Convex Sets part 2

Sunday, January 24, 2021 9:04 PM

(still following Boyd + Vandenberghe ch. 2)

Finishing up geometric objects

Recall a set of points $\{x_i\}_{i=1}^n$ is linearly independent

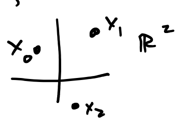
if $\sum_{i=1}^n t_i x_i = 0 \implies t_i = 0 \ \forall i$ (x_i vectors, t_i scalars)

and of course in n-dim. vector spaces, at most n points can be lin. independent

Similarly, define $\{x_i\}_{i=0}^n$ to be affinely independent

if $\sum_{i=1}^n t_i (x_i - x_0) = 0 \implies t_i = 0 \ \forall i$ (of course, it doesn't really matter which vector of the set you label to be x_0)

and in $\mathbb{R}^n$, at most $n+1$ points can be affinely independent



and define the affine dimension of an affine space to be the dimension when $\underset{\text{usual vector space dimension}}{}$ we "homogenize" the space, i.e., affine space W, "homogenized" space $V = W - w_0 := \{x - w_0 : x \in W\}$ for any $w_0 \in W$

There's always a "+1" difference (so we use 0-based indices)

Vector Space: $\text{span}(\{\boxed{0,} x_1, x_2, \dots, x_n\})$ has dimension n

Affine Space: $\text{aff}(\{\boxed{x_0,} x_1, x_2, \dots, x_n\})$ has affine dimension n

An important class of polyhedra is the simplex.

For any set of $k+1$ points $\{x_i\}_{i=0}^k$ in $\mathbb{R}^n$, which are affinely independent, ($\underset{\text{so}}{\frown}$ $k \le n$)

they determine a simplex $C = \text{conv}(\{x_i\}_{i=0}^k) = \{x = \sum_{i=0}^k t_i x_i : t_i \ge 0, \sum_{i=0}^k t_i = 1\}$

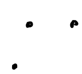
Since x_i are affinely independent, "all the fat has been trimmed", and more canonical

Affine-dim of $k+1$ point simplex is k, so call it a "k-dim simplex"

Ex: 1-dim simplex = $\text{conv}(\{x_0, x_1\})$ = line segment, in $\mathbb{R}^n$, $n \ge 1$

2-dim simplex = $\text{conv}(\{x_0, x_1, x_2\})$ = triangle (w/ interior), in $\mathbb{R}^n$, $n \ge 2$

3-dim ... = tetrahedron in $\mathbb{R}^n$ ($n \ge 3$) 

in $\mathbb{R}^2$,  doesn't generate a simplex (not affinely independent)

In $\mathbb{R}^n$, the unit simplex is the simplex generated by $\{0, e_1, e_2, \dots, e_n\}$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, etc. (unit vectors / canonical basis) component-wise

aka $\mathbb{1}^T x \le \mathbb{1}$

ones vector (NOT identity)

$$= \{ x \in \mathbb{R}^n : x \geq 0, \ \sum_i x_i \leq 1 \}$$

and it has affine dimension n.

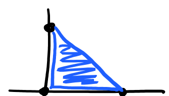
In $\mathbb{R}^n$, the $(n-1)$-dim probability simplex is generated by $\{e_1, e_2, \ldots, e_n\}$

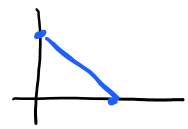
(ie., same as unit simplex but w/o the 0)

Q1: Give a nice description of the probability simplex

A: $= \{ x \in \mathbb{R}^n : x \geq 0, \ \mathbb{1}^T x = \mathbb{1} \}$

vs $\ldots \leq \ldots$ for unit simplex

The 2D unit simplex is  (in $\mathbb{R}^2$)

whereas the 1D- probability simplex is 

See Venkat C., Gongguo Tang ...
for atomic norms (which generalize simplices), and gauges

Operations that preserve convexity (§2.3 in BV04)

⓪ Cartesian / direct products

If $C_1 \subseteq \mathbb{R}^{n_1}$ and $C_2 \subseteq \mathbb{R}^{n_2}$ both convex, then $C_1 \times C_2 \subseteq \mathbb{R}^{n_1+n_2}$ is convex

① Intersections (arbitrary #, even uncountable)

C_1, C_2 convex $\Rightarrow$ $C_1 \cap C_2$ convex

NOT true for unions  not convex

② Image of an affine function $f(x) = A \cdot x + b$

$C \subseteq \mathbb{R}^n$

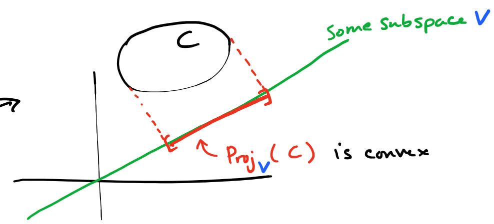
1) $f(C) := \{ f(x) : x \in C \}$ is convex if C is (and f affine)

2) and $f^{-1}(C) := \{ x : f(x) \in C \}$ is convex if $C \subseteq \mathbb{R}$ is (f^{-1} need not exist as a function!)

implies

2a	Scaling	$f(x) = 3 \cdot x$
2b	translation	$f(x) = x + b$
2c	rotation	$f(x) = Qx$
2d	✱ projection	$f(x) = Px$

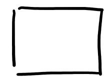
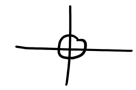
(and combinations thereof) preserve convexity

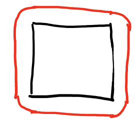


2e Sum (Minkowski sum)

$$C_1 + C_2 := \{ x + y : x \in C_1, \ y \in C_2 \}$$

Think of it like a convolution.

