

# Convex Sets part 2

Sunday, January 24, 2021

9:04 PM

(still following Boyd + Vandenberghe ch. 2)

Finishing up geometric objects

Recall a set of points  $\{x_i\}_{i=1}^n$  is linearly independent

$$\text{if } \sum_{i=1}^n t_i x_i = 0 \Rightarrow t_i = 0 \quad \forall i \quad (x_i \text{ vectors, } t_i \text{ scalars})$$

and of course in  $n$ -dim. vector spaces, at most  $n$  points can be lin. independent

Similarly, define  $\{x_i\}_{i=0}^n$  to be affinely independent

$$\text{if } \sum_{i=1}^n t_i (x_i - x_0) = 0 \Rightarrow t_i = 0 \quad \forall i \quad (\text{of course, it doesn't really matter which vector of the set you label to be } x_0)$$

and in  $\mathbb{R}^n$ , at most  $n+1$  points can be affinely independent



and define the **affine dimension** of an affine space to be the dimension when we "homogenize" the space, i.e., affine space  $W$ , "homogenized" space  $V = W - w_0 := \{x - w_0 : x \in W\}$  for any  $w_0 \in W$

There's always a "+1" difference (so we use offset indices)

Vector Space:  $\text{span}(\{0, x_1, x_2, \dots, x_n\})$  has dimension  $n$

Affine Space:  $\text{aff}(\{x_0, x_1, x_2, \dots, x_n\})$  has affine dimension  $n$

An important class of polyhedra is the **simplex**.

$\underbrace{\text{so}}_{K \leq n}$

For any set of  $K+1$  points  $\{x_i\}_{i=0}^K$  in  $\mathbb{R}^n$ , which are affinely independent, they determine a simplex  $C = \text{conv}(\{x_i\}_{i=0}^K) = \left\{ x = \sum_{i=0}^K t_i x_i : t_i \geq 0, \sum_{i=0}^K t_i = 1 \right\}$

Since  $x_i$  are affinely independent, "all the fat has been trimmed", and more canonical

Affine-dim of  $K+1$  point simplex is  $K$ , so call it a " $K$ -dim simplex"

Ex: 1-dim simplex =  $\text{conv}(\{x_0, x_1\})$  = line segment, in  $\mathbb{R}^n$ ,  $n \geq 1$

2-dim simplex =  $\text{conv}(\{x_0, x_1, x_2\})$  = triangle ( $\omega$ , interior), in  $\mathbb{R}^n$ ,  $n \geq 2$



3-dim ... = tetrahedron in  $\mathbb{R}^n$  ( $n \geq 3$ )



in  $\mathbb{R}^2$ ,  $\begin{array}{c} \bullet \\ \cdot \\ \cdot \end{array}$  doesn't generate a simplex (not affinely independent)

In  $\mathbb{R}^n$ , the **unit simplex** is the simplex generated by  $\{0, e_1, e_2, \dots, e_n\}$

where  $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots$ , etc. (unit vectors / canonical basis)

$$= \{ x \in \mathbb{R}^n : x \geq 0, \sum_i x_i \leq 1 \}$$

component-wise  
aka  $\mathbb{I}^T x \leq 1$   
↑ ones vector  
(NOT identity)

and it has affine dimension  $n$ .

In  $\mathbb{R}^n$ , the  $(n-1)$ -dim probability simplex is generated by  $\{e_1, e_2, \dots, e_n\}$   
(ie., same as unit simplex but w/o the 0)

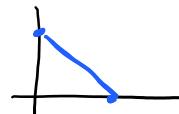
Q1: Give a nice description of the probability simplex

A:  $= \{ x \in \mathbb{R}^n : x \geq 0, \mathbb{I}^T x = 1 \}$   
vs ... for unit simplex

The 2D unit simplex is



whereas the 1D probability simplex is



See Venkat C., Ganguly T...  
for atomic norms (which generalize simplices), and gauges

Operations that preserve convexity (§ 2.3 in BVb4)

