Ch 21 Online Learning

Friday, March 27, 2020 3:05 PM

In Shalev-Shwartz + Ben-David, ch 21 Online Learning is first ch of "Part 3: Additional Learning Models"
(21: online learning, 22: clustering, 23: dimensionality reduction, 24: generative models (MLE, Bayes), 25: Feature selection)
↑ most similar to PAC... — mostly orthogonal ideas, or (eg ch 24) you'd see in a standard Stat. class

Online Learning

Before, we had training data, then (phase 2) could apply classifier to test data
With online learning, no ① separate phases: train and test on same data
(many things naturally operate this way, eg., SPAM filters, medical knowledge, ...)

Also, theory can be completely ② distribution free: instead, allow it to change, or even be adversarial
↳ connections to game theory

We'll have to change how we measure "learning" (don't use risk L_D anymore)

Many similarities to PAC learning though (notions similar to VC-dimension,
agnostic or not,
online-to-batch conversion)

Popular recently ('05-'15 very hot)
Many online methods are cheap to implement, ie., can deal w/ data streams

History:
Hannan '57, Rosenblatt '58, Novikoff '62
Modern start w/ Littlestone + Warmuth '89

Plan: ① Binary classification — realizable / agnostic

② Regression or surrogate loss (need convexity)

1a Online Classification, realizable (focus on learning, not computational complexity)

Setup: T rounds ($t \in [T]$), we assume $T \in \mathbb{N}$ but will take $\lim_{T\to\infty}$ sometimes
Each round, observe data/feature x_t
you (or the alg.) predicts p_t ... then true answer y_t is revealed (for now, $y_t \in \{0,1\}$)
Goal: make as few mistakes as possible (ie. 0-1 loss)
So... want a good prediction now and to learn so we make a good prediction in the future

(general examples: · SPAM filtering
· Restaurant problem (we'll see again in Reinforcement Learning)
example of "reveal": y_t = whether you liked the food x_t)

How are x_t and y_t generated? Deterministic, Stochastic, adversarial ← our analysis, since distribution-free, worst-case

Of course, need some assumptions
(otherwise, choose $y_t = \neg p_t$ and it's hopeless)

Realizable Case Assume (x_t) is arbitrary/adversarial, but $\exists\, h^* \in \mathcal{H}$ s.t. $(\forall t)\ y_t = h^*(x_t)$

Def $M_A(\mathcal{H})$ is the maximum # of mistakes $(p_t \neq y_t)$ made by algorithm A on a sequence of data $(x_t, y_t)_{t=1}^T$, over all (x_t) and all $h^* \in \mathcal{H}$

Def Given $S = ((x_1, h^*(x_1)), \dots, (x_T, h^*(x_T)))$, $h^* \in \mathcal{H}$, then $M_A(S)$ is # mistakes algo A makes on S, and $M_A(\mathcal{H})$ is supremum of $M_A(S)$ over all such S (arbitrary length), and a bound of the form $M_A(\mathcal{H}) \leq B\ (\forall T)$ is a mistake bound, and $\mathcal{H}$ is online learnable if $\exists$ algo A w/ a mistake bound for $\mathcal{H}$

Warmup: let $|\mathcal{H}| < \infty$ (like we did for PAC learning)

For PAC learning, our main tool was Empirical Risk Minimization (ERM)
We have something similar:

Algorithm: "Consistent" $|\mathcal{H}| < \infty$

Init.: $V_1 = \mathcal{H}$
for $t = 1, 2, \dots, T$
  receive x_t → so this is an ERM for $S = (x_1, y_1), \dots, (x_{t-1}, y_{t-1})$
  choose any $h \in V_t$, predict $p_t = h(x_t)$
  receive true label y_t
  update (prune) $V_{t+1} = \{h \in V_t : \underbrace{h(x_t) = y_t}_{\text{consistent}}\}$

Corollary 21.2 A = Consistent then $M_A(\mathcal{H}) \leq |\mathcal{H}| - 1$

proof Every time we make a mistake, $|V_{t+1}| \leq |V_t| - 1$
So for M mistakes,
$$1 \leq |V_t| \leq |\mathcal{H}| - M$$
(1 ≤ by realizability) □

Is that good? No. Easy fix:

