6. Multivariate Normal Distributions

Thursday, January 20, 2022 4:30 PM

Random Variables (i.e., scalars)

Notation: $\mu_X = \mathbb{E}[X]$, $\sigma_X = \sqrt{Var[X]}$

Def Covariance

$$Cov(X,Y) = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \in \mathbb{R}$$
$$= \mathbb{E}[XY] - \mu_X \mu_Y$$

Def Correlation

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} =: \rho_{XY} \in [-1,1]$$

(unitless)

Facts

- $Var(X) = Cov(X,X)$
- $Cov(aX+b, Y) = a \cdot Cov(X,Y)$
- $Cov(X,Y) = Cov(Y,X)$ symmetry
- $Cov(X+Y, Z) = Cov(X,Z) + Cov(Y,Z)$ distributive

Random Vectors

Let $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ $\vec{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$

Def (vector) covariance $\vec{X} \in \mathbb{R}^m$, $\vec{Y} \in \mathbb{R}^n$

$Cov(\vec{X}, \vec{Y})$ is an $m \times n$ matrix,

(i,j) entry is $Cov(X_i, Y_j)$

Facts

- $Var(\vec{X}) := Cov(\vec{X}, \vec{X})$, often denoted Σ or Σ_X

  Σ is a symmetric matrix and non-negative definite

- $Cov(A\vec{X} + \vec{\mu}, B\vec{Y} + \vec{\nu}) = A \cdot Cov(\vec{X}, \vec{Y}) \cdot B^T$

  matrix vector

- $Cov(\vec{X}, \vec{Y}) = Cov(\vec{Y}, \vec{X})^T$

Def (vector) correlation

$$[Cor(\vec{X}, \vec{Y})]_{ij} = \frac{[Cov(\vec{X}, \vec{Y})]_{ij}}{\sigma_{X_i} \sigma_{Y_j}} = Cor(X_i, Y_j)$$

Normal Distribution (aka Gaussian)

If $X \sim N(\mu, \sigma^2)$ then it's probability density function, f,

is $f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\left(-\frac{1}{2}(x-\mu)^2\right)$

Multivariate Normal Distribution

$\vec{X} \in \mathbb{R}^n$ has a multivariate normal distribution if its pdf f

is $$f(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n \, |\Sigma|}} \cdot \exp\Big(\underbrace{-\frac{1}{2} (\underbrace{\vec{x}-\vec{\mu}}_{\text{vector}})^T \overset{\text{matrix}}{\Sigma^{-1}} (\underbrace{\vec{x}-\vec{\mu}}_{\text{vector}})}_{\text{scalar}} \Big)$$

$|\Sigma| := \det(\Sigma)$

written $\vec{X} \sim N(\vec{\mu}, \Sigma)$ or $\sim N_n(\vec{\mu}, \Sigma)$

or $\sim MVN(\vec{\mu}, \Sigma)$

N.B. Books/papers usually use **bold** to denote vectors

also, $\vec{X} \sim N(\mu, \Sigma)$ then:

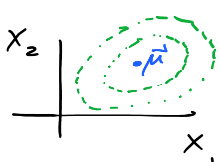
$X_1 \sim N(\mu_1, \Sigma_{11})$

$X_i \sim N(\mu_i, \Sigma_{ii})$

and in fact all marginals are (multi-variate) normal

2D case, $\vec{x} \in \mathbb{R}^2$

$f(\vec{x})$



level sets are ellipsis, determined by properties of Σ

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Generally:

$\vec{X} \sim N(\vec{\mu}, \Sigma)$

If Σ is diagonal (e.g., $\Sigma = \sigma^2 \cdot \overset{\text{identity}}{I}$)

this means X_i and X_j are $(i \neq j)$ uncorrelated

but in fact for multivariate normal, X_i and X_j are actually independent

So $\vec{X} \sim N(\vec{0}, \sigma^2 I)$

$\Rightarrow X_i$ are iid $N(0, \sigma^2)$

Extra info on Multivariate Normal distrubtions MATH/STAT 4540/5540 Spr 2022 Time Series

Instructor: Prof. Becker

These properties are all standard and can be found in numerous textbooks (e.g., Appendix A.3 of our Brockwell and Davis 3rd ed. text) as well as wikipedia's MVN page.

Affine transformation of multivariate normal (MVN) If $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a n-dimensional MVN and $\boldsymbol{Y} = \boldsymbol{c} + B\boldsymbol{X}$ is an affine transformation of $\boldsymbol{X}$ (here, B is a $m \times n$ matrix for any positive integer m, and $\boldsymbol{c}$ is a vector of length m), then $\boldsymbol{Y}$ is also MVN but with adjusted parameters: $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{c} + B\boldsymbol{\mu}, B\boldsymbol{\Sigma}B^T)$.

Conditional distrubution of MVN Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be a n-dimensional MVN, and partition $\boldsymbol{X}$ as

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}$$

where $\boldsymbol{X}_1$ is length q and $\boldsymbol{X}_2$ is length $q' \stackrel{\text{def}}{=} n - q$. Similarly partition $\boldsymbol{\mu}$ into $[\boldsymbol{\mu}_1, \boldsymbol{\mu}_2]^T$, and partition the covariance matrix into

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}^T & \boldsymbol{\Sigma}_{22} \end{bmatrix} \text{ with sizes } \begin{bmatrix} q \times q & q \times q' \\ q' \times q & q' \times q' \end{bmatrix}.$$

Then the conditional distribution of $\boldsymbol{X}_1$ conditioned on $\boldsymbol{X}_2$ is also MVN, as follows: $\boldsymbol{X}_1 \mid \boldsymbol{X}_2 \sim \mathcal{N}(\bar{\boldsymbol{\mu}}, \overline{\boldsymbol{\Sigma}})$ where

$$\bar{\boldsymbol{\mu}} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{X}_2 - \boldsymbol{\mu}_2)$$

and covariance matrix

$$\overline{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^T.$$

Marginal distribution of MVN Under the same partition as above, the marginal distribution is very easy to find: just drop the irrelevant variables. Thus $\boldsymbol{X}_1 \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$.