Review of linear algebra

Saturday, November 21, 2020 9:28 AM

Some review of what you learned in linear algebra, some new stuff

Parts from ch 6.1 & ch 6.3 but mostly misc. material

Matrices and vectors

$A = \begin{bmatrix} 3 & 4 & 6 \\ 2 & 1 & 8 \end{bmatrix}$ is a 2×3 matrix, and sometimes write $A = [a_{ij}]$ to denote $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

usually capital letters for matrices

Indices a_{ij} a nearly universal convention is that i is the row index, j is the column index

Vectors: a "vector" in a vector space V is an abstract object that you should have learned about in your lin. alg. class.

For us now, "vector" is a specific representation of a vector in a finite dimensional vector space, given a specific basis

i.e., just a list of numbers $x = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \in \mathbb{R}^3$

Sometimes, not always, write as $\vec{x}$ or $\vec{x}$ or $\underline{x}$ or (when typeset) in bold. Usually lowercase.

For computation, "vector" = "column vector" though sometimes we write it as a row vector just because it's easier to typeset

(a rigorous way to do this is $x^T = [3, 4, 5]$)

transpose

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

⚠ Numpy distinguishes a vector (1D) from a row or column vector ($1 \times n$ or $n \times 1$)

If $A = A^T$, we call the matrix symmetric

A^* means $\bar{A}^T$ (aka A^+ in physics notation, though in numerical analysis, "dagger" is used for pseudo-inverse)

"adjoint" complex conjugate "dagger"

"Matlab notation"

"end" keyword in Matlab is useful

$A(2,:)$ means the 2^{nd} row of A (":" means "everything")

$A(:,4)$ means the 4^{th} column of A

Numpy has similar, but not identical, syntax

How is a matrix stored on a computer?

Let's start w/ a vector. In python, an array $x=[2,3,9]$ might be stored as a linked list. Good for appending, horrible for numerical computation.

Numpy and matlab store as a contiguous array, much like in C

$x=[2,3,9]$ $\longrightarrow$ stores metadata, like length

index a in memory — index $a+\Delta$ — $a+2\Delta$

where Δ is size of the datatype (eg. 8 bytes)

Now a matrix

$$A = \begin{bmatrix} 2 & 3 & 9 \\ 4 & 6 & -2 \end{bmatrix}$$

Store metadata (# rows, columns)

then 2 main classes: row-major order: $\text{vec}(A) = [2\ 3\ 9\ \ 4\ 6\ -2]$ and store this vector

Numpy, C

or column-major-order: $\text{vec}(A) = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 6 \\ 9 \\ -2 \end{bmatrix}$ and store this vector

Fortran, Matlab

Sparse matrix storage

A "sparse" matrix is one that has many 0 entries, like

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}.$$

Mathematically, "many 0's" is vague.

Computationally, it is not vague: you must tell your system (Numpy...) to use a sparse format (eg. Matlab: $A = \text{sparse}(A)$)

Otherwise it stores the 0's.

Won't go into detail. It's not always efficient: need a lot of 0 entries to be worth it.

(google compressed sparse row (CSR) or compressed sparse column (CSC))

Matlab uses

Special matrices / types-of-matrices

- Diagonal

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots 0 & a_{nn} \end{bmatrix}$$

often use notation D

- Lower triangular

$$A = \begin{bmatrix} a_{11} & 0 & & 0 \\ a_{21} & a_{22} & \ddots & \\ \vdots & & & 0 \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

often use L

Upper triangular is the opposite

$$A = \begin{bmatrix} & \\ 0 & \end{bmatrix}$$

often use U

- Identity

$$I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$

or $I_{3\times 3}$ to make size clear if needed

$$I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- All ones vector

$$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

- All ones matrix ($\neq I$) is

$$\mathbb{1}\mathbb{1}^T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & \\ \vdots & & & & \\ 1 & & \cdots & & 1 \end{bmatrix}$$

- Canonical basis vectors aka unit vectors

$$e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th} \text{ spot}$$

<u>ex</u>: in $\mathbb{R}^3$, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- Diagonally dominant if $\forall i$,

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

if $>$ then "strictly" diag. dom.

e.g.

$$A = \begin{pmatrix} 13 & 2 & 1 \\ 5 & 19 & -2 \\ 3 & 6 & -20 \end{pmatrix}$$

check:

$13 > 2+1$ ✓

$19 > 5 + |-2|$ ✓

$|-20| > 3+6$ ✓

so A is strictly diag. dominant

- 0 matrix

$$0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$$

usually the size is determined by the context

- Block matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

we do this a lot w/ 0 blocks $\boxed{A_{11}} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$, $A_{12} = \ldots$

