Linear Systems of Equations

Tuesday, November 24, 2020 10:04 PM

i.e. things like Reduced Row Echelon Form (RREF) and Gaussian Elimination

Linear Systems

Ex:
$$\underbrace{\begin{aligned} 3x + 4y &= 7 \\ -2x + 5y &= 0 \end{aligned}}$$

NOT
$$3x^2 + 4\cdot\sin(y) = 7$$
$$x\cdot y = 2$$

#1 skill: be able to write in matrix form

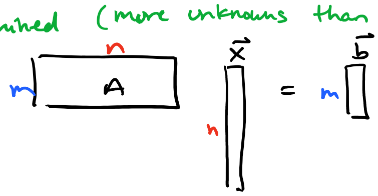
$$\underbrace{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 0 \end{bmatrix}}_{\vec{b}}$$

More generally
$$\begin{aligned} a_{11}\cdot x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{bmatrix}}_{\substack{A \\ m\times n}} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

- Underdetermined (more unknowns than equations)

$m < n$



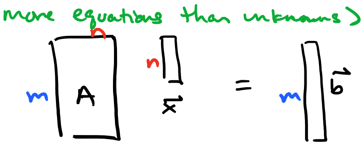
Facts: null(A) is non-trivial

There need not be any solutions (ex: $A=0$, $\vec{b}\neq 0$)

but if there is a solution, it is not unique

- Overdetermined (more equations than unknowns)

$m > n$



Facts: There need not be any solutions. If there are solutions, then some equations are redundant.

The solution is unique (if it exists) if $\text{rank}(A) = n$ "full rank"

- Exactly determined (square): one equation per unknown
  We'll focus on this case in our class — simplest

$m = n$

$$m\underbrace{\boxed{\quad A \quad}}_{}^{\,n} \; {}^{n}\!\left[\vec{x}\right] = {}^{m}\!\left[\vec{b}\right]$$

Facts: There need not be a solution, nor does it need to be unique.

The solution (exists and is unique) occurs

iff A is full rank ($\text{rank}(A) = n$)

iff A is invertible.

If A is invertible, $\vec{x} = \underbrace{A^{-1}\vec{b}}$.

⚠ Computationally we don't do this

Solving Square systems of equations

Observations: ① Solving

$$3x + 4y = 2$$
$$5x - 3y = 9$$

is the same as solving

$$5x - 3y = 9$$
$$3x + 4y = 2$$

① Re-arrange / swap

② $3x + 4y = 2 \iff \alpha(3x+4y) = \alpha(2)$ for some $\alpha \neq 0$

precise mathematical meaning: iff

② Scale

(if $\alpha = 0$ then $\Rightarrow$ but not $\Leftarrow$)

means "implies"

③ $3x + 4y = 2 \iff 3x + 4y + \beta = 2 + \beta$ for some β

③ Add

These are the tricks behind Gaussian Elimination

Linear Algebra 101 method to solve a square linear system of equations

$$3x_1 + 4x_2 + 5x_3 = 6$$
$$6x_2 + 0 \cdot x_2 - 3x_3 = 4$$
$$9x_3 + 6x_2 + 4x_3 = -2$$

① Form the augmented matrix $[A \mid b]$

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 6 & 0 & -3 & 4 \\ 9 & 6 & 4 & -2 \end{array}\right]$$

② Apply $\underbrace{\text{elementary row operations}}_{\text{essentially}}$ to the augmented matrix

essentially swap, scale, add

Start on upper left

① scale it to a 1

(2) add a multiple of top row to other rows in order to make their 1^{st} entry a 0

(3) move-on: repeat (1) and (2) on all but 1^{st} row + 1^{st} column

(only swap if step (1) impossible due to 0 entry)

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 6 & 0 & -3 & 4 \\ 9 & 6 & 4 & -2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/3} \left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ 6 & 0 & -3 & 4 \\ 9 & 6 & 4 & -2 \end{array}\right]$$

$R_2 \leftarrow R_2 - 6 \cdot R_1$

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ 0 & -8 & \underbrace{-3-10}_{-13} & \underbrace{4-12}_{-8} \\ 9 & 6 & 4 & -2 \end{array}\right] \longrightarrow$$

$R_3 \leftarrow R_3 - 9 \cdot R_1$

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ 0 & -8 & -13 & -8 \\ 0 & \underbrace{6-12}_{-6} & \underbrace{4-15}_{-11} & \underbrace{-2-18}_{-20} \end{array}\right]$$

$R_2 \leftarrow -R_2/8$

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ 0 & 1 & 13/8 & 1 \\ 0 & -6 & -11 & -20 \end{array}\right] \longrightarrow$$

$R_3 \leftarrow R_3 + 6 \cdot R_2$

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ 0 & 1 & 13/8 & 1 \\ 0 & 0 & \underbrace{-11+\frac{69}{8}}_{-19/8} & \underbrace{-20+6}_{-14} \end{array}\right]$$

