9. ARMA models

See §2.1–2.3 and §3.1

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PLAN
- Review
- ARMA(1,1)
- Linear Processes
- ARMA(p,q)
- Causality, invertibility

REVIEW: AR (Autoregressive) and MA (Moving average)

AR(1) Assume $\{X_t\}$ is stationary, and $X_t = \phi X_{t-1} + Z_t$, $t \in \mathbb{Z}$,

and $\{Z_t\} \sim WN(0, \sigma^2)$, $|\phi| < 1$, and $Cor(Z_t, X_s) = 0 \ \forall s < t$

we computed $\rho(h) = \phi^{|h|}$

MA(q) $X_t = Z_t + \theta_1 \cdot Z_{t-1} + \dots + \theta_q Z_{t-q} \quad \forall t \in \mathbb{Z}$

$\{Z_t\} \sim WN(0, \sigma^2)$, $\theta_1, \dots, \theta_q$ are constants.

ARMA(1,1)

Def $\{X_t\}$ is an Autoregressive Moving Average Process of order (1,1) ARMA(1,1)

if it is stationary and satisfies $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\phi + \theta \neq 0$.

⚠ We do not a priori assume any uncorrelated between X_s and Z_t

Question: this is an implicit definition, like an ODE.

Does a (stationary) solution exist?

If so, is it unique?

Analysis

Write in language/notation of the backshift operator B

$B X_t := X_{t-1}$

Define $\phi(B) = 1 - \phi \cdot B$, $\theta(B) = 1 + \theta \cdot B$

so our equation is $\boxed{\phi(B) X_t = \theta(B) Z_t}$

For grad students: this is the Neumann Series

Case 1 $|\phi| < 1$

Note: if $|x| < 1$, $f(x) = \frac{1}{1-x}$, Taylor Series gives $\frac{1}{1-x} = 1 + x + x^2 + \dots$

Write $\frac{1}{\phi(x)} = \sum_{j=0}^{\infty} \phi^j x^j \equiv \chi(x)$ chi

$\chi = \phi^{-1}$

i.e. $\chi(B) = \sum_{j=0}^{\infty} \phi^j B^j$. You can show this is a well-defined operator (section 2.2 roughly) and $\chi(B)\phi(B) = 1$

so

$$X_t = \underbrace{\chi(B)\theta(B)}_{\psi(B) := \sum_{j=0}^{\infty} \psi^j B^j} Z_t$$

$$\psi(B) := \sum_{j=0}^{\infty} \psi^j B^j = (1 + \phi B + \phi^2 B + \dots)(1 + \theta B)$$

$$= 1 + (\phi + \theta) B + (\phi^2 + \theta\phi) B^2 + \dots$$

i.e.

$$\psi^0 = 1, \quad \psi^j = (\phi + \theta)\phi^{j-1} \quad j \geq 1$$

and

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}, \quad \text{"explicit".}$$

This is a MA(∞) process and (since $\sum_{j=0}^{\infty} \phi^{j-1} < \infty$) it is stationary (cf. Prop. 2.2.1). Also unique since explicit

Case 2 $|\phi| > 1$, so $|\phi^{-1}| < 1$

idea: reverse time $X_t - \phi X_{t-1} = RHS(t)$, first $\div$ by $-\phi$:

$$-\frac{1}{\phi} X_t + X_{t-1} = -\frac{1}{\phi} RHS(t) \qquad \forall t \in \mathbb{N}$$

intuition

let $\tilde{X}_t := X_{-t}$ be $\{X_t\}$ backwards

so $\tilde{X}_{-t+1} = X_{t-1}$ so... $-\frac{1}{\phi}\tilde{X}_{-t} + \tilde{X}_{-t+1} = -1/\phi \, RHS(t)$

Now just relabel, $s = -t$, so $-1/\phi \, \tilde{X}_s + \tilde{X}_{s+1} = -1/\phi \, RHS(-s)$

or $\boxed{\tilde{X}_s - \frac{1}{\phi}\tilde{X}_{s-1} = -1/\phi \, RHS(-s+1)}$

$\tilde{\phi} = \frac{1}{\phi}$ $\qquad |\phi| > 1 \iff |\tilde{\phi}| < 1$

Redefine $\chi(x) := -\sum_{j=1}^{\infty} \phi^{-j} x^{-j}$ (see Problem 2.7) negative exponents!

So still have $\chi(B)\phi(B) = 1$ but just using power series w/ negative exponents.

Get

$$X_t = \frac{-\theta}{\phi} Z_t - (\theta + \phi) \sum_{j=1}^{\infty} \frac{1}{\phi^{j+1}} Z_{t+j}$$

"future"

also "explicit" and MA(∞)

Case 3 $|\phi| = 1$, i.e., ± 1

$\phi(B) = 1 - B$ or $1 + B$, not "invertible"

You can show that there is no stationary solution!