Ex: $C_1 =$  $C_2 =$ 

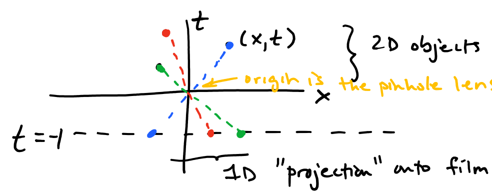
then $C_1 + C_2 =$  $=$ 

③ Linear-Fractional and Perspective Function

Perspective $P: \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$, $\quad P(\vec{z}, \underset{\text{scalar}}{t}) = \vec{z}/t$

think of as normalizing then projection, $(\vec{z}, t) \to (\vec{z}/t, t/t) \to (\vec{z}/t)$

or pin-hole camera, in 2D or 3D



Since the line is $\alpha(x,t)$, $\alpha \in \mathbb{R}$

so projecting to $t=-1$ means

$(\alpha x, \alpha t)$ s.t. $\alpha t = -1 \Rightarrow \alpha = -1/t$

so $(\underbrace{-x/t}_{\text{just the (negative) perspective}}, -1)$

Then

1) C convex in $\mathbb{R}^{n+1}$, P the perspective function $\Rightarrow P(C)$ is convex (in $\mathbb{R}^n$)

2) and also $C = \mathbb{R}^n$ convex $\Rightarrow P^{-1}(C)$ is convex (in $\mathbb{R}^{n+1}$)

Linear-Fractional Function is composing the $\underbrace{\text{perspective}}_{P}$ w/ an $\underbrace{\text{affine function}}_{g}$

$$g(x) := \begin{bmatrix} A \\ c^T \end{bmatrix} \cdot x + \begin{bmatrix} b \\ d \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^{m+1}$$

$(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, d \in \mathbb{R})$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$f = P \circ g$, $\quad f(x) = \dfrac{Ax+b}{c^Tx+d}$ w/ domain $\{x : c^Tx + d > 0\}$

Then 1) if C is convex in $\mathbb{R}^n$, $f(C)$ is convex in $\mathbb{R}^m$

Arises in probability, e.g., conditional probabilities (see Ex. 2.13 book)

Generalized Inequalities (§2.4 BV'04)

Any $\underline{\text{proper cone}}$ K induces a partial order $x \preceq_K y$ if $y - x \in K$

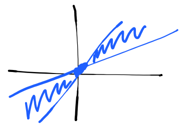
not written when can be inferred. Usually psd

$x \prec_K y$ if $y - x \in \text{int}(K)$

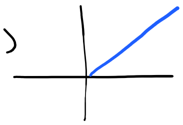
1) convex
2) closed
3) solid ($\text{int}(K) \neq \emptyset$)
4) pointed (contains no line)

Q2 Which (if any) of above 4 properties violated. All in $\mathbb{R}^2$

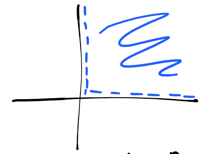
(a)



(b)



(c)



Ex: $\{x \in \mathbb{R}^2 : x_1, x_2 \text{ both } \geq 0 \text{ or both } \leq 0\}$

Ex: $\{t \cdot x_0 : t \geq 0\}$

Ex: $\mathbb{R}^n_{++} = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, x_i > 0 \right\} \cup \{0\}$

A: (a) 1) (not) convex
4) (not) pointed

(b) 3) (not) solid

(c) 2) (not) closed

Dual Cones

Def If K is a set, its **dual cone** is $K^* = \{ y : \langle x, y \rangle \geq 0 \ \forall x \in K \}$

(Related: the polar cone $K^\circ = \{ y : \langle x, y \rangle \leq 1 \ \forall x \in K \}$

$\Rightarrow \langle x, y \rangle \leq 0 \ \forall x \in K, y \in K^\circ$ if K a cone, due to scaling property of cones

so some authors define $K^\circ = \{ y : \langle x, y \rangle \leq 0 \ \forall x \in K \}$ in which case $K^\circ = -K^*$

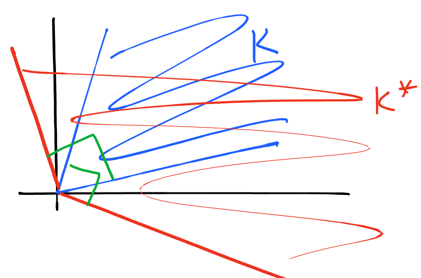
Facts

- K^* is a cone (even if K isn't a cone!)
- K^* is convex (even if K isn't convex!)
- $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$
- $K^{**} = K$ iff K is a ~~proper cone~~ closed and convex

corrected 1/31/21

Ex If K is a subspace, $K^* = K^\perp$

Ex



Ex $\mathbb{R}^n_+ = (\mathbb{R}^n_+)^*$

Ex $K = S^n_+$ (PSD matrices, $X \in S^n_+ \Rightarrow X = GG^T$) [aka $X \succeq 0$]

Calculate $K^* = \{ Y \in S^n : \underbrace{\langle Y, X \rangle}_{\mathrm{tr}(Y^T X)} \geq 0 \ \forall X \succeq 0 \}$

$\mathrm{tr}(Y^T X) = \mathrm{tr}(Y X)$ since $Y = Y^T$

$= \mathrm{tr}(Y G G^T)$

$= \mathrm{tr}(\underbrace{G^T Y G}) \geq 0 \ \forall$ matrices G

iff $Y \succeq 0$.

$\therefore$ So S^n_+ is self-dual.