Improper Integrals

Sunday, October 18, 2020 3:30 PM

* Proper vs Improper is NOT Definite vs Indefinite
$\int_a^b f(x)dx$ $\int f(x)dx$

What is a proper integral?

It is $\int_a^b f(x)\,dx$ where f is bounded and a, b are finite.

It need not exist for all functions... when it does exist, we call f integrable on $[a,b]$

Thm If a,b finite, f is bounded, then f continuous* is sufficient for f to be integrable.

* it can be piecewise continuous w/ a countable number of discontinuities

Ex. of a proper integral: $\int_{-1}^{1} x^2\,dx$ and other similarly boring things

So what's an improper integral? (sometimes called singular integrals)
It violates at least one of the following:

(1) Violates Finite domain (ie, if domain is ∞ then its an improper integral)

Ex: $\int_1^\infty \frac{1}{x^2}\,dx$ is improper. We define it as the limit of proper integrals: (if the limit exists)

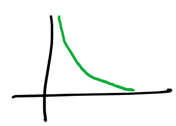
$$\int_1^\infty \frac{1}{x^2}\,dx := \lim_{R\to\infty} \underbrace{\int_1^R \frac{1}{x^2}\,dx}_{\text{proper}} = \lim_{R\to\infty} \left.\frac{-1}{x}\right|_1^R$$

$$= \lim_{R\to\infty} \frac{-1}{R} - \frac{-1}{1} = 1$$

Ex: $\int_1^\infty \frac{1}{x}\,dx$ isn't integrable, even in the improper sense: Does Not Exist

(2) violates f being bounded

Ex $\int_0^1 \frac{1}{\sqrt{x}}\,dx$, $f(x) = \frac{1}{\sqrt{x}}$



it's continuous on $(0,1)$ ✓
but not bounded ✗

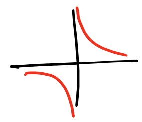
doesn't this violate EVT? No. Domain isn't closed.

define the improper integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to0^+} \overbrace{\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx}^{\text{proper}} = \lim_{\varepsilon\to0^+} 2-2\sqrt{\varepsilon} = 2$$

(*) violates being "integrable" (Riemann sums don't converge)

Ex $$\int_{-1}^{1} \frac{1}{x}\,dx = \lim_{\delta\to0^+}\int_{-1}^{-\delta} \frac{1}{x}\,dx + \lim_{\varepsilon\to0^+}\int_\varepsilon^1 \frac{1}{x}\,dx$$



$= -\infty + \infty$

meaning we say the improper integral Does Not Exist "DNE"

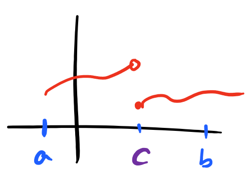
If we require $\delta = \varepsilon$ then we get cancellation, so you could make sense of $\int_{-1}^{1} \frac{1}{x}\,dx$ this way: this is called the principal value integral and we won't be discussing. "Even less proper than an improper integral"

Back to quadrature

All our techniques required f to be continuous on the closed, bounded interval $[a,b]$ ($\Rightarrow f$ is automatically bounded via EVT)

Let's start relaxing these

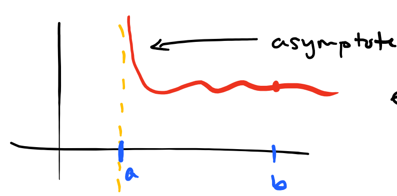
① Suppose f is piecewise continuous, $[a,b]$ is still finite. This is still a proper integral. Suppose we know f isn't continuous at c



Then simple fix:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

② left-endpoint singularity



e.g., f is continuous on $(a,b]$ and isn't bounded as $x \to a^+$ so this is an improper integral

Note: without loss of generality, we can assume $a=0$ (if not, shift f by a)

"WLOG"

⚠ The integral may or may not exist: $\int_0^1 \frac{1}{x^2}\,dx$ DNE, $\int_0^1 \frac{1}{x^1}\,dx$ DNE

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ does exist, $\int_0^1 \frac{1}{x^{1-\varepsilon}}\,dx$ does exist if $\varepsilon > 0$

Technique #1 Write $f(x) = \dfrac{g(x)}{x^k}$ for some k with g continuous on $[0,b]$ (or bounded)

(if this is possible). If $0 < k < 1$ then the improper integral exists

So, with $g(x) = x^k \cdot f(x)$, Taylor expand g, to any order

$$P(x) = g(0) + g'(0)\cdot x + g''(0)\,x^2/2! + g'''(0)\,x^3/3! + g^{(4)}(0)\,x^4/4!$$