Algorithm: "Halving" $|\mathcal{H}| < \infty$

Init.: $V_1 = \mathcal{H}$
for $t = 1, 2, \dots, T$
  receive x_t
  predict $p_t = \underset{r \in \{0,1\}}{\operatorname{argmax}} \left| \{h \in V_t : h(x_t) = r\} \right|$ → i.e. pick majority vote
  receive truth $y_t = h^*(x_t)$
  update $V_{t+1} = \{h \in V_t : h(x_t) = y_t\}$ → same consistent update

Thm 21.3 $M_{\text{Halving}}(\mathcal{H}) \le \log_2(|\mathcal{H}|)$

Proof Every time we make a mistake, we can reduce size of V_t a lot, since at least half got it wrong

$|V_{t+1}| \le \frac{1}{2}|V_t|$ so $1 \le |V_{T+1}| \le |\mathcal{H}|\, 2^{-M}$ (by realizability) □

General case ($\mathcal{H}$ infinite is OK)

We want a measure of complexity of $\mathcal{H}$, analogous to VC dimension

(Recall: $VCdim(\mathcal{H}) \ge m \Rightarrow \mathcal{H}$ can shatter m points, i.e., realize all possible dichotomies)

Analogy/example ("A theory that explains everything, explains nothing")

I create a new physics theory. Of course, it involves some unknown parameters, so in fact I actually have a class of theories $\mathcal{H} = \{h_1, \dots, h_k\}$.

What's the fewest number of experiments needed for you to falsify my theory? (I am your adversary) or, you are my adversary

Also, I am unscrupulous (or, "you can make the data say whatever you want")

i.e., you propose an experiment x_t with outcome $\neg p_t$, then I claim $y_t = p_t$

... but I have to be self-consistent (if you ask x_k again, I can't change my answer)

and I've claimed $y_t = h^*(x_t)$ for some $h^* \in \{h_1, \dots, h_k\}$

You need at least $\log_2(k)$ experiments before you could prove that I'm faking it (or that my theory is false)

(more generally)

Get in the mindset of an adversary.

Your annoying sibling has 1 piece of candy, puts it behind their back, and asks you to choose which hand

You choose "left hand". Probability you're correct? ~~50%~~? $\circ\ \sigma_0$?

Because they're your sibling, they cheat as much as possible without you being able to prove it

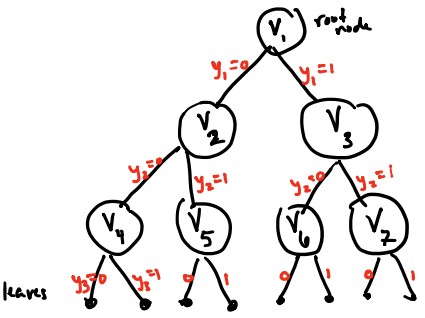
(Notes written during coronavirus quarantine w/ two daughters who just got Easter candy)

Our notion will be $Ldim(\mathcal{H})$ (after Nick Littlestone)

think of learner vs. environment: environment chooses x_t, learner chooses p_t, then environment chooses y_t but must be realizable by some $h \in \mathcal{H}$

In fact, we only care about rounds where we make mistakes, so let $y_t = \neg p_t$ (i.e. $\neg p_t = 1 - p_t$)

To characterize what an adversarial environment can do, make a binary tree



$V_i \in X$ ($V_2 = V_3$ is OK, though $V_2 = V_1$ is silly (non-optimal))

depth = T (ex: 3 in this case)

$2^T - 1$ nodes

(don't put V nodes on leaves)

Environment chooses x_t (i.e. v_i), but you choose p_t (so force $y_t = 1 - p_t$ if it's a mistake)

so you choose how to traverse the tree from root to leaf.

Def A shattered tree of depth T is a tree w/ nodes $v_1, v_2, \dots, v_{2^T-1}$ such that any path from root to leaf (via nodes $(v_{i_1}=v_1, v_{i_2}, \dots, v_{i_T})$ w/ labels $(y_1, \dots, y_T)$ can be realized by some $h \in H$ s.t. $h(x_{i_t}) = y_t \quad \forall t \in [T]$. ($x_{i_t}$ or v_{i_t}; $v_{i_2} = x_{i_2}$ ...)

