Basics of Measure Theory

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MATH 4900 Senior Seminar

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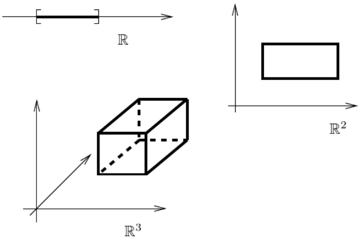
Measure

The *measure* $m(E)$ of a solid body E in one or more dimensions is fundamental to Euclidean geometry.

In one, two and three dimensions, measure is referred to as *length*, *area*, and *volume* of E, respectively.



Suppose we have a sequence of intervals in

$$[0,1]$$

of total length less than 1.

Measure

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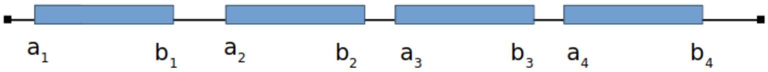
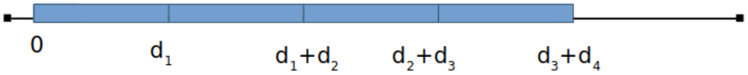


$\mathbb{R}$

Measure

Why not just re-arrange?



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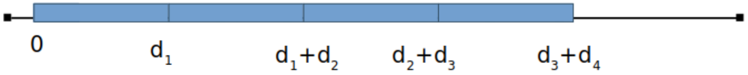

$\mathbb{R}$

Does not cover past $\sum d_n$

$\mapsto$ Do the original intervals cover?

Measure

Replace $[0,1]$ with rational interval

$$[0,1] \cap \mathbb{Q}$$

If our intervals have lengths 1/4, 1/8, 1/16 … then the left-to-right intervals will only cover the interval

$$[0, \frac{1}{2}] \cap \mathbb{Q}$$

Measure

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If our intervals have lengths 1/4, 1/8, 1/16 … then the left-to-right intervals will only cover the interval

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but the original intervals could cover since we can enumerate the rationals.

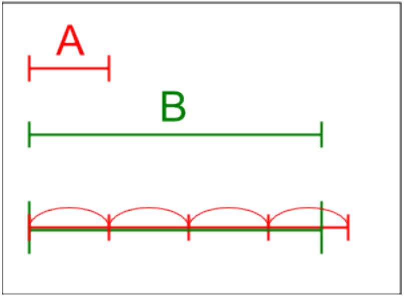
Therefore, measure must depend on properties of the reals not shared by the rationals.

Measure

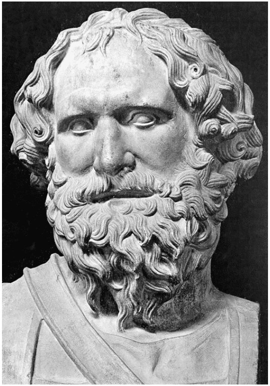
We want to define *length* for general sets of reals, i.e. $\mathbb{Q}$.

We want a notion of length that applies to all the sets we know and are used to and that is additive.

Brief history



Archimedean property



Brief history

1872 – Weierstrass's construction of a nowhere differentiable function.

1881 – Introduction of functions of bounded variation by Jordan and later (1887) connection with rectifiability.

1883 – Cantor's ternary set.

1890 – Construction of a space-filling curve by Peano.

1898 – Borel's measurable sets.

1902 – Lebesgue's theory of measure and integration.

1905 – Construction of non-measurable sets by Vitali.

1906 – Fatou's application of Lebesgue theory to complex analysis.

Main theorems

Axiom of Choice

An arbitrary Cartesian product of non-empty sets remains non-empty.

Ernst Zermelo

If E is any subset of $\mathbb{R}^d$, the **exterior measure** of E is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.

Main theorems

A subset E of $\mathbb{R}^d$ is **measurable** if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and

$$m_*(\mathcal{O} - E) \leq \epsilon$$

If E is measurable, we define its **measure** $m(E)$ by

$$m(E) = m_*(E)$$

Properties of measurable sets

Property 1 Every open set in $\mathbb{R}^d$ is measurable.
Property 2 If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.
Property 3 A countable union of measurable sets is measurable.
Property 4 Closed sets are measurable.
Property 5 The complement of a measurable set is measurable.
Property 6 A countable intersection of measurable sets is measurable.

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$\mapsto$ The collection of measurable sets is closed under the familiar operations of set theory.

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(II) A measure $\mu : \mathcal{M} \to [0, \infty]$ with the following defining property: if $E_1, E_2, \ldots$ is a countable family of disjoint sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu\left(E_n\right)$$

Main theorems

A σ-**algebra** of sets is a family of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements.

