Forward Euler (or just "Euler's Method")

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Goal: find numerical solution to

IVP
$$\boxed{\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha}$$

We really want a function $y(t)$

but we'll settle for an estimate of $y(t_i)$ for $i = 0, 1, 2, \ldots, n$
(and could always interpolate later)

Notation ① $y(t_i)$ aka y_i is the true solution $y(t)$ evaluated at $t = t_i$

② w_i is our approximation to y_i (Some books use y_i vs. $\hat{y}_i$ ...)

First question: which time points t_i to use?
Later we'll allow it to be adaptive (and non-uniform) but for now, require it to be uniform

$t_i = a + i \cdot h$, ie.,
$$t_0 = a$$
$$t_1 = a + h \qquad \text{where } n = \frac{b-a}{h}$$
$$\vdots$$
$$t_n = b$$

Deriving Euler's method ← Rarely used in practice. Mostly a warmup for us

Taylor expand (assuming $y \in C^2$)

$$(*) \quad y(t_{i+1}) := y(t_i + h) = y(t_i) + \underbrace{y'(t_i) \cdot h}_{\text{what is this?}} + y''(\xi)\frac{h^2}{2!}$$

exact (unknown) solution

our ODE: $y'(t) = f(t,y)$

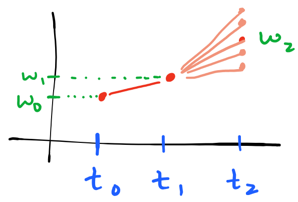
$$= y(t_i) + h \cdot f(t_i, y(t_i)) + \underbrace{y''(\xi)\frac{h^2}{2!}}_{\text{error term}}$$

so Forward Euler (aka "Euler's method")

Ex. of a difference equation (discrete time)
$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h \cdot f(t_i, w_i), \quad i = 1, 2, \ldots, n \end{cases} \qquad \left(h = \frac{b-a}{n} \right)$$

(What if we had Taylor expanded about t_{i+1} instead of t_i? That's also valid, and we'd get "Backward Euler". More on that later in this chapter)

Graphically



what value for w_2? adjust slope $\dfrac{w_2 - w_1}{t_2 - t_1}$ so it matches $\underbrace{f(t_1, w_1)}_{y'(t_1)}$

Error Analysis of Euler's Method

Euler is $w_{i+1} = w_i + h \cdot f(t_i, w_i)$

More generally, many methods can be written as

$$w_{i+1} = w_i + h \cdot \phi(t_i, w_i)$$

Define the local truncation error τ as

$y_i = y(t_i)$ NOT w_i

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\,\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i)$$

Interpretation: how well does the exact solution $y(t)$ satisfy the difference equation (ie., the numerical method). A bit odd.

So for Forward Euler this is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) \overset{\text{via } (*)}{=} y''(\xi) \cdot \frac{h}{2!} = O(h)$$

More Interpretation: suppose (unlikely) that $\underset{\text{approx}}{w_i} = \underset{\text{exact}}{y(t_i)}$ } this is why we call it local

then $w_{i+1} = y_i + h \cdot f(t_i, y_i)$ (Forward Euler formula)

$$y_{i+1} = y_i + h \cdot f(t_i, y_i) + h\,\tau_{i+1}(h)$$

$$= w_{i+1} + O(h^2)$$

So with perfect data at t_i, error $|y_{i+1} - w_{i+1}| = O(h^2)$

But since we don't actually have $w_i = y(t_i)$,

won't errors accumulate?

