SGD - Random Variable Convergence Worksheet

Wednesday, April 17, 2024 11:17 AM

Let $\Omega = [0,1]$ w/ $\mathbb{P}$ the uniform measure on Ω, i.e. $\mathbb{P}[\,[a,b]\,] = \mathbb{P}[\,(a,b)\,] = b-a$ $\forall$ $0 \le a \le b \le 1$

Let $e_n(\omega) := \begin{cases} n^\alpha & 0 \le \omega < 1/n \\ 0 & 1/n \le \omega \le 1 \end{cases}$

so $(*)$ $e_n = \begin{cases} n^\alpha & \text{w.p. } 1/n \\ 0 & \text{w.p. } 1-1/n \end{cases}$

Activity: fill in the table (answers in red)

α	L^p (which p?)	almost sure	in probability
0	$\forall p \in [1,\infty)$ not $p=\infty$	yes	yes
$\frac{1}{2}$	$\forall p \in [1,2)$	yes	yes
1	No p	yes	yes

justification: either direct proof or follows from "a.s."

Q: Can you define f_n that also satisfy $(*)$ but don't converge in the same way?

A: let $f_n = e_n$ if n even, $f_n(\omega) = \begin{cases} n^\alpha & 1-1/n < \omega \le 1 \\ 0 & 0 \le \omega \le 1-1/n \end{cases}$ if n odd

Then same L^p convergence but doesn't converge a.s. for any α

(so for $\alpha = 0, 1/2$ it still converges in probability. For $\alpha = 1$ it also converges in prob. via direct calculation)