The **Borel σ-algebra** in $\mathbb{R}^d$, denoted by $\mathcal{B}_{\mathbb{R}^d}$ is the smallest σ-algebra that contains all open sets. Elements of this σ-algebra are called **Borel sets**.

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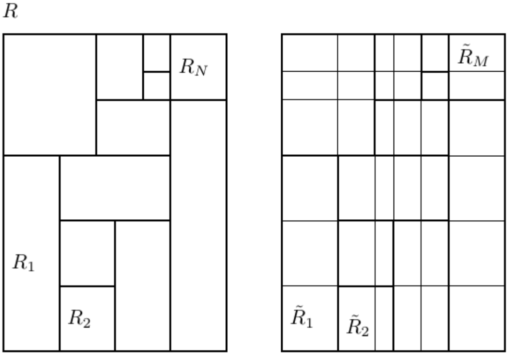
The Lebesgue measurable sets arise as the completion of the σ-algebra of Borel sets, by adjoining all subsets of Borel sets of measure zero.

Examples

Theorem: If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \bigcup_{k=1}^{N} R_k$, then

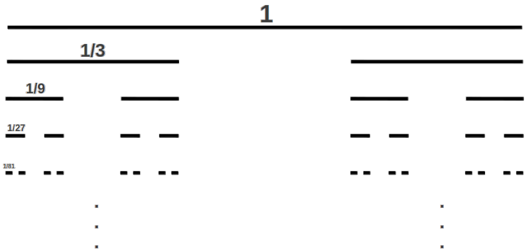
$$|R| = \sum_{k=1}^{N} |R_k|$$



$$|R| = \sum_{j=1}^{M} \left|\tilde{R}_j\right| = \sum_{k=1}^{N} \sum_{j \in J_k} \left|\tilde{R}_j\right| = \sum_{k=1}^{N} |R_k|$$

Examples

The Cantor set C has measure zero but is uncountable.



From the construction of $\mathcal{C}$, we know that $\mathcal{C} \subset C_k$, where each C_k is a disjoint union of 2^k closed intervals, each of length 3^{-k}.

Consequently, $m_*(\mathcal{C}) \leq (2/3)^k$ for all k, hence $m_*(\mathcal{C}) = 0$

Banach-Tarski paradox

The unit ball

$$B := \left\{(x, y, z) \in \mathsf{R}^3 : x^2 + y^2 + z^2 \leq 1\right\}$$

can be disassembled into a finite number of pieces and then be reassembled to form two disjoint copies of the ball B.

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These pathological decomposition pieces require the *axiom of choice*.

Rather than study every subset E of R^d, we settle for only non-pathological subsets called measurable sets.

A **probability space** is a measure space $(\Omega, \mathcal{F}, \mathsf{P})$ of total measure 1 : $\mathsf{P}(\Omega) = 1$.

The measure P is known as a probability measure.

Applications

Change in notation from $(X, \mathcal{B}, \mu)$ to $(\Omega, \mathcal{F}, \mathsf{P})$:

(i) The space Ω is known as the sample space

(ii) The σ -algebra $\mathcal{F}$ is known as the event space

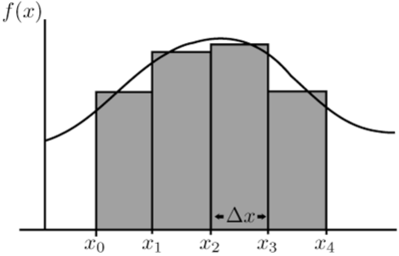
(iii) The measure $\mathsf{P}(E)$ of an event is known as the probability of that event.

Applications

When we can construct measures on on infinite-dimensional spaces, we can set up probability spaces associated to both discrete and continuous *random processes* even with infinite length.

Music and speech are *random processes*.

Georg Friedrich Bernhard Riemann



Applications

Lebesgue integral

$\int_{\mathsf{R}^d} f(x)$

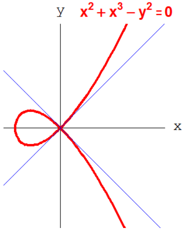
$f : \mathsf{R}^d \to [0, +\infty]$

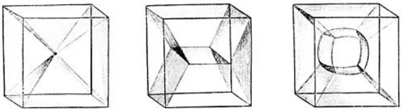
Open problems

A topology on X is a *Hausdorff space* when if $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$

Open problem: Find a topological characterization of Hausdorff topological spaces containing absolutely non-measurable subsets.

Open problem: Are the tangent cones to area minimizing surfaces unique?





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