### ① Cartesian/direct products

If  $C_1 \subseteq \mathbb{R}^{n_1}$  and  $C_2 \subseteq \mathbb{R}^{n_2}$  both convex, then  $C_1 \times C_2 \subseteq \mathbb{R}^{n_1+n_2}$  is convex

### ② Intersections (arbitrary #, even uncountable)

$C_1, C_2$  convex  $\Rightarrow C_1 \cap C_2$  convex

NOT true for unions



not convex

### ③ Image of an affine function $f(x) = Ax + b$

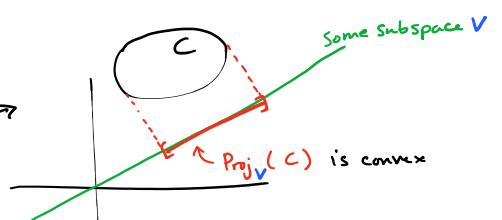
i)  $f(C) := \{ f(x) : x \in C \}$  is convex if  $C \subseteq \mathbb{R}^n$  is (and  $f$  affine)

ii) and  $f^{-1}(C) := \{ x : f(x) \in C \}$  is convex if  $C \subseteq \mathbb{R}$  is ( $f^{-1}$  need not exist as a function!)

implies

- 2a Scaling  $f(x) = 3x$
- 2b translation  $f(x) = x + b$
- 2c rotation  $f(x) = Qx$
- 2d ~~projection~~  $f(x) = Px$

(and combinations thereof) preserve convexity



### 2e Sum (Minkowski sum)

$$C_1 + C_2 := \{ x + y : x \in C_1, y \in C_2 \}$$

Think of it like a convolution.

$$\text{Ex: } C_1 = \boxed{\phantom{00}} \quad C_2 = \boxed{\phantom{0}}$$

then

$$C_1 + C_2 = \boxed{\begin{array}{|c|c|} \hline \text{0} & \text{0} \\ \hline \text{0} & \text{0} \\ \hline \end{array}} = \boxed{\phantom{00}}$$

### ③ Linear-Fractional and Perspective Function

$$\text{Perspective } P: \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}_{++} \rightarrow \mathbb{R}^n, \quad P(\vec{z}, t) = \frac{\vec{z}}{t}$$

think of as normalizing then projection,  $(\vec{z}, t) \rightarrow (\vec{z}/t, t/t) \rightarrow (\vec{z}/t)$

or pin-hole camera, in 2D  $\rightsquigarrow$  3D

Since the line

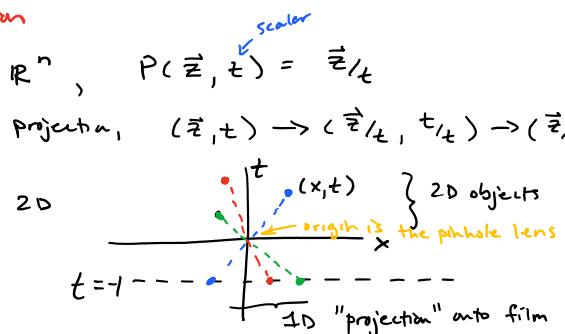
is  $\alpha(x, t), \alpha \in \mathbb{R}$

so projecting to  $t=-1$  means

$$(\alpha x, \alpha t) \text{ s.t. } \alpha t = -1 \Rightarrow \alpha = -1/t$$

$$\text{so } (-x/t, -1)$$

$\rightsquigarrow$  just the (negative) perspective



Then  
1)  $C$  convex in  $\mathbb{R}^{n+1}$ ,  $P$  the perspective function  $\Rightarrow P(C)$  is convex (in  $\mathbb{R}^n$ )

2) and also  $C = \mathbb{R}^n$  convex  $\Rightarrow P^{-1}(C)$  is convex (in  $\mathbb{R}^{n+1}$ )

