

# Stability (examples)

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Recall our 3 notions: (all for  $h \rightarrow 0$ )

- 1) Consistency Does the method locally approximate the ODE?
- 2) Convergence Does the approximation  $w_i$  converge to  $y(t_i)$ ? MAIN GOAL
- 3) Stability Links consistency to convergence

This lecture: examples of stability/convergence for multistep methods

Setup IVP:  $y' = f(t, y)$ ,  $a \leq t \leq b$ ,  $y(a) = \alpha$

$$w_0 = \alpha, w_1 = \alpha_1, w_2 = \alpha_2, \dots, w_{m-1} = \alpha_{m-1} \quad ] \text{ initialization}$$

$$\begin{aligned} w_{i+1} &= a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m} && \text{generic linear} \\ &+ h \cdot \underbrace{\left( b_m f_{i+1} + b_{m-1} f_i + \dots + b_0 f_{i+1-m} \right)}_{F(t_i, h, w_{i+1}, w_i, \dots, w_{i+1-m})}, \quad f_i := f(t_i, w_i) \end{aligned}$$

Def The characteristic polynomial  $P(\lambda)$  associated w/ the numerical method  $(*)$  (it does not depend on what  $F$  looks like) is

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - a_{m-3}\lambda^{m-3} - \dots - a_0$$

$$\text{example } m=3 \quad P(\lambda) = \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$$

Def Root condition and strongly stable and weakly stable

Let  $\{\lambda_j\}_{j=1}^m$  be the roots of the characteristic polynomial  $P$  associated with a multi-step scheme. Then we say the multi-step method satisfies the root condition if

$$\textcircled{A} \quad |\lambda_j| \leq 1 \quad \forall j$$

$$\text{and } \textcircled{B} \quad |\lambda_j| = 1 \Rightarrow \lambda_j \text{ is a simple root}$$

and if a method satisfies the root condition, then we call it either

strongly stable if  $|\lambda_j| \Rightarrow \lambda_j = 1$

weakly stable otherwise.

? root condition  
= zero-stability

If it doesn't satisfy the root condition, we call it unstable

Thm 5.24 "Equivalence Thm" i.e. output depends continuously on input data

① A multistep method is stable iff it satisfies the root condition

② If the multistep method is consistent, then it is convergent iff it's stable

## New examples

① Forward (explicit) Euler  $w_{i+1} = w_i + h f(t_i, w_i)$

Is this a convergent method?

1<sup>st</sup>, check if it's consistent (if it isn't, then it's not convergent)

i.e., is  $\lim_{h \rightarrow 0} \epsilon(h) = 0$ ? yes, we've already seen  $\epsilon(h) = O(h)$

In extra detail: i.e., is  $\lim_{h \rightarrow 0} \frac{y(t+h) - y(t) + h f(t, y(t))}{h} = 0$ ?

$$= \lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h} - f(t, y(t))$$

$$= y'(t) - f(t, y(t)) = 0 \quad \text{since } y' = f(t, y) \\ (\text{this is our ODE!})$$

2<sup>nd</sup>, check is it stable

i.e., check the root condition, so form the characteristic polynomial

As  $h \rightarrow 0$ , so ignoring  $h f(t_i, w_i)$ , we have  $w_{i+1} = w_i$ .

$$\text{i.e., } 1w_{i+1} - 1w_i = 0$$

So  $P(\lambda) = 1 \cdot \lambda - 1$ , which has only one root,  $\lambda = 1$

(recall  $\lambda = 1$  is always a root if method is consistent)

Quick sanity check:  $\deg(P) = m$  (# steps)

$m = 1$  since it's a 1-step method

so, it satisfies the root condition

and is in fact strongly stable

3<sup>rd</sup>, deduce it's convergent via the Dahlquist equivalence theorem

Note: for forward Euler, we already knew this from specially theory theorems  
(Thm. 5.9 for Euler, or Thm. 5.20 for all 1-step methods)

② Backward (implicit) Euler  $w_{i+1} = w_i + h f(t_{i+1}, w_{i+1})$

You can check this is consistent, so then convergence follows either from Thm 5.20 or since we know it has the same characteristic polynomial as forward Euler

$$(3) \text{ AB4 (Adams-Basforth 4th order / 4 step)} \quad f_i := f(t_i, w_i)$$

$$w_{i+1} = w_i + h \underbrace{\left( 55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3} \right)}_{F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}, w_{i-3})}$$

This is consistent since we know it's  $\mathcal{E}(h) = O(h^4)$

Does it satisfy the root condition? TRY ON YOUR OWN!

The characteristic polynomial is  $P(\lambda) = \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1)$

$$(*) \quad w_{i+1} - w_i = 0 \Rightarrow \lambda = 1$$

$$\lambda = 0 \text{ (with multiplicity 3)}$$

⚠ It's tempting to write

$$P(\lambda) = \lambda - 1 \text{ since } (*)$$

is just like it was for Euler's.

root condition ✓

strongly stable ✓

In this case it wouldn't have led to a false conclusion, but it's not the right thing.  $P$  should always have degree  $m$

( $m = \# \text{steps} = \text{how far back we go}$ )

(4) All Adams methods (Adams-Basforth, Adams-Moulton)

$$\text{For } m\text{-steps, } P(\lambda) = \lambda^m - \lambda^{m-1} = \lambda^{m-1}(\lambda - 1)$$

$$\Rightarrow \lambda = 1$$

$$\lambda = 0 \text{ (with multiplicity } m-1)$$

$\Rightarrow$  All Adams method satisfy the root condition and are strongly stable hence convergent since they're all consistent

(5) 4<sup>th</sup> order Milne's method

$$w_{i+1} = w_{i-3} + h \underbrace{\frac{4}{3} \left( 2f_i - f_{i-1} + 2f_{i-2} \right)}_F$$

This is consistent since we know it's 4<sup>th</sup> order, i.e.,  $\mathcal{E}(h) = O(h^4)$

$$\text{so } \lim_{h \rightarrow 0} \mathcal{E}(h) = 0$$

Does it satisfy the root condition?

