Minimizers, intro to convexity

Tuesday, January 19, 2021 5:26 PM

First, go over discussion questions from last time

Solutions

Generic problem: $\min_{x \in C} f(x)$, $C \subseteq \mathbb{R}^n$

- A feasible point x means $x \in C$
- A solution or minimizer or global minimizer x^* means
  1) $x^* \in C$
  2) $\forall y \in C,\ f(x^*) \le f(y)$ (NOT $f(x^*) < f(y)$)

  This is what we usually really want

  i.e., $x^* \in \operatorname{argmin}_{x \in C} f(x)$

⚠ Solutions may not exist ... even for convex problems

Ex $\min_{x \in \mathbb{R}} x$ Ex $\min_{x \in (0,1)} x^2$ abuse of notation. it's more proper to write $\inf_{x \in (0,1)} x^2$ but optimizers are too lazy

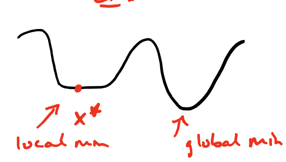
⚠ Solutions need not be unique ... even for convex problems

Ex $\min_{x \in \mathbb{R}} f(x)$ where $f(x) = 0 \ \forall x$

- x^* is a local minimizer if x^* is feasible and $\exists\, \varepsilon > 0$ s.t.

  $f(x^*) \le f(y) \quad \forall y \in C \cap \underbrace{B_\varepsilon(x^*)}_{\text{open ball of radius } \varepsilon} := \{ y : \|y - x^*\| < \varepsilon \}$

Ex:



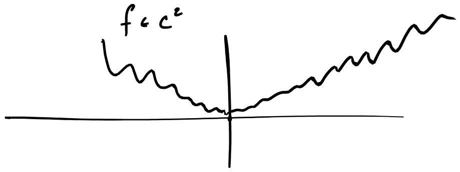
- x^* is a strict local minimizer if x^* is feasible and $\exists\, \varepsilon > 0$ s.t.

  $f(x^*) < f(y) \quad \forall\, y \in B_\varepsilon(x^*) \cap C$

- x^* is an isolated (and strict) local min if a local min, and no other local min are nearby

  FAIRLY ESOTERIC

Ex $f(x) = \begin{cases} x^4 \cos(1/x) + 2x^4 & x \ne 0 \\ 0 & x = 0 \end{cases}$ $x = 0$ is a strict but not isolated local min



i.e., $\exists$ sub-optimal local min x_j w/ $x_j \to 0$

B/w, notation

$f \in C^3$ means f, f', f'', f''' all exist and are continuous

or $f \in C^3(\mathbb{R}^n)$ means $f, \nabla f, \nabla^2 f, \nabla^3 f$ all exist and are continuous

Connections w/ Calc. I

$\min\limits_{x \in [a,b]} f(x)$

The procedure: 1) find all points x_i such that $f'(x_i) = 0$ "critical numbers" or where f is discontinuous or not differentiable

2) throw in $\{a, b\}$ boundary

3) minimize $\{a, b, x_i\}$

in textbook exercises, $\{x_i\}$ was always finite

Ex: minimize $\{3, -2, 5, 17, -31, 21\}$
it's easy: it's -31

generalization

- A critical or stationary point of $f(x)$ is a point where $\nabla f(x) = 0$

... but this isn't enough usually.

1) it doesn't take into account constraints.

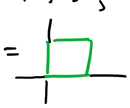
If $x \in \partial C$ then don't expect $\nabla f(x) = 0$ here

boundary, $= \overline{C} \setminus \mathrm{int}(C)$, $\mathrm{int}(C)$ = largest open set in C $= \{x \in C : \exists\, \varepsilon > 0 \;\; B_\varepsilon(x) \subseteq C\}$

$\overline{C}$ = closure(C) = smallest closed set containing C

2) we can't just check boundary separately

in $\mathbb{R}$, if $C = [0,1]$, $\partial C = \{0, 1\}$, so $|\partial C| = 2$ ← cardinality

in $\mathbb{R}^2$, if $C = [0,1]^2$, $\partial C =$  so $|\partial C| = \infty$

3) often there are an ∞ # (or at least a large #) of critical points, so can't check all

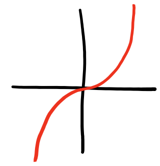
- If x^* is a local or global minimizer and $C = \mathbb{R}^n$ (i.e., unconstrained) then x^* is a critical point ← "necessary"

minimizer $\Rightarrow$ crit. pt.

