Convex Functions, part 2

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First-order conditions §3.1.3 BV'04

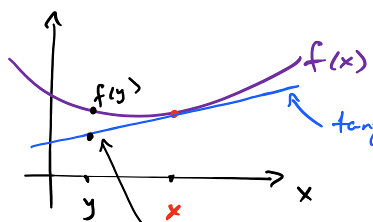
i.e., characterize convexity of f by looking at f' (i.e., ∇f)

why? differentiability at boundary is sketchy

Fact 1 If $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable on dom(f), and if dom(f) is open and convex,

then f is convex iff (1) $\underbrace{\forall x, y \in \text{dom}(f),\ f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle}_{\text{i.e.,}}$

Taylor approximation is an underestimate



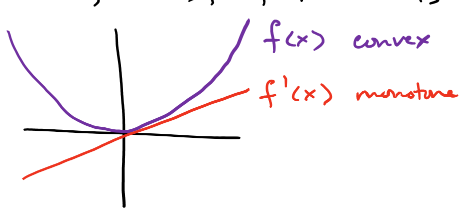
Fact 2 "..." (same setup)

then f is convex iff (2) $\forall x, y \in \text{dom}(f)$,

$$\langle x - y, \nabla f(x) - \nabla f(y) \rangle \geq 0$$

i.e., $\boxed{\nabla f \text{ is monotone}}$

Recall, in 1D, $f: \mathbb{R} \to \mathbb{R}$ is convex if slope is non-decreasing, i.e., monotone



i.e., if $x \geq y$

then $f'(x) \geq f'(y)$

i.e., $x - y \geq 0 \Rightarrow f'(x) - f'(y) \geq 0$

$x - y \leq 0 \Rightarrow f'(x) - f'(y) \leq 0$

and since positive · positive ≥ 0

negative · negative ≥ 0

$$(x - y) \cdot (f'(x) - f'(y)) \geq 0$$

So we can see the connection w/ our definition of monotone.

Fact 3 (2nd order condition) $f: \mathbb{R}^n \to \mathbb{R}$

If $\nabla^2 f(x)$ ("Hessian", like f'') exists for all $x \in \text{dom}(f)$

then (3a) f is convex iff $\nabla^2 f(x) \succeq 0 \quad \forall x \in \text{dom}(f)$ (pos. semidef) and (3b) f is μ-strongly convex (w.r.t. $\|\cdot\|_2$) iff $\nabla^2 f(x) \succeq \mu \cdot I$.

⚠ $\nabla^2 f(x) \succ 0 \implies$ strictly convex (sufficient but not necessary)

Remark Change the $\leq$ to $<$ for (1) and (2) for strict convexity

⚠ f can be convex but ∇f and $\nabla^2 f$ need not exist!

So... what about when f isn't differentiable?

Def Let $f: \mathbb{R}^n \to (-\infty, \infty]$ be proper, then we define the subdifferential of f at x to be

$$\underbrace{\partial f(x)}_{\text{subdiff.}} := \{ \underbrace{d}_{\text{subgradient}} \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, \; f(y) \geq f(x) + \langle d, y - x \rangle \}$$

(this may be an empty set... but :

Fact If f is proper and convex then (if domain is $\mathbb{R}^n$)

if $x \in \text{relint}(\text{dom}(f)) \implies \partial f(x) \neq \emptyset$)

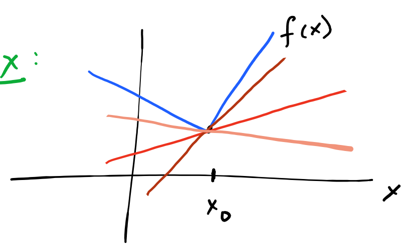
(Related to separating/supporting hyperplanes)

This should remind you of (1). In fact,

roughly, $\partial f(x)$ is a singleton, i.e. $|\partial f(x)| = 1$, iff f differentiable at x

