Duality practice problem

Monday, March 10, 2025 10:45 AM

Solve: $\min_{x\in\mathbb{R}^2} \frac{1}{2}\| \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \cdot x - \underbrace{\begin{bmatrix} 3 \\ 3 \end{bmatrix}}_{b} \|^2$ subject to $\sum x_i \le \beta$ (e.g. $\beta = 2$)

$A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T b = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

Solution $\mathcal{L}(x,\lambda) = \frac{1}{2}\| Ax - b\|^2 + \lambda \cdot (\mathbb{1}^T x - \beta)$

For fixed λ, $\min_x \mathcal{L}$ is solvable:

$$0 = A^T(Ax - b) + \lambda \cdot \mathbb{1}$$

i.e. $$x = (A^T A)^{-1}(A^T b - \lambda \cdot \mathbb{1})$$

$$= \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 - \lambda \\ 3 - \lambda \end{bmatrix} = \begin{bmatrix} \frac{6-\lambda}{4} \\ 3 - \lambda \end{bmatrix} \quad x_\lambda$$

check: is $\lambda = 0$ feasible? (i.e. $x = \begin{bmatrix} 3/2 \\ 3 \end{bmatrix}$ is unconstrained solution).

Yes, if $\beta \geq 4\frac{1}{2}$

No, if $\beta < 4\frac{1}{2}$. Assuming $\beta < 4.5$, let's continue...

How to find value of λ? Use λ to enforce $\sum x_i = \beta$ (if $\lambda \neq 0$ can't have $\sum x_i < \beta$)

so... $\mathbb{1}^T x_\lambda = \beta$, i.e. $\frac{6-\lambda}{4} + \frac{12 - 4\lambda}{4} = \beta$

or $18 - 5\lambda = 4\beta$, $5\lambda = 18 - 4\beta$. Suppose $\beta = 2$

... $5\lambda = 18 - 8 = 10 \Rightarrow \lambda = 2$

So $x_\lambda = \begin{bmatrix} \frac{6-2}{4} \\ 3 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *.