Duality practice problem

Monday, March 10, 2025 10:45 AM

Solve: $\min\limits_{x \in \mathbb{R}^2} \frac{1}{2} \| \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \cdot x - \underbrace{\begin{bmatrix} 3 \\ 3 \end{bmatrix}}_{b} \|^2$ subject to $\sum x_i \le \beta$ (eg. $\beta = 2$)

$A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

Solution $\mathcal{L}(x, \lambda) = \frac{1}{2} \| Ax - b \|^2 + \lambda \cdot (\mathbb{1}^T x - \beta)$

$A^T b = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

For fixed λ, $\min\limits_x \mathcal{L}$ is solvable:

$$0 = A^T(Ax - b) + \lambda \cdot \mathbb{1}$$

i.e. $$x = (A^T A)^{-1} (A^T b - \lambda \cdot \mathbb{1})$$

$$= \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 - \lambda \\ 3 - \lambda \end{bmatrix} = \begin{bmatrix} \frac{6-\lambda}{4} \\ 3 - \lambda \end{bmatrix} \quad x_\lambda$$

check: is $\lambda = 0$ feasible? (i.e. $x = \begin{bmatrix} 3/2 \\ 3 \end{bmatrix}$ is unconstrained solution).

Yes, if $\beta \geq 4\frac{1}{2}$

No, if $\beta < 4\frac{1}{2}$. Assuming $\beta < 4.5$, let's continue...

How to find value of λ? Use λ to enforce $\sum x_i = \beta$ (if $\lambda \neq 0$ can't have $\sum x_i < \beta$)

So... $\mathbb{1}^T x_\lambda = \beta$, i.e. $\frac{6-\lambda}{4} + \frac{12 - 4\lambda}{4} = \beta$

or $18 - 5\lambda = 4\beta$, $5\lambda = 18 - 4\beta$. Suppose $\beta = 2$

... $5\lambda = 18 - 8 = 10 \Rightarrow \lambda = 2$

So $x_\lambda = \begin{bmatrix} \frac{6-2}{4} \\ 3 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\ast$.

Another duality practice problem

Wednesday, April 23, 2025 5:07 PM

(4/14/25)

In-class excercise

Solve: $\min_{x \in \mathbb{R}^2} \frac{1}{2}\|Ax-b\|_2^2$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

s.t. $c^T x = 0$

Note: unconstrained sol'n is $x_{LS} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 17 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Solution ①

Form Lagrangian $\mathcal{L}(x, \nu) = \frac{1}{2}\|Ax-b\|^2 + \nu \cdot c^T b$

KKT equations: ① "stationarity": $0 = \nabla_x \mathcal{L}(x,\nu) = A^T(A\underset{\text{unknown}}{x}-b) + \underset{\text{unknown}}{\nu} \cdot c$

② primal feasibility: $c^T \underset{\text{unknown}}{x} = 0$

③ ④ N/A since no inequality constraints

i.e. Solve $A^TA \cdot x + c \cdot \nu = A^Tb$
$c^T \cdot x = 0$

System of linear eq'n

$$\begin{bmatrix} A^TA & c \\ c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} A^Tb \\ 0 \end{bmatrix}$$

solve via Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 17 & -1 & 5 \\ 1 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 16 & -2 & 4 \\ 0 & -2 & -1 & -1 \end{array}\right] \xrightarrow{\text{swap}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 16 & -2 & 4 \end{array}\right] \quad R3 \leftarrow R3 - 8R2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -10 & -4 \end{array}\right]$$

Backsub: $\nu = 4/10 = .4$
$2x_2 = 1 - \nu = .6$
$x_2 = .3$
$x_1 = 1 - x_2 - \nu = 1 - .3 - .4 = .3$

So $\nu = .4$
$x = \begin{bmatrix} .3 \\ .3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$. ✱

Method ②

$c^T x = 0 \iff x_1 = x_2$ so then it becomes unconstrained,

$$\min_{x_1 \in \mathbb{R}} \frac{1}{2}\left\| \begin{bmatrix} 2x_1 \\ 4x_1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|^2 = \frac{1}{2}\left\| \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{A} x_1 - \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b} \right\|^2$$

$A^TA = [2 \ 4]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 4 + 16$

So $x_1 = (A^TA)^{-1}A^Tb = (20)^{-1} \cdot (2+4) = \frac{6}{20} = \frac{3}{10}$

$x_2 = x_1 = 3/10$.

Moral of the story:

You already knew (or should have known) that minimizing a convex quadratic reduces to linear algebra.

Adding linear equality constraints... it's still just linear algebra.