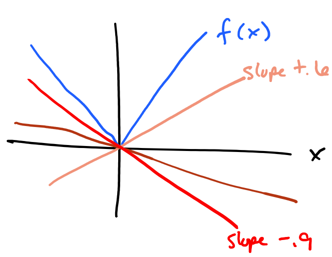
(precisely, cf Prop. 17.26 Bauschke + Combettes '11, f proper and convex, $x \in \text{dom}(f)$, then
1) f differentiable at $x \implies \partial f(x) = \{\nabla f(x)\}$
2) f continuous at x (e.g., $x \in \text{relint}(\text{dom}(f))$), and $\partial f(x) = \{u\}$, then f differentiable at x and $\nabla f(x) = u$)

Ex:



For $x \neq x_0$, $\partial f(x) = \{f'(x)\}$ since differentiable

For $x = x_0$, $\partial f(x_0)$ has more than one entry



More specifically, let $f(x) = |x|$, $f: \mathbb{R}^1 \to \mathbb{R}$

Then if $x \neq 0$, $f'(x) = \text{sign}(x)$, and $\partial f(x) = \{f'(x)\}$

If $x = 0$, $f'(0)$ DNE

but $\partial f(0) = [-1, 1]$

Fundamental Theorem of Convex Optimization*

*no one else calls it this

Thm: Fermat's Rule If f is a proper function, then $\operatorname{argmin}_x f(x) = \underbrace{\{ x : 0 \in \partial f(x) \}}$

means $\forall y,\ f(y) \geq f(x) + \langle 0, y-x \rangle$

i.e., $f(x) \leq f(y) \quad \forall y$

i.e., x is a global minimizer

proof: triviality

[This covers constrained optimization too, since f can be extended valued]

Generalizes 1D idea of findinding x s.t. $f'(x)=0$

or smooth and unconstrained notion of finding x st. $\nabla f(x) = 0$.

Subdifferentials are a global notion ... yet gradients are a local notion. For convex functions, the convexity (a global notion) links the two.

... so, all we need to do is invert ∂f, i.e., $\operatorname{argmin} f(x) = \partial f^{-1}(0)$

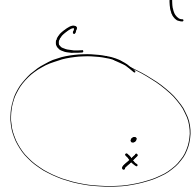
this is usually not directly possible for interesting problems (though it may be possible for subproblems).

Ex Let $C \neq \emptyset$ be convex, so I_C is a proper convex function. Then $\partial I_C = N_C$, the normal cone

Def The normal cone to a set C at the point x is

$$N_C(x) = \begin{cases} \{ d \mid \langle d, y-x \rangle \leq 0 \quad \forall y \in C \} & \text{if } x \in C \\ \emptyset & \text{if } x \notin C \end{cases}$$

Ex



$x \in \operatorname{int}(C) \Rightarrow N_C(x) = \{0\}$

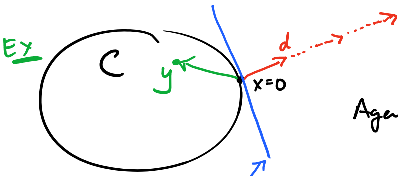
why? wlog, shift so $x=0$

If $\langle d, y \rangle \leq 0$ for all $y \in C$

then $x \in \operatorname{int}(C) \Rightarrow y = \varepsilon d \in C$ for ε sufficiently small but nonzero

$\Rightarrow \varepsilon \|d\|^2 \leq 0 \Rightarrow d = 0$.

Ex



overloaded notation! means "boundary"

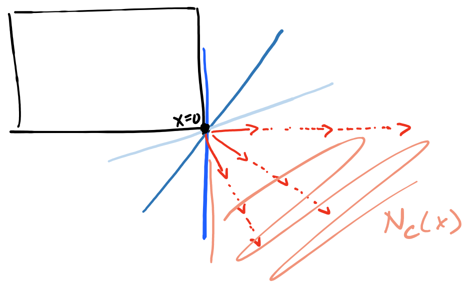
$x \in \partial C$

Again wlog let $x=0$, want d s.t. $\langle d, y \rangle \le 0 \ \forall y \in C$

(In fact, equivalent definition, p.66 BV'04
$N_C(x)$ = set of all vectors that define a supporting hyperplane to C, passing through x)

