

9. ARMA models

See § 2.1-2.3 and § 3.1

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PLAN

- Review
- ARMA(1,1)
- Linear Processes
- ARMA(p,q)
- Causality, invertibility

REVIEW: AR (Autoregressive) and MA (Moving average)

AR(1) Assume $\{X_t\}$ is stationary, and $X_t = \phi X_{t-1} + Z_t, t \in \mathbb{Z}$,
and $\{Z_t\} \sim WN(0, \sigma^2)$, $|\phi| < 1$, and $\text{Cor}(Z_t, X_s) = 0 \forall s < t$

we computed $\rho(h) = \phi^{|h|}$

MA(q) $X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q} \quad \forall t \in \mathbb{Z}$
 $\{Z_t\} \sim WN(0, \sigma^2)$, $\theta_1, \dots, \theta_q$ are constants.

ARMA(1,1)

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Def $\{X_t\}$ is an Autoregressive Moving Average Process of order (1,1)

if it is stationary and satisfies $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\phi + \theta \neq 0$.

Δ We do not a priori assume any uncorrelated between X_s and Z_t

Question: this is an implicit definition, like an ODE.

Does a (stationary) solution exist?

If so, is it unique?

Analysis

Write in language/notation of the backshift operator \mathcal{B}

$$\mathcal{B} X_t := X_{t-1}$$

$$\text{Define } \phi(\mathcal{B}) = 1 - \phi \cdot \mathcal{B}, \quad \theta(\mathcal{B}) = 1 + \theta \cdot \mathcal{B}$$

so our equation is $\boxed{\phi(B) X_t = \theta(B) Z_t}$

For grad students:

this is the Neumann Series

case 1 $|\phi| < 1$

Note: if $|x| < 1$, $f(x) = \frac{1}{1-x}$, Taylor series gives

$$\text{Write } \frac{1}{\phi(x)} = \sum_{j=0}^{\infty} \phi^j x^j \equiv \chi(x)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

i.e. $\chi(B) = \sum_{j=0}^{\infty} \phi^j B^j$. You can show this is a well-defined operator
and $\chi(B) \phi(B) = 1$

so

$$X_t = \underbrace{\chi(B) \theta(B)}_{\Psi(B)} Z_t$$

$$\Psi(B) := \sum_{j=0}^{\infty} \psi^j B^j = (1 + \phi B + \phi^2 B + \dots)(1 + \theta B)$$

i.e.

$$= 1 + (\phi + \theta)B + (\phi^2 + \theta\phi)B + \dots$$

$$\psi^0 = 1, \quad \psi^j = (\phi + \theta)\phi^{j-1} \quad j \geq 1$$

and

$$X_t = Z_t + (\phi + \theta) \sum_{j=0}^{\infty} \phi^{j-1} Z_{t-j}, \text{ "explicit".}$$

This is a MA(∞) process and (since $\sum_{j=0}^{\infty} \phi^{j-1} < \infty$)

it is stationary (cf. Prop. 2.2.1).

Also unique since explicit

case 2 $|\phi| > 1$, so $|\phi^{-1}| < 1$

idea: reverse time $X_t - \phi X_{t-1} = \text{RHS}(t)$, first \div by $-\phi$:

$$-\frac{1}{\phi} X_t + X_{t-1} = -\frac{1}{\phi} \text{RHS}(t) \quad \forall t \in \mathbb{N}$$

let $\tilde{X}_{-t} := X_t$ be $\{X_t\}$ backwards

$$\text{so } \tilde{X}_{-t+1} = X_{t-1} \quad \text{so...} \quad -\frac{1}{\phi} \tilde{X}_{-t} + \tilde{X}_{-t+1} = -\frac{1}{\phi} \text{RHS}(t)$$

Now just relabel, $s = -t$, so $-\frac{1}{\phi} \tilde{X}_s + \tilde{X}_{s+1} = -\frac{1}{\phi} \text{RHS}(-s)$

$$\text{or } \boxed{\tilde{X}_s - \frac{1}{\phi} \tilde{X}_{s-1} = -\frac{1}{\phi} \text{RHS}(-s+1)}$$

$$\Rightarrow \tilde{\phi} = \frac{1}{\phi} \quad |\phi| > 1 \Leftrightarrow |\phi| < 1$$

intuition

Redefine $\chi(x) := -\sum_{j=1}^{\infty} \phi^{-j} x^{-j}$ (see Problem 2.7)

negative exponents!

So still have $\chi(B) \phi(B) = 1$ but just using power series

w/ negative exponents.