So $$\int_0^b f(x)\,dx = \int_0^b \frac{g(x)}{x^k}\,dx = \underbrace{\int_0^b \frac{p(x)}{x^k}\,dx}_{A} + \underbrace{\int_0^b \frac{g(x)-p(x)}{x^k}\,dx}_{B}$$

A $$\int_0^b \frac{p(x)}{x^k}\,dx = g(0)\int_0^b \frac{1}{x^k}\,dx + g'(0)\int_0^b \frac{x}{x^k}\,dx$$
$$+ \frac{g''(0)}{2!}\int_0^b \frac{x^2}{x^k}\,dx + \frac{g'''(0)}{3!}\int_0^b \frac{x^3}{x^k}\,dx + \frac{g^{(4)}(0)}{4!}\int_0^b \frac{x^4}{x^k}\,dx$$

These would be $\frac{(x-a)^j}{(x-a)^k}$ if we hadn't said wlog $a=0$

and each integral has a closed form $\int_0^b x^{j-k}\,dx = \frac{1}{j-k+1}\, b^{j-k+1}$

(no issues since $k<1$)

B $$\int_0^b \underbrace{\frac{g(x)-p(x)}{x^k}}_{=G(x)}\,dx$$ (and define $G(0)=0$)

Then $G(x)$ does not blowup to $\pm\infty$ as $x \to 0$ because by Taylor's remainder theorem, $\forall x$ $g(x)-p(x) = g^{(j+1)}(\xi(x))\cdot\frac{x^{j+1}}{(j+1)!}$

(for this to work, we'll need to assume $g \in C^{j+1}[0,b]$)

So $|G(x)| \le \left(\max_{\xi} |g^{(j+1)}(\xi)|\right)\cdot \frac{x^{j+1-k}}{(j+1)!}$

and for $j \ge 0$ and $k<1$ then x^{j+1-k} is bounded as $x \to 0$

So.. just use your normal quadrature rule on $G(x)$ on $[0,b]$
e.g., composite Simpson, etc.

Technique #2 we'll also split into 2 parts, but a different way
(ref. Quarteroni et al., "Numerical Math")

Same setup as before, $f(x) = \frac{g(x)}{x^k}$, $k<1$, and

also as before, do the Taylor series of $g(x)$. Call this Taylor polynomial $p(x)$ (also as before.)

But instead of $\int_0^b \frac{g(x)}{x^k}\,dx = \int_0^b \frac{p(x)}{x^k}\,dx + \int_0^b \frac{g(x)-p(x)}{x^k}\,dx$

do this instead $$\int_0^b \frac{g(x)}{x^k}\,dx = \underbrace{\int_0^\varepsilon \frac{p(x)}{x}\,dx}_{A'} + \underbrace{\int_\varepsilon^b f(x)\,dx}_{B'}$$

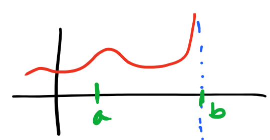
and do (A') just as we did earlier
(but with ε instead of b)

and do (B') via your favorite quadrature, since the integrand no longer blows up.

Tune ε so both terms have similar amount of error

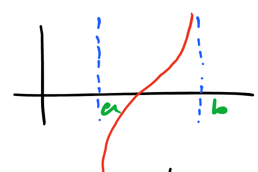
Technique #3 Doubly exponential transformation
we'll come back to this

(2') Right endpoint singularity



Same as left endpoint singularity, just do change-of-variables
$\tilde{x} = -x$, $d\tilde{x} = -dx$

(2'') singularities at both endpoints



eg. $\tan(x)$

Rewrite $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for some $c \in (a,b)$

(3) Unbounded integrals

Technique 1 Use Gauss-Hermite (for $\int_{-\infty}^{\infty} f(x)\,dx$)

Gauss-Laguerre (for $\underbrace{\int_0^{\infty}}_{\text{or } \int_a^{\infty} \text{ or } \int_{-\infty}^{b}} f(x)\,dx$)

via change-of-variable

Works well when we can write
$$f(x) = g(x)e^{-x^2} \quad \text{G.-Hermite}$$
or
$$f(x) = g(x)e^{-x} \quad \text{G.-Laguerre}$$

and $g(x)$ doesn't grow (or at least not quickly)
as $x \to \infty$ or $x \to \pm\infty$

Technique 2
Approximate $\int_0^{\infty} f(x)\,dx$ with $\int_0^R f(x)\,dx$ for a large R

Simple (and often effective enough). Works for $\int_{-\infty}^{\infty} f(x)dx$ too.

