Convex Sets (following B&V)

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Chapter 2 in Boyd + Vandenberghe's book

§2.1 Affine and convex sets

Def Let $x, y \in \mathbb{R}^n$ (or any vector space), then

1) $tx + (1-t)y$, $t \in [0,1]$ is a convex combination (of x, y)
2) $tx + (1-t)y$, $t \in \mathbb{R}$ is a linear combination (of x, y)
3) $tx + sy$, $t \geq 0, s \geq 0$ is a conic combination (of x, y) — a more fitting name would be "convex conic" combo

and more generally, for k points $\{x_i\}_{i=1}^k$, $\sum_{i=1}^k t_i x_i$ is a ...

convex combination if $\sum t_i = 1$, $t_i \geq 0$

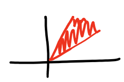
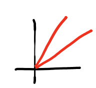
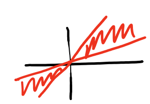
linear combination if $\sum t_i = 1$

conic cobination if $t_i \geq 0$.

* updated notes 1/31/21 to fix earlier errors

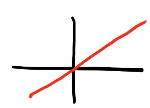
Def A set $C \subseteq \mathbb{R}^n$ is

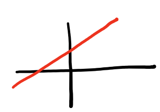
1) convex if $\forall x, y \in C$, it contains all convex combinations of x, y
2) * affine if $\forall x, y \in C$, it contains all linear combinations of x, y
3) a cone if $\forall x \in C$, it contains the ray $\{ sx : s \geq 0 \}$
4) a convex cone if it's convex and a cone, i.e., $\forall x, y \in C$, contains all conic combinations of x, y
   or equiv., $\forall x, y \in C$, $\forall t, s \geq 0$, $tx + sy \in C$

↳ Ex: convex cone  non-convex cones  

(more examples later)

* An affine set (aka affine subspace) is like a subspace (i.e., a linear subspace) except it's possible shifted (so need not include 0)

Ex: 1D subspace in $\mathbb{R}^2$ looks like 

a 1D affine space in $\mathbb{R}^2$ might look like 

Usually "affine" properties very similar to "linear", and people often say "linear" when it would technically be correct to say "affine"

Ex: $\underbrace{\text{dim(affine space)}}_{W}$ easily defined by shifting the space (pick $w_0 \in W$, $V := \{ x - w_0 : x \in W \}$), V is now linear, define $\dim(W) = \dim(V)$.

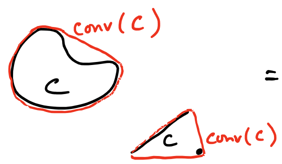
Just like in real analysis we define things like "closure" ($\bar{A}$:= set A together w/ all limit points
= smallest closed set containing A
= intersection of all closed sets containing A)

we can define:

affine hull, aff(C) := smallest affine set containing C

and

convex hull, conv (C) := smallest convex set containing C
= intersection of all convex sets containing C
$= \{ t_1 x_1 + t_2 x_2 + t_3 x_3 + \ldots + t_k x_k : x_i \in C, \; t_i \geq 0, \; \sum_{i=1}^{k} t_i = 1 \}$ *
Sometimes/often this is the definition of convex combination
= "stretch rubber band around C, include everything inside"



* for "any $k \in \mathbb{N}$" though you can prove $k = n+1$ is sufficient if $C \subseteq \mathbb{R}^n$

and conic hull is smallest conic and convex set containing C, etc.

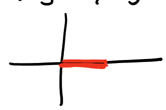
Recall, interior of a set, $\text{int}(C) := \{ x \in C : \exists \varepsilon > 0, \; B_\varepsilon(x) \subseteq C \}$

New concept: relative interior of a set, $\text{relint}(C) := \{ x \in C : \exists \varepsilon > 0, \; B_\varepsilon(x) \cap \text{aff}(C) \subseteq C \}$
or $\text{ri}(C)$

makes geometric sense

Ex: let $C = [0,1] \subseteq \mathbb{R}^1$, $\text{int}(C) = (0,1)$.
$\text{relint}(C) = (0,1)$.

but
same shape, but embedded in higher dimension
let $C = [0,1] \times \{0\} \subseteq \mathbb{R}^2$, $\text{int}(C) = \emptyset$
$\text{relint}(C) = (0,1) \times \{0\}$ ✓



since $\text{aff}(C) = \mathbb{R} \times \{0\} \cong \mathbb{R}^1$

Ex. of common geometric objects

Hyperplane   For $a \in \mathbb{R}^n, b \in \mathbb{R}$, $\{ x \in \mathbb{R}^n : a^T x = b \}$. $n-1$ dimensional, affine

