SGD - commuting gradient and expectation

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Let $L_D(\omega) = \underset{z\sim D}{\mathbb{E}}\, \ell(\omega, z)$. Is $\frac{d}{d\omega} L_D(\omega) = \underset{z\sim D}{\mathbb{E}} \frac{\partial}{\partial\omega} \ell(\omega, z)$?

Answer: Sometimes

Example $z \sim \text{Unif}(\{\pm 1\})$, $\ell(\omega, 1)$ isn't differentiable, $\ell(\omega, -1) := -\ell(\omega, 1)$

so $L_D(\omega) = 0$ which is differentiable

Example let $L_D(\omega) = |\omega|$ for $\omega \in (-\pi, \pi)$

the Fourier series is $= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)\omega)$

So

$$\underbrace{\frac{\pi}{2} - |\omega|}_{\substack{\text{not differentiable} \\ \text{at } \omega = 0}} = \frac{4}{\pi} \cdot c \cdot \underset{z\sim D}{\mathbb{E}} \cos((2z-1)\cdot\omega)$$

where $\mathbb{P}[z = n] = \frac{1}{c\cdot(2n-1)^2}$ c a normalization constant

$\ell(\omega, z) = \cos((2z-1)\omega)$ is differentiable.

OR...

use $L_D(\omega) = \omega$ which has Fourier series $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin^2(n\omega)}{n^2}$ $\omega \in (0, \pi)$

↑ not periodic   periodic

... so discontinuities at $0, \pi$ ...

OR... $L_D(\omega) = \frac{\omega^2}{4} - \frac{\pi}{2}\omega + \frac{\pi^2}{6}$, Fourier Series $\sum_{n=1}^{\infty} \frac{\cos(n\omega)}{n^2}$ $\omega \in [0, 2\pi]$

Thm (1D case) cf. Folland Thm 2.2.7

Let $\ell : \mathcal{H} \times Z \to \mathbb{R}$ where $\mathcal{H} = [a, b]$. If $\forall \omega \in \mathcal{H}$, $\mathbb{E}|\ell(\omega, z)| < \infty$

and $\frac{\partial \ell}{\partial \omega}$ exists, and $\exists\, g : Z \to \mathbb{R}$ st.

$(\forall \omega, z)$ $\left| \frac{\partial \ell}{\partial \omega}(\omega, z) \right| \leq g(z)$ then

1) $L_D(\omega) := \underset{z\sim D}{\mathbb{E}} \ell(\omega, z)$ is differentiable, and

2) $\frac{d}{d\omega} L_D(\omega) = \underset{z}{\mathbb{E}} \frac{\partial \ell}{\partial \omega}(\omega, z)$

proof: Lebesgue's Dominated Convergence Thm