Since $n = \dfrac{b-a}{h}$, after n steps, each with $O(h^2)$ error,

aka h^{-1}

it would seem we get $n \cdot O(h^2) = \boxed{O(h) \text{ error.}}$ ← Global error

But that's not rigorous, since our local truncation error doesn't help with perturbed starting points

Rigorous global analysis of Forward Euler

Lemma 5.7 $\forall x \geq -1, \quad 0 \leq 1+x \leq e^x$

(proof: "$\leq$" obvious
"$\geq$" $f(x)=e^x$, Taylor expand, $f''(\xi) \geq 0$)

and also $\forall m > 0$

$$0 \leq (1+x)^m \leq e^{mx}$$

(proof: if $m>0$, $a \leq b$ iff $a^m \leq b^m$)

Lemma 5.8 "Discrete Grönwall's Lemma"

let $s, t > 0$ and $(a_i)_{i=0}^{k}$ be a sequence with $a_0 \geq -t/s$

and $(**)$ $a_{i+1} \leq (1+s)a_i + t \quad (\forall i = 0, 1, \dots, k-1)$

then

$$a_{i+1} \leq e^{(i+1)s} \cdot \left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

proof

$$a_{i+1} \leq (1+s)a_i + t \qquad \text{by } (**)$$

$$\leq \underbrace{(1+s)((1+s)\overbrace{a_{i-1} + t}^{a_i \leq \dots}) + t} \qquad \text{by } (**) \text{ again}$$

Geometric Series

$$\sum_{j=0}^{i} \beta^j = \frac{1-\beta^{i+1}}{1-\beta}$$

$$= (1+s)^2 a_{i-1} + (1 + (1+s))t$$

$$\leq \cdots$$

$$\leq (1+s)^{i+1} a_0 + \Big(\overbrace{1 + (1+s) + (1+s)^2 + \dots + (1+s)^i}^{\text{geometric series}} \Big) t$$

$$= \frac{1-(1+s)^{i+1}}{1-(1+s)} = \frac{1}{s}\left((1+s)^{i+1} - 1\right)$$

$$= (1+s)^{i+1}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

$$\leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s} \qquad \text{via Lemma 5.7} \quad \square$$

Theorem 5.9 Global error bound for Forward Euler

Suppose f is continuous and has Lipschitz constant L with respect to y, uniformly in t on $D = \{(t,y) : a \leq t \leq b, -\infty \leq y \leq \infty\}$, and $\exists\, M < \infty$ s.t. $|y''(t)| \leq M \quad \forall t \in [a,b]$ where $y(t)$ is the unique solution to the IVP. Let $(w_i)_{i=0}^{n}$ be via Forward Euler. Then $(\forall i = 0, 1, \dots, n)$

$$|y(t_i) - w_i| \leq \frac{h \cdot M}{2L}\left(e^{L(t_i - a)} - 1\right).$$

Interpretation: the global error bound is $O(h)$, though it grows exponentially with t.] UNFORTUNATE proof:

via Taylor expansion (i.e. (*)),

$$y_{i+1} = y_i + h \cdot f(t_i, y_i) + y''(\xi) \cdot \frac{h^2}{2}$$

and $\quad w_{i+1} = w_i + h \cdot f(t_i, w_i) \quad$ (Euler's method)

so

did a triangle inequality, $|a+b+c| \le |a|+|b|+|c|$

$$\underbrace{|y_{i+1} - w_{i+1}|}_{a_{i+1}} \le \underbrace{|y_i - w_i|}_{a_i} + h \underbrace{|f(t_i, y_i) - f(t_i, w_i)|}_{\text{Lipschitz}} + \frac{M}{2} h^2$$

$$\le L \cdot \underbrace{|y_i - w_i|}_{a_i}$$

$$a_{i+1} \le (1 + \underbrace{hL}_{s}) a_i + \underbrace{\frac{M}{2} h^2}_{t}$$

so apply Lemma 5.8

$a_0 = 0 \ge -t/s$ ✓

So

$$|y_{i+1} - w_{i+1}| \le e^{\underbrace{(i+1)hL}}\left(a_0 = 0 + \frac{\frac{M}{2} h^2}{hL}\right) - \frac{\frac{M}{2} h^2}{hL}$$

$(i+1)h = t_{i+1} - a$

$$= \frac{Mh}{2L}\left(e^{(t_{i+1} - a)L} - 1\right). \quad \square$$

How to use this global error bound?

