Convex Sets (following B&V)

Tuesday, January 19, 2021 10:34 PM

Chapter 2 in Boyd & Vandenberghe's book

§2.1 Affine and convex sets

Def Let $x, y \in \mathbb{R}^n$ (or any vector space), then

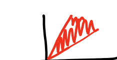
1) $tx + (1-t)y$, $t \in [0,1]$ is a convex combination (of x, y)

2) $tx + (1-t)y$, $t \in \mathbb{R}$ is a linear combination (of x, y)

3) tx, $t \geq 0$ is a conic combination (of x)

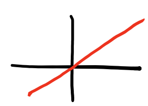
Def A set $C \subseteq \mathbb{R}^n$ is

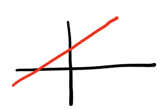
1) convex if $\forall x, y \in C$, it contains all convex combinations of x, y
2)* affine if $\forall x, y \in C$, it contains all linear combinations of x, y
3) a cone if $\forall x \in C$, it contains all conic combinations of x
4) a convex cone if it's convex and a cone,
   or equiv., $\forall x, y \in C$, $\forall t, s \geq 0$, $tx + sy \in C$

↳ Ex: convex cone  non-convex cones 

(more examples later)

*An affine set (aka affine subspace) is like a subspace (i.e., a linear subspace) except it's possible shifted (so need not include 0)

Ex: 1D subspace in $\mathbb{R}^2$ looks like 

a 1D affine space in $\mathbb{R}^2$ might look like 

Usually "affine" properties very similar to "linear", and people often say "linear" when it would technically be correct to say "affine"

Ex: $\dim(\underbrace{\text{affine space}}_{W})$ easily defined by shifting the space (pick $w_0 \in W$, $V := \{x - w_0 : x \in W\}$), V is now linear, define $\dim(W) = \dim(V)$.

Just like in real analysis we define things like "closure" ($\bar{A} :=$ set A together w/ all limit points

= smallest closed set containing A
= intersection of all closed sets containing A)

we can define:

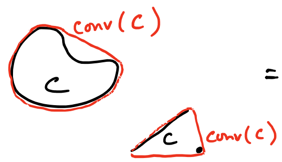
affine hull, $\mathrm{aff}(C)$:= smallest affine set containing C

and

convex hull, $\mathrm{conv}(C)$:= smallest convex set containing C
= intersection of all convex sets containing C
$= \{ t_1 x_1 + t_2 x_2 + t_3 x_3 + \ldots + t_k x_k : x_i \in C,\ t_i \geq 0,\ \sum_{i=1}^{k} t_i = 1 \}$ *

sometimes/often this is the definition of convex combination

= "stretch rubber band around C, include everything inside"



* for "any $k \in \mathbb{N}$" though you can prove $k = n+1$ is sufficient if $C \subseteq \mathbb{R}^n$

and conic hull defined similarly.

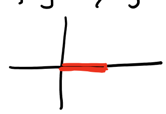
Recall, interior of a set, $\mathrm{int}(C) := \{ x \in C : \exists\, \varepsilon > 0,\ B_\varepsilon(x) \subseteq C \}$

New concept: relative interior of a set, $\mathrm{relint}(C) := \{ x \in C : \exists\, \varepsilon > 0,\ B_\varepsilon(x) \cap \mathrm{aff}(C) \subseteq C \}$
or $\mathrm{ri}(C)$

makes geometric sense

Ex: let $C = [0,1] \subseteq \mathbb{R}^1$, $\mathrm{int}(C) = (0,1)$.
$\mathrm{relint}(C) = (0,1)$.

but
same shape, but embedded in higher dimensi
let $C = [0,1] \times \{0\} \subseteq \mathbb{R}^2$, $\mathrm{int}(C) = \emptyset$
$\mathrm{relint}(C) = (0,1) \times \{0\}$ ✓



since $\mathrm{aff}(C) = \mathbb{R} \times \{0\} \cong \mathbb{R}^1$

Ex. of common geometric objects

Hyperplane For $a \in \mathbb{R}^n, b \in \mathbb{R}$, $\{ x \in \mathbb{R}^n : a^T x = b \}$. $n-1$ dimensional, affine