Hw exerc.

but if x^* is a critical point, it need not be a local or global minimizer (so not sufficient) in which case we often call it a saddle point

Ex: $f(x) = x^3$



$x^* = 0$ is a saddle point

($f'(0) = 3x^2\big|_{x=0} = 0$ so it's a crit. pt., but not a local min.)

19th century analysis theorem

Sometimes called "Weierstrass thm":

ie., in $\mathbb{R}^n$, means closed and bounded

if f is continuous and C is compact then f achieves its infimum over C, ie., $\inf_{x \in C} f(x) = \min_{x \in C} f(x)$

(our first existence of minimizers result)

proof

Q1

on your own, using these ingredients:

1) in $\mathbb{R}^n$, compact $\Leftrightarrow$ sequentially compact (i.e., every sequence in C has a convergent subseq. whose limit is in C)

2) in $\mathbb{R}^n$ w/ usual topology (ie., induced by a norm),

a function f is cts $\Leftrightarrow$ sequentially cts (ie., $x_n \to x \Rightarrow f(x_n) \to f(x)$)

shorthand for continuous

One way to prove it:

First, note the set $f(C) := \{ f(x), x \in C \}$ is bounded (compact in fact)

since if not bounded, pick $(\alpha_k) \subseteq f(C)$ s.t. $|\alpha_k| \geq k$

but then $\exists\ x_k$ s.t. $\alpha_k = f(x_k)$, and $\exists$ convergent subseq. $x_{k_j} \to x \in C$

so f cts $\Rightarrow$ $\lim_{j\to\infty} \underbrace{f(x_{k_j})}_{= \lim \alpha_{k_j}} = f(x)$ ← limit

$= \lim \alpha_{k_j}$ ← unbounded

Convergent sequences are bdd. Contradiction

So this guarantees $\inf_{x \in C} f(x) =: f^*$ is finite.

So, second step, by "infimum", pick $\alpha_k \in f(C)$ s.t. $\alpha_k \to f^*$

again, $\exists\ x_k$ st $\alpha_k = f(x_k)$, again pick convergent subseq. x_{k_j},

again use continuity of f. ∎

So... it's nice when our constraints C are compact

It turns out that's often too much to ask for, but at the very least, we like our constraint sets to be closed

Ex: $\min \|x\|^2$ (1) $\min \|x\|^2$ (2)

s.t. $\|Ax - b\| < \varepsilon$ s.t. $\|Ax - b\| \leq \varepsilon$

very similar, but (1) probably doesn't have a solution, whereas (2) does.

Other high-level properties of the feasible set C:

- If $C = \emptyset$, the problem is "infeasible"

  Not always obvious to spot, ex: $C = \{ x : Ax \leq b \}$ may or may not be empty.

• In this class, C will usually be convex … in particular, not integral. ($\mathbb{Z}^n$)

ex: $C = \{ x \in \mathbb{R}^2,\ x \geq 0,\ 7 \cdot x_1 + 4 \cdot x_2 \leq 13,\ \overbrace{x_1, x_2 \in \mathbb{Z}}^{\text{makes it hard}} \}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

max $\underbrace{21 x_1 + 11 x_2}_{f(x)}$ s.t. $x \in C$ "integer linear program"

the line $7x_1 + 4x_2 = 13$ is drawn in green (intercepts: $x_2 = 0$, $x_1 = \frac{13}{7} \approx 1.85$; $x_1 = 0$, $x_2 = 13/4 = 3.25$)

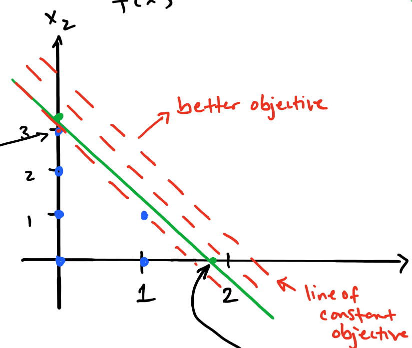
C shown in blue

level sets of objective $f(x)$ in red

eg. $f(x) = \alpha$

$\alpha = 33$ $\quad 21 x_1 + 11 x_2 = 33$

intercepts $(0, 3)$ and $(\frac{33}{21}, 0)$ ≈ 1.57



w/o $\mathbb{Z}^2$ constraints, max is at $(13/7, 0)$

w/ $\mathbb{Z}^2$ constraint, max is at $(0, 3)$

Far away.