Get

$$X_t = \frac{-\theta}{\phi} Z_t - (\theta + \phi) \sum_{j=0}^{\infty} \frac{1}{\phi^{j+1}} Z_{t+j}$$

also "explicit"
and MAT(r)

"future"

case 3 $|\phi| = 1$, i.e., ± 1

$$\phi(B) = 1 - B \text{ or } 1 + B, \text{ not "invertible"}$$

You can show that there is no stationary solution!
(Compute ACVF and find issues)

(Simpler: there can't be a unique stationary sol'n when $\phi=1$ since you can pick and mean μ you want)

Recap

① $|\phi| < 1$ case: $\exists!$ stationary solution

$$X_t = Z_t + (\phi + \theta) \sum_{j=0}^{\infty} \phi^{j-1} Z_{t-j}$$

Def "causal" we say $\{X_t\}$ is causal (a "causal function of $\{Z\}$ "
or "a causal autoregressive process")

because X_t only depends on current and past values of Z_t .

Generally, $\{X_t\}$ is a linear process if it can be written

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \quad \text{where } \{Z_j\} \sim WN(0, \sigma^2)$$

↑ coefficients (not random) and $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

abs. summable

and a linear process is causal if $\psi_j = 0 \quad \forall j < 0$
else it is noncausal.

② $|\phi| > 1$, also $\exists!$ unique solution

$$X_t = \frac{-\theta}{\phi} Z_t - (\theta + \phi) \sum_{j=0}^{\infty} \frac{1}{\phi^{j+1}} Z_{t+j}$$

Also a linear process, but this is noncausal

Invertibility

For linear models $X_t = \sum_{j \in \mathbb{Z}} \psi_j Z_{t+j}$, it's conceivable you could solve

$$\text{and rewrite as } Z_t = \sum_{j \in \mathbb{Z}} \pi_j X_{t-j}$$

As a special case, if you can make Z_t a causal process in $\{X_t\}$

$$\text{i.e. } Z_t = \sum_{j \geq 0} \pi_j X_{t-j}, \text{ we say the original linear model}$$

(or the ARMA process which gave rise to it) to be invertible.

Observe

$$\text{ARMA}(1,1) : X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \quad \phi + \theta \neq 1$$

X and Z aren't interchangeable (we assume $Z_t \sim WN(0, \sigma^2)$), but the equation almost is. It turns out

$$|\phi| < 1 \Rightarrow \text{causal}$$

$$|\phi| > 1 \Rightarrow \text{non-causal}$$

and

$$|\theta| < 1 \Rightarrow \text{invertible} \quad Z_t = X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^j X_{t-j}$$

$$|\theta| > 1 \Rightarrow \text{non-invertible} \quad Z_t = -\phi \theta^{-1} X_t + (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t+j}$$

ARMA(p, q) (§3.1 in book)

Def $\{X_t\}$ is an ARMA(p, q) process if it is ^①stationary

and ^② $(\forall t \in \mathbb{Z}) X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$

$$\text{i.e. } \phi(B) X_t = \theta(B) Z_t, \quad B = \text{backshift (as always)}$$

$$\text{where } \phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$$

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$$

$$\text{and } ③ \{Z_t\} \sim WN(0, \sigma^2)$$

and ^④ $\phi(\cdot)$ and $\theta(\cdot)$ polynomials have no common factors.
(i.e. no shared roots)

Remarks

- For ARMA(1,1) we required $\phi_1 + \theta_1 \neq 0$, i.e., $\phi_1 \neq -\theta_1$,

$$\phi(z) = 1 - \phi_1 z, \quad \theta(z) = 1 + \theta_1 z$$

if $\phi_1 = -\theta_2$ then $\phi(z) = \theta(z)$ and they share a common factor
so condition (4) generalizes our old condition

- If $\theta(z) = 1$ ($q=0$) we say it's autoregressive AR(p)
- If $\phi(z) = 1$ ($p=0$) we say it's moving avg. process MA(q)
- What about existence/uniqueness of a stationary solution?

For ARMA(1,1), $\exists!$ stationary sol'n iff $|1-\phi_1| < 1$ or $|1-\theta_1| > 1$
if and only if

$$\text{and } \phi(z) = 1 - \phi_1 z.$$

$$\underbrace{\text{roots:}}_{z = \frac{1}{\phi_1}}$$

Note: if $|\phi_1| = 1$ it means the root $z = \frac{1}{\phi_1}$
also has $|z|=1$

Theorem: A stationary solution to ② $\phi(B)X_t = \theta(B)Z_t$ exists

(and is also unique) iff $\phi(\cdot)$ has no roots on the complex unit circle.
i.e. $\phi(z) \neq 0 \quad \forall |z|=1$

Causality is unchanged: $\{X_t\}$ is causal if we can write

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad (\forall t \in \mathbb{Z})$$

$$(\text{and require } \sum_{j=0}^{\infty} |\psi_j| < \infty)$$

The (ψ_j) coefficients can be computed

[see book for now]

Theorem An ARMA process is causal iff $\phi(\cdot)$ has no roots
in the complex unit ball $\{z : |z| \leq 1\}$

proof

[next class]