3. Examples of stationary time series

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Classic Examples §1.4 Brockwell + Davis

Ex iid Noise

independent, identically distributed

$\{X_t\}$ iid noise, written $\{X_t\} \sim \text{iid}(0, \sigma^2)$

means ① All X_t have the same distribution (and this distribution has 0 mean and variance σ^2)

② All X_t, X_s are independent $(t \neq s)$

If our residual is iid Noise, it means we've successfully extracted out all useful information / structure.
(that would be great, though usually we'll be less ambitious)

Is this stationary? ① Is mean constant (independent of t)? Yes.

② $\gamma_X(t+h, t) := \text{Cov}(X_{t+h}, X_t) = \underbrace{\begin{cases} \sigma^2 & h=0 \\ 0 & h \neq 0 \end{cases}}$

doesn't depend on t, so this **is** stationary

Ex White noise $\{X_t\} \sim WN(0, \sigma^2)$

means each X_t can have a different distribution, as long as that distribution has ① 0 mean, ② σ^2 variance

and

③ X_t, X_s $(t \neq s)$ are uncorrelated

So this is less strict than iid noise ($IID(0,\sigma^2)$ is also $WN(0,\sigma^2)$ but not vice-versa)

Just like $IID(0,\sigma^2)$, this is also stationary.

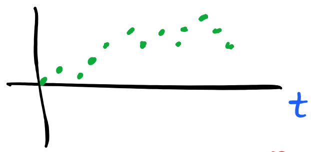
This is what we'll settle for making our residuals look like
(if residual is $WN(0,\sigma^2)$ it means we've extracted most of the signal, "as best we can tell")
i.e., it's not easy to distinguish observations of $IID(0,\sigma^2)$ from $WN(0,\sigma^2)$

Note: as a preview (or for anyone whose had an EE linear systems course) the spectrum of WN is flat

Ex Random Walk

Let $\{X_t\} \sim IID(0, \sigma^2)$, define $S_t = \sum_{i=1}^{t} X_i$ "random walk"

Classic ex: $X_t \sim \text{Bernoulli}(\tfrac{1}{2}) \in \{\pm 1\}$



Is S_t stationary? ① Check if mean is constant. $E(X_t) = 0$

$$E(S_t) = E\left(\sum_{i=1}^{t} X_i\right) \overset{E \text{ is linear}}{=} \sum_{i=1}^{t} E(X_i)$$

$= 0$ independent of t ✓

② Check covariance.

If it is stationary, then $Var(S_t) = Var(S_1)\ \forall t$

$$Var(S_1) = Var(X_1) = \underline{\sigma^2}$$ not equal for all t

$$Var(S_t) = Var(X_1 + X_2 + \dots + X_t)$$

$$= Var(X_1) + \dots + Var(X_t)$$ } since X_1, X_2 are uncorrelated

$$= \underline{t \cdot \sigma^2}$$ So not stationary

Ex MA(1) i.e. First-Order Moving Average

Let $\{Z_t\} \sim WN(0, \sigma^2)$. Not very interesting, and Z_t gives (almost no) help to predict Z_{t+1}

The MA(1) process is $\{X_t\}$ defined by

$$X_t = Z_t + \theta Z_{t-1} \quad \forall t \in \mathbb{Z}$$

Fixed constant

Ex: $\theta = 1$ then this is averaging pairs

Is $\{X_t\}$ stationary?

① Check for constant mean

$$E[X_t] = E[Z_t + \theta Z_{t-1}]$$

$$= E[Z_t] + \theta E[Z_{t-1}]$$

$$= 0 + \theta \cdot 0 = 0.$$ Independent of t ✓

② check covariance

$$\gamma(t, t+h) = Cov(X_t, X_{t+h})$$

$$= Cov(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1})$$

$$= \underbrace{Cov(Z_t, Z_{t+h} + \theta Z_{t+h-1})}_{\boxed{Cov(Z_t, Z_{t+h}) + \theta \cdot Cov(Z_t, Z_{t+h-1})}}$$

$$+ \theta \cdot \underbrace{Cov(Z_{t-1}, Z_{t+h} + \theta \cdot Z_{t+h-1})}_{\boxed{\theta \cdot Cov(Z_{t-1}, Z_{t+h}) + \theta^2 \cdot Cov(Z_{t-1}, Z_{t+h-1})}}$$

Use: $Z_t \sim WN(0, \sigma^2)$

so $Cov(Z_t, Z_s) = \begin{cases} \sigma^2 & t = s \\ 0 & t \neq s \end{cases}$

ACVF → $= \begin{cases} \sigma^2 + \theta^2\sigma^2 & h=0 \\ \theta\cdot\sigma^2 & h=+1 \text{ or } -1 \\ 0 & \text{else} \end{cases}$ independent of t so it is stationary

To summarize (we'll use this later),

the ACF for MA(1) is $\rho(h) = \begin{cases} 1 & h=0 \\ \frac{\theta}{1+\theta^2} & h=\pm 1 \\ 0 & |h|>1 \end{cases}$ $\}$ $\rho(0)=1$ always!

Ex AR(1) aka 1^{st} order Auto Regressive process

MA(1) is straightforward since it's explicit

AR(1) is weirder at first since it's self-referential

... think of it as a difference equation, which is analogous to an ODE for continuous time.

Again, let ① $\{Z_t\} \sim WN(0,\sigma^2)$

but now assume $\{X_t\}$ is a ② stationary series that satisfies

③ $X_t = \phi X_{t-1} + Z_t$ } existence, uniqueness of a solution? Be patient

and ④ Z_t is uncorrelated w/ X_s for $s<t$

⑤ $|\phi| < 1$ (like a noisy contraction)

We've assumed it's stationary, so $\mathbb{E}[X_t] = \mathbb{E}[X_{t-1}] = \mu$

What is μ?

By $\mathbb{E}[$ ③ $]$ (using linearity) we have

$\mu = \phi\mu + 0$, i.e. $(1-\phi)\mu = 0$

So since $\phi \neq 1$, $\mu = 0$.

What is ACVF γ or the ACF ρ?

Apply $\mathrm{Cov}(X_{t-h}, \cdot)$ to ③ so

via linearity

$$\underbrace{\mathrm{Cov}(X_{t-h}, X_t)}_{\underbrace{\gamma(-h)}_{\gamma(h)}} \overset{③}{=} \phi\,\mathrm{Cov}(X_{t-h}, X_{t-1}) + \mathrm{Cov}(X_{t-h}, Z_t)$$

$$= \phi\,\gamma(h-1) + 0$$

$$= \phi^2 \gamma(h-2)$$

$$= \phi^h \gamma(0)$$

Since $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$ this means we know ρ:

$$\rho(h) = \phi^{|h|} \quad \forall h \in \mathbb{Z}$$

(see book: easy to work out $\gamma(0) = Var(X_t) = \frac{\sigma^2}{1-\phi^2}$)

i.e., if $\sigma^2 = 0$ so no noise, then $\{X_t\}$ is deterministic