Our IVP is $y' = f(t,y)$, $a \le t \le b$, $y(0) = \alpha$

We are given f, so we could figure out its Lipschitz constant L

But what about M, our bound on $|y''|$? We don't know y!

Sometimes we can bound it:

$y = y(t)$ Chain rule

$$y''(t) = \frac{d}{dt}(y') = \frac{d}{dt}(f(t,y)) = \frac{\partial}{\partial t} f(t,y) + \frac{\partial}{\partial y} f(t,y) \cdot \underbrace{\frac{d}{dt} y}_{f}$$

so if we can bound $\frac{\partial}{\partial t} f$ and $\left(\frac{\partial}{\partial y} f\right) \cdot f$ (f is known) for all inputs t, y

then we have a bound for y'' (y is unknown).

Ex: $y' = \underbrace{y - t^2 + 1}_{f(t,y)}$ on $0 \le t \le 2$, $y(0) = 1/2$

(1) Lipschitz constant: Find $\frac{\partial f}{\partial y} = 1$. So $L = 1$ ✓

(2) $|y''| \le M$? $\quad y'' = \frac{\partial}{\partial t} f + \left(\frac{\partial}{\partial y} f\right)(f)$

$$= \underbrace{-2t + 1\cdot(y - t^2 + 1)}_{\text{We can't easily bound this!}} \qquad 0 \le t \le 2, \quad -\infty \le y \le \infty.$$

So doesn't always work.

Roundoff Error

For differentiation, as $h \to 0$, finite difference formulas are unstable

For integration, composite Newton-Cotes is stable as $n \to \infty$
i.e. $h \to 0$

For ODE?

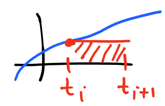
We have derivatives, so is it like differentiation? Not exactly.

It's somewhat like integration

$$y(t_{i+1}) = \int_{t_i}^{t_{i+1}} y'(t)\,dt + y(t_i) \qquad \text{F.T.C.}$$

... but it does accumulate!

$$\approx \underbrace{y'(t_i)\cdot(t_{i+1} - t_i)}_{\text{left endpoint rule}} + y(t_i)$$



Formally, if

$$\tilde{w}_0 = \alpha + \delta_0$$
$$\tilde{w}_{i+1} = \tilde{w}_i + h f(t_i, \tilde{w}_i) + \delta_{i+1}$$

roundoff errors due to floating point numbers

then using the discrete Grönwall Lemma again ...

Thm 5.10 Global error bound, in floating pt., of Forward Euler

Under the same conditions as Thm 5.9, with $|\delta_i| \le \delta \quad (\forall i)$

then $(\forall\, i = 0, 1, \dots, n)$

$$\underline{|y(t_i) - \tilde{w}_i| \le \frac{1}{L}\Big(\underbrace{\frac{hM}{2} + \frac{\delta}{h}}\Big)\cdot\left(e^{L(t_i - a)} - 1\right) + |\delta_0|\, e^{L(t_i - a)}}$$

In terms of h, $\longrightarrow \infty$ as $h \to 0$

Smallest error for $h = \sqrt{\frac{2\delta}{M}}$. We can't make error arbitrarily small by taking $h \to 0$

However, a fundamental difference:

for finite differences, we wanted to take $h \to 0$

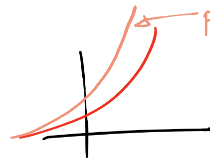
for ODEs, we don't want $h \to 0$ since then $n \to \infty$ and our computation time increases.

So for ODE, usually computation time not roundoff error is what limits our accuracy.

⚠ Our error bound is just a <u>bound</u>.

<u>Ex</u>: $y' = 5 \cdot y \implies L = 5$,

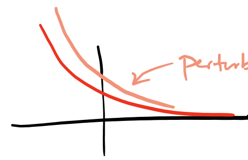
$y = e^{5t}$



perturbed sol'n can grow away from true sol'n

<u>Ex</u>: $y' = -5y \implies L = 5$

$y = e^{-5t}$



perturbed sol'n gets <u>closer</u> to true sol'n over time!