4. Properties of the ACF, GPs, & strict stationarity

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ACF Theorem (§ 2.1 book)

1) $\rho(0) = 1$
2) $|\rho(h)| \le 1$
3) $\rho(h) = \rho(-h)$

} immediate

4) ρ is nonnegative definite which means for any finite sequence $\vec{a} = (a_1, a_2, \dots, a_k)$ and any sequence of k time points $(t_1, \dots, t_k)$ it holds that

aka positive semi-definite

$$\sum_{i=1}^{k} \sum_{j=1}^{k} a_i \, \rho(t_i - t_j) \, a_j \geq 0$$

Similarly, since ρ and γ are related by a positive constant, γ (ACVF) is also non-negative definite: $\forall \vec{a}, t \dots$

$$\sum_{i=1}^{k} \sum_{j=1}^{k} a_i \, \gamma(t_i - t_j) \, a_j \geq 0$$

In matrix notation, letting Γ be the $k \times k$ matrix, $\Gamma_{ij} = \gamma(t_i - t_j)$

this is $\vec{a}^T \cdot \Gamma \cdot \vec{a} \geq 0$ ($\Leftrightarrow$ all eigenvalues of Γ are non-negative)

proof sketch: covariance matrices are always non-negative semi-definite

5) [essentially Thm. 2.1.1]

Any function ρ defined on $\mathbb{Z}$ which satisfies 1) – 4) properties is an ACF, meaning $\exists$ a stationary process $\{X_t\}$ which has an ACF given by ρ

proof is beyond our scope: show $\exists$ a Gaussian Process $\{X_t\}$ and use Kolmogorov's Thm

Def Gaussian Process ($\underbrace{}_{GP}$) (our book calls it $\overbrace{\text{a Gaussian time series}}^{\text{Def. A.3.2}}$)

$\{X_t\}$ is a GP if all its joint distributions are multivariate normal (see A.3 for refresher)

i.e., $\forall\, n \in \mathbb{N}$ and $\forall\, (i_1, \dots, i_n) \in \mathbb{Z}^n$

then $(X_{i_1}, \dots, X_{i_n}) \sim$ multivariate normal

(need not be stationary)

Strict Stationarity

Def $\{X_t\}$ is **strictly stationary** (aka strongly stationary)
if $\forall n \in \mathbb{Z}$, $\forall t_i \in \mathbb{Z}$ $i=1,\dots,n$, $\forall h \in \mathbb{Z}$

$$(X_{t_1}, \dots, X_{t_n}) \overset{d}{=} (X_{t_1+h}, \dots, X_{t_n+h})$$

More strict than weak: any set of $\{t_i\}$, and distributions equal, not just 1^{st} 2 moments

Thm If $\{X_t\}$ is strictly stationary, then

① $\{X_t\}$ is identically distributed (not necessarily i.i.d)

② $(X_t, X_{t+h}) \overset{d}{=} (X_1, X_{1+h})$ $\forall\, t, h \in \mathbb{Z}$

③ $\{X_t\}$ is weakly stationary if $\mathbb{E}[X_t^2] < \infty$

④ weak $\not\Rightarrow$ strong

⑤ in particular, IID sequences are strictly stationary

⑥ for GP, strict $\Longleftrightarrow$ weak
(related to: if multivariate normal,
independent $\Longleftrightarrow$ uncorrelated)

Remark
(Any kind of) stationarity is about the process $\{X_t\}$ and not the data $\{x_t\}$.

For analysis we can assume stationarity and see if we like the results we get