↳ i.e. defeats all learners, or all learners make at least T mistakes

Def $\mathrm{Ldim}(H)$ is the maximal integer T s.t. $\exists$ a shattered tree of depth T (environment's strategy)

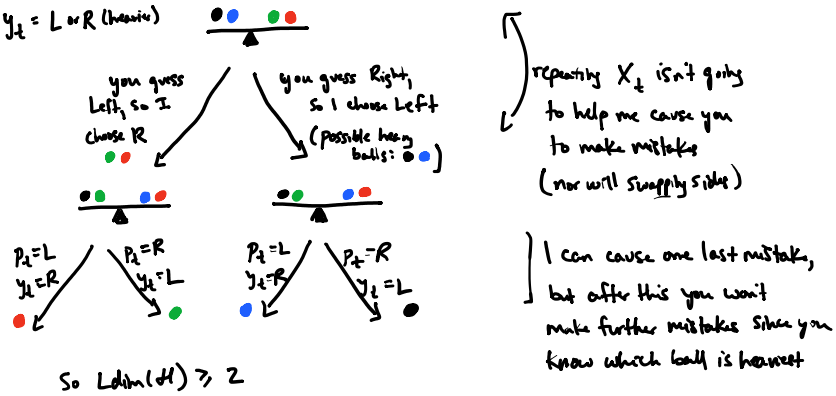
Immediately proves

Lemma 21.6: $\forall$ alg A, $M_A(H) \geq \mathrm{Ldim}(H)$

Ex 4 balls ●●●● 3 identical, 1 is heavier i.e. $|H| = 4$

x_t = weighing partition

y_t = L or R (heavier)



repeatedly x_t isn't going to help me cause you to make mistakes (nor will swapping sides)

I can cause one last mistake, but after this you won't make further mistakes since you know which ball is heaviest

So $\mathrm{Ldim}(H) \geq 2$

Ex $\mathrm{Ldim}(H) \leq \log_2(|H|)$ if H finite

(either just see it, or prove via HALVING Alg)

Ex Unit vectors $X = \{1, \dots, d\}$, $H = \{h_1, \dots, h_d\}$ $h_j(x) = \begin{cases} 1 & x = j \\ 0 & x \neq j \end{cases}$

Then $\mathrm{Ldim}(H) = 1$

Why? take $v_1 = 1$, wlog (due to symmetry)

If alg says $p_1 = 1$, then no mistake and its learned h

If alg says $p_1 = 0$, they make a mistake, but they see $y_1 = 1$ and learn h

Either way, the alg won't make any more mistakes

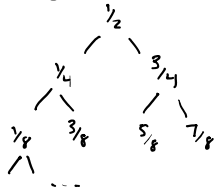
Since d can be arbitrarily large, this shows

$\mathrm{Ldim}(H) \ll |H|$ is possible.

Ex $X = [0,1] \subseteq \mathbb{R}$, $H = \{ \mathbb{1}_{[x \leq a]}, a \in [0,1] \}$

Recall $\mathrm{VCdim}(H) = 1$

b.t $\mathrm{Ldim}(H) = \infty$. Choose nodes



$x_1 = 1/2$, you guess 1 ($x^* \le 1/2$)

So adversary decides $x^* > 1/2$

$x_2 = 3/4$, you guess 0 ($x^* > 3/4$)

So adversary decides $x^* \le 3/4$

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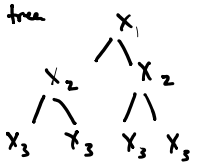
you'll make unbounded # of mistakes.

Ldim vs VCdim: $\forall\ \mathcal{H}$, $\mathrm{VCdim}(\mathcal{H}) \le \mathrm{Ldim}(\mathcal{H})$, but $<$ possible (i.e. above example)

proof: let $d = \mathrm{VCdim}(\mathcal{H})$, and $(x_1, \dots, x_d)$ a shattered set

make the tree



Now, we know $m_A(\mathcal{H}) \ge \mathrm{Ldim}(\mathcal{H})$. What about $\exists A$ s.t. $m_A(\mathcal{H}) = \mathrm{Ldim}(\mathcal{H})$?