Think of difference equation

$$w_{i+1} + 0 \cdot w_i + 0 \cdot w_{i-1} + 0 \cdot w_{i-2} - w_{i-3} = 0$$

$$P(\lambda) = \lambda^4 - 1 \Rightarrow \text{roots are } \lambda = \{\pm 1, \pm \sqrt{-1}\}$$

I won't use "i" here since we use i as an index

$$\text{All } |\lambda| \leq 1, \text{ no multiple roots}$$

$\Rightarrow$  root condition ✓

but... it's only weakly stable not strongly stable

So, people usually prefer something like AB4 over Milne's

## (6) Backward Differentiation

Name	Order	Steps m	$a_{m-1}$	$a_{m-2}$	$a_{m-3}$	$a_{m-4}$	$b_m$
BD1	1	1	1				1
BD2	2	2	$4/3$	$-1/3$			$2/3$
BD3	3	3	$18/11$	$-9/11$	$2/11$		$6/11$
BD4	4	4	$48/25$	$-36/25$	$16/25$	$-3/25$	$12/25$

All are consistent

BD1,  $P(\lambda) = \lambda - 1 \Rightarrow \lambda = 1 \Rightarrow$  strongly stable as we already knew

BD3,  $P(\lambda) = \lambda^3 - \frac{18}{11}\lambda^2 + \frac{9}{11}\lambda^1 - 2/11 \Rightarrow \lambda = ?$  How do you solve a cubic?

(1) cubic formula (Cardano, 1545, but due to del Ferro, Tartaglia)  
tried to keep secret

(2) Mathematica, sympy, etc

(3) Tricks

we know  $\lambda = 1$  is a root since BD is consistent

wlog work with  $11\lambda^3 - 18\lambda^2 + 9\lambda - 2 = 0$

factor out  $(\lambda - 1)$

$$\begin{array}{r}
 11\lambda^2 - 7\lambda + 2 \\
 \hline
 (\lambda - 1) \overline{) 11\lambda^3 - 18\lambda^2 + 9\lambda - 2} \\
 11\lambda^3 - 11\lambda^2 \\
 \hline
 -7\lambda^2 + 9\lambda \\
 -7\lambda^2 + 7\lambda \\
 \hline
 2\lambda - 2 \\
 2\lambda - 2 \\
 \hline
 0
 \end{array}$$

$$\begin{aligned}
 \text{so } 11\lambda^3 - 18\lambda^2 + 9\lambda - 2 &= (\lambda - 1) \underbrace{(11\lambda^2 - 7\lambda + 2)}_{\substack{-b \pm \sqrt{b^2 - 4ac} \\ \text{find roots via quadratic formula}}} \\
 &\quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
 \end{aligned}$$

$$\frac{7 \pm \sqrt{49 - 88}}{22} = \frac{7 \pm \sqrt{-39}}{22}$$

Also, what is  $\left| \frac{7 \pm \sqrt{-39}}{22} \right| = \left| \frac{7}{22} \pm \frac{\sqrt{39}}{22} i \right|$

$$\begin{aligned} i &= \sqrt{-1} \text{ for now (not an index)} \\ &= \sqrt{\frac{7^2}{22^2} + \frac{39}{22^2}} \\ &= \sqrt{\frac{49+39}{22^2}} = \sqrt{\frac{88}{22 \cdot 22}} \\ &= \sqrt{\frac{8}{2 \cdot 22}} \\ &= \sqrt{\frac{2}{11}} \\ &= \frac{\sqrt{22}}{11} \approx 0.426 < 1 \end{aligned}$$

So, roots are  $\lambda \in \{1, \frac{7 \pm \sqrt{-39}}{22}\}$  and is strongly stable

It turns out BD1, BD2, ..., BD6 are strongly stable

BD7 and higher are unstable which is why you never see them!

⑦ LIAF (this is the name given to it by Driscoll + Braun...  
it doesn't really have a name because no one uses it!)

$$w_{i+1} = -4w_i + 5w_{i-1} + h(4f_i + 2f_{i-1})$$

is 3<sup>rd</sup> order accurate, so consistent. It has just m=2 steps  
yet 3<sup>rd</sup> order... nice! Better order of accuracy than Adams or BD  
(to get this better accuracy, it uses all available  $a_i, b_i$  terms)

But... 1<sup>st</sup> Dahlquist stability barrier:

$$\text{order of accuracy} \leq \begin{cases} m+2 & \text{if } m \text{ even} \\ m+1 & \text{if } m \text{ odd} \\ m & \text{if explicit} \end{cases} \quad \text{if the method is } \underline{\text{stable}}$$

$\Rightarrow$  LIAF can't be stable.

$$\text{Verify: } P(\lambda) = \lambda^2 + 4\lambda - 5 = (\lambda - 1)(\lambda + 5)$$

$$\Rightarrow \lambda \in \{1, -5\} \Rightarrow \text{UNSTABLE}$$