$n=2$, these are lines
$n=3$, these are planes

They split $\mathbb{R}^n$ into 2 sides

Half-space $\{ x \in \mathbb{R}^n : a^T x \geq b \}$ (or another parameterization, $a^T(x - x_0) \leq 0$
i.e., $b = a^T x_0$)

Not affine (so no "dimension")
but convex

Euclidean ball
open ball $B_\varepsilon(x) = \{ y \in \mathbb{R}^n : \|y - x\| < \varepsilon \}$
closed ball $\overline{B_\varepsilon}(x) = \{ y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon \}$

→ Euclidean norm unless otherwise specified

* oddly, B+V define $B_\varepsilon(x)$ to be the closed ball by default, nonstandard in real analysis. We won't follow B+V

Ellipsoid $\mathcal{E} = \{ x : (x-x_0)^T P^{-1} (x-x_0) \le 1 \}$

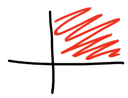
for some matrix $\underbrace{P \succ 0}$ means P is <u>positive definite</u> (see supplemental notes)

Includes balls
(choose $P = \varepsilon^2 I$)

Always convex

Cones

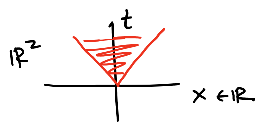
- Positive orthant in $\mathbb{R}^n$, $\mathbb{R}^n_+ = \{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : x_i \ge 0 \}$, is a cone.



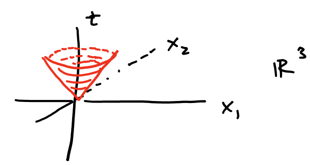
$\mathbb{R}^n_{++} = \{ \dots : x_i > 0 \}$ isn't a cone
(doesn't include 0, but 0 is always a conic combination)

- Lorentz cone / 2^{nd} order cone / "ice cream cone"

$C = \{ (x,t) \in \mathbb{R}^{n+1}, \ \substack{x \in \mathbb{R}^n \\ t \in \mathbb{R}} : \|x\|_2 \le t \}$



or



- Positive semidefinite (PSD) matrices. Most important non-polyhedral cone

Recall $\mathbb{R}^{m \times n} = \{ m \times n \text{ matrices} \}$ is a mn dimensional linear space (aka vector space)

and let $S^n = \{ X \in \mathbb{R}^{n \times n} : X = X^T \}$ be square, symmetric matrices
(or $X = X^*$, if working w/ $\mathbb{C}^{n \times n}$)

(often S^{n-1} means the sphere in $\mathbb{R}^n$, but we'll usually use it to mean symm. matrices)

S^n is a linear space

Define PSD cone to be $S^n_+ = \{ X \in S^n : \underbrace{X \succeq 0} \}$ means X is psd

(Pos. Def. set $S^n_{++} = \{ X \in S^n : X \succ 0 \}$ isn't a cone and isn't closed)

Polyhedron

Very <u>common</u> in math so there are conflicting definitions

Def: polyhedron $P \subseteq \mathbb{R}^n$ is a set of the intersection of a <u>finite</u> number of half-spaces and half-planes

$P = \{ x \in \mathbb{R}^n : \ a_j^T x \le b_j \ , \ j = 1, \dots, m$
$\qquad c_j^T x = d_j \ , \ j = 1, \dots, p \}$

Equiv. def'n: A set is polyhedral if it can be written as the convex hull of a <u>finite</u> # of points

∞ # half-spaces/planes <u>isn't</u> necessarily a polyhedron

<u>Ex</u>: $\{ x \in \mathbb{R}^n : a_j^T x \le 1, \ a_j \in \text{unit circle} \}$
defines the closed ball $\overline{B_1(0)}$, <u>not</u> polyhedral

Polyhedra are <u>always convex</u>

Some authors call a bounded polyhedron a polytope

(others define a polytope as we did a polyhedra, and define a polyhedra to mean a bounded polytope! completely switched!)

In other fields / wikipedia,

polygon means in $\mathbb{R}^2$
polyhedron means in $\mathbb{R}^3$
polytope means higher dimension.

We won't use these definitions.

I'll probably use all 3 interchangeably... sorry!

To prove a set is polyhedral...

1) show it's the finite intersection of polyhedra

or 2) show it can be written as $\text{conv}(V)$ for a finite set of vertices V

3) lift to a higher dimensional space, show that set is polyhedral, and the original set is the projection of that polyhedron.