



Basics of Measure Theory

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MATH 4900 Senior Seminar

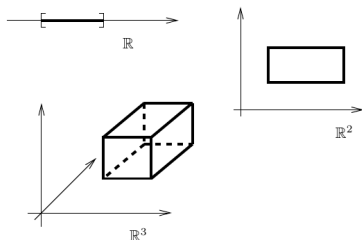
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Measure

The *measure* $m(E)$ of a solid body E in one or more dimensions is fundamental to Euclidean geometry.

In one, two and three dimensions, measure is referred to as *length*, *area*, and *volume* of E , respectively.



Topic: Measure

Suppose we have a sequence of intervals in

$$[0, 1]$$

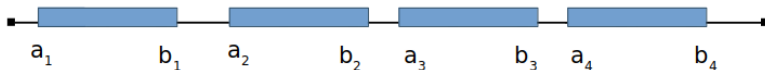
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Measure

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\mathbb{R}

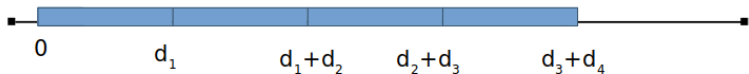
Measure

Why not just re-arrange?



\mathbb{R}

Why not just re-arrange?



\mathbb{R}

Does not cover past $\sum d_n$

↪ Do the original intervals cover?

Replace $[0, 1]$ with rational interval

$$[0, 1] \cap \mathbb{Q}$$

If our intervals have lengths $1/4, 1/8, 1/16 \dots$ then the left-to-right intervals will only cover the interval

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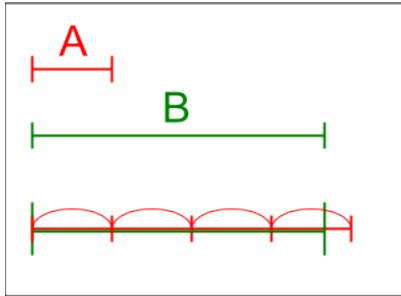
but the original intervals could cover since we can enumerate the rationals.

Therefore, measure must depend on properties of the reals not shared by the rationals.

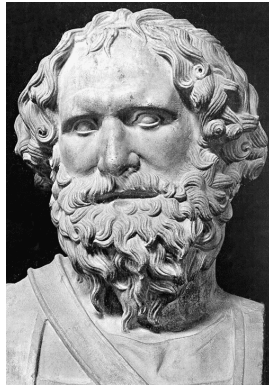
We want to define *length* for general sets of reals, i.e. \mathbb{Q} .

We want a notion of length that applies to all the sets we know and are used to and that is additive.

Brief history



Archimedean property



- 1872 – Weierstrass's construction of a nowhere differentiable function.
- 1881 – Introduction of functions of bounded variation by Jordan and later (1887) connection with rectifiability.
- 1883 – Cantor's ternary set.
- 1890 – Construction of a space-filling curve by Peano.
- 1898 – Borel's measurable sets.
- 1902 – Lebesgue's theory of measure and integration.
- 1905 – Construction of non-measurable sets by Vitali.
- 1906 – Fatou's application of Lebesgue theory to complex analysis.

Axiom of Choice

An arbitrary Cartesian product of non-empty sets remains non-empty.



Ernst Zermelo

If E is any subset of \mathbb{R}^d , the **exterior measure** of E is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.

A subset E of \mathbb{R}^d is **measurable** if for any $\epsilon > 0$ there exists an open set \mathcal{O} with $E \subset \mathcal{O}$ and

$$m_*(\mathcal{O} - E) \leq \epsilon$$

If E is measurable, we define its **measure** $m(E)$ by

$$m(E) = m_*(E)$$

Properties of measurable sets

Property 1 Every open set in \mathbb{R}^d is measurable.

Property 2 If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.

Property 3 A countable union of measurable sets is measurable.

Property 4 Closed sets are measurable.

Property 5 The complement of a measurable set is measurable.

Property 6 A countable intersection of measurable sets is measurable.

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⇒ The collection of measurable sets is closed under the familiar operations of set theory.

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(I) A σ -algebra \mathcal{M} of "measurable" sets, which is a non-empty collection of subsets of X closed under complements and countable unions and intersections.

(II) A measure $\mu : \mathcal{M} \rightarrow [0, \infty]$ with the following defining property: if E_1, E_2, \dots is a countable family of disjoint sets in \mathcal{M} , then

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n)$$

A **σ -algebra** of sets is a family of subsets of \mathbb{R}^d that is closed under countable unions, countable intersections, and complements.