Rounding $(13/7, 0)$ to nearest integers is not optimal

Done all the time in practice.

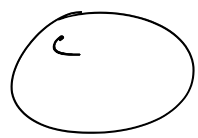
Sometimes it works sorta ok… often it doesn't work well at all.

(More on this later in semester)

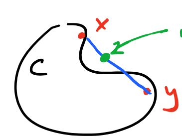
Convexity

Def. A set C $\overbrace{\text{inside any vector space}}^{\text{from now on, this is always implicitly assumed in this class}}$ is convex if $\forall\, x, y \in C$ and $\forall\, t \in [0,1]$, then $t \cdot x + (1-t) y \in C$

i.e., $\overline{xy} = \{ t x + (1-t) y : t \in [0,1] \}$



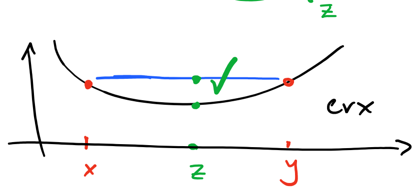
convex



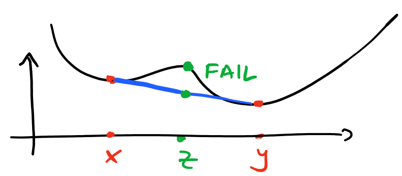
not convex

Def A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\forall\, x, y \in \mathbb{R}^n$, $\forall\, t \in [0,1]$

then $f(\underbrace{t x + (1-t) y}_{z}) \leq t \cdot f(x) + (1-t) f(y)$



cvx

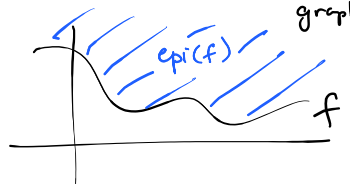


not convex

or, since mathematicians don't like to write/talk more than necessary,

a function f is convex iff the set "epi(f)" is convex where $\mathrm{epi}(f) = \mathrm{epigraph}(f) = \{ (x,s) : x \in \mathbb{R}^n,\ s \in \mathbb{R},\ s \geq f(x) \}$

$\mathrm{graph}(f) = \{ (x,s) : \text{''} \ldots \text{''} \quad s = f(x) \}$



Why do we like convexity?

Basic thm #1: If f is convex and C is convex, then any local minimizer of $\min_{x \in C} f(x)$ is in fact global. (Also, $X := \operatorname{argmin}_{x \in C} f(x)$, the set of global sol'n, is convex, an in particular, connected)

proof

$\boxed{Q2}$ on your own (scroll down for answer)

proof: let x be a local sol'n. For contradiction, assume it's not global, so $\exists\ y \in C$ s.t. $f(y) < f(x)$.

Since $x \in C$ is a local sol'n, $\exists\ \varepsilon$ st. $f(x) \leq f(z)\ \forall\ z \in B_\varepsilon(x) \cap C$

Pick $t \in [0,1]$ sufficiently close to 1 to guarantee $z = tx + (1-t)y$ is within ε of x

Since C is convex, $z \in C$, thus by we know $f(x) \leq f(z)$. $(*)$

But by convexity of f,

$$
\begin{aligned}
f(z) &= f(tx + (1-t)y) \\
&\leq t \cdot f(x) + (1-t) f(y) \\
&< t \cdot f(x) + (1-t) f(x) \quad \text{since } (1-t) \geq 0 \text{ and } f(\ ) < f(\ ) \\
&= f(x)
\end{aligned}
$$

so $f(z) < f(x)$ which contradicts $(*)$.