Sometimes this approach doesn't work well

Ex $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$, decay is $\approx x^{-2}$, $\underbrace{\int_R^{\infty} x^{-2}\,dx = R^{-1}}_{\text{error}}$

so for error $\approx 10^{-6}$, need $R \approx 10^6$

For a stepsize $h \ll 1$, this means millions to billions of nodes

Technique 3

Something like $\int_0^{\infty} f(x)\,dx = \underbrace{\int_0^R f(x)\,dx}_{\text{easy}} + \int_R^{\infty} f(x)\,dx$

and for $\int_R^{\infty} f(x)\,dx$ to the change-of-variables

$t = 1/x$, $dt = -1/x^2\,dx$ so $dx = -t^{-2}\,dt$

and $x = R \Rightarrow t = \underbrace{1/R}_{\varepsilon}$, $x = \infty \Rightarrow t = 0$

so ... $= \underbrace{\int_0^{\varepsilon = 1/R} f(1/t)\,t^{-2}\,dt}$ (minus sign is gone since we swap order $[\varepsilon, 0]$ to $[0, \varepsilon]$)

unbounded as $t \to 0$, so use one of the techniques for 2 that we discussed.

Technique 4 Comparison with a known function

not in many books

Consider $\int_1^{\infty} \frac{1}{1+x^2}\,dx$. (There is an exact antiderivative but pretend we don't know that)

Write $\int_1^{\infty} \frac{1}{1+x^2}\,dx = \int_1^{\infty} \frac{1}{1+x^2} - \underbrace{\frac{1}{x^2}}_{g(x)}\,dx + \int_1^{\infty} \underbrace{\frac{1}{x^2}}\,dx$

More generally, pick g such that
1) g is near f
2) $\int g$ is known in closed form

$\approx \int_1^{R} \underbrace{\frac{1}{1+x^2} - \frac{1}{x^2}}_{\text{small}}\,dx + \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}_{\text{closed form}}$

Technique 5 Double exponential transformations

(Ref. ch 9.7 Driscoll and Braun)

Use hyperbolic functions

$\sinh(t) = \frac{e^t - e^{-t}}{2}$, $\cosh(t) = \frac{e^t + e^{-t}}{2}$

Note: $i = \sqrt{-1}$

$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$

$\cos(t) = \frac{e^{it} + e^{-it}}{2}$

For large t, $\sinh(t) \approx \pm e^{|t|}$ as $t \to \pm\infty$

$\cosh(t) \approx e^{|t|}$ as $t \to \pm\infty$

since Euler's identity:

$$e^{it} = \cos(t) + i\sin(t)$$

Facts: $\sinh' = \cosh$
$\cosh' = \sinh$
$\cosh^2 - \sinh^2 = 1$

$$\tanh(t) := \frac{\sinh(t)}{\cosh(t)} \to \pm 1 \text{ as } t \to \pm\infty$$

Often the double exponential change-of-variables

$x = \sinh\left(\frac{\pi}{2}\sinh(t)\right)$ is helpful

so $x \approx \pm \frac{1}{2} e^{\pi/4 e^{|t|}}$ as $t \to \pm\infty$

and $x = 0 \Rightarrow t = 0$

$x = \pm\infty \Rightarrow t = \pm\infty$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x(t))\frac{dx}{dt}\,dt$$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty} \underbrace{f(x(t))}_{\substack{\text{decays} \\ \text{even more quickly}}} \cdot \underbrace{\cosh\left(\frac{\pi}{2}\sinh(t)\right)\cosh(t)}_{\text{grows}}\,dt$$

New integral decays so quickly that we can safely truncate at $\int_{-R}^{R} \ldots$ for reasonable choices of R

i.e., we're condensing the x-axis

Similar tricks are used for Technique #3 for ② left-endpt. singularities or other singularities

Do

$$x = \tanh\left(\frac{\pi}{2}\sinh(t)\right)$$

so

$$\frac{dx}{dt} = \frac{\pi}{2} \cdot \frac{\cosh(t)}{\cosh^2(\pi/2 \sinh(t))}$$

transforms $x \in (-1,1)$ to $t \in (-\infty,\infty)$

$$\int_{-1}^{1} f(x)\,dx = \int_{-\infty}^{\infty} \underbrace{f(x(t))\frac{dx}{dt}}_{\text{decays quickly}}\,dt$$

so truncate to $\int_{-R}^{R} \ldots$

Doubly exponential transformations not in many books

Takahasi and Mori, 1974