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MATH 467-900
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Russell's Paradox: Chapter 2 Exercise 19 (p. 95)

(a) Let S be the following self-referential statement: "Statement S is false." Show that S is true iff S is false. This is the liar paradox. Does it imply that some statements are neither true nor false? (Kurt Godel used the variant "This statement is unprovable" as the starting point for his famous incompleteness theorem in mathematical logic.)

Let S be the following statement: "Statement S is false." We aim to prove that S is true if and only if (iff) S is false.

By the law of excluded middle, either S is true or S is false. Let's suppose S is true. Now we have a contradiction, because the statement itself claims, "Statement S is false." S cannot be both true and false at the same time, so our supposition that S is true must have been wrong, and S must be false instead. Therefore, S is true implies that S is false.

But what if we had supposed S to be false instead? Let's suppose S is false so that "Statement S is false" is a false statement. If the statement S claims S to be false, but the statement itself S is false, then we must negate S. Now our statement essentially says, "Statement S is not false." If S is not false, then S is true. But since we supposed S to be false, we now have a contradiction following from our supposition, and so said supposition is wrong, and we conclude that S is true. Therefore, S is false implies that S is true.

Since we found S is true to imply S is false, and S is false to imply S is true, then we conclude that S is true if and only if S is false. Let's make a truth table to represent a normal "if and only if" table and a truth table to represent our conclusion about S:

p	q	p⇔q
T	T	T
T	F	F
F	T	F
F	F	T

vs.

S	~S	S⇔~S
T	***F***	***F***
T	F	F
F	T	F
F	***T***	***F***

Normally, p if and only if q is true two out of four times, namely when p and q share the same truth value. However, in our case, S if and only if ~S is never true, because S and ~S can never share the same truth value. Therefore, the statement S is never true, showing us that there exist sentences that cannot be true regardless of its assigned truth value. S is a self-defeating, self-contradicting statement; it is a paradox.

"Paradox," in this sense, should not be confused with a *literary* paradox such as "O brawling love! O loving hate!" from Shakespeare's *Romeo and Juliet* or "I can resist anything but temptation." from Oscar Wilde's *Lady Windermere's Fan*. These statements seem contradictory, yet may *perhaps* be true. On the contrary, we are concerned with the *logical* paradox, which can *never* be true. Merriam-Webster defines a logical paradox to be "an argument that apparently derives self-contradictory conclusions by valid deduction from acceptable premises."

Here is another example of a logical paradox inspired by philosopher Kurt Gödel's First Incompleteness Theorem: "This statement is unprovable." If such a statement were indeed unprovable, then it would be a true statement and thus prove and defeat itself. If logical paradoxes, such as Gödel's and the liar paradox, defeat themselves, then is it true that some statements are neither true nor false?

Subjective statements, for one, are neither true nor false. They are outside the realm of what is verifiable. They do not make factual, provable claims, but express merely opinions. So what sort of statements *are* verifiable? We will use an example.

First, let's claim, "The juice is orange," and call that statement 'J'. Then, let's claim, "The statement J is true," and call that statement 'S'. The statements S and J make a reference or a claim about something outside of itself. S is dependent on J, and J is dependent on 'the juice'. The statement S is distinctly different from the statement J. Therefore, we should conclude neither S = J nor ~S = ~J.

Our new statement S is different from our old statement S, which says, "The statement S is false." Our new S made a claim about something outside of itself, namely J. However, our old S does not make a verifiable, provable claim about reality. It is a self-referential statement which is completely dependent upon itself. It is a line of thinking that does not go anywhere but back upon itself in a vicious cycle[1] completely detached from anything relevant or valuable. We may assign it an arbitrary truth value, but that says nothing about its actual, verifiable truth. So, we see that the old statement S is neither true nor false in a similar sense to the way subjective statements are neither true nor false.

> (b) A set is intuitively any collection of things, and those things are the elements of that set. Suppose we collect all the sets S with the property that S ∉ S and only those sets. Call that set C. By the law of excluded middle, either C ∈ C or C ∉ C. Show that in either case, a contradiction can be deduced. This is Bertrand Russell's paradox. Does it imply that set theory is inconsistent?

Russell's Paradox says that we should let S be the set of all sets that are not members of themselves. S, therefore would not be part of this set, because S cannot contain a member of itself. However, now we have that S ∉ S, which means that S should be included in S. But as soon as S becomes a member of itself, we must remove it again. This cycle may continue on and on ad infinitum.

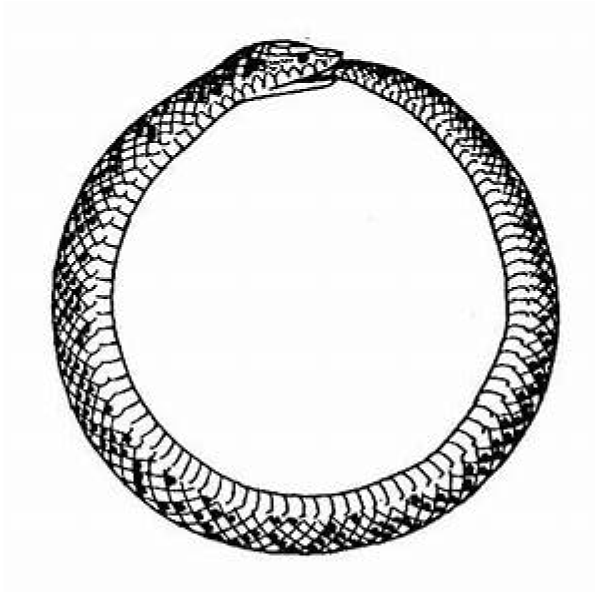
Similarly, if we call the set containing all the sets, set C, we can say C = {S:S $\notin$ S}. Does C contain itself? It either does or does not. If C $\notin$ C, then C $\notin$ {S:S $\notin$ S}, and therefore S $\in$ S. This is a contradiction, because we already claimed S $\notin$ S.

Russell has a similar paradox known as the Barber Paradox. This paradox states that in a certain village, a barber shaves everyone that does not shave himself. The barber either shaves himself or not. If he shaves himself, then we have a contradiction. If he does not shave himself, then he becomes someone he must shave. But as soon as he shaves himself, we arrive again at our contradiction.

The Barber Paradox is a bit different from the Set Paradox, because we can easily squirm out of the contradictions by saying that such a village does not exist. In the same way, if we had the statement, "in a certain village, 2+2=5," we could easily say such a village does not exist. The Set Paradox, on the other hand, is harder to wiggle out of. It's implied contradictions stab at the heart of set theory. But there is a solution.

In this problem, a set was defined as "any collection of things, and those things are the elements of that set." This is the understanding of a set given by Naive Set Theory, invented by Georg Cantor. Due to Russell's Paradox, we know this theory is flawed. We need a solution: a nuanced definition for "set" and new axioms. The Zermelo Fraenkel (ZFC) Set Theory provides such a solution.

ZFC has an axiom that sets constraints on a set as follows: $\exists y \forall x(x \in y \Leftrightarrow x \in A \land \varphi)$. If this paper were longer, we may derive that a set must have an element already in A that is not an element of itself… and see how such a constraint would prevent Russell's paradox. But instead, we will conclude with this: Set theory *is* inconsistent, *if* we are operating under Naive Set Theory. But thanks to Russell's Paradox, mathematics was forced to refine the definition of a set and so improve Set Theory.



[1]**Figure 1.** *Ouroboros*

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