(Compute ACVF and find issues)

(Simpler: there can't be a unique stationary sol'n when $\phi = 1$ since you can pick and mean μ you want)

Recap

① $|\phi| < 1$ case: $\exists!$ stationary solution

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

Def "causal"

we say $\{X_t\}$ is causal (a "causal function of $\{Z\}$" or "a causal autoregressive process")

because X_t only depends on current and past values of Z_t.

Generally, $\{X_t\}$ is a linear process if it can be written

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \quad \text{where } \{Z_j\} \sim WN(0, \sigma^2)$$

coefficients (not random) and $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

abs. summable

and a linear process is causal if $\psi_j = 0 \;\; \forall j < 0$

else it is noncausal.

② $|\phi| > 1$, also $\exists!$ unique solution

$$X_t = \frac{-\theta}{\phi} Z_t - (\theta + \phi) \sum_{j=1}^{\infty} \frac{1}{\phi^{j+1}} Z_{t+j}$$

Also a linear process, but this is noncausal

Invertibility

For linear models $X_t = \sum_{j\in\mathbb{Z}} \psi_j Z_{t-j}$, it's conceivable you could solve

and rewrite as $Z_t = \sum_{j\in\mathbb{Z}} \pi_j X_{t-j}$

As a special case, if you can make Z_t a causal process in $\{X_t\}$

i.e. $Z_t = \sum_{j\geq 0} \pi_j X_{t-j}$, we say the original linear model

(or the ARMA process which gave rise to it) to be invertible.

__Observe__

ARMA(1,1): $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$, $\phi + \theta \neq 1$

X and Z aren't interchangeable (we assume $Z_t \sim WN(0,\sigma^2)$),
but the equation almost is. It turns out

$|\phi| < 1 \Rightarrow$ causal

$|\phi| > 1 \Rightarrow$ non-causal

and

$|\theta| < 1 \Rightarrow$ invertible $\quad Z_t = X_t - (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^j X_{t-j}$

$|\theta| > 1 \Rightarrow$ non-invertible $\quad Z_t = -\phi\theta^{-1}X_t + (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{-j-1}X_{t+j}$

ARMA(p,q) (§3.1 in book)

__Def__ $\{X_t\}$ is an ARMA(p,q) process if it is ① __stationary__

and ② $(\forall t\in\mathbb{Z})$ $X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$

i.e. $\phi(B)X_t = \theta(B)Z_t$, B = backshift (as always)

where $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$

$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$

and ③ $\{Z_t\} \sim WN(0,\sigma^2)$

and ④ $\phi(\cdot)$ and $\theta(\cdot)$ polynomials have __no common factors__.
(i.e. no shared roots)

__Remarks__

- For ARMA(1,1) we required $\phi_1 + \theta_1 \neq 0$, i.e., $\phi_1 \neq -\theta_1$

  $\phi(z) = 1 - \phi_1 z$, $\theta(z) = 1 + \theta_1 z$

if $\phi_1 = -\theta_2$ then $\phi(z) = \theta(z)$ and they share a common factor

so condition (4) generalizes our old condition

- If $\theta(z) = 1$ ($q = 0$) we say it's autoregressive $AR(p)$
- If $\phi(z) = 1$ ($p = 0$) we say it's moving avg. process $MA(q)$

- What about existence/uniqueness of a stationary solution?

For $ARMA(1,1)$, $\exists!$ stationary sol'n $\underset{\text{if and only if}}{\text{iff}}$ $\left(|\phi_1| < 1 \text{ or } |\phi_1| > 1 \right)$

and $\underbrace{\phi(z) = 1 - \phi_1 z}_{\text{roots: } z = \frac{1}{\phi_1}}$.

Note: if $|\phi| = 1$ it means the root $z = 1/\phi_1$ also has $|z| = 1$

Theorem: A stationary solution to (2) $\phi(B) X_t = \theta(B) Z_t$ exists (and is also unique) iff $\underbrace{\phi(\cdot) \text{ has no roots on the complex unit circle.}}_{\text{i.e. } \phi(z) \neq 0 \ \forall\, |z| = 1}$

Causality is unchanged: $\{X_t\}$ is causal if we can write

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \qquad (\forall t \in \mathbb{Z})$$

$$\left(\text{and require } \sum_{j=0}^{\infty} |\psi_j| < \infty \right)$$

The (ψ_j) coefficients can be computed

[see book for now]

Theorem An ARMA process is causal iff $\phi(\cdot)$ has no roots in the complex unit ball $\{z : |z| \leq 1\}$

proof

[next class]