Ex. where $N_C(x)$ is solid

Ex



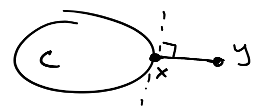
Ex If C is a vector (sub-)space, $N_C(x) = \begin{cases} C^{\perp} & x \in C \\ \phi & x \notin C \end{cases}$

Fact If $C \ne \phi$ is closed and convex then

$x = P_C(y)$ iff $y - x \in N_C(x)$

orthogonal projection

cf Prop. 6.46 Bauschke + Combettes '11



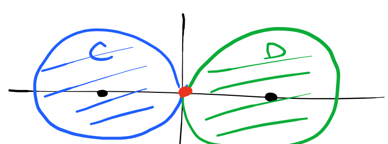
Calculus means a set of rules we can use to calculate

"Rule #1 from Calc. I that is so simple you never even have to cite it"

$$\frac{d}{dx}(f+g) = f' + g', \quad \text{or in } \mathbb{R}^n, \quad \nabla(f+g) = \nabla f + \nabla g$$

Is it true that $\partial(f+g) = \partial f + \partial g$? No! (but often true)

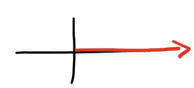
Ex: $f = I_C$, $g = I_D$ in $\mathbb{R}^2$



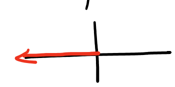
C, D balls of radius 1 centred at $(-1,0)$ and $(+1,0)$, resp.

Then $\partial(f+g)(x) = \partial f(x) + \partial g(x)$ for all x except at $x=0$

At $x=0$, $\partial f(0) = N_C(0) = \mathbb{R}_+ \times \{0\}$

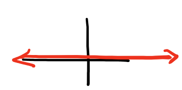


$\partial g(0) = N_D(0) = \mathbb{R}_- \times \{0\}$



so $\partial f(0) + \partial g(0) = \mathbb{R} \times \{0\}$ (A)

Minkowski sum



$C \cap D = \{\vec{0}\}$

but... $\partial(f+g)(0) = N_{C\cap D}(0) = N_{\{\vec{0}\}}$

$$:= \{ d \mid \underbrace{\langle d, y - \vec{0} \rangle \le 0 \ \forall y \in \{0\}}_{\text{vacuous constraint}} \}$$

$= \mathbb{R}^2$ (B) (A) $\ne$ (B)

It's often true $\partial(f+g) = \partial f + \partial g$.

Sufficient conditions to guarantee when this is true are called constraint qualifications. "CQ"

i.e., ex: Slater

Cor. 16.38 (iv) Bauschke + Combettes '11

If $f, g \in \Gamma_0(\mathcal{H})$, and $\mathcal{H} = \mathbb{R}^n$, then if

1) $\text{relint}(\text{dom}(f)) \cap \text{relint}(\text{dom}(g)) \neq \emptyset$

or 2) $\text{dom}(f) \cap \text{int}(\text{dom}(g)) \neq \emptyset$

} fancy

or 3) * either f or g has full domain (all of $\mathbb{R}^n$) } most commonly used

then $\partial(f+g) = \partial f + \partial g$.

... back to example $f = I_C$, $g = I_D$. This did not satisfy a CQ

$\text{dom}(f) = C$

$\text{dom}(g) = D$, $C \cap D = \vec{0}$

$\text{int}(C) \cap \text{int}(D) = \emptyset$

none of 1) 2) 3) above hold.

⚠ A lot of people are unaware of CQ and their importance.

Lack of CQ lead to "degenerate" optimization problems.

You hear, "MAXCUT, a NP-Hard problem, has a SDP relaxation (which is solvable in polynomial time) that guarantees a good approximation."

but is only "sorta" true: degenerate SDP are not efficiently solvable

(also, "polynomial time" depends on your model of computation, and if LP / SDP are strongly polynomial is an open question)

More cones

We defined the normal cone, closely related to the tangent cone (they are polars of each other, i.e. neg. duals)

Other cones come up (we'll discuss if needed, e.g., recession / asymptotic cone and barrier cone.