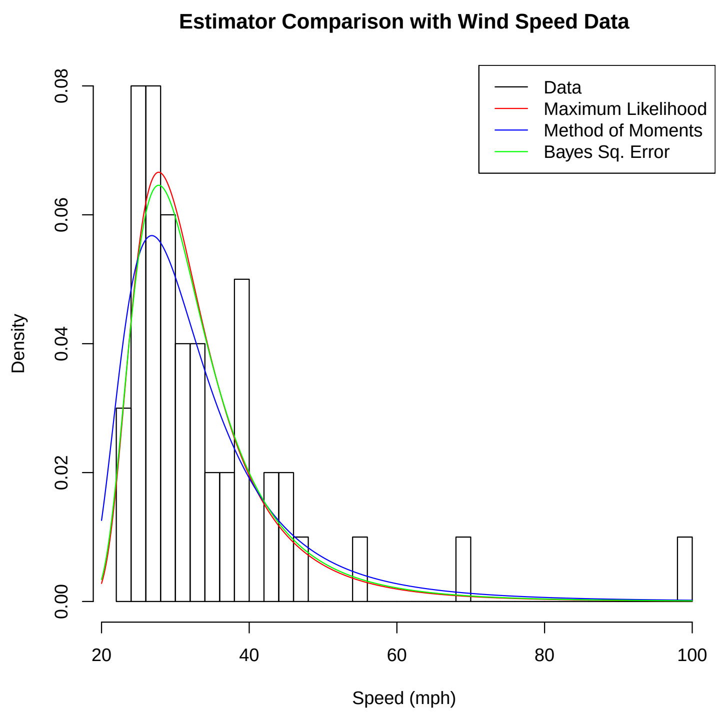


5 Data Analysis

5.1 Wind Speed Data

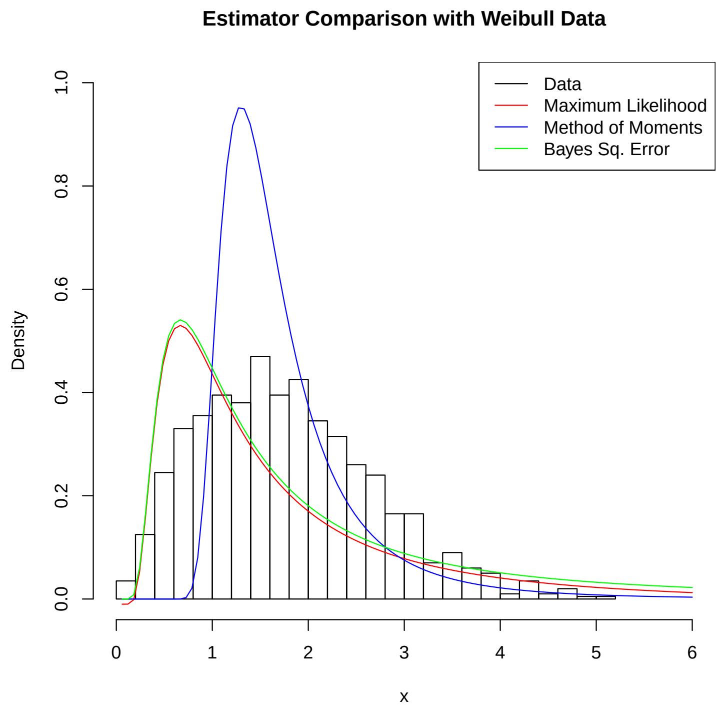
Data of extreme wind speeds in mph was used to test the parameter estimation techniques. This data is Fréchet distributed and taken from *Extreme*

Value Theory in Engineering (Castillo). However, description of the data was quite generic with little detail, and there are only 50 observations.



5.2 Model Misspecification

Performance of estimators was also tested under a misspecified model. Here, data was generated from a Weibull Distribution with shape: $\lambda = 2$ and Scale: $\kappa = 2$.The Weibull Distribution is an appropriate distribution to misspecify data from as it is the inverse Fréchet Distribution and the type III GEV model.



6 Conclusion

Simulation In the simulation, the Maximum Likelihood estimator and Bayes estimator performed almost identically, and both clearly outperformed the Method of Moments estimator. The bias varied without a clear trend, but MSE increased with β. Surprisingly, the change in sample size produced little effect on both bias and MSE.

Data Comparison The estimators performed well with the wind speed data, offering a decent fit. Even better results would probably be attained with a larger data set. However, under model misspecification, the estimators did not perform quite as well. Overall, both comparisons continued to show that the Maximum Likelihood estimator and Bayes estimator perform almost identically, and both outperform the Method of Moments estimator.