The **Borel σ -algebra** in \mathbb{R}^d , denoted by $\mathcal{B}_{\mathbb{R}^d}$ is the smallest σ -algebra that contains all open sets. Elements of this σ -algebra are called **Borel sets**.

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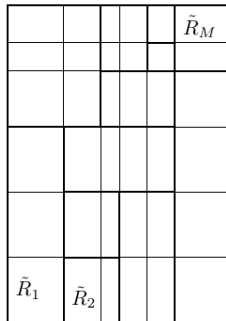
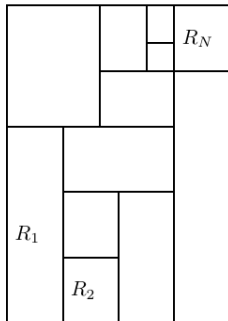
The Lebesgue measurable sets arise as the completion of the σ -algebra of Borel sets, by adjoining all subsets of Borel sets of measure zero.

Theorem: If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \bigcup_{k=1}^N R_k$, then

$$|R| = \sum_{k=1}^N |R_k|$$

Examples

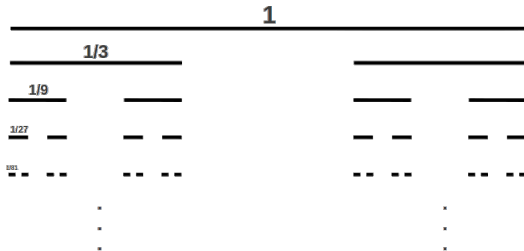
R



$$|R| = \sum_{j=1}^M |\tilde{R}_j| = \sum_{k=1}^N \sum_{j \in J_k} |\tilde{R}_j| = \sum_{k=1}^N |R_k|$$

Examples

The Cantor set \mathcal{C} has measure zero but is uncountable.



From the construction of \mathcal{C} , we know that $\mathcal{C} \subset C_k$, where each C_k is a disjoint union of 2^k closed intervals, each of length 3^{-k} .

Consequently, $m_*(\mathcal{C}) \leq (2/3)^k$ for all k , hence $m_*(\mathcal{C}) = 0$

Banach-Tarski paradox

The unit ball

$$B := \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 \leq 1\}$$

can be disassembled into a finite number of pieces and then be reassembled to form two disjoint copies of the ball B .

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These pathological decomposition pieces require the *axiom of choice*.

Rather than study every subset E of \mathbf{R}^d , we settle for only non-pathological subsets called measurable sets.

A **probability space** is a measure space $(\Omega, \mathcal{F}, \mathbf{P})$ of total measure 1 : $\mathbf{P}(\Omega) = 1$.

The measure \mathbf{P} is known as a probability measure.

Change in notation from (X, \mathcal{B}, μ) to $(\Omega, \mathcal{F}, \mathbf{P})$:

(i) The space Ω is known as the sample space

(ii) The σ -algebra \mathcal{F} is known as the event space

(iii) The measure $\mathbf{P}(E)$ of an event is known as the probability of that event.

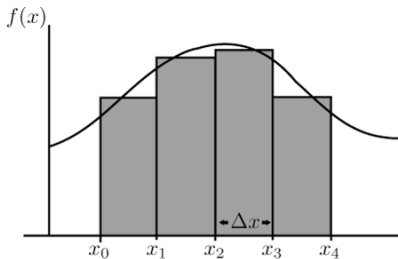
When we can construct measures on infinite-dimensional spaces, we can set up probability spaces associated to both discrete and continuous *random processes* even with infinite length.

Music and speech are *random processes*.

Applications



Georg Friedrich Bernhard Riemann



Lebesgue integral

$$\int_{\mathbb{R}^d} f(x)$$

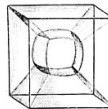
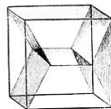
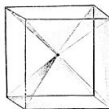
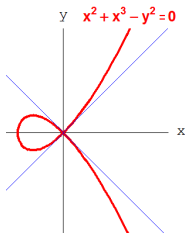
$$f : \mathbb{R}^d \rightarrow [0, +\infty]$$

A topology on X is a *Hausdorff space* when if $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$

Open problem: Find a topological characterization of Hausdorff topological spaces containing absolutely non-measurable subsets.

Open problems

Open problem: Are the tangent cones to area minimizing surfaces unique?



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