Linear-Fractional Function is composing the perspective w/ an affine function  $\underbrace{P}_g$

$$g(x) := \begin{bmatrix} A \\ c^T \end{bmatrix} \cdot x + \begin{bmatrix} b \\ d \end{bmatrix} : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \quad (A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, d \in \mathbb{R})$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f = P \circ g, \quad f(x) = \frac{Ax+b}{c^T x + d} \quad \text{w/ domain } \{x: c^T x + d > 0\}$$

Then D if  $C$  is convex in  $\mathbb{R}^n$ ,  $f(C)$  is convex in  $\mathbb{R}^m$

Arises in probability, e.g., conditional probabilities (see Ex. 2.13 book)

Generalized Inequalities (§ 2.4 BV'04)

Any proper cone  $K$  induces a partial order

1) convex

2) closed

3) solid ( $\text{int}(K) \neq \emptyset$ )

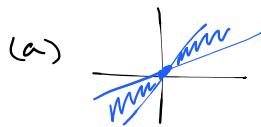
4) pointed (contains no lines)

not written when can be inferred. Usually psd

$$x \leq_K y \quad \text{if } y-x \in K$$

$$x <_K y \quad \text{if } y-x \in \text{int}(K)$$

Q2 Which (if any) of above 4 properties violated. All in  $\mathbb{R}^2$



$$Ex: \left\{ \mathbf{x} \in \mathbb{R}^2 : x_1, x_2 \text{ both } \geq 0 \text{ or both } \leq 0 \right\}$$



$$Ex: \left\{ t \cdot \mathbf{x}_0 : t \geq 0 \right\}$$



$$Ex: \mathbb{R}_{++}^n = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, x_i > 0 \right\} \cup \{ \mathbf{0} \}$$

A: (a) 1) (not) convex  
4) (not) pointed

(b) 3) (not) solid

(c) 2) (not) closed

### Dual Cones

Def If  $K$  is a set, its **dual cone** is  $K^* = \{ y : \langle x, y \rangle \geq 0 \ \forall x \in K \}$

(Related: the **polar cone**  $K^\circ = \{ y : \langle x, y \rangle \leq 0 \ \forall x \in K \}$ )

$\Rightarrow \langle x, y \rangle \leq 0 \ \forall x \in K, y \in K^\circ$  if  $K$  a cone, due to scaling property of cones

#### Facts

- $K^*$  is a cone (even if  $K$  isn't a cone!)
- $K^*$  is convex (even if  $K$  isn't convex!)
- $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$
- $K^{**} = K$  iff  $K$  is a proper cone

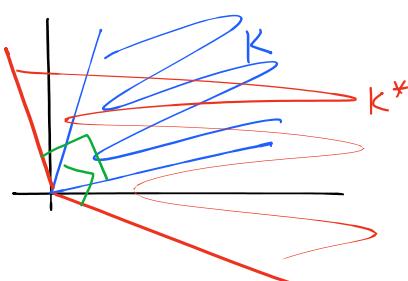
so some authors define

$$K^\circ = \{ y : \langle x, y \rangle \leq 0 \ \forall x \in K \}$$

in which case  $K^\circ = -K^*$

Ex If  $K$  is a subspace,  $K^* = K^\perp$

Ex



$$Ex \quad \mathbb{R}_+^n = (\mathbb{R}_+^n)^*$$

Ex  $K = S_+^n$  (PSD matrices,  $\underbrace{\mathbf{x} \in S_+^n \Rightarrow \mathbf{x} = \mathbf{G}\mathbf{G}^T}_{\text{aka } \mathbf{x} \geq 0}$ )

Calculate  $K^* = \{ Y \in S^n : \underbrace{\langle Y, X \rangle \geq 0}_{\text{since } Y = Y^T} \ \forall X \geq 0 \}$

$$\begin{aligned} \text{tr}(Y^T X) &= \text{tr}(Y X) \quad \text{since } Y = Y^T \\ &= \text{tr}(Y G G^T) \\ &= \text{tr}(G^T Y G) \end{aligned}$$

... so  $S_+^n$  is self-dual.

$\geq 0 \ \forall \text{ matrix } G$   
iff  $Y \geq 0$ .