$n=2$, these are lines
$n=3$, these are planes

They split $\mathbb{R}^n$ into 2 sides

Half-space $\{ x \in \mathbb{R}^n : a^T x \leq b \}$ (or another parameterization, $a^T(x - x_0) \leq 0$
i.e., $b = a^T x_0$)

Not affine (so no "dimension")
but convex

Euclidean ball

open ball $B_\varepsilon(x) = \{ y \in \mathbb{R}^n : \| y - x \| < \varepsilon \}$ → Euclidean norm unless otherwise specified

closed ball $\overline{B_\varepsilon}(x) = \{ y \in \mathbb{R}^n : \| y - x \| \leq \varepsilon \}$

* oddly, B+V define $B_\varepsilon(x)$ to be the closed ball by default, nonstandard in real analysis. We won't follow B+V

Ellipsoid $\mathcal{E} = \{ x : (x - x_0)^T P^{-1} (x - x_0) \leq 1 \}$

for some matrix $P \succ 0$

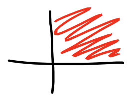
$\underbrace{}$ means P is positive definite (see supplemental notes)

Includes balls
(choose $P = \varepsilon^2 I$)

Always convex

Cones

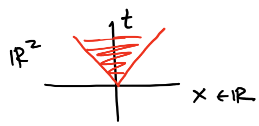
- Positive orthant in $\mathbb{R}^n$, $\mathbb{R}^n_+ = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : x_i \geq 0 \right\}$, is a cone.



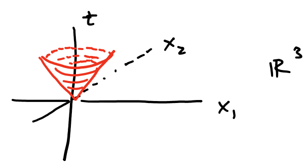
$\mathbb{R}^n_{++} = \{ \ \ldots \ : x_i > 0 \}$ isn't a cone
(doesn't include 0, but 0 is always a conic combination)

- Lorentz cone / 2nd order cone / "ice cream cone"

$C = \{ (x, t) \in \mathbb{R}^{n+1}, \ \begin{matrix} x \in \mathbb{R}^n \\ t \in \mathbb{R} \end{matrix} : \|x\|_2 \leq t \}$



or



- Positive Semidefinite matrices. Most important non-polyhedral cone
  (PSD)

Recall $\mathbb{R}^{m \times n} = \{ m \times n \text{ matrices} \}$ is a $m \cdot n$ dimensional linear space
aka vector space

and let $S^n = \{ X \in \mathbb{R}^{n \times n} : X = X^T \}$ be square, symmetric matrices
(or $X = X^*$, if working w/ $\mathbb{C}^{n \times n}$)

⚠ (often S^{n-1} means the sphere in $\mathbb{R}^n$, but we'll usually use it to mean symm. matrices)

S^n is a linear space

Define PSD cone to be $S^n_+ = \{ X \in S^n : \underbrace{X \succeq 0}_{\text{means } X \text{ is psd}} \}$

(Pos. Def. set $S^n_{++} = \{ X \in S^n : X \succ 0 \}$
isn't a cone and isn't closed)

Polyhedron

⚠ Very common in math so there are conflicting definitions

Def: polyhedron $P \subseteq \mathbb{R}^n$ is a set of the intersection of a finite number of half-spaces and half-planes

$$P = \{ x \in \mathbb{R}^n : \begin{matrix} a_j^T x \leq b_j, & j = 1, \ldots, m \\ c_j^T x = d_j, & j = 1, \ldots, p \end{matrix} \}$$

⚠ ∞ # half-spaces/planes isn't necessarily a polyhedra

Ex: $\{ x \in \mathbb{R}^n : a_j^T x \leq 1, \ a_j \in \text{unit circle} \}$
defines the closed ball $\overline{B_1(0)}$, not polyhedral

Polyhedra are always convex

Some authors call a bounded polyhedron a polytope
(others define a polytope as we did a polyhedra, and define a polyhedra to mean a bounded polytope! completely switched!)

In other fields / wikipedia,

polygon means in $\mathbb{R}^2$

polyhedron means in $\mathbb{R}^3$

polytope means higher dimension.

We <u>won't</u> use these definitions.

I'll probably use all 3 interchangeably... sorry!