Yes, constructive (generalize HALVING):

Algo: "Standard Optimal Algorithm" (SOA) for $\mathcal{H}$ ($|\mathcal{H}| = \infty$ is OK)

Initialize $V_1 = \mathcal{H}$

for $t = 1, 2, \dots, T$

receive x_t

predict $p_t = \underset{r \in \{0,1\}}{\operatorname{argmax}}\ \mathrm{Ldim}(\{h \in V_t : h(x_t) = r\})$ (only difference; $V_t^{(r)}$)

receive true label y_t

update $V_{t+1} = \{h \in V_t : h(x_t) = y_t\}$ (same as before)

Lemma 21.7 $M_{SOA}(\mathcal{H}) \le \mathrm{Ldim}(\mathcal{H})$

proof If we make a mistake, $\mathrm{Ldim}(V_{t+1}) \le \mathrm{Ldim}(V_t) - 1$. wlog, say we chose $p_t = 0$ (so actually $y_t = 1$)

If not, then $\mathrm{Ldim}(V_{t+1}) = \mathrm{Ldim}(V_t)$

$= V_t^{(1)}$ by update rule

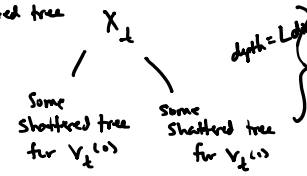
But also $\mathrm{Ldim}(V_t) \ge \mathrm{Ldim}(V_t^{(0)}) \ge \mathrm{Ldim}(V_t^{(1)}) \ge \mathrm{Ldim}(V_t)$

$\mathrm{Ldim}(A) \ge \mathrm{Ldim}(B)$ if $A \supseteq B$

by how we chose r

but then if $\mathrm{Ldim}(V_t^{(0)}) = \mathrm{Ldim}(V_t^{(1)}) = \mathrm{Ldim}(V_t)$

can make a new shattered tree



depth $= \mathrm{Ldim}(V_t) + 1$

$\Rightarrow \mathrm{Ldim}(V_t) \ge \mathrm{Ldim}(V_t) + 1$

Contradiction □

Corollary $\mathcal{H}$ has a mistake bound iff $\mathrm{Ldim}(\mathcal{H}) < \infty$

(16) Online classification, unrealizable case

If we were realizable, it's possible to have a bounded # of mistakes even as $T \to \infty$

Not true if unrealizable, so need new measure of learning

Want our algo competitive with best fixed predictor, i.e., "regret"

Def $\text{Regret}_A(h, T) = \sup_{(x_1,y_1),\dots,(x_T,y_T)} \sum_{t=1}^{T} |p_t - y_t| - \sum_{t=1}^{T} |h(x_t) - y_t|$

← this is for 1' loss, but can be defined more generally for an arbitrary loss

Def $\text{Regret}_A(\mathcal{H}, T) = \sup_{h \in \mathcal{H}} \text{Regret}_A(h, T) = \sup_h \sup_{(x_i,y_i)} \sum |p_t - y_t| - \sum |h(x_t) - y_t|$

$$= \sup_{(x_i,y_i)} \left(\sum |p_t - y_t| + \sup_h - \sum |h(x_t) - y_t| \right)$$

$$= \sup_{(x_i,y_i)} \left(\sum |p_t - y_t| - \inf_h \sum |h(x_t) - y_t| \right)$$

i.e. best expert

We usually have $\inf_{h \in \mathcal{H}} \sum_{t=1}^{T} |h(x_t) - y_t|$ grow sublinearly with T

(if not, then best expert is about as good as always choosing $p_t = 1$)

So if we want our algo to be useful, also need it to be sublinear in T

and want the regret to grow sublinearly in T

($\Rightarrow$ the difference between our learner and best hypothesis (in hindsight) goes to 0 as $T \to \infty$)

Unfortunately, this isn't possible (Cover '65)

Ex $\mathcal{H} = \{h_0, h_1\}$ $h_0(x) = 0 \ \forall x$, $h_1(x) = 1 \ \forall x$

Not realizable, so adversary has no constraint. You guess $(p_1, p_2, \dots, p_T)$

and it sets $y_1 = \neg p_1, \dots, y_T = \neg p_T$, so you make T mistakes.

The best-in-class makes $\leq T/2$ mistakes, so regret $\geq T - T/2 = T/2$

which isn't sublinear.

So must limit adversary's power

Allow learner to randomize, and look at expectation

Assume randomness is independent of adversary, i.e., you flip a coin, and adversary knows your general strategy but not the outcome of the coin flip.

In the binary case, output label $\hat{y} = \begin{cases} 1 & \text{w.p. } p_t \\ 0 & \text{w.p. } 1 - p_t \end{cases}$

Then expected loss is

$$\mathbb{E}\,|\hat{y} - y| = \mathbb{P}[\hat{y} \neq y] = \begin{cases} 1 - p_t & y = 1 \\ p_t & y = 0 \end{cases} = |p_t - y_t|$$