- $A = A^T$ is symmetric
  $A = A^* \; (:= \bar{A}^T)$ is self-adjoint = Hermitian

} same thing if all $\mathbb{R}$ entries and not $\mathbb{C}$

Matrix Algebra

The dot product aka inner product of two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$

$$\vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i y_i \qquad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Sometimes written $\langle x, y \rangle$ or $\langle x | y \rangle$ in physics

and if $\vec{x}, \vec{y} \in \mathbb{C}^n$, it is $\vec{x}^* \vec{y} = \sum_{i=1}^{n} \bar{x}_i y_i$

$$(x_1 \;\; x_2 \; \cdots \; x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

[Note: the outer product is $\vec{x} \cdot \vec{y}^T$ (physics $|x\rangle\langle y|$)

i.e. $\begin{bmatrix} 3 \\ 1 \end{bmatrix} [2 \;\; 4] = \begin{bmatrix} 6 & 12 \\ 2 & 4 \end{bmatrix}$]

If a matrix A is $m \times n$ w/ rows $A = \begin{bmatrix} - \vec{a}_1^T - \\ \vdots \\ - \vec{a}_m^T - \end{bmatrix}$

then for $\vec{y} \in \mathbb{R}^n$, $A\vec{y} \in \mathbb{R}^m$, $(A\vec{y})_i = \vec{a}_i^T \vec{y}$ "matrix-vector product" mat-vec

If a matrix B is $n \times k$, w/ colums $B = \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{bmatrix}$

then $(AB)_{ik} = \vec{a}_i^T \vec{b}_k = \underbrace{\sum_{j=1}^{n} a_{ij} b_{jk}}$

you might have learned via indices like this

but I encourage you to think of via dot products and higher-level concepts

Ex $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

$c_{21} = [1 \;\; 2 \;\; 6] \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -1 + 4 + 0$
$= 3$

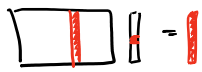
Remember, if $\underset{m \times k}{C} = \underset{m \times n}{A} \times \underset{n \times k}{B}$

(n and n) must match

Facts: • In general, matrix multiplication is not commutative

$AB \neq BA$ (in fact, the sizes might not even work!)

- $A \cdot e_j = A(:,j)$ i.e., j^{th} column



- $A \cdot I = A$ and $I \cdot A = A$

- $A = B \Rightarrow CA = CB$ (if C is the right size)

and $A = B \Rightarrow AC = BC$ (if C is the right size)

- If A is a <u>square matrix</u> (ie $n \times n$),

  we say it is invertible aka non-singular if there is a matrix (which we denote as A^{-1}) such that $A \cdot A^{-1} = I$ ← identity

  (and it turns out this implies $A^{-1}A = I$ as well)

  $(A^{-1})^{-1} = A$

  If not invertible, we call it singular

- $(AB)^T = B^T A^T$ and $(AB)^* = B^* A^*$

- $(AB)^{-1} = B^{-1}A^{-1}$ if AB is invertible

- Sherman-Morrison / Woodbury / Matrix-Inversion Lemma

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1} U)^{-1} V^T A^{-1}$$

eg., if u, v are vectors

$$(A + \underbrace{\varepsilon u v^T}_{\text{rank 1 perturbation}}) = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1/\varepsilon + v^T A^{-1} u}$$

- Matrix multiplication <u>is</u> associative and distributive

$$A(BC) = ABC = (AB)C$$

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

- The inverse of a lower (upper) triangular matrix, if it exists, is also lower (upper) triangular

- The product of two lower (upper) triangular matrices is also lower (upper) triangular

- Not going over rank, determinant, trace, etc. but hopefully you've seen in linear algebra

Addendum

The determinant of a matrix, written $\det(A)$ or $|A|$, can be defined by cofactor expansion (annoying) or, if A is diagonalizable, then it's the product of all the eigenvalues, $\det(A) = \prod_{i=1}^{n} \lambda_i$

(and $\operatorname{trace}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i$ is the sum of all eigenvalues)

The meaning of $\det(A)$ is how much the linear transformation $T(\vec{x}) = A\vec{x}$ shrinks or expands space. Hence we use it in multivariate change-of-variables for integrals.

★ "A" is singular (= not invertible) iff $\det(A) = 0$

Facts (Thm 6.16)

- If A has an all zero row or column, or duplicate rows or columns, then $\det(A) = 0$
- $\det(\lambda A) = \lambda^n \det(A)$, and more (see book)
- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = 1/\det(A)$
- If A is diagonal, upper or lower-triangular, then $\det(A) = \prod_{i=1}^{n} a_{ii}$