$R_3 \leftarrow -8/19 \cdot R_3$

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ 0 & 1 & 13/8 & 1 \\ 0 & 0 & 1 & \underbrace{-14 \cdot \frac{-8}{19}}_{112/19} \end{array}\right]$$

STOP

This is row echelon form

and what we did was Gaussian Elimination

You might recall reduced row echelon form (RREF) (and its procedure is called Gauss-Jordan Elimination)

which continues and gets it to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \tilde{b}_1 \\ 0 & 1 & 0 & \tilde{b}_2 \\ 0 & 0 & 1 & \tilde{b}_3 \end{array}\right]$

meaning
$$\begin{aligned} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 &= \tilde{b}_1 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 &= \tilde{b}_2 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 &= \tilde{b}_3 \end{aligned} \implies \begin{aligned} x_1 &= \tilde{b}_1 \\ x_2 &= \tilde{b}_2 \\ x_3 &= \tilde{b}_3 \end{aligned}$$

Instead, do BACK SUBSTITUTION

Note: we didn't have to make the pivot a 1

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ 0 & 1 & 13/8 & 1 \\ \underbrace{0 \quad 0 \quad 1}_{\text{observe this is upper triangular}} & & & \underbrace{-14 \cdot \frac{-8}{19}}_{112/19} \end{array}\right]$$

means

$$1 \cdot x_1 + 4/3\, x_2 + 5/3\, x_3 = 2$$
$$1 \cdot x_2 + 13/8\, x_3 = 1$$
$$1 \cdot x_3 = 112/19 \quad \text{] Solve last equation first}$$

① $x_3 = 112/19$

② plug x_3 into equation 2

$$1 \cdot x_2 + 13/8 \cdot (112/19) = 1$$

so $x_2 = 1 - \frac{13 \cdot 112}{8 \cdot 19} \approx 8.58$

③ plug x_2 and x_3 into equation 1

$$1 \cdot x_1 + 4/3\,(8.58) + 5/3\,(112/19) = 2$$

so now we easily solve for x_1. ✓

That's Gaussian Elimination with Back Substitution

Ch 6.1 in the book gives a more formal algorithm, but most important is to understand what/why.

FLOP counts

① For Gaussian Elimination

Ignoring possible swaps.

1. For 1^{st} column, we scale entire row, and then add to rest of the rows

$$n \left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right] \quad (n \text{ columns})$$

$O(n^2)$ operations

2. For 2^{nd} column, same thing but exclude 1^{st} row and 1^{st} column

$$\left[\begin{array}{c|cc|c} * & * & * & * \\ \hline & * & * & * \\ & * & * & * \end{array}\right]$$

$O(\,(n-1)^2\,)$ operations

3. etc.

So total # operations is $O\left(\sum_{j=1}^{n} (n-j)^2\right) = O\left(\sum_{j=1}^{n} (n^2 - 2nj + j^2)\right)$

$$= O\left(n^3 - 2n \sum_{j=1}^{n} j + \sum_{j=1}^{n} j^2\right)$$

$$= O(n^3) - 2n\,O(n^2) + O(n^3)$$

Recall $\sum_{j=1}^{n} j = \frac{n(n+1)}{2} = O(n^2)$

$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} = O(n^3)$

$= \underline{O(n^3)}$ (book does it much more carefully)

② For back-substitution:

last equation: $O(1)$ work

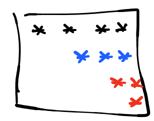
2^{nd}-to-last equation: $O(2)$ work

⋮

k^{th}-to-last eq'n $O(k)$ work

⋮

n^{th}-to-last eq'n $O(n)$ work
aka 1^{st} equation



So $\sum_{j=1}^{n} O(j)$ flops, $= O(n^2)$ flops

Message

① Gaussian Elimination on a $n \times n$ matrix takes $O(n^3)$ flops

② Back substitution (on an upper triangular matrix) takes $O(n^2)$ flops

Solving n linear equations in n unknowns takes $O(n^3)$ time

but solving a <u>triangular</u> system takes just $O(n^2)$ time

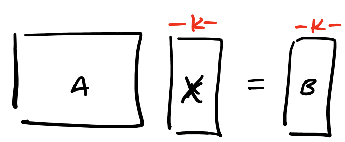
(and solving a <u>diagonal</u> system takes $O(n)$ time)

Relatively slow, which is why we pay attention carefully!

Multiple right-hand-sides

Solve $A\vec{x}_1 = \vec{b}_1$ and $A\vec{x}_2 = \vec{b}_2$

or, generally, K



$B = \begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix}$

① make augmented matrix

$[\, A \,\vdots\, B \,]$

② Apply Gaussian elimination

$O(n^2(n+K))$

③ Do back substitution

$O(n(n+K))$

Finding A^{-1}

Set $B = I$, so $AX = I$ solve for X, $X = A^{-1} \cdot I = A^{-1}$

So to solve $A\vec{x} = \vec{b}$, if we did $\vec{x} = A^{-1}\vec{b}$, that involved a lot of unnecessary work, and it's less accurate

☆ Don't use A^{-1} unless you really really need to

Standard